Analytic well-posedness of periodic gKdV

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Abstract

In the periodic case, it is proved that the Cauchy problem for the generalized Korteweg-de Vries equation (gKdV) is locally well-posed in a class of analytic functions that can be extended holomorphically in a symmetric strip of the complex plane around the $x-$axis. Thus, the uniform analyticity radius of the solution does not change as time progresses. Also, information about the regularity of the solution in the time variable is provided.

1 Introduction

For $k = 1, 2, 3, \ldots$ we consider the initial value problem for the generalized Korteweg-de Vries equation (gKdV)

$$\begin{cases} \partial_t u + \partial_x^3 u + u^k \partial_x u = 0, \quad x \in \mathbb{T}, \ t \in \mathbb{R}, \\ u(x,0) = \varphi(x), \end{cases} \tag{1.1}$$

with initial data $\varphi(x)$ belonging in a class of 2π-*periodic* analytic functions that can be extended holomorphically in a symmetric strip $S_\delta \doteq \{x + iy : |y| < \delta\}$, $\delta > 0$, of the complex plane around the $x-$axis. Our main result is that the gKdV equation is locally well-posed in these spaces. A consequence of this result is that the uniform analyticity radius of the solution $u(x,t)$ during its local lifespan is the same as that of the initial data $u(x,0)$. The motivation for this work comes from the results in [HHP2], where well-posedness of gKdV was studied in Gevrey and analytic spaces, but no precise information about the uniform analyticity radius of the solution was provided.

Concerning well-posedness for the *nonperiodic* gKdV equation, Kato and Masuda [KM] showed that if the initial data of the gKdV equation has an analytic continuation which is bounded in a strip containing the real axis, then the solution has the same property for all time, although the width of the strip might *decrease* with time. Results of this type have been also obtained by Hayashi [H].

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Using the analytic spaces $G^{\delta,s}$ introduced by Foias and Temam [FT] which are defined by the norm

$$||\varphi||^2_{G^{\delta,s}} = \int_{\mathbf{R}} (1+|\xi|)^{2s} e^{2\delta(1+|\xi|)} |\widehat{\varphi}(\xi)|^2 d\xi < \infty, \tag{1.2}$$

Grujić and Kalisch [GK] showed that for given initial data that are analytic in a symmetric strip $\{z = x+iy : |y| < \delta\}$ in the complex plane of width 2δ, there exists a time $T > 0$ such that the corresponding gKdV solution is analytic in the same strip during the time period $[0, T]$. In other words, in the nonperiodic case, the uniform radius of spatial analyticity does not shrink as time progresses. Further results on the uniform radius of spatial analyticity have been established by Bona, Grujić and Kalisch [BGK].

As we have mentioned earlier, in the periodic case, the Cauchy problem for gKdV was studied in [HHP2] using Bourgain type analytic and Gevrey spaces. However, the uniform radius of spatial analyticity obtained could decrease immediately after the initial time. In this paper, using a periodic version of the analytic spaces used in [GK] and a modified version of the Bourgain spaces used by Colliander, Keel, Staffilani, Takaoka, and Tao [CKSTT1], we show that the uniform radius of analyticity does not shrink as time progresses. This is a consequence of the following well-posedness result, which for simplicity we state for sufficiently small initial data so that the corresponding lifespan is equal to one.

Theorem 1.1 *Let $s \geq 1/2$ and $\delta > 0$. For small initial data in the space*

$$G^{\delta,s}(\mathbb{T}) = \{f \in L^2(\mathbb{T}) : ||f||^2_{G^{\delta,s}(\mathbb{T})} = \sum_{k\in\mathbb{Z}} |k|^{2s} e^{2\delta|k|} |\widehat{f}(k)|^2 < \infty\}, \tag{1.3}$$

the Cauchy problem for gKdV (1.1) *is locally well-posed in the space* $C([-1,1], G^{\delta,s}(\mathbb{T}))$.

Spacial analyticity of the periodic KdV equation was proved first by Trubowitz [Tr]. Construction of non-analytic solutions in time with analytic initial data for KdV have been obtained in [BH] and for gKdV in [GH2] and [HP]. When the initial data are analytic then the time-regularity of the solutions is of class Gevrey three. Furthermore, this result is sharp in the sense that there exist initial data that are analytic and 2π-*periodic* but the corresponding solution to (1.1) does belong to $G^r(\mathbb{R})$ for $1 \leq r < 3$ [HHP2]. New such examples are presented here using the spaces $G^{\delta,s}(\mathbb{T})$ defined by (1.3).

For additional results concerning well-posedness and regularity properties of gKdV we refer the reader to Bourgain [B], De Bouard, Hayashi and Kato [DHK], Kato [K], Kato and Masuda [KM], Kato and Ogawa [KO], Kenig, Ponce and Vega [KPV1], [KPV2] and the references therein.

Finally, we would like to point out that there are many works treating the analytic smoothing effect, that is to say, the phenomenon that solutions become analytic with respect to the spacial variable even if the initial data is not analytic but decays rapidly enough. For instance, Tarama [Ta] demonstrated this phenomenon for the KdV on the line. More precisely, he showed that if the initial data $\varphi \in L^2(\mathbb{R})$ satisfy the decay conditions

$$\int_{-\infty}^{\infty} (1+|x|)|\varphi(x)|\, dx < \infty \quad \text{and} \quad \int_0^{\infty} e^{-\delta|x|^{1/2}} |\varphi(x)|^2\, dx < \infty, \tag{1.4}$$

where δ is some positive number, then the corresponding KdV solution $u(x,t)$ becomes analytic with respect to the variable x for any $t>0$. In the periodic case the decay conditions (1.4) do not give any additional information for the initial data φ beyond that it belongs to $L^2[0,2\pi]$. Taking the initial data such that $\widehat{\varphi}(k)=1/(1+k^2)$ we have $\varphi\in L^2[0,2\pi]$. The solution of the linear part of KdV that corresponds to this initial data has Fourier transform with respect to x equal to $e^{ik^3t}/(1+k^2)$. Therefore, it is **not** analytic for any $t>0$. Adding the nonlinearity $u\partial_x u$ to this equation, which gives KdV, should not change this conclusion. For more information about analytic smoothing effect of dispersive equations and also for nonlinear dispersive equations we refer the reader to [DHK], [CKS], [HK1], [HK2] and [KO].

The paper is organized as follows. Section 2 is dedicated to the proof of Theorem 1.1. In subsection 2.1 we introduce the spaces needed and prove existence of solution. Then, in subsection 2.2 we prove uniqueness, and in subsection 2.3 we prove continuous dependence on initial data. In section 3 we demonstrate that the uniform radius of analyticity does not change as time progresses. Finally, in the last section we discuss regularity in the time variable.

2 Proof of Theorem 1.1

2.1 Existence

Taking the Fourier transform with respect to x in (1.1), solving the resulting differential equation in t and using the inverse Fourier transform reduces the Cauchy problem (1.1) to the following integral equation

$$u(x,t)=W(t)\varphi(x)-\int_0^t W(t-\tau)w(x,\tau)d\tau, \tag{2.5}$$

where $W(t)=e^{-t\partial_x^3}$ and $w=u^k\partial_x u$. Next we localize in the time variable by using a cut-off function $\psi(t)\in C_0^\infty(-2,2)$ with $0\le\psi\le 1$ and such that $\psi(t)\equiv 1$ for $|t|<1$. Multiplying (2.5) by ψ, we have

$$\psi(t)u(x,t)=\psi(t)W(t)\varphi(x)-\psi(t)\int_0^t W(t-\tau)w(x,\tau)d\tau, \tag{2.6}$$

which can be written as

$$\begin{aligned}\psi(t)u(x,t)&=\psi(t)\sum_{n\in\mathbb{Z}}e^{i(nx+n^3t)}\widehat{\varphi}(n)\\&\quad+i\psi(t)\sum_{n\in\mathbb{Z}}e^{i(nx+n^3t)}\int_{-\infty}^{\infty}\frac{e^{i(\lambda-n^3)t}-1}{\lambda-n^3}\widehat{w}(n,\lambda)d\lambda.\end{aligned} \tag{2.7}$$

Here, for simplicity, we shall restrict our attention to mean zero data

$$\widehat{\varphi}(0)=\frac{1}{2\pi}\int_{\mathbb{T}}\varphi(x)dx=0. \tag{2.8}$$

Next, we define the right hand-side of the integral equation (2.6) by Tu and decompose it as follows

$$Tu(x,t) = \psi(t) \sum_{n\in\mathbb{Z}^*} e^{i(nx+n^3t)} \widehat{\varphi}(n) \tag{2.9}$$

$$+i\sum_{j=1}^{\infty} \frac{i^j t^j}{j!} \psi(t) \sum_{n\in\mathbb{Z}^*} e^{i(nx+n^3t)} \int_{\mathbb{R}} \psi(\lambda - n^3)(\lambda - n^3)^{j-1} \widehat{w}(n,\lambda)\, d\lambda \tag{2.10}$$

$$+i\psi(t) \sum_{n\in\mathbb{Z}^*} e^{inx} \int_{\mathbb{R}} \frac{(1-\psi)(\lambda - n^3)}{\lambda - n^3} e^{i\lambda t} \widehat{w}(n,\lambda)\, d\lambda \tag{2.11}$$

$$-i\psi(t) \sum_{n\in\mathbb{Z}^*} e^{i(nx+n^3t)} \int_{\mathbb{R}} \frac{(1-\psi)(\lambda - n^3)}{\lambda - n^3} \widehat{w}(n,\lambda)\, d\lambda, \tag{2.12}$$

where

$$\widehat{w}(n,\lambda) = \widehat{u^k \partial_x u} \simeq (\underbrace{\widehat{u} * \widehat{u} \cdots * \widehat{u}}_{k} * \widehat{\partial_x u})(n,\lambda)$$

and

$$\mathbb{Z}^* = \mathbb{Z} \backslash \{0\}.$$

Now, our goal is to solve the equation $Tu = u$. We begin by defining the spaces needed. They can be thought as the periodic version of the spaces used in [GK].

The Spaces. Let $\delta > 0$, $s \geq 0$. We set

$$X_{\delta,s} = X_{\delta,s}(\mathbb{T}\times\mathbb{R}) = \{v \in L^2(\mathbb{T}\times\mathbb{R}) : ||v||^2_{X_{\delta,s}} < \infty\},$$

where

$$||v||^2_{X_{\delta,s}} = \sum_{k\in\mathbb{Z}} \int_{\mathbb{R}} (1 + |\tau - k^3|)|k|^{2s} e^{2\delta|k|} |\widehat{v}(k,\tau)|^2 d\tau.$$

The space $X_{\delta,s}(\mathbb{T}\times\mathbb{R})$ is the natural periodic extension of the space $X_{\delta,s,1/2}$ (see [GK]) for the nonperiodic case. It is well known that in the periodic case we really need $b = 1/2$ in order to prove multi-linear estimates in the spaces X_s, which correspond to the case $\delta = 0$ (see [KPV2]). But when $b = 1/2$ we no longer have the continuous embedding $X_{\delta,s}(\mathbb{T}\times\mathbb{R}) \hookrightarrow C([0,T], G^{\delta,s}(\mathbb{T}))$ used in [GK]. In order to fix it, we shall introduce the following family of spaces.

Definition 2.1 *Let $\delta > 0$, $s \geq 0$. We set*

$$Y_{\delta,s} = Y_{\delta,s}(\mathbb{T}\times\mathbb{R}) = \{v \in L^2(\mathbb{T}\times\mathbb{R}) : ||v||^2_{Y_{\delta,s}} < \infty\},$$

where

$$||v||_{Y_{\delta,s}} = ||v||_{X_{\delta,s}} + \Big(\sum_{k\in\mathbb{Z}} |k|^{2s} e^{2\delta|k|} \Big[\int_{\mathbb{R}} |\widehat{v}(k,\tau)| d\tau \Big]^2 \Big)^{\frac{1}{2}}.$$

Choosing $\delta = 0$ gives the Y_s spaces introduced by Colliander, Keel, Staffilani, Takaoka, and Tao for studying well-posedness of KdV, mKdV in [CKSTT1] and gKdV in [CKSTT2]. The spaces $Y_{\delta,s}$ possess the following important property.

Lemma 2.2 $Y_{\delta,s}(\mathbb{T}\times\mathbb{R}) \hookrightarrow C([-T,T], G^{\delta,s}(\mathbb{T}))$ *for any* $T>0$.

Proof. Let $T>0$ be given. Recalling that the norm of $u \in C([-T,T], G^{\delta,s}(\mathbb{T}))$ is given by

$$|u|_{C_{T,\delta,s}} = \sup_{|t|\le T} ||u(\cdot,t)||_{G^{\delta,s}(\mathbb{T})},$$

we have

$$\begin{aligned} ||u(\cdot,t)||_{G^{\delta,s}(\mathbb{T})} &= \Big(\sum_{k\in\mathbb{Z}} |k|^{2s} e^{2\delta|k|} \Big| \frac{1}{2\pi}\int_{\mathbb{R}} e^{i\tau t}\widehat{u}(k,\tau)d\tau\Big|^2\Big)^{\frac{1}{2}} \\ &\le \frac{1}{2\pi}\Big(\sum_{k\in\mathbb{Z}} |k|^{2s} e^{2\delta|k|}\Big(\int_{\mathbb{R}} |\widehat{u}(k,\tau)|d\tau\Big)^2\Big)^{\frac{1}{2}} \le \frac{1}{2\pi}||u||_{Y_{\delta,s}}. \end{aligned}$$

Thus, $||u(\cdot,t)||_{G^{\delta,s}(\mathbb{T})} \le \frac{1}{2\pi}||u||_{Y_{\delta,s}}, \ \forall\ t\in\mathbb{R}$, which implies that

$$|u|_{C_{T,\delta,s}} = \sup_{|t|\le T} ||u(\cdot,t)||_{G^{\delta,s}(\mathbb{T})} \le \frac{1}{2\pi}||u||_{Y_{\delta,s}}.$$

This completes the proof of Lemma 2.2. □

Next, computing the $Y_{\delta,s}$ norm of Tu, as in [GH1], we obtain the following result.

Lemma 2.3 *If* $s \ge 1/2$ *then there is a constant* $c_\psi > 0$ *such that*

$$\|Tu\|_{Y_{\delta,s}} \le c_\psi \left(\|w\|_{Z_{\delta,s}} + \|\varphi\|_{G^{\delta,s}(\mathbb{T})}\right), \quad \textit{for all} \ \ u \in Y_{\delta,s}, \tag{2.13}$$

where $w = u^k\partial_x u$ *and*

$$\begin{aligned} \|w\|_{Z_{\delta,s}} \doteq &\left(\sum_{n\in\mathbb{Z}^*} |n|^{2s} e^{2\delta|n|}\int_{\mathbb{R}} \frac{|\widehat{w}(n,\lambda)|^2}{1+|\lambda-n^3|}d\lambda\right)^{\frac{1}{2}} \\ &+ \left(\sum_{n\in\mathbb{Z}^*} |n|^{2s} e^{2\delta|n|}\left(\int_{\mathbb{R}} \frac{|\widehat{w}(n,\lambda)|}{1+|\lambda-n^3|}d\lambda\right)^2\right)^{\frac{1}{2}}. \end{aligned} \tag{2.14}$$

Proposition 2.4 *For* $s\ge 1/2$ *and* $v_1, v_2, ..., v_{k+1} \in Y_{\delta,s}$, *we have*

$$\|v_1\cdot v_2\cdot \ldots \cdot v_k \cdot \partial_x(v_{k+1})\|_{Z_{\delta,s}} \lesssim \|v_1\|_{Y_{\delta,s}}\|v_2\|_{Y_{\delta,s}}\cdots\|v_{k+1}\|_{Y_{\delta,s}}. \tag{2.15}$$

Proof. First, we observe that the operator A defined by

$$\widehat{Au}(n,\lambda) = e^{\delta|n|}\widehat{u}(n,\lambda)$$

satisfies the relation

$$||u||_{Y_{\delta,s}} = ||Au||_{Y_s}, \quad \text{for all } \ u \in Y_{\delta,s}. \tag{2.16}$$

Also, for any $v_1\cdot\ldots\cdot v_{k+1} \in Y_{\delta,s}$ it satisfies the relation

$$||v_1\cdot\ldots\cdot v_k\cdot\partial_x v_{k+1}||_{Z_{\delta,s}} = ||A(v_1\cdot\ldots\cdot v_k\cdot\partial_x v_{k+1})||_{Z_s}. \tag{2.17}$$

Therefore, to prove (2.15) it suffices to show the following inequality

$$||A(v_1 \cdot \ldots \cdot v_k \cdot \partial_x v_{k+1})||_{Z_s} \leq ||(Av_1) \cdot \ldots \cdot (Av_k) \cdot (\partial_x (Av_{k+1}))||_{Z_s}. \tag{2.18}$$

In fact, applying Proposition 1 from [CKSTT2] we obtain

$$||(Av_1) \cdot \ldots \cdot (Av_k) \cdot (\partial_x (Av_{k+1}))||_{Z_s} \leq C ||Av_1||_{Y_s} \cdot \ldots \cdot ||Av_{k+1}||_{Y_s}, \tag{2.19}$$

which combined with identity (2.16) gives the desired multilinear estimate (2.15).

Let us prove (2.18). Letting $w = v_1 \cdot \ldots \cdot v_k \cdot \partial_x v_{k+1}$ we have

$$\begin{aligned}
&||A(v_1 \cdot \ldots \cdot v_k \cdot \partial_x v_{k+1})||_{Z_s} = ||Aw||_{Z_s} \\
&= \left(\sum_{n \in \mathbb{Z}^*} |n|^{2s} \int_{\mathbb{R}} \frac{\left| \widehat{Aw}\,(n,\lambda) \right|^2}{1 + |\lambda - n^3|} d\lambda \right)^{\frac{1}{2}} + \left(\sum_{n \in \mathbb{Z}^*} |n|^{2s} \left(\int_{\mathbb{R}} \frac{\left| \widehat{Aw}\,(n,\lambda) \right|}{1 + |\lambda - n^3|} d\lambda \right)^2 \right)^{\frac{1}{2}} \\
&= \left(\sum_{n \in \mathbb{Z}^*} |n|^{2s} \int_{\mathbb{R}} \frac{\left| e^{\delta |n|} \widehat{w}\,(n,\lambda) \right|^2}{1 + |\lambda - n^3|} d\lambda \right)^{\frac{1}{2}} + \left(\sum_{n \in \mathbb{Z}^*} |n|^{2s} \left(\int_{\mathbb{R}} \frac{\left| e^{\delta |n|} \widehat{w}\,(n,\lambda) \right|}{1 + |\lambda - n^3|} d\lambda \right)^2 \right)^{\frac{1}{2}}.
\end{aligned}$$

Next, we analyze the first part of the last sum. The analysis of the second part is similar. By the definition of $\widehat{w}(n,\lambda) \simeq (\widehat{v_1} * \widehat{v_2} * \ldots * \widehat{v_k} * \widehat{\partial_x v_{k+1}})(n,\lambda)$ as a convolution we have

$$\begin{aligned}
&\left(\sum_{n \in \mathbb{Z}^*} |n|^{2s} \int_{\mathbb{R}} \frac{\left| e^{\delta |n|} \widehat{w}\,(n,\lambda) \right|^2}{1 + |\lambda - n^3|} d\lambda \right)^{\frac{1}{2}} \\
&= \Big(\sum_{n \in \mathbb{Z}^*} |n|^{2s} \int_{\mathbb{R}} \Big| e^{\delta |n|} \sum_{n_1 \in \mathbb{Z}} \int_{\mathbb{R}_1} \ldots \sum_{n_{k-1} \in \mathbb{Z}} \int_{\mathbb{R}_{k-1}} \sum_{n_k \in \mathbb{Z}} \int_{\mathbb{R}_k} \widehat{v_1}(n - n_1, \lambda - \lambda_1) \ldots \\
&\quad \widehat{v_k}(n_{k-1} - n_k, \lambda_{k-1} - \lambda_k) n_k \widehat{v_{k+1}}\,(n_k, \lambda_k)\, d\lambda_1 \ldots d\lambda_k \Big|^2 (1 + \left| \lambda - n^3 \right|)^{-1} d\lambda \Big)^{\frac{1}{2}} \\
&\leq \Big(\sum_{n \in \mathbb{Z}^*} |n|^{2s} \int_{\mathbb{R}} \Big| \sum_{n_1 \in \mathbb{Z}} \int_{\mathbb{R}_1} \ldots \sum_{n_{k-1} \in \mathbb{Z}} \int_{\mathbb{R}_{k-1}} \sum_{n_k \in \mathbb{Z}} \int_{\mathbb{R}_k} e^{\delta |n - n_1|} \widehat{v_1}(n - n_1, \lambda - \lambda_1) \ldots \\
&\quad e^{\delta |n_{k-1} - n_k|} \widehat{v_k}(n_{k-1} - n_k, \lambda_{k-1} - \lambda_k) n_k e^{\delta |n_k|} \widehat{v_{k+1}}\,(n_k, \lambda_k)\, d\lambda_1 \ldots d\lambda_k \Big|^2 \frac{d\lambda}{1 + |\lambda - n^3|} \Big)^{\frac{1}{2}} \\
&\leq \Big(\sum_{n \in \mathbb{Z}^*} |n|^{2s} \int_{\mathbb{R}} \Big| \sum_{n_1 \in \mathbb{Z}} \int_{\mathbb{R}_1} \ldots \sum_{n_{k-1} \in \mathbb{Z}} \int_{\mathbb{R}_{k-1}} \sum_{n_k \in \mathbb{Z}} \int_{\mathbb{R}_k} \widehat{Av_1}(n - n_1, \lambda - \lambda_1) \ldots \\
&\quad \widehat{Av_k}(n_{k-1} - n_k, \lambda_{k-1} - \lambda_k) n_k \widehat{Av_{k+1}}\,(n_k, \lambda_k)\, d\lambda_1 \ldots d\lambda_k \Big|^2 (1 + \left| \lambda - n^3 \right|)^{-1} d\lambda \Big)^{\frac{1}{2}} \\
&\leq \Big(\sum_{n \in \mathbb{Z}^*} |n|^{2s} \int_{\mathbb{R}} \Big| (\widehat{Av_1} * \ldots * \widehat{Av_k} * \partial_x \widehat{(Av_{k+1})})(n,\lambda) |^2 (1 + \left| \lambda - n^3 \right|)^{-1} d\lambda \Big)^{\frac{1}{2}}.
\end{aligned}$$

Letting $w_A = (Av_1) \cdot \ldots \cdot (Av_k) \cdot (\partial_x (Av_{k+1}))$, it follows from the above that

$$\left(\sum_{n \in \mathbb{Z}^*} |n|^{2s} \int_{\mathbb{R}} \frac{\left| e^{\delta |n|} \widehat{w}\,(n,\lambda) \right|^2}{1 + |\lambda - n^3|} d\lambda \right)^{\frac{1}{2}} \leq \Big(\sum_{n \in \mathbb{Z}^*} |n|^{2s} \int_{\mathbb{R}} \frac{|(\widehat{w_A}(n,\lambda)|^2}{1 + |\lambda - n^3|} d\lambda \Big)^{\frac{1}{2}}.$$

Similarly, we obtain

$$\left(\sum_{n\in\mathbb{Z}^*}|n|^{2s}\left(\int_{\mathbb{R}}\frac{\left|e^{\delta|n|}\widehat{w}\,(n,\lambda)\right|}{1+|\lambda-n^3|}d\lambda\right)^2\right)^{\frac{1}{2}}\leq\Big(\sum_{n\in\mathbb{Z}^*}|n|^{2s}\Big(\int_{\mathbb{R}}\frac{|(\widehat{w_A}(n,\lambda)|}{1+|\lambda-n^3|}d\lambda\Big)^2\Big)^{\frac{1}{2}}.$$

Adding the above two inequalities gives (2.18). This completes the proof of Proposition 2.4. □

Combining Lemma 2.3 and Proposition 2.4 gives the following result.

Proposition 2.5 *If $s\geq 1/2$ then there is a constant $c_\psi>0$ such that*

$$\|Tu\|_{Y_{\delta,s}}\leq c_\psi\|u\|^{k+1}_{Y_{\delta,s}}+c_\psi\|\varphi\|_{G^{\delta,s}(\mathbb{T})},\quad u\in Y_{\delta,s},\tag{2.20}$$

and

$$\|Tu-Tv\|_{Y_{\delta,s}}\leq c_\psi\Big(\sum_{\ell=0}^{k}\|u\|^{k-\ell}_{Y_{\delta,s}}\|v\|^{\ell}_{Y_{\delta,s}}\Big)\|u-v\|_{Y_{\delta,s}},\quad u,v\in Y_{\delta,s}.\tag{2.21}$$

Proof. Estimate (2.20) follows from (2.13) and (2.15) applied with $v_1=\cdots=v_{k+1}=u$. To prove estimate (2.21) we observe that

$$Tu-Tv=i\psi(t)\sum_{n\in\mathbb{Z}}e^{i(nx+n^3t)}\int_{\mathbb{R}}\frac{e^{i(\lambda-n^3)t}-1}{\lambda-n^3}\widehat{w}(n,\lambda)\;d\lambda,\tag{2.22}$$

where w now is given by

$$w=\frac{1}{k+1}\partial_x(u^{k+1}-v^{k+1})=\frac{1}{k+1}\partial_x\Big[(u-v)\sum_{\ell=0}^{k}u^{k-\ell}v^{\ell}\Big].\tag{2.23}$$

Thus, applying estimate (2.13) with $\varphi=0$ and multilinear estimate (2.15) for each one of the $(k+1)$ terms of the sum (2.23) we obtain (2.21) completing the proof of Proposition 2.5. □

The next proposition shows that our map T is in fact a contraction.

Proposition 2.6 *Let $s\geq 1/2$. For initial data φ satisfying the smallness condition*

$$||\varphi||_{G^{\delta,s}(\mathbb{T})}\leq\frac{3^k-1}{3^{k+1}c_\psi^{\frac{k+1}{k}}(k+1)^{\frac{k+1}{k}}}\tag{2.24}$$

if we choose the ball $\mathbb{B}(0,r)\doteq\{u\in Y_{\delta,s}:||u||_{Y_{\delta,s}}\leq r\}$ with radius $r=1/[3c_\psi^{1/k}(k+1)^{1/k}]$ then $T:\mathbb{B}(0,r)\to\mathbb{B}(0,r)$ is a contraction.

Proof: In fact, applying Proposition 2.5 we get

$$\begin{aligned}||Tu||_{Y_{\delta,s}}&\leq c_\psi||u||^{k+1}_{Y_{\delta,s}}+c_\psi||\varphi||_{G^{\delta,s}(\mathbb{T})}\\&\leq c_\psi\left(\frac{1}{3c_\psi^{\frac{1}{k}}(k+1)^{\frac{1}{k}}}\right)^{k+1}+c_\psi\left(\frac{3^k-1}{3^{k+1}c_\psi^{\frac{k+1}{k}}(k+1)^{\frac{k+1}{k}}}\right)\\&=\frac{c_\psi\cdot 3^k}{3^{k+1}c_\psi^{\frac{k+1}{k}}(k+1)^{\frac{k+1}{k}}}\leq\frac{1}{3c_\psi^{\frac{1}{k}}(k+1)^{\frac{1}{k}}}=r.\end{aligned}$$

Thus T maps $\mathbb{B}(0,r)$ into $\mathbb{B}(0,r)$. Also, it is a contraction, since

$$\begin{aligned}
||Tu - Tv||_{Y_{\delta,s}} &\leq c_\psi \Big(\sum_{\ell=0}^{k} ||u||_{Y_{\delta,s}}^{k-\ell} ||v||_{Y_{\delta,s}}^{\ell} \Big) ||u-v||_{Y_{\delta,s}} \\
&\leq c_\psi \left(\sum_{\ell=0}^{k} r^{k-\ell} r^{\ell} \right) ||u-v||_{Y_{\delta,s}} \\
&= c_\psi r^k (k+1) ||u-v||_{Y_{\delta,s}} \\
&= c_\psi \Big(\frac{1}{3 c_\psi^{\frac{1}{k}} (k+1)^{\frac{1}{k}}} \Big)^k (k+1) ||u-v||_{Y_{\delta,s}} \\
&= \Big(\frac{1}{3} \Big)^k ||u-v||_{Y_{\delta,s}}. \qquad \Box
\end{aligned}$$

End of Proof of Existence. By Proposition 2.6 we see that for $\|\varphi\|_{G^{\delta,s}(\mathbb{T})}$ sufficiently small, the operator T is a contraction on a small ball centered at the origin in $\|\cdot\|_{Y_{\delta,s}}$. Hence the transformation T has a unique fixed point u in a $\|\cdot\|_{Y_{\delta,s}}$-neighbourhood of 0. Since $\psi(t) = 1$, $|t| \leq 1$ it follows that $u(x,t)$ solves the gKdV initial value problem (1.1). Finally, thanks to Lemma 2.2, with $T = 1$, we have proved existence of a solution to our Cauchy problem which belongs to the space $C([-1,1], G^{\delta,s}(\mathbb{T}))$.

2.2 Uniqueness

Uniqueness of the solution in $C([-1,1], G^{\delta,s}(\mathbb{T}))$ can be proved by the following standard argument.

Lemma 2.7 *Suppose that $u, v \in C([-1,1], G^{\delta,s}(\mathbb{T}))$ are solutions to the gKdV Cauchy problem (1.1) with $u(\cdot,0) = v(\cdot,0)$ in $G^{\delta,s}(\mathbb{T})$ and $s \geq 1/2$. Then $u = v$.*

Proof. Setting $w = u - v$, we see that w solves the Cauchy problem

$$\partial_t w + \partial_x^3 w + \frac{1}{k+1} \partial_x (f w) = 0, \quad w(0) = 0, \tag{2.25}$$

where

$$f = u^k + u^{k-1} v + \cdots u v^{k-1} v + v^k. \tag{2.26}$$

Then using equation (2.25) we form the following identity for the L^2-energy of w

$$\frac{1}{2} \frac{d}{dt} ||w(t)||_{L^2(\mathbb{T})}^2 = - \int_{\mathbb{T}} w \partial_x^3 w dx - \frac{1}{k+1} \int_{\mathbb{T}} w \partial_x (f w) dx. \tag{2.27}$$

Integrating by parts we obtain that $\int_{\mathbb{T}} w \partial_x^3 w dx = 0$. Using this and again integrating by parts, from equation (2.27) we get

$$\frac{d}{dt} ||w(t)||_{L^2(\mathbb{T})}^2 = - \frac{1}{(k+1)} \int_{\mathbb{T}} \partial_x f \cdot w^2 dx,$$

from which we deduce the inequality

$$\Big|\frac{d}{dt}||w(t)||^2_{L^2(\mathbb{T})}\Big| \leq \frac{1}{(k+1)}\|\partial_x f\|_{L^\infty}\|w(t)\|^2_{L^2(\mathbb{T})}. \tag{2.28}$$

Since $u, v \in C([-1,1], G^{\delta,s}(\mathbb{T}))$ we have that u and v are continuous in t on the compact set $[-1,1]$ and are C^∞ in x on the compact torus. Thus the L^∞ norm of both u and v is finite. This implies that the L^∞ norm of $\partial_x f$ is finite, that is

$$\|\partial_x f(t)\|_{L^\infty} \leq c_0 < \infty. \tag{2.29}$$

Therefore, from (2.28) and (2.29) we obtain the differential inequality

$$\Big|\frac{d}{dt}||w(t)||^2_{L^2(\mathbb{T})}\Big| \leq c||w(t)||^2_{L^2(\mathbb{T})}, \quad |t| \leq 1, \tag{2.30}$$

where $c = \frac{c_0}{k+1}$. Solving it gives

$$||w(t)||^2_{L^2(\mathbb{T})} \leq e^c||w(0)||^2_{L^2(\mathbb{T})}, \quad |t| \leq 1. \tag{2.31}$$

Since $||w(0)||_{L^2(\mathbb{T})} = 0$, from (2.31) we obtain that $w(t) = 0$ or $u = v$. □

2.3 Continuous dependence of the initial data

The next result implies the continuity of the data-to-solution map.

Lemma 2.8 *Suppose that u and v are solutions to* (1.1) *corresponding to initial data φ and θ respectively with the norms $||\varphi||_{G^{\delta,s}(\mathbb{T})}, ||\theta||_{G^{\delta,s}(\mathbb{T})}$ small and $s \geq 1/2$. Then*

$$|u-v|_{C_1,\delta,s} \leq \frac{3c_\psi}{2}||\varphi-\theta||_{G^{\delta,s}(\mathbb{T})}.$$

Proof. We have

$$\begin{aligned} |u-v|_{C_1,\delta,s} &= \sup_{t\in[-1,1]} ||(u-v)(\cdot,t)||_{G^{\delta,s}(\mathbb{T})} \\ &\leq ||u-v||_{Y_{\delta,s}} = ||Tu-Tv||_{Y_{\delta,s}}, \end{aligned} \tag{2.32}$$

taking u and v in a small $||\cdot||_{Y_{\delta,s}}$- neighbourhood of 0 we have

$$||u-v||_{Y_{\delta,s}} = ||Tu-Tv||_{Y_{\delta,s}} \leq \frac{1}{3}||u-v||_{Y_{\delta,s}} + c_\psi||\varphi-\theta||_{G_{\delta,s}(\mathbb{T})}.$$

Thus,

$$||u-v||_{Y_{\delta,s}} \leq \frac{3}{2}c_\psi||\varphi-\theta||_{G_{\delta,s}(\mathbb{T})}. \tag{2.33}$$

It follows from (2.32) and (2.33) that

$$|u-v|_{C_T,\delta,s} \leq \frac{3}{2}c_\psi||\varphi-\theta||_{G_{\delta,s}(\mathbb{T})}$$

and therefore the proof of continuous dependence is complete. □

This completes de proof of Theorem 1.1.

3 Uniform Radius of Analyticity

Next we show that the radius of analyticity of the solution $u(\cdot,t)$ does not change as time progresses. If $\varphi \in G^{\delta,s}(\mathbb{T})$, then it follows from the definition of the space $G^{\delta,s}(\mathbb{T})$ that there exists a positive constant L such that the following inequality holds true

$$|\hat{\varphi}(n)| \leq Le^{-\delta|n|},\ \forall\ n \in \mathbb{Z}. \tag{3.34}$$

Then, from (3.34) it follows that $\psi(x) \doteq \sum_{n=0}^{\infty} \widehat{\varphi}(n)e^{inx} \in C^{\omega}(\mathbb{T})$. Furthermore, $\widehat{\psi}(n) = \widehat{\varphi}(n),\ \forall\ n \in \mathbb{Z}$. Since $\varphi \in L^2(\mathbb{T})$, we conclude that $\varphi \in C^{\omega}(\mathbb{T})$. We also have the following analytic continuation result.

Lemma 3.1 *If $\varphi \in G^{\delta,s}(\mathbb{T})$, then φ has an analytic extension in a symmetric strip around the real axis and its width is equal to δ.*

Proof. Since by (3.34) we can write

$$\varphi(x) = \sum_{n\in\mathbb{Z}} e^{inx}\hat{\varphi}(n),$$

we define

$$\tilde{\varphi}(x+iy) = \sum_{n\in\mathbb{Z}} e^{in(x+iy)}\hat{\varphi}(n) = \sum_{n\in\mathbb{Z}} e^{inx}e^{-yn}\hat{\varphi}(n),$$

which gives $\tilde{\varphi}(x+i0) = \varphi(x)$. Next we show that $\tilde{\varphi}$ is holomorphic in the strip $|y| < \delta$. In fact, given y such that $|y| < \delta$ there exists $L > 0$ such that

$$\begin{aligned} |\tilde{\varphi}(x+iy)| &\leq \sum_{n\in\mathbb{Z}} e^{-yn}|\hat{\varphi}(n)| \leq \sum_{n\in\mathbb{Z}} e^{|y||n|}Le^{-\delta|n|} \\ &= L\sum_{n\in\mathbb{Z}} e^{-(\delta-|y|)|n|} < \infty. \end{aligned}$$

Differentiating the series defining $\tilde{\varphi}(x+iy)$, we can show similarly that the resulting series converges absolutely. Therefore, we can apply the Cauchy-Riemann operator $\bar{\partial}$ term by term to see that $\bar{\partial}\tilde{\varphi} = 0$, which shows that $\tilde{\varphi}$ is analytic in $|y| < \delta$ and and $2\pi-$periodic in x. This completes the proof of Lemma 3.1. □

Following the lines of the proof of Lemma 3.1 one can easily show that the radius of analyticity of the solution $u(\cdot,t)$ does not change as time progresses.

4 Regularity in Time Variable

It follows from Theorem 1.1 that for the initial data $\varphi \in G^{\delta,s}(\mathbb{T})$, the solution $u(x,t)$ is analytic in $x \in \mathbb{T}$ and is only continuous in the time t variable. But this regularity can be improved by using the results proved in [HHP2], section 4. In fact, for initial data $\varphi \in G^{\delta,s}(\mathbb{T})$ the results in [HHP2] guarantee that the solution $u(x,t)$ is Gevrey of order 3 in the time variable t near zero, i.e., $u(x,\cdot) \in G^3$.

In [HP] non-analytic in time solutions to the Cauchy problem (1.1) with real-valued analytic initial data are constructed when k is not a multiple of four. Next, we provide a few new examples where the initial data are in $G^{\delta,s}(\mathbb{T})$. If for $k \in \{1,3,5,\ldots\}$ we define

$$\varphi(x) = -\mathrm{Re}\left(\sum_{n=1}^{\infty} e^{-2\delta n} e^{inx}\right),$$

and for $k = 4r+2$, $r = 0,1,2,\ldots$ we define

$$\psi(x) = \mathrm{Re}\left(i\sum_{n=1}^{\infty} e^{-2\delta n} e^{inx}\right),$$

then it is easily seen that φ and ψ belong to the space $G^{\delta,s}(\mathbb{T})$. Furthermore, following [HP] one can prove that the solutions to the Cauchy problem (1.1) with initial data φ and ψ, respectively, are not G^σ in t, for t near zero and for $1 \le \sigma < 3$.

For $k = 4\ell, \ell = 1,2,\ldots$ the following modification of an example given in [HHP1]

$$\theta(x) = i^{1/2\ell}\sum_{n=1}^{\infty} e^{-2\delta n} e^{inx}$$

provides initial data $\theta \in G^{\delta,s}(\mathbb{T})$. Now, following [HHP2] one can prove that the solution to the Cauchy problem (1.1) with initial data θ is not G^σ in t, for t near zero and for $1 \le \sigma < 3$.

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