Asymptotic spectrum for Dirichlet Laplacian in thin deformed tubes with scaled geometry

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Abstract

It is known that geometric properties does not play a role in the asymptotics of the eigenvalues of the Laplacian restricted to thin Dirichlet deformed tubular regions. In this work it is shown that, by properly scaling geometric properties of the tubes, a limit where the geometry is on equal footing with the deformation function is obtained; such result gives an indication of how subtle confining limits can be.

Short title: Deformed tubes with scaled geometry

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1 Introduction

The confinement of quantum particles has been studied in several situations [1, 3, 4, 5, 6, 7, 8, 9, 10, 15, 14], and a common procedure is to consider this particles in certain regions of the space $\mathbb{R}^3$ or the plane $\mathbb{R}^2$, which are naturally modeled by the Dirichlet Laplacian. In space it is common to consider tubular domains constructed in the following form. Let $I \subseteq \mathbb{R}$ be an interval and S an open, bounded, simply connected and nonempty subset of $\mathbb{R}^2$. Consider the domain Ω generated by the cross section S which rotates along a curve $r : I \to \mathbb{R}^3$ parametrized by arc-length x. Let $k(x)$ and $\tau(x)$ denote the curvature and torsion of $r(x)$ at the point x, respectively. At each point x the cross section S is rotated by an angle $\alpha(x)$. A situation of interest is to study what happens when the region Ω_ε is squeezed to the curve $r(x)$, i.e., one considers the sequence of tubes generated by the cross section εS and analyzes the behaviour of certain operators in the limit $\varepsilon \downarrow 0$. This procedure is a typical instance of *reduction of dimension*.

Potential applications of such confining limits include electronic motion in nanostructures, particularly in quantum wires and carbon nanotubes (see, for instance, [16]), the quantum studies of the motion of valence electrons in aromatic molecules [2], and studies of quantum graphs [11], in which the limiting dynamics to a graph is expected to be described by an effective Schrödinger operator. The results of this article give an indication of how subtle such confining limits and effective operators can be.

Let $-\Delta_\varepsilon$ be the Dirichlet Laplacian in Ω_ε, that is, its domain is the Sobolev space $\mathcal{H}_0^1(\Omega_\varepsilon)$. If I is a bounded interval, then $-\Delta_\varepsilon$ has compact resolvent and consequently its spectrum is discrete. Considering this case, in [3] it was analyzed the convergence of the eigenvalues $\{\lambda_i^\varepsilon : i \in \mathbb{N}\}$ of the problem

$$-\Delta_\varepsilon \psi_i^\varepsilon = \lambda_i^\varepsilon \psi_i^\varepsilon, \qquad \psi_i^\varepsilon \in \mathcal{H}_0^1(\Omega_\varepsilon), \tag{1}$$

and shown that

$$\lambda_i^\varepsilon = \frac{\nu_0}{\varepsilon^2} + \mu_i^\varepsilon, \quad \mu_i^\varepsilon \to \mu_i \quad (\varepsilon \to 0),$$

where ν_0 is the first eigenvalue of the Laplacian in $\mathcal{H}_0^1(S)$,

$$(-\Delta u_0)(y) = \nu_0 u_0(y), \quad u_0 \in \mathcal{H}_0^1(S), \quad \int_S |u_0|^2 \mathrm{d}y = 1, \tag{2}$$

and μ_i the eigenvalues of the one-dimensional operator

$$w(x) \mapsto -w''(x) + \left[(\tau + \alpha')^2(x) C(S) - \frac{k(x)^2}{4}\right] w(x), \tag{3}$$

acting in $\mathrm{L}^2(I)$. $C(S)$ is a non negative constant that depends on S. For unbounded tubes this situation is more delicate. For such tubes, in [14] it is shown that if $(\tau + \alpha')(x) = 0$ and $k(x) \neq 0$, then the discrete spectrum is nonempty, whereas if $(\tau + \alpha')(x) \neq 0$ and $k(x) = 0$, then the discrete spectrum is empty. In [5], by using Γ-convergence, a strong resolvent convergence was proven and the same action (3) for the respective effective operator (now acting in $\mathrm{L}^2(\mathbb{R})$) was found as $\varepsilon \to 0$.

By the following modification of the boundary $\partial\Omega_\varepsilon$ (which has been called a *deformed tube*), it is possible to ensure the existence of eigenvalues for problem (1) for both bounded and unbounded tubes. Let $J = [-a, b]$, $0 < a, b \leq \infty$ and $h(x) > 0$ be a continuous function satisfying (if J is unbounded, for simplicity we suppose $J = \mathbb{R}$):

(i) $h(x)$ is a C^1 function in $J \setminus \{0\}$ and $\|h'/h\|_\infty < \infty$;

(ii) near the origin h behaves as

$$h(x) = M - x^2 + O(|x|^3), \qquad M > 0, \tag{4}$$

and $x = 0$ is a single point of global maximum for h;

(iii) in case $J = \mathbb{R}$, it is assumed that $\limsup_{|x|\to\infty} h(x) < M$.

Again, let $r(x)$ be a curve in $\mathbb{R}^3$ parametrized by its arc-length x. In [7] it was considered the sequence of tubes generated by the cross section $\varepsilon h(x)S$ that rotates along $r(x)$. It was shown that, for ε small enough, the discrete spectrum of the Laplacian is always nonempty and its eigenvalues $\lambda_j(\varepsilon)$ have the following behaviour

$$\mu_j = \lim_{\varepsilon\to 0} \varepsilon \left(\lambda_j(\varepsilon) - \frac{\nu_0}{\varepsilon^2 M^2}\right),$$

where μ_j are the eigenvalues of the operator in $\mathrm{L}^2(\mathbb{R})$ (it acts on a subspace of $\mathrm{L}^2(\mathbb{R})$, independently if the interval J is bounded or not) given by

$$(Tw)(x) = -w''(x) + 2\frac{\nu_0}{M^3}\, x^2\, w(x). \tag{5}$$

In this circumstance, T has been called *weakly effective operator*.

Note that, in such situation, there is a competition between the tube deformation, represented by the function h, and the curvature, rotation and torsion of the reference curve r. The main contribution of [7] was the proof that the effects due to the tube deformation predominates so that the geometric quantities k, τ, α do not contribute to the weakly effective operator (5) in the limit $\varepsilon \to 0$.

It is a natural question if there are any particular situation where, even in the presence of tube deformation, such geometric features actually contribute to the effective operator. To present an affirmative answer to this point is the main goal of this work. We will combine the above construction with appropriate scaled geometry, as described in the sequel. In this vein, new families of effective operators and asymptotic limits of the Dirichlet Laplacian eigenvalues are obtained. For this, it is convenient to recall results of [1, 4, 8], where the authors have analyzed the problem of approximating a smooth quantum waveguide by a quantum graph.

In [1] it was considered a planar curve with compactly supported curvature $k(x)$ (which is nonzero only near $x = 0$) and a strip of constant width around the curve. It was constructed a sequence of strips depending on $\varepsilon > 0$ and whose curvature of each strip is rescaled of the form

$$k_\delta(x) := \frac{1}{\delta} k\left(\frac{x}{\delta}\right),$$

where $\delta = \delta(\varepsilon)$ and $\delta(\varepsilon) \to 0$ as $\varepsilon \to 0$. Thus, the strip can be approximated by a singular limit curve consisting of one vertex and two infinite, straight edges, i.e., a *broken line*. The continuations of the half-lines to the left and to the right of that support joint at the origin with an angle θ; this angle is exactly the integral

$$\theta = \int_{\mathbb{R}} k_\delta(x)\mathrm{d}x = \int_{\mathbb{R}} k(x)\mathrm{d}x. \tag{6}$$

The problem is reduced to the study of the sequence of one-dimensional operators

$$(H_\delta u)(x) := -u''(x) - \frac{k_\delta^2(x)}{4} u(x), \quad \operatorname{dom} H_\delta = \mathcal{H}^2(\mathbb{R}).$$

It turns out that details of the limit $\delta \to 0$ depend on a possible resonance of the operator $(\overline{H}u)(x) = -u''(x) - k^2(x)/4u(x)$ at zero or not [1, 4].

A similar situation was considered in [5], but for tubular regions in space; the curvature, torsion and the speed of the rotation angle α' were compactly supported in $(-1, 1)$ and scaled as

$$k_\delta(x) := \frac{1}{\delta} k\left(\frac{x}{\delta}\right), \quad \tau_\delta(x) = \frac{1}{\delta}\tau\left(\frac{x}{\delta}\right), \quad \alpha'_\delta(x) := \frac{1}{\delta}\alpha'\left(\frac{x}{\delta}\right), \tag{7}$$

$\delta = \delta(\varepsilon) \to 0$ as $\varepsilon \to 0$; condition (6) was also assumed to hold. The problem was again reduced to a sequence of one-dimensional operators [5]

$$(H_\delta u)(x) := -u''(x) + V_\delta(x)u(x), \quad \operatorname{dom} H_\delta = \mathcal{H}^2(\mathbb{R}),$$

with

$$V_\delta(x) := (\tau_\delta(x) - \alpha'_\delta(x))^2 \, C(S) - \frac{k_\delta(x)^2}{4}.$$

In this work we combine different situations: we consider the sequence of tubes Ω_ε defined at the beginning of this introduction, but we multiply its boundary $\partial\Omega_\varepsilon$ by the deformation function $h(x)$ (satisfying (i), (ii) and (iii) above), and simultaneously rescale the curvature, torsion and rotation angle as in (7). Specifically, we consider $\delta(\varepsilon) := \varepsilon^{1/2}$. Thus, we shall obtain a new sequence of tubes which we denote by Λ_ε (details appear in Section 2), and we study the spectral problem

$$-\Delta_\varepsilon \psi = l(\varepsilon)\psi, \qquad \psi \in \mathcal{H}_0^1(\Lambda_\varepsilon).$$

We will show that, for $\varepsilon > 0$ small enough, this problem admits eigenvalues and we find an asymptotic behaviour for them. Our main result may be stated as follows:

Theorem 1. *Assume that conditions* (I) *and* (II) *(see Section 3) hold true. Let $I = [-a, b]$, $0 < a, b \le \infty$. If $l_j(\varepsilon)$ denote the eigenvalues of the Dirichlet Laplacian $-\Delta_\varepsilon$ in the deformed tube Λ_ε, then, the limits*

$$\mu_j = \lim_{\varepsilon \to 0} \varepsilon \left(l_j(\varepsilon) - \frac{\nu_0}{\varepsilon^2 M^2} \right) \tag{8}$$

exist, where μ_j are the eigenvalues of the self-adjoint operator T acting in $\mathrm{L}^2(\mathbb{R})$, whose action is

$$(Tu)(x) = -u''(x) + \left(\frac{2\nu_0}{M^3}|x|^2 + C(S)\left(\tau + \alpha'\right)^2(x) - \frac{k^2(x)}{4M^2} \right) u(x).$$

Theorem 1 shows that appropriate rescaled curvature, torsion and rotation angle result in a weakly effective operator that depends on such geometric properties, as well as on the deformation function h.

In Section 2 we present a detailed construction of the sequence of tubes Λ_ε. Our main tools rely on a study of the sequence of quadratic forms

$$F_\varepsilon(\psi) = \int_{\Lambda_\varepsilon} \left(|\nabla\psi|^2 - \frac{\nu_0}{\varepsilon^2 M^2}|\psi|^2 \right) \mathrm{d}x, \quad \mathrm{dom}\, F_\varepsilon = \mathcal{H}_0^1(\Lambda_\varepsilon); \tag{9}$$

in the next sections it will become clear the technique used and why we subtract terms of the form $\nu_0/\varepsilon^2 M^2$ from the quadratic forms. In Section 3 we also perform a series of change of variables to simplify calculations and, among them, a change so that the integration region and the corresponding domains of (9) remain fixed. In Section 4 we show that the analysis can be restricted to a specific subspace and we will see that this subspace can be identified with the Sobolev space $\mathcal{H}^1(\mathbb{R})$. In Section 5 we complete the proof of Theorem 1, whereas some technicalities are presented in Appendix A.

Our results can be adapted to include more general deformation functions (as discussed in [9, 10]), such that near the unique global maximum at the origin they behave as

$$h(x) = \begin{cases} M - c_+|x|^m + O(|x|^{m+1}), & \text{if } x > 0; \\ M - c_-|x|^m + O(|x|^{m+1}), & \text{if } x < 0. \end{cases}$$

for some positive numbers $M, m, c_\pm$. For simplicity, in Eq. (4) we have chosen $m = 2$ and $c_+ = c_- = 1$.

Remark 1. *For bounded intervals $I = [-a, b]$, $0 < a, b < \infty$, it is also possible to consider the Laplacian in Λ_ε but with Neumann condition at the vertical part of the boundary $\partial\Lambda_\varepsilon$, that is, $(\{-a\} \times S) \cup (\{b\} \times S)\}$ (with Dirichlet condition otherwise), and the same conclusions of Theorem 1 are found. We will not discuss the proof here, since it can be adapted from the proof of Theorem 1 and ideas discussed in [10, 7].*

Remark 2. *A word on geometric interpretations. Theorem 1 gives the asymptotic behaviour of the eigenvalues of the Laplacian inside our tubes as $\varepsilon \to 0$. If both $\tau(x)$ and $\alpha'(x)$ are null functions, with $k(x) \neq 0$, then our curve is planar and approaches a broken line with the half-lines joined with angle θ as in (6). In case $\tau(x)$ is not the null function either, then for small $\varepsilon > 0$ the curve is approximately described by the same broken line as before, but the plane containing both half-lines rotates with angular velocity diverging as $\varepsilon \to 0$ (thus, up to rotations we again have a broken line). We have found no particular interpretation in case $\alpha'(x) \neq 0$.*

2 The tubes

Let $r : I \to \mathbb{R}^3$ be a simple C^3 curve in $\mathbb{R}^3$ parametrized by its arc-length parameter x. The curvature of r at the position x is defined by $k(x) :=$

$|r''(x)|$. We choose the orthonormal triad of vector fields $\{T, N, B\}$, called the tangent, normal and binormal vectors, respectively, moving along the curve and given by

$$T = r', \qquad N = k^{-1}T', \qquad B = T \times N. \tag{10}$$

To justify the construction (10), it is assumed that $k > 0$, but if r is a piece of a straight line (i.e., $k = 0$ identically in this piece), one can choose a constant Frenet frame instead. It is possible to combine constant Frenet frames with the Frenet frame (10) to include other types of curves, for instance, curves with $k > 0$ only on a compact interval of values of x (and so obtaining a global C^2 Frenet frame; see [13], Theorem 1.3.6).

In each situation above we assume that the Frenet frame exists and that the Frenet equations are satisfied, that is,

$$\begin{pmatrix} T' \\ N' \\ B' \end{pmatrix} = \begin{pmatrix} 0 & k & 0 \\ -k & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix} \begin{pmatrix} T \\ N \\ B \end{pmatrix}, \tag{11}$$

where τ is the torsion of r, actually defined by (11).

Now we rescale the curvature and torsion defining

$$k_\varepsilon(x) := \frac{1}{\varepsilon^{1/2}} k\left(\frac{x}{\varepsilon^{1/2}}\right), \quad \tau_\varepsilon(x) := \frac{1}{\varepsilon^{1/2}} \tau\left(\frac{x}{\varepsilon^{1/2}}\right), \quad \varepsilon > 0. \tag{12}$$

Let $r_\varepsilon(x) \in \mathbb{R}^3$ be the curve corresponding to k_ε and τ_ε. We denote by T_ε, N_ε and B_ε the tangent, normal and binormal vectors of the curve r_ε, respectively.

Let $\alpha : I \to \mathbb{R}$ be a bounded C^1 function so that $\alpha(0) = 0$. For each $\varepsilon > 0$ we consider the function $\alpha_\varepsilon : I \to \mathbb{R}$ so that its derivative is scaled as

$$\alpha'_\varepsilon(x) := \frac{1}{\varepsilon^{1/2}} \alpha'\left(\frac{x}{\varepsilon^{1/2}}\right). \tag{13}$$

Let S be an open, bounded, simply connected and nonempty subset of $\mathbb{R}^2$, and let $h(x)$ be the deformation function presented in the introduction of this work. For $\varepsilon > 0$ small enough and $y = (y_1, y_2) \in S$, write

$$\vec{\mathbf{x}}_\varepsilon(x, y) = r_\varepsilon(x) + \varepsilon\, h(x)\, y_1 N_{\alpha_\varepsilon}(x) + \varepsilon\, h(x)\, y_2 B_{\alpha_\varepsilon}(x)$$

and consider the domain

$$\Lambda_\varepsilon = \left\{ \vec{\mathbf{x}}_\varepsilon(x, y) \in \mathbb{R}^3 : x \in I, y = (y_1, y_2) \in S \right\},$$

where

$$\begin{aligned} N_{\alpha_\varepsilon}(x) &:= \cos\alpha_\varepsilon(x) N_\varepsilon(x) + \sin\alpha_\varepsilon(x) B_\varepsilon(x), \\ B_{\alpha_\varepsilon}(x) &:= -\sin\alpha_\varepsilon(x) N_\varepsilon(x) + \cos\alpha_\varepsilon(x) B_\varepsilon(x). \end{aligned}$$

This tube Λ_ε is obtained by putting the region $\varepsilon h(x)S$ along the curve $r_\varepsilon(x)$, which is simultaneously rotated by an angle $\alpha_\varepsilon(x)$ with respect to the cross section at the position $x = 0$.

In this work we study the behaviour of a free quantum particle that moves in Λ_ε, and with Dirichlet condition at the boundary $\partial\Lambda_\varepsilon$. Thus, we initially consider the family of quadratic forms

$$b_\varepsilon(\psi) := \int_{\Lambda_\varepsilon} |\nabla\psi|^2 \mathrm{d}x, \qquad \operatorname{dom} b_\varepsilon = \mathcal{H}_0^1(\Lambda_\varepsilon), \tag{14}$$

which is associated with the Dirichlet Laplacian operator $-\Delta_\varepsilon$ in Λ_ε. The symbol $\nabla = (\partial_x, \nabla_y)$, with $\nabla_y = (\partial_{y_1}, \partial_{y_2})$, denotes the gradient in the coordinates (x, y_1, y_2) in $\mathbb{R}^3$. Sometimes we denote $\partial_x\psi$ simply by ψ'.

3 Changes of variables

In this section we first perform a "traditional" change of variables so that the integration region in (14), and consequently the domains of the quadratic forms, become independent of $\varepsilon > 0$. For the singular limit $\varepsilon \to 0$, customary "regularizations" will be employed. In this process we propose here additional change of variables we have found handy for the specific problem we consider in this work; the final result is the expression for the quadratic form t_ε at the end of this section.

Consider the mapping

$$\begin{array}{rccl} f_\varepsilon: & I \times S & \to & \Lambda_\varepsilon \\ & (x, y_1, y_2) & \mapsto & r_\varepsilon(x) + \varepsilon\, h(x)\,(y_1 N_{\alpha_\varepsilon}(x) + y_2 B_{\alpha_\varepsilon}(x)), \end{array}$$

and suppose the boundedness $\|k\|_\infty < \infty$. This condition is to guarantee that f_ε will be a diffeomorphism for small ε. With this change of variables we work with a fixed region for all $\varepsilon > 0$; more precisely, in the new variables the domain of the quadratic form (14) turns out to be $\mathcal{H}_0^1(I \times S)$. On the other hand, the price to be paid is a nontrivial Riemannian metric $G = G_\varepsilon^{\alpha_\varepsilon}$ which is induced by f_ε, i.e.,

$$G = (G_{ij}), \qquad G_{ij} = \langle e_i, e_j\rangle = G_{ji}, \qquad 1 \le i, j \le 3,$$

where

$$e_1 = \frac{\partial f_\varepsilon}{\partial x}, \qquad e_2 = \frac{\partial f_\varepsilon}{\partial y_1}, \qquad e_3 = \frac{\partial f_\varepsilon}{\partial y_2}.$$

Some calculations show that in the Frenet frame

$$J = \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} = \begin{pmatrix} \beta_\varepsilon & \sigma_\varepsilon & \delta_\varepsilon \\ 0 & \varepsilon h \cos\alpha_\varepsilon & \varepsilon h \sin\alpha_\varepsilon \\ 0 & -\varepsilon h \sin\alpha_\varepsilon & \varepsilon h \cos\alpha_\varepsilon \end{pmatrix},$$

where $\beta_\varepsilon(x,y) = 1 - \varepsilon^{1/2} h(x) k\left(x/\varepsilon^{1/2}\right) \langle z_{\alpha_\varepsilon}, y\rangle$, and

$$\begin{array}{rcl} \sigma_\varepsilon(x,y) & = & -\varepsilon h(x)(\tau_\varepsilon + \alpha_\varepsilon')(x)\langle z_{\alpha_\varepsilon}^\perp(x), y\rangle + \varepsilon h'(x)\langle z_{\alpha_\varepsilon}(x), y\rangle, \\ \delta_\varepsilon(x,y) & = & \varepsilon h(x)(\tau_\varepsilon + \alpha_\varepsilon')(x)\langle z_{\alpha_\varepsilon}(x), y\rangle + \varepsilon h'(x)\langle z_{\alpha_\varepsilon}^\perp(x), y\rangle, \\ z_{\alpha_\varepsilon}(x) & = & (\cos\alpha_\varepsilon(x), -\sin\alpha_\varepsilon(x)), \quad z_{\alpha_\varepsilon}^\perp(x) = (\sin\alpha_\varepsilon(x), \cos\alpha_\varepsilon(x)). \end{array}$$

The inverse matrix of J is given by

$$J^{-1} = \begin{pmatrix} \beta_\varepsilon^{-1} & \tilde{\sigma_\varepsilon} & \tilde{\delta}_\varepsilon \\ 0 & (\varepsilon h)^{-1}\cos\alpha_\varepsilon & -(\varepsilon h)^{-1}\sin\alpha_\varepsilon \\ 0 & (\varepsilon h)^{-1}\sin\alpha_\varepsilon & (\varepsilon h)^{-1}\cos\alpha_\varepsilon \end{pmatrix},$$

where

$$\tilde{\sigma}_\varepsilon(x,y) = \frac{1}{\beta_\varepsilon(x,y)}\left[(\tau_\varepsilon + \alpha_\varepsilon')(x) y_2 - \frac{h'(x)}{h(x)} y_1\right],$$

$$\tilde{\delta}_\varepsilon(x,y) = \frac{1}{\beta_\varepsilon(x,y)}\left[-(\tau_\varepsilon + \alpha_\varepsilon')(x) y_1 - \frac{h'(x)}{h(x)} y_2\right].$$

Note that $JJ^t = G$ and $\det J = |\det G|^{1/2} = \varepsilon^2 h^2(x)\beta_\varepsilon(x,y)$. Since k and h are bounded functions, for $\varepsilon > 0$ small enough, β_ε does not vanish in $I \times S$. Thus, $\beta_\varepsilon > 0$ and f_ε is a local diffeomorphism. By requiring that f_ε is injective (that is, the tube is not self-intersecting), a global diffeomorphism is obtained.

Introducing the notation

$$||\psi||_G^2 := \int_{I\times S} |\psi(x,y)|^2 \varepsilon^2 h^2(x)\beta_\varepsilon(x,y)\,\mathrm{d}x\mathrm{d}y,$$

we obtain a sequence of quadratic forms

$$\tilde{b}_\varepsilon(\psi) := \left|\left|J^{-1}\nabla\psi\right|\right|_G, \qquad \mathrm{dom}\,\tilde{b}_\varepsilon = \mathcal{H}_0^1(I \times S, G).$$

More precisely, the above change of coordinates was obtained by a unitary transformation

$$\begin{array}{rccc} U_\varepsilon : & \mathrm{L}^2(\Lambda_\varepsilon) & \to & \mathrm{L}^2(I\times S, G) \\ & \phi & \mapsto & \phi\circ f_\varepsilon \end{array}.$$

However, we still denote $U_\varepsilon\psi$ by ψ.

Recall that ν_0 is the lowest eigenvalue of the negative Laplacian with Dirichlet boundary conditions in the cross section region S, and $u_0 \geq 0$ the corresponding eigenfunction of this restricted problem. This eigenfunction u_0 is directly related to transverse oscillations in Λ_ε. Due to this fact, in [3, 5] the authors have removed the diverging energy ν_0/ε^2 from their quadratic forms. In our case, as the boundary of the tubes were multiplied by $h(x)$, we shall subtract terms of the form $\nu_0/(\varepsilon M)^2$, i.e., since $0 < h(x) \leq M$, for all $x \in I$, we eliminate the "least transverse energy" (a normalization).

Therefore, we turn to the study of the sequence of quadratic forms

$$\tilde{t}_\varepsilon(\psi) := \left(\left|\left|J^{-1}\nabla\psi\right|\right|_G^2 - \frac{\nu_0}{\varepsilon^2 M^2}\|\psi\|_G^2 + \frac{c}{\varepsilon}\|\psi\|_G^2 \right),$$

where c is a positive constant to be conveniently chosen later on. After the norms are written out, we obtain

$$\begin{aligned} \tilde{t}_\varepsilon(\psi) &= \varepsilon^2 \int_{I\times S} \Big(\frac{1}{\beta_\varepsilon^2(x,y)} \left| \psi' + \nabla_y\psi \cdot Ry\,(\tau_\varepsilon + \alpha_\varepsilon')(x) - \nabla_y\psi \cdot y\,\frac{h'(x)}{h(x)} \right|^2 \\ &+ \frac{|\nabla_y\psi|^2}{\varepsilon^2 h(x)^2} - \frac{\nu_0}{\varepsilon^2 M^2}|\psi|^2 + \frac{c}{\varepsilon}|\psi|^2 \Big) h(x)^2 \beta_\varepsilon(x,y)\,\mathrm{d}x\mathrm{d}y\,, \end{aligned}$$

where R is the rotation matrix $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Note that $\operatorname{dom}\tilde{t}_\varepsilon = \mathcal{H}_0^1(I\times S)$ is a subspace of $\mathrm{L}^2(I\times S, h(x)^2\beta_\varepsilon(x,y))$.

Consider the isometry

$$\begin{array}{ccc} \mathrm{L}^2(I\times S,\beta_\varepsilon) & \to & \mathrm{L}^2(I\times S, h(x)^2\beta_\varepsilon) \\ v & \mapsto & vh^{-1} \end{array}.$$

This change of variables and the division by the global factor ε^2 (a common singular factor due to the "change of dimension" as $\varepsilon \to 0$) leads to

$$\begin{aligned} \check{t}_\varepsilon(v) &:= \int_{I\times S} \Big(\frac{1}{\beta_\varepsilon(x,y)} \left| v' - v\,\frac{h'(x)}{h(x)} + \nabla_y v \cdot Ry\,(\tau_\varepsilon + \alpha_\varepsilon')(x) - \nabla_y v \cdot y\,\frac{h'(x)}{h(x)} \right|^2 \\ &+ \frac{\beta_\varepsilon(x,y)}{\varepsilon^2 h(x)^2}|\nabla_y v|^2 - \nu_0 \frac{\beta_\varepsilon(x,y)}{\varepsilon^2 M^2}|v|^2 + \frac{c}{\varepsilon}\beta_\varepsilon(x,y)|v|^2 \Big)\,\mathrm{d}x\mathrm{d}y, \end{aligned}$$

with $\operatorname{dom}\check{t}_\varepsilon = \mathcal{H}_0^1(I\times S)$ as a subspace of $\mathrm{L}^2(I\times S, \beta_\varepsilon(x,y))$.

Now we consider an additional change of variables and so we get a quadratic form defined in $\mathrm{L}^2(I\times S)$. We consider the isometry

$$\begin{array}{ccc} \mathrm{L}^2(I\times S) & \to & \mathrm{L}^2(I\times S,\beta_\varepsilon(x,y)) \\ \psi & \mapsto & \beta_\varepsilon^{-1/2}\psi \end{array},$$

so that in the new variables the above quadratic form reads

$$\begin{aligned} \hat{t}_\varepsilon(\psi) = \int_{I\times S} \frac{1}{\beta_\varepsilon^2} \Big| &\frac{\partial\psi}{\partial x} + \psi\,\beta_\varepsilon^{1/2}\frac{\partial}{\partial x}\left(1/\beta_\varepsilon^{1/2}\right) - \psi\frac{h'(x)}{h(x)} + \nabla_y\psi \cdot Ry\,(\tau_\varepsilon + \alpha_\varepsilon')(x) \\ &+ \psi\,\beta_\varepsilon^{1/2}\,\nabla_y\left(1/\beta_\varepsilon^{1/2}\right)\cdot Ry\,(\tau_\varepsilon + \alpha_\varepsilon') - \nabla_y\psi \cdot y\,\frac{h'(x)}{h(x)} - \psi\,\beta_\varepsilon^{1/2}\nabla_y\left(1/\beta_\varepsilon^{1/2}\right)\cdot y\frac{h'(x)}{h(x)} \Big|^2 \mathrm{d}x\mathrm{d}y \\ &+ \int_{I\times S} \frac{\beta_\varepsilon}{\varepsilon^2 h(x)^2} \left(\left|\nabla_y\left(\psi/\beta_\varepsilon^{1/2}\right)\right|^2 - \nu_0\left|\psi/\beta_\varepsilon^{1/2}\right|^2 \right)\mathrm{d}x\mathrm{d}y \end{aligned}$$

$$+\int_{I\times S}\frac{\nu_0}{\varepsilon^2}\left(\frac{1}{h(x)^2}-\frac{1}{M^2}\right)|\psi|^2\mathrm{d}x\mathrm{d}y+\int_{I\times S}\frac{c}{\varepsilon}|\psi|^2\mathrm{d}x\mathrm{d}y,$$

and $\operatorname{dom}\hat{t}_\varepsilon=\mathcal{H}_0^1(I\times S)$ is a subset of $\mathrm{L}^2(I\times S)$. Observe that, beyond the change of variables, we have added and subtracted the term $\nu_0/\varepsilon^2 h(x)^2|\psi|^2$ in $\hat{t}_\varepsilon$. This will also be convenient later on.

Now we are going to rewrite the second integral in the definition of $\hat{t}_\varepsilon$ above. This step is very important because in this way the curvature will appear explicitly in the quadratic form. First observe that

$$\int_S\frac{\beta_\varepsilon}{\varepsilon^2}\left[\frac{\partial}{\partial y_1}\left(\psi/\beta_\varepsilon^{1/2}\right)\right]^2\mathrm{d}y$$

$$=\int_S\left[\frac{1}{\varepsilon^2}\left(\frac{\partial\psi}{\partial y_1}\right)^2+\frac{\beta_\varepsilon}{\varepsilon^2}\left[\frac{\partial}{\partial y_1}\left(\frac{1}{\beta_\varepsilon^{1/2}}\right)\right]^2|\psi|^2+2\frac{\beta_\varepsilon^{1/2}}{\varepsilon^2}\frac{\partial}{\partial y_1}\left(\frac{1}{\beta_\varepsilon^{1/2}}\right)\frac{\partial\psi}{\partial y_1}\psi\right]\mathrm{d}y$$

$$=\int_S\left[\frac{1}{\varepsilon^2}\left(\frac{\partial\psi}{\partial y_1}\right)^2-\frac{k_\varepsilon^2(x)}{4}\frac{\cos^2\alpha_\varepsilon(x)}{\beta_\varepsilon^2}|\psi|^2\right]\mathrm{d}y$$

because an integration by parts shows that

$$\int_S\frac{\beta_\varepsilon}{\varepsilon^2}\left[\frac{\partial}{\partial y_1}\left(\frac{1}{\beta_\varepsilon^{1/2}}\right)\right]^2|\psi|^2\mathrm{d}y$$

$$=\int_S\left[-\frac{k_\varepsilon^2(x)}{4}\frac{\cos^2\alpha_\varepsilon(x)}{\beta_\varepsilon^2}|\psi|^2-2\frac{\beta_\varepsilon^{1/2}}{\varepsilon^2}\frac{\partial}{\partial y_1}\left(\frac{1}{\beta_\varepsilon^{1/2}}\right)\frac{\partial\psi}{\partial y_1}\psi\right]\mathrm{d}y.$$

Similarly

$$\int_S\frac{\beta_\varepsilon}{\varepsilon^2}\left[\frac{\partial}{\partial y_2}\left(\psi/\beta_\varepsilon^{1/2}\right)\right]^2\mathrm{d}y=\int_S\left[\frac{1}{\varepsilon^2}\left(\frac{\partial\psi}{\partial y_2}\right)^2-\frac{k_\varepsilon^2(x)}{4}\frac{\sin^2\alpha_\varepsilon(x)}{\beta_\varepsilon^2}|\psi|^2\right]\mathrm{d}y.$$

Therefore,

$$\int_{I\times S}\frac{\beta_\varepsilon}{\varepsilon^2}\left(\left|\nabla_y\left(\psi/\beta_\varepsilon^{1/2}\right)\right|^2-\nu_0\left|\psi/\beta_\varepsilon^{1/2}\right|^2\right)\mathrm{d}x\mathrm{d}y$$

$$=\int_{I\times S}\left[\frac{1}{\varepsilon^2}(|\nabla_y\psi|^2-\nu_0|\psi|^2)-\frac{k_\varepsilon^2(x)}{4\beta_\varepsilon^2}|\psi|^2\right]\mathrm{d}x\mathrm{d}y.$$

Finally, the quadratic form $\hat{t}_\varepsilon$ can be rewritten as

$$\hat{t}_\varepsilon(\psi)=\int_{I\times S}\frac{1}{\beta_\varepsilon^2}\left|\frac{\partial\psi}{\partial x}+\psi\,\beta_\varepsilon^{1/2}\frac{\partial}{\partial x}\left(1/\beta_\varepsilon^{1/2}\right)-\psi\frac{h'(x)}{h(x)}+\nabla_y\psi\cdot Ry\,(\tau_\varepsilon+\alpha_\varepsilon')(x)\right.$$

$$\left.+\psi\,\beta_\varepsilon^{1/2}\nabla_y\left(1/\beta_\varepsilon^{1/2}\right)\cdot Ry\,(\tau_\varepsilon+\alpha_\varepsilon')-\nabla_y\psi\cdot y\,\frac{h'(x)}{h(x)}-\psi\beta_\varepsilon^{1/2}\nabla_y\left(1/\beta_\varepsilon^{1/2}\right)\cdot y\frac{h'(x)}{h(x)}\right|^2\mathrm{d}x\mathrm{d}y$$

$$
+\int_{I\times S}\left(\frac{|\nabla_y\psi|^2}{\varepsilon^2 h^2(x)}-\nu_0\frac{|\psi|^2}{\varepsilon^2 h^2(x)}\right)\mathrm{d}x\mathrm{d}y-\int_{I\times S}\frac{k_\varepsilon^2(x)}{4\beta_\varepsilon^2 h^2(x)}|\psi|^2\mathrm{d}x\mathrm{d}y
$$

$$
+\int_{I\times S}\frac{\nu_0}{\varepsilon^2}\left(\frac{1}{h(x)^2}-\frac{1}{M^2}\right)|\psi|^2\mathrm{d}x\mathrm{d}y+\int_{I\times S}\frac{c}{\varepsilon\beta_\varepsilon^2}|\psi|^2\mathrm{d}x\mathrm{d}y,
$$

and with $\operatorname{dom}\hat{t}_\varepsilon=\mathcal{H}_0^1(I\times S)$.

By replacing the global multiplicative factors β_ε by the constant 1 in the first, third and last integral in the expression of $\hat{t}_\varepsilon$, we arrive now at the quadratic form

$$
\begin{aligned}
t_\varepsilon(\psi) \;&=\; \int_{I\times S}\Bigg|\frac{\partial\psi}{\partial x}+\psi\,\beta_\varepsilon^{1/2}\frac{\partial}{\partial x}\left(1/\beta_\varepsilon^{1/2}\right)-\psi\frac{h'(x)}{h(x)}+\nabla_y\psi\cdot Ry\,(\tau_\varepsilon+\alpha_\varepsilon')(x)\\
&+\;\psi\,\beta_\varepsilon^{1/2}\nabla_y\left(1/\beta_\varepsilon^{1/2}\right)\cdot Ry\,(\tau_\varepsilon+\alpha_\varepsilon')\\
&-\;\nabla_y\psi\cdot y\,\frac{h'(x)}{h(x)}-\psi\beta_\varepsilon^{1/2}\nabla_y\left(1/\beta_\varepsilon^{1/2}\right)\cdot y\frac{h'(x)}{h(x)}\Bigg|^2\mathrm{d}x\mathrm{d}y\\
&+\;\int_{I\times S}\left(\frac{|\nabla_y\psi|^2}{\varepsilon^2 h^2(x)}-\nu_0\frac{|\psi|^2}{\varepsilon^2 h^2(x)}\right)\mathrm{d}x\mathrm{d}y-\int_{I\times S}\frac{k_\varepsilon^2(x)}{4h^2(x)}|\psi|^2\mathrm{d}x\mathrm{d}y\\
&+\;\int_{I\times S}\frac{\nu_0}{\varepsilon^2}\left(\frac{1}{h(x)^2}-\frac{1}{M^2}\right)|\psi|^2\mathrm{d}x\mathrm{d}y+\int_{I\times S}\frac{c}{\varepsilon}|\psi|^2\mathrm{d}x\mathrm{d}y,
\end{aligned}
\tag{15}
$$

$\operatorname{dom} t_\varepsilon=\mathcal{H}_0^1(I\times S)$.

We add the condition $h^{-1}\in L^\infty$. If one chooses c so that $\|k_\varepsilon^2/(4h^2)\|_\infty < c/\varepsilon$, the quadratic forms $\hat{t}_\varepsilon$ and t_ε become positive. Let $\hat{T}_\varepsilon$ and T_ε the self-adjoint operators associated with $\hat{t}_\varepsilon$ and t_ε, respectively.

Until now we only suppose that $k, h^{-1}\in \mathrm{L}^\infty(I)$. For the next results we are going to add the following hypotheses:

(I) k, k', α, α', α'', τ, $\tau'\in\mathrm{L}^\infty(I)$;

(II) $h(x), h^{-1}(x), \dfrac{h'(x)}{h(x)}, \dfrac{\partial}{\partial x}\left(\dfrac{h'(x)}{h(x)}\right)\in\mathrm{L}^\infty(I)$.

Theorem 2. *Suppose that conditions* (I) *and* (II) *are satisfied. Then, there exist numbers $C, C'>0$ so that*

$$
(1-C\varepsilon^{1/2})\,t_\varepsilon(\psi)\le\hat{t}_\varepsilon(\psi)\le(1+C\varepsilon^{1/2})\,t_\varepsilon(\psi),\tag{16}
$$

for all $\psi\in\mathcal{H}_0^1(I\times S)$, and

$$
\|\hat{T}_\varepsilon^{-1}-T_\varepsilon^{-1}\|\le C'\,\varepsilon^{1/2},\tag{17}
$$

for $\varepsilon>0$ small enough.

Proof. First observe that there exists a number $c_1 > 0$ so that

$$t_\varepsilon(\psi) \geq \int_{I\times S} \left(-\frac{k_\varepsilon^2(x)}{4h(x)} + \frac{c}{\varepsilon} \right) |\psi|^2 \mathrm{d}x\mathrm{d}y \geq \frac{c_1}{\varepsilon} \int_{I\times S} |\psi|^2 \mathrm{d}x\mathrm{d}y,$$

for all $\psi \in \mathcal{H}_0^1(I \times S)$ (just take $c > 0$ large enough).

Due to conditions (I) and (II) there exists $C_1 > 0$ so that

$$\left| \frac{1}{\beta_\varepsilon^2(x,y)} - 1 \right| \leq C_1 \varepsilon^{1/2}, \quad \forall (x,y) \in I \times S.$$

Thus, there exist numbers $C_2, C > 0$ so that

$$\begin{aligned}
|\hat{t}_\varepsilon(\psi) - t_\varepsilon(\psi)| \quad &\leq \quad \int_{I\times S} \left| \frac{1}{\beta_\varepsilon^2} - 1 \right| \left| \frac{\partial \psi}{\partial x} + \psi\, \beta_\varepsilon^{1/2} \frac{\partial}{\partial x} \left(1/\beta_\varepsilon^{1/2} \right) - \psi \frac{h'(x)}{h(x)} \right. \\
&+ \quad \nabla_y \psi \cdot Ry\, (\tau_\varepsilon + \alpha_\varepsilon')(x) + \psi\, \beta_\varepsilon^{1/2} \nabla_y \left(1/\beta_\varepsilon^{1/2} \right) \cdot Ry\, (\tau_\varepsilon + \alpha_\varepsilon') \\
&- \quad \left. \nabla_y \psi \cdot y\, \frac{h'(x)}{h(x)} - \psi \beta_\varepsilon^{1/2} \nabla_y \left(1/\beta_\varepsilon^{1/2} \right) \cdot y \frac{h'(x)}{h(x)} \right|^2 \mathrm{d}x\mathrm{d}y \\
&+ \quad \int_{I\times S} \left| \frac{c}{\varepsilon} - \frac{k_\varepsilon^2(x)}{4h(x)} \right| \left| \frac{1}{\beta_\varepsilon^2} - 1 \right| |\psi|^2 \mathrm{d}x\mathrm{d}y \\
&\leq \quad C_1\, \varepsilon^{1/2}\, t_\varepsilon(\psi) + C_2\, \varepsilon^{1/2} \int_{I\times S} \frac{|\psi|^2}{\varepsilon} \mathrm{d}x\mathrm{d}y \\
&\leq \quad C_1\, \varepsilon^{1/2}\, t_\varepsilon(\psi) + \frac{C_2}{c_1}\, \varepsilon^{1/2} t_\varepsilon(\psi) \\
&\leq \quad C\, \varepsilon^{1/2} t_\varepsilon(\psi),
\end{aligned}$$

for all $\psi \in \mathcal{H}_0^1(I \times S)$. This proves (16).

The proof of inequality (17) is similar to proof of Theorem 3.1 in [7] and will not be present here. Note that due to the scaled geometry the power on the right-hand-side of (17) is 1/2 in contrast to the power 1 in Theorem 3.1 in [7]. □

The quadratic form t_ε will be our starting point in the discussion of the dimensional reduction in Section 4.

4 Reduction of dimension

Recall that $u_0(y)$ is the positive and normalized eigenfunction corresponding to the first eigenvalue ν_0 of the Laplacian in $\mathcal{H}_0^1(S)$. Let $\mathcal{L}$ be the subspace of $\mathrm{L}^2(I \times S)$ generated by all functions of the form $w(x)u_0(y)$, $w \in \mathrm{L}^2(I)$. By considering the orthogonal decomposition

$$\mathrm{L}^2(I \times S) = \mathcal{L} \oplus \mathcal{L}^\perp, \tag{18}$$

for $\psi \in \mathrm{L}^2(I \times S)$ we can write

$$\psi(x,y) = w(x)u_0(y) + \eta(x,y), \tag{19}$$

with $w \in \mathrm{L}^2(I)$ and $\eta \in \mathcal{L}^\perp$. Note that $wu_0 \in \mathcal{H}_0^1(I \times S)$ if $w \in \mathcal{H}_0^1(I)$. Correspondingly, for $\psi \in \mathcal{H}_0^1(I \times S)$, write $\psi = wu_0 + \eta$ with $w \in \mathcal{H}_0^1(I)$ and $\eta \in \mathcal{H}_0^1(I \times S) \cap \mathcal{L}^\perp$.

Now we are going to study the quadratic form t_ε restricted to the subspace $\mathcal{H}_0^1(I \times S) \cap \mathcal{L}$. Assuming conditions (I) and (II) in the previous section, for $w \in \mathcal{H}_0^1(I)$ some *long* calculations (see Appendix A) show that

$$\begin{aligned} t_\varepsilon(wu_0) &= \int_I \left[\left| \frac{\partial w}{\partial x} \right|^2 + g_\varepsilon(x)|w|^2 + b_\varepsilon(x)|w|^2 + \frac{1}{\varepsilon^{1/2}} q_\varepsilon\left(\frac{x}{\varepsilon^{1/2}}\right)|w|^2 \right] \mathrm{d}x \\ &+ \int_I \frac{C(S)}{\varepsilon} (\tau + \alpha')^2 \left(\frac{x}{\varepsilon^{1/2}}\right) |w|^2 \mathrm{d}x \\ &+ \int_I \frac{\nu_0}{\varepsilon^2} \left(\frac{1}{h(x)^2} - \frac{1}{M^2} \right) |w|^2 \mathrm{d}x \\ &- \int_I \frac{1}{4\varepsilon h^2(x)} k^2 \left(\frac{x}{\varepsilon^{1/2}}\right) |w|^2 \mathrm{d}x + \frac{c}{\varepsilon} \int_I |w|^2 \mathrm{d}x, \end{aligned} \tag{20}$$

where $C(S)$ is a constant that depends on S, $g_\varepsilon(x) \to 0$ uniformly as $\varepsilon \to 0$; $b_\varepsilon(x), q_\varepsilon(x) \in \mathrm{L}^\infty(I)$ and there exists $D > 0$ so that $\|b_\varepsilon\|_\infty, \|q_\varepsilon\|_\infty < D$, for $\varepsilon > 0$ small enough, say $0 < \varepsilon < \varepsilon_0$ for some fixed ε_0. Put

$$\begin{aligned} W_\varepsilon(x) &:= g_\varepsilon(x) + b_\varepsilon(x) + \frac{1}{\varepsilon^{1/2}} q_\varepsilon\left(\frac{x}{\varepsilon^{1/2}}\right) + \frac{C(S)}{\varepsilon}(\tau + \alpha')^2 \left(\frac{x}{\varepsilon^{1/2}}\right) \\ &+ \frac{\nu_0}{\varepsilon^2}\left(\frac{1}{h(x)^2} - \frac{1}{M^2} \right) - \frac{1}{4\varepsilon h^2(x)} k^2 \left(\frac{x}{\varepsilon^{1/2}}\right) + \frac{c}{\varepsilon}, \end{aligned}$$

and choose the constant c so that $c/\varepsilon > \max\{\|k_\varepsilon^2/(4h^2)\|_\infty, (E+1)/\varepsilon\}$ for some

$$E > \left\| \varepsilon g_\varepsilon(x) + \varepsilon b_\varepsilon(x) + \varepsilon^{1/2} q_\varepsilon\left(\frac{x}{\varepsilon^{1/2}}\right) - \frac{1}{4h^2(x)} k^2 \left(\frac{x}{\varepsilon^{1/2}}\right) \right\|_\infty,$$

for all ε small enough. Thus, for such values of ε one has

$$W_\varepsilon(x) \geq \frac{\nu_0}{\varepsilon^2}\left(\frac{1}{h(x)^2} - \frac{1}{M^2} \right).$$

This inequality will be important ahead.

Taking into account the isometry

$$\mathcal{L} \ni wu_0 \mapsto w \in \mathrm{L}^2(I), \qquad \int_{I\times S} |w(x)u_0(y)|^2 \,\mathrm{d}x\mathrm{d}y = \int_I |w(x)|^2 \,\mathrm{d}x,$$

we may think of t_ε restricted to $\mathrm{L}^2(I)$, that is, $t_\varepsilon(w) := t_\varepsilon(wu_0)$. The self-adjoint operator associated with t_ε in $\mathrm{L}^2(I)$ is

$$(A_\varepsilon w)(x) := -w''(x) + W_\varepsilon(x)w(x), \quad \mathrm{dom}\, A_\varepsilon = \mathcal{H}^2(I) \cap \mathcal{H}_0^1(I), \tag{21}$$

and the above choice of c implies that zero is not a spectral point of A_ε.

Lemma 4.1. Let $I = [-a, b]$, $0 < a, b \leq \infty$. Then, for $\varepsilon > 0$ small enough, there exists $F > 0$ so that

$$\|(A_\varepsilon)^{-1}\| \leq F\varepsilon.$$

By noting that

$$\frac{\varepsilon^2 W_\varepsilon(x)}{x^2} \geq \frac{\nu_0}{x^2}\delta\left(\frac{1}{h(x)^2} - \frac{1}{M^2}\right),$$

for some $\delta > 0$, the proof of Lemma 4.1 is similar to the proof of Lemma 2.1 in [9] and it will not be reproduced here.

Theorem 3. *Suppose that conditions* (I) *and* (II) *are satisfied. Let* $I = [a, b]$, $0 < a, b \leq \infty$. *Then, there exists* $K > 0$ *so that, for* ε *small enough,*

$$\left|\left|(T_\varepsilon)^{-1} - \left((A_\varepsilon)^{-1} \oplus 0\right)\right|\right| \leq K\,\varepsilon^{3/2},$$

where 0 *denotes the null operator on the subspace* $\mathcal{L}^\perp$.

Proof. For $\psi_1, \psi_2 \in \mathcal{H}_0^1(I \times S)$, denote by $t_\varepsilon(\psi_1, \psi_2)$ the sesquilinear form associated with the t_ε. Recall the decompositions (18) and (19), and let ν_1 be the second eigenvalue of the Laplacian in $\mathcal{H}_0^1(S)$. We are going to show that, for $0 < \varepsilon < \varepsilon_0$, the form t_ε satisfy the following conditions:

$$t_\varepsilon(wu_0) \geq \frac{F'}{\varepsilon}\|wu_0\|^2, \quad \forall w \in \mathcal{H}_0^1(I), \tag{22}$$

for some $F' > 0$;

$$t_\varepsilon(\eta) \geq \frac{\nu_1 - \nu_0}{\varepsilon^2}\|\eta\|^2, \quad \forall \eta \in \mathcal{L}^\perp \cap \mathcal{H}_0^1(I \times S); \tag{23}$$

and for some $L > 0$,

$$t_\varepsilon(wu_0, \eta) \leq L\,\varepsilon^{1/2}\, t_\varepsilon(wu_0)\, t_\varepsilon(\eta), \quad \forall \psi \in \mathcal{H}_0^1(I \times S), \tag{24}$$

so that the theorem follows by applying Proposition 3.1 in [10].

The statement of Lemma 4.1 is equivalent to

$$t_\varepsilon(wu_0) \geq \frac{1}{F\varepsilon}\int_I |wu_0|^2 \mathrm{d}x\mathrm{d}y, \quad \forall w \in \mathcal{H}_0^1(I). \tag{25}$$

Thus, condition (22) is satisfied taking $F' = 1/F$.

For $\eta \in \mathcal{L}^{\perp} \cap \mathcal{H}_0^1(I \times S)$,

$$\int_{I\times S} \frac{|\nabla_y \eta|^2}{\varepsilon^2} \mathrm{d}x\mathrm{d}y \geq \frac{\nu_1}{\varepsilon^2} \int_{I\times S} |\eta|^2 \mathrm{d}x\mathrm{d}y,$$

and then,

$$\int_{I\times S} \left(\frac{|\nabla_y \eta|^2}{\varepsilon^2} - \frac{\nu_0}{\varepsilon^2}|\eta|^2 \right) \mathrm{d}x\mathrm{d}y \geq \frac{\nu_1 - \nu_0}{\varepsilon^2} \int_{I\times S} |\eta|^2 \mathrm{d}x\mathrm{d}y,$$

so that

$$t_\varepsilon(\eta) \geq \frac{\nu_1 - \nu_0}{\varepsilon^2} \int_{I\times S} |\eta|^2 \mathrm{d}x\mathrm{d}y, \tag{26}$$

and since $\nu_1 - \nu_0 > 0$ (recall that ν_0 is a simple eigenvalue), condition (23) holds true. The condition (24) is more delicate and is discussed in Appendix B. $\square$

5 Proof of Theorem 1

From now on we are going to study the sequence of operators A_ε. For $I = [-a, b]$, $0 < a, b \leq \infty$, introduce the family of segments

$$I_\varepsilon = (-\varepsilon^{-1/2} a, \varepsilon^{-1/2} b), \quad \varepsilon > 0,$$

and the family of unitary operators $J_\varepsilon : \mathrm{L}^2(I) \to \mathrm{L}^2(I_\varepsilon)$ generated by the dilation $x \mapsto \varepsilon^{1/2} x$,that is,

$$(J_\varepsilon \psi)(x) = \varepsilon^{1/4} \psi(\varepsilon^{1/2} x).$$

Set $\hat{A}_\varepsilon := \varepsilon J_\varepsilon A_\varepsilon J_\varepsilon^{-1}$, which is self-adjoint in $\mathrm{L}^2(I_\varepsilon)$. The operator

$$(T^c w)(x) := -w''(x) + \left(\frac{2\nu_0}{M^3}|x|^2 + C(S)(\tau + \alpha')^2(x) - \frac{k^2(x)}{4M^2} + c \right) w(x).$$

has purely discrete spectrum $\{\mu_j^c : j \in \mathbb{N}\}$, and it is essentially the operator T defined in the Introduction, which also has purely discrete spectrum $\{\mu_j : j \in \mathbb{N}\}$. Observe that, for all j, one has $\mu_j = \mu_j^c - c$.

Theorem 4. *Assume conditions* (I) *and* (II). *If* $I = [-a, b]$, $0 < a, b < \infty$, *then*

$$\left\| \left[(\hat{A}_\varepsilon)^{-1} \oplus 0 \right] - (T^c)^{-1} \right\| \to 0, \quad as \quad \varepsilon \to 0, \tag{27}$$

where 0 *is the null operator on the subspace* $\mathrm{L}^2(\mathbb{R} \backslash I_\varepsilon)$. *If* $I = \mathbb{R}$,

$$\left\| (\hat{A}_\varepsilon)^{-1} - (T^c)^{-1} \right\| \to 0, \quad as \quad \varepsilon \to 0. \tag{28}$$

Proof. We will rely on results in [9, 10]. Some calculations show that

$$\begin{aligned}\varepsilon J_\varepsilon W_\varepsilon(x) J_\varepsilon^{-1} &= \varepsilon W_\varepsilon(\varepsilon^{1/2}x)\\ &= \varepsilon g_\varepsilon(\varepsilon^{1/2}x) + \varepsilon b_\varepsilon(\varepsilon^{1/2}x) + \varepsilon^{1/2} q_\varepsilon(x) + C(S)(\tau+\alpha')^2(x)\\ &+ \frac{2\nu_0}{M^3}|x|^2 + \nu_0\varepsilon^{1/2}\rho(\varepsilon^{1/2}x)|x|^3 - \frac{1}{4h^2(\varepsilon^{1/2}x)}k^2(x) + c,\end{aligned}$$

where $\rho(x)$ is a bounded function on any bounded interval. Observe that

$$\lim_{\varepsilon\to 0} \varepsilon W_\varepsilon(\varepsilon^{1/2}x) = 2\nu_0/M^3|x|^2 + C(S)(\tau+\alpha')^2(x) - (1/4M^2)k^2(x) + c,$$

with uniform convergence in each bounded interval. Since

$$C(S)(\tau+\alpha')^2(x) - (1/4M^2)k^2(x) + c > 0,$$

the proof of the convergence (27) is similar to the proof of Theorem 1.3 in [9], and the proof of convergence (28) to the proof of Theorem 5.2 in [10]. □

Finally, the proof of our main result.

Proof. (**Theorem 1**) Let $\lambda_j(\varepsilon)$, $l_j(A_\varepsilon)$ and $l_j(\hat{A}_\varepsilon)$ denote the eigenvalues of T_ε, A_ε and $\hat{A}_\varepsilon$, respectively. Observe also that the nonzero eigenvalues of $(A_\varepsilon)^{-1}\oplus 0$ are exactly the eigenvalues of $(A_\varepsilon)^{-1}$. Hence, by Theorem 4.10, page 291, in [12] and Theorem 3 we have

$$\begin{aligned}\left|\frac{1}{\lambda_j(\varepsilon)} - \frac{1}{l_j(A_\varepsilon)}\right| &\le \left\|(T_\varepsilon)^{-1} - \big((A_\varepsilon)^{-1}\oplus 0\big)\right\|\\ &\le K\,\varepsilon^{3/2}.\end{aligned}$$

Thus,

$$\left|\frac{1}{\varepsilon\lambda_j(\varepsilon)} - \frac{1}{\varepsilon l_j(A_\varepsilon)}\right| \le K\,\varepsilon^{1/2}. \tag{29}$$

As $l_j(\hat{A}_\varepsilon) = \varepsilon\, l_j(A_\varepsilon)$, by Theorem 4, it is found that

$$\varepsilon l_j(A_\varepsilon) \to \mu_j^c, \quad \varepsilon\to 0, \tag{30}$$

and, then,

$$\varepsilon\lambda_j(\varepsilon) \to \mu_j^c, \quad \varepsilon\to 0. \tag{31}$$

For $j\ge 1$, let D_j denote a subspace of dimension j of $\mathrm{L}^2(I\times S)$. By the min-max Principle we can characterize the eigenvalues $\lambda_j(\varepsilon)$ as

$$\lambda_0(\varepsilon) = \inf_{\substack{\psi\in \operatorname{dom} T_\varepsilon\\ \|\psi\|=1}} \langle\psi, T_\varepsilon\psi\rangle,$$

$$\lambda_j(\varepsilon) = \sup_{D_j} \inf_{\substack{\psi\in D_j^{\perp}\cap \operatorname{dom} T_\varepsilon\\ \|\psi\|=1}} \langle\psi, T_\varepsilon\psi\rangle, \qquad j\ge 1,$$

with similar expressions for the eigenvalues $\hat{\lambda}_j(\varepsilon)$ of the operator $\hat{T}_\varepsilon$.

By (16) (see Theorem 2) we have

$$(1 - C\,\varepsilon^{1/2})\,\varepsilon\,\lambda_j(\varepsilon) \le \varepsilon\,\hat{\lambda}_j(\varepsilon) \le (1 + C\,\varepsilon^{1/2})\,\varepsilon\,\lambda_j(\varepsilon), \qquad j \ge 0,$$

and by taking into account (31), we conclude that

$$\varepsilon\hat{\lambda}_j(\varepsilon) \to \mu_j^c = \mu_j + c, \qquad \varepsilon \to 0.$$

Now recall that $l_j(\varepsilon)$ denote the eigenvalues of $-\Delta_\varepsilon$ in Λ_ε. Thus, since $\hat{\lambda}_j(\varepsilon) = l_j(\varepsilon) - \nu_0/\varepsilon^2 M^2 + c/\varepsilon$, the equality (8) follows. □

A The restriction of t_ε to $\mathcal{L}$

Here we restrict the quadratic form t_ε, defined in (15), to the subspace $\mathcal{L}$ in order to verify (20). First, observe that

$$\int_{I\times S} \left(\frac{|\nabla_y w u_0|^2}{\varepsilon^2 h(x)^2} - \nu_0 \frac{|w u_0|^2}{\varepsilon^2 h(x)^2} \right) \mathrm{d}x\mathrm{d}y = 0,$$

because u_0 is an eigenfunction of the Laplacian in $\mathcal{H}_0^1(S)$. Using that $\int_S |u_0|^2 \mathrm{d}y = 1$, it follows that

$$\int_{I\times S} \left[-\frac{k_\varepsilon^2(x)}{4h^2(x)} |w u_0|^2 + \frac{\nu_0}{\varepsilon^2} \left(\frac{1}{h^2(x)} - \frac{1}{M^2} \right) |w u_0|^2 + \frac{c}{\varepsilon} |w u_0|^2 \right] \mathrm{d}x\mathrm{d}y$$

$$= \int_I \left[-\frac{k_\varepsilon^2(x)}{4h^2(x)} |w|^2 + \frac{\nu_0}{\varepsilon^2} \left(\frac{1}{h^2(x)} - \frac{1}{M^2} \right) |w|^2 + \frac{c}{\varepsilon} |w|^2 \right] \mathrm{d}x.$$

Now we are going to analyze the first integral that appears in the definition of t_ε. Some rather long calculations show that

$$\int_{I\times S} \left(\frac{\partial}{\partial x}(w u_0) + w u_0 \beta_\varepsilon^{1/2} \frac{\partial}{\partial x} \left(1/\beta_\varepsilon^{1/2} \right) - w u_0 \frac{h'(x)}{h(x)} + w \nabla_y(u_0) \cdot R y (\tau_\varepsilon + \alpha_\varepsilon') \right.$$

$$+ \beta_\varepsilon^{1/2} w u_0 \nabla_y (1/\beta_\varepsilon^{1/2}) \cdot R y (\tau_\varepsilon + \alpha_\varepsilon') - w \nabla_y(u_0) \cdot y \frac{h'(x)}{h(x)}$$

$$\left. - w u_0 \beta_\varepsilon^{1/2} \nabla_y (1/\beta_\varepsilon^{1/2}) \cdot y \frac{h'(x)}{h(x)} \right)^2 \mathrm{d}x\mathrm{d}y$$

$$= \int_I \left(\left| \frac{\partial w}{\partial x} \right|^2 + m_\varepsilon(x) \right) \mathrm{d}x,$$

where

$$
\begin{aligned}
m_\varepsilon(x) &= \int_S |u_0|^2 \beta_\varepsilon \left[\frac{\partial}{\partial x}\left(1/\beta_\varepsilon^{1/2}\right)\right]^2 \mathrm{d}y \\
&+ \left[\frac{h'(x)}{h(x)}\right]^2 \\
&+ \int_S \left(\frac{\partial u_0}{\partial y_1}\right)^2 \frac{y_2^2}{\varepsilon}(\tau+\alpha')^2\left(\frac{x}{\varepsilon^{1/2}}\right) \mathrm{d}y && \text{(A.1)} \\
&+ \int_S |u_0|^2 \beta_\varepsilon \left[\frac{\partial}{\partial y_1}\left(1/\beta_\varepsilon^{1/2}\right)\right]^2 \frac{y_2^2}{\varepsilon}(\tau+\alpha')^2\left(\frac{x}{\varepsilon^{1/2}}\right) \mathrm{d}y && \text{(A.2)} \\
&+ \int_S \left(\frac{\partial u_0}{\partial y_2}\right)^2 \frac{y_1^2}{\varepsilon}(\tau+\alpha')^2\left(\frac{x}{\varepsilon^{1/2}}\right) \mathrm{d}y && \text{(A.3)}
\end{aligned}
$$

$$
\begin{aligned}
&+ \quad \int_S |u_0|^2 \beta_\varepsilon \left[\frac{\partial}{\partial y_2}\left(1/\beta_\varepsilon^{1/2}\right)\right]^2 \frac{y_1^2}{\varepsilon}(\tau+\alpha')^2\left(\frac{x}{\varepsilon^{1/2}}\right) \mathrm{d}y \\
&+ \quad \left(\frac{h'(x)}{h(x)}\right)^2 \int_S \left(\left|\frac{\partial u_0}{\partial y_1}\right|^2 y_1^2 + \left|\frac{\partial u_0}{\partial y_2}\right|^2 y_2^2\right) \mathrm{d}y \\
&+ \quad \int_S |u_0|^2 \beta_\varepsilon \left[\frac{\partial}{\partial y_1}\left(1/\beta_\varepsilon^{1/2}\right)\right]^2 y_1^2 \left(\frac{h'(x)}{h(x)}\right)^2 \mathrm{d}y \qquad \text{(A.4)} \\
&+ \quad \int_S |u_0|^2 \beta_\varepsilon \left[\frac{\partial}{\partial y_2}\left(1/\beta_\varepsilon^{1/2}\right)\right]^2 y_2^2 \left(\frac{h'(x)}{h(x)}\right)^2 \mathrm{d}y \\
&- \quad \int_S |u_0|^2 \frac{\partial}{\partial x}\left(\beta_\varepsilon^{1/2}\right) \frac{\partial}{\partial x}\left(1/\beta_\varepsilon^{1/2}\right) \mathrm{d}y \\
&- \quad \int_S |u_0|^2 \beta_\varepsilon^{1/2} \frac{\partial^2}{\partial x^2}\left(1/\beta_\varepsilon^{1/2}\right) \mathrm{d}y \\
&+ \quad \frac{\partial}{\partial x}\left(\frac{h'(x)}{h(x)}\right) \\
&- \quad \int_S u_0 \frac{\partial u_0}{\partial y_1} \frac{y_2}{\varepsilon}(\tau'+\alpha'')\left(\frac{x}{\varepsilon^{1/2}}\right) \mathrm{d}y \qquad \text{(A.5)} \\
&- \quad \int_S |u_0|^2 \frac{\partial}{\partial x}\left(\beta_\varepsilon^{1/2}\right) \frac{\partial}{\partial y_1}\left(1/\beta_\varepsilon^{1/2}\right) \frac{y_2}{\varepsilon^{1/2}}(\tau+\alpha')\left(\frac{x}{\varepsilon^{1/2}}\right) \mathrm{d}y \\
&- \quad \int_S |u_0|^2 \beta_\varepsilon^{1/2} \frac{\partial^2}{\partial x \partial y_1}\left(1/\beta_\varepsilon^{1/2}\right) \frac{y_2}{\varepsilon^{1/2}}(\tau+\alpha')\left(\frac{x}{\varepsilon^{1/2}}\right) \mathrm{d}y \\
&- \quad \int_S \beta_\varepsilon^{1/2} |u_0|^2 \frac{\partial}{\partial y_1}\left(1/\beta_\varepsilon^{1/2}\right) \frac{y_2}{\varepsilon}(\tau'+\alpha'')\left(\frac{x}{\varepsilon^{1/2}}\right) \mathrm{d}y \qquad \text{(A.6)} \\
&+ \quad \int_S u_0 \frac{\partial u_0}{\partial y_2} \frac{y_1}{\varepsilon}(\tau'+\alpha'')\left(\frac{x}{\varepsilon^{1/2}}\right) \mathrm{d}y \qquad \text{(A.7)} \\
&+ \quad \int_S |u_0|^2 \frac{\partial}{\partial x}\left(\beta_\varepsilon^{1/2}\right) \frac{\partial}{\partial y_2}\left(1/\beta_\varepsilon^{1/2}\right) \frac{y_1}{\varepsilon^{1/2}}(\tau+\alpha')\left(\frac{x}{\varepsilon^{1/2}}\right) \mathrm{d}y \\
&+ \quad \int_S |u_0|^2 \beta_\varepsilon^{1/2} \frac{\partial^2}{\partial x \partial y_2}\left(1/\beta_\varepsilon^{1/2}\right) \frac{y_1}{\varepsilon^{1/2}}(\tau+\alpha')\left(\frac{x}{\varepsilon^{1/2}}\right) \mathrm{d}y \\
&+ \quad \int_S |u_0|^2 \beta_\varepsilon^{1/2} \frac{\partial}{\partial y_2}\left(1/\beta_\varepsilon^{1/2}\right) \frac{y_1}{\varepsilon}(\tau'+\alpha'')\left(\frac{x}{\varepsilon^{1/2}}\right) \mathrm{d}y \\
&+ \quad \int_S u_0 \frac{\partial u_0}{\partial y_1} y_1 \frac{\partial}{\partial x}\left(\frac{h'(x)}{h(x)}\right) \mathrm{d}y \\
&+ \quad \int_S |u_0|^2 \beta_\varepsilon^{1/2} \frac{\partial^2}{\partial x \partial y_1}\left(1/\beta_\varepsilon^{1/2}\right) y_1 \frac{h'(x)}{h(x)} \mathrm{d}y \\
&+ \quad \int_S |u_0|^2 \frac{\partial}{\partial x}\left(\beta_\varepsilon^{1/2}\right) \frac{\partial}{\partial y_1}\left(1/\beta_\varepsilon^{1/2}\right) y_1 \frac{h'(x)}{h(x)} \mathrm{d}y
\end{aligned}
$$

$$
\begin{aligned}
&+ \int_S |u_0|^2 \beta_\varepsilon^{1/2} \frac{\partial}{\partial y_1}\left(1/\beta_\varepsilon^{1/2}\right) y_1 \frac{\partial}{\partial x}\left(\frac{h'(x)}{h(x)}\right) \mathrm{d}y \\
&+ \int_S u_0 \frac{\partial u_0}{\partial y_2} y_2 \frac{\partial}{\partial x}\left(\frac{h'(x)}{h(x)}\right) \mathrm{d}y \\
&+ \int_S |u_0|^2 \frac{\partial}{\partial x}\left(\beta_\varepsilon^{1/2}\right) \frac{\partial}{\partial y_2}\left(1/\beta_\varepsilon^{1/2}\right) y_2 \frac{h'(x)}{h(x)} \mathrm{d}y \\
&+ \int_S |u_0|^2 \beta_\varepsilon^{1/2} \frac{\partial^2}{\partial x \partial y_2}\left(1/\beta_\varepsilon^{1/2}\right) y_2 \frac{h'(x)}{h(x)} \mathrm{d}y \\
&+ \int_S |u_0|^2 \beta_\varepsilon^{1/2} \frac{\partial}{\partial y_2}\left(1/\beta_\varepsilon^{1/2}\right) y_2 \frac{\partial}{\partial x}\left(\frac{h'(x)}{h(x)}\right) \mathrm{d}y \\
&- 2\int_S |u_0|^2 \beta_\varepsilon^{1/2} \frac{\partial}{\partial x}\left(1/\beta_\varepsilon^{1/2}\right) \frac{h'(x)}{h(x)} \mathrm{d}y \\
&+ 2\int_S u_0 \frac{\partial u_0}{\partial y_1} \beta_\varepsilon^{1/2} \frac{\partial}{\partial x}\left(1/\beta_\varepsilon^{1/2}\right) \frac{y_2}{\varepsilon^{1/2}} (\tau+\alpha')\left(\frac{x}{\varepsilon^{1/2}}\right) \mathrm{d}y \\
&+ 2\int_S |u_0|^2 \beta_\varepsilon \frac{\partial}{\partial x}\left(1/\beta_\varepsilon^{1/2}\right) \frac{\partial}{\partial y_1}\left(1/\beta_\varepsilon^{1/2}\right) \frac{y_2}{\varepsilon^{1/2}} (\tau+\alpha')\left(\frac{x}{\varepsilon^{1/2}}\right) \mathrm{d}y \\
&- 2\int_S u_0 \frac{\partial u_0}{\partial y_2} \beta_\varepsilon^{1/2} \frac{\partial}{\partial x}\left(1/\beta_\varepsilon^{1/2}\right) \frac{y_1}{\varepsilon^{1/2}} (\tau+\alpha')\left(\frac{x}{\varepsilon^{1/2}}\right) \mathrm{d}y \\
&- 2\int_S |u_0|^2 \beta_\varepsilon \frac{\partial}{\partial x}\left(1/\beta_\varepsilon^{1/2}\right) \frac{\partial}{\partial y_2}\left(1/\beta_\varepsilon^{1/2}\right) \frac{y_1}{\varepsilon^{1/2}} (\tau+\alpha')\left(\frac{x}{\varepsilon^{1/2}}\right) \mathrm{d}y \\
&- 2\int_S u_0 \frac{\partial u_0}{\partial y_1} \beta_\varepsilon^{1/2} \frac{\partial}{\partial x}\left(1/\beta_\varepsilon^{1/2}\right) y_1 \frac{h'(x)}{h(x)} \mathrm{d}y \\
&- 2\int_S |u_0|^2 \beta_\varepsilon \frac{\partial}{\partial x}\left(1/\beta_\varepsilon^{1/2}\right) \frac{\partial}{\partial y_1}\left(1/\beta_\varepsilon^{1/2}\right) y_1 \frac{h'(x)}{h(x)} \mathrm{d}y \\
&- 2\int_S u_0 \frac{\partial u_0}{\partial y_2} \beta_\varepsilon^{1/2} \frac{\partial}{\partial x}\left(1/\beta_\varepsilon^{1/2}\right) y_2 \frac{h'(x)}{h(x)} \mathrm{d}y \\
&- 2\int_S |u_0|^2 \beta_\varepsilon \frac{\partial}{\partial x}\left(1/\beta_\varepsilon^{1/2}\right) \frac{\partial}{\partial y_1}\left(1/\beta_\varepsilon^{1/2}\right) y_2 \frac{h'(x)}{h(x)} \mathrm{d}y \\
&- 2\int_S u_0 \frac{\partial u_0}{\partial y_1} \frac{h'(x)}{h(x)} \frac{y_2}{\varepsilon^{1/2}} (\tau+\alpha')\left(\frac{x}{\varepsilon^{1/2}}\right) \mathrm{d}y \\
&- 2\int_S |u_0|^2 \beta_\varepsilon^{1/2} \frac{h'(x)}{h(x)} \frac{\partial}{\partial y_1}\left(1/\beta_\varepsilon^{1/2}\right) \frac{y_2}{\varepsilon^{1/2}} (\tau+\alpha')\left(\frac{x}{\varepsilon^{1/2}}\right) \mathrm{d}y \\
&+ 2\int_S u_0 \frac{\partial u_0}{\partial y_2} \frac{h'(x)}{h(x)} \frac{y_1}{\varepsilon^{1/2}} (\tau+\alpha')\left(\frac{x}{\varepsilon^{1/2}}\right) \mathrm{d}y \\
&+ 2\int_S |u_0|^2 \beta_\varepsilon^{1/2} \frac{h'(x)}{h(x)} \frac{\partial}{\partial y_2}\left(1/\beta_\varepsilon^{1/2}\right) \frac{y_1}{\varepsilon^{1/2}} (\tau+\alpha')\left(\frac{x}{\varepsilon^{1/2}}\right) \mathrm{d}y
\end{aligned}
$$

$$
\begin{aligned}
&+ \quad 2\int_S u_0 \frac{\partial u_0}{\partial y_1} y_1 \left[\frac{h'(x)}{h(x)}\right]^2 \mathrm{d}y \\
&+ \quad 2\int_S |u_0|^2 \beta_\varepsilon^{1/2} \frac{\partial}{\partial y_1}\left(1/\beta_\varepsilon^{1/2}\right) y_1 \left[\frac{h'(x)}{h(x)}\right]^2 \mathrm{d}y \\
&+ \quad 2\int_S u_0 \frac{\partial u_0}{\partial y_2} y_2 \left[\frac{h'(x)}{h(x)}\right]^2 \mathrm{d}y \\
&+ \quad 2\int_S |u_0|^2 \beta_\varepsilon^{1/2} \frac{\partial}{\partial y_2}\left(1/\beta_\varepsilon^{1/2}\right) y_2 \left[\frac{h'(x)}{h(x)}\right]^2 \mathrm{d}y \\
&+ \quad 2\int_S \frac{\partial u_0}{\partial y_1} u_0 \beta_\varepsilon^{1/2} \frac{\partial}{\partial y_1}\left(1/\beta_\varepsilon^{1/2}\right) \frac{y_2^2}{\varepsilon}(\tau+\alpha')^2 \left(\frac{x}{\varepsilon^{1/2}}\right) \mathrm{d}y \\
&- \quad 2\int_S \frac{\partial u_0}{\partial y_1}\frac{\partial u_0}{\partial y_2} \frac{y_1 y_2}{\varepsilon}(\tau+\alpha')^2 \left(\frac{x}{\varepsilon^{1/2}}\right) \mathrm{d}y \qquad\qquad \text{(A.8)} \\
&- \quad 2\int_S \frac{\partial u_0}{\partial y_1} u_0 \beta_\varepsilon^{1/2} \frac{\partial}{\partial y_2}\left(1/\beta_\varepsilon^{1/2}\right) \frac{y_1 y_2}{\varepsilon}(\tau+\alpha')^2 \left(\frac{x}{\varepsilon^{1/2}}\right) \mathrm{d}y \\
&- \quad 2\int_S \left[\frac{\partial u_0}{\partial y_1}\right]^2 \frac{y_1 y_2}{\varepsilon^{1/2}}(\tau+\alpha') \left(\frac{x}{\varepsilon^{1/2}}\right) \frac{h'(x)}{h(x)} \mathrm{d}y \\
&- \quad 2\int_S u_0 \frac{\partial u_0}{\partial y_1} \beta_\varepsilon^{1/2} \frac{\partial}{\partial y_1}\left(1/\beta_\varepsilon^{1/2}\right) \frac{y_1 y_2}{\varepsilon^{1/2}}(\tau+\alpha') \left(\frac{x}{\varepsilon^{1/2}}\right) \frac{h'(x)}{h(x)} \mathrm{d}y \\
&- \quad 2\int_S \frac{\partial u_0}{\partial y_1}\frac{\partial u_0}{\partial y_2} \frac{h'(x)}{h(x)} \frac{y_2^2}{\varepsilon^{1/2}}(\tau+\alpha') \left(\frac{x}{\varepsilon^{1/2}}\right) \mathrm{d}y \\
&- \quad 2\int_S u_0 \frac{\partial u_0}{\partial y_1} \beta_\varepsilon^{1/2} \frac{\partial}{\partial y_2}\left(1/\beta_\varepsilon^{1/2}\right) \frac{h'(x)}{h(x)} \frac{y_2^2}{\varepsilon^{1/2}}(\tau+\alpha') \left(\frac{x}{\varepsilon^{1/2}}\right) \mathrm{d}y \\
&- \quad 2\int_S u_0 \frac{\partial u_0}{\partial y_2} \beta_\varepsilon^{1/2} \frac{\partial}{\partial y_1}\left(1/\beta_\varepsilon^{1/2}\right) \frac{y_1 y_2}{\varepsilon}(\tau+\alpha')^2 \left(\frac{x}{\varepsilon^{1/2}}\right) \mathrm{d}y \\
&- \quad 2\int_S |u_0|^2 \beta_\varepsilon^{1/2} \frac{\partial}{\partial y_1}\left(1/\beta_\varepsilon^{1/2}\right) \frac{\partial}{\partial y_2}\left(1/\beta_\varepsilon^{1/2}\right) \frac{y_1 y_2}{\varepsilon}(\tau+\alpha')^2 \left(\frac{x}{\varepsilon^{1/2}}\right) \mathrm{d}y \\
&- \quad 2\int_S u_0 \frac{\partial u_0}{\partial y_1} \beta_\varepsilon^{1/2} \frac{\partial}{\partial y_1}\left(1/\beta_\varepsilon^{1/2}\right) \frac{y_1 y_2}{\varepsilon^{1/2}}(\tau+\alpha') \left(\frac{x}{\varepsilon^{1/2}}\right) \frac{h'(x)}{h(x)} \mathrm{d}y \\
&- \quad 2\int_S |u_0|^2 \beta_\varepsilon \left[\frac{\partial}{\partial y_1}\left(1/\beta_\varepsilon^{1/2}\right)\right]^2 \frac{y_1 y_2}{\varepsilon^{1/2}}(\tau+\alpha') \left(\frac{x}{\varepsilon^{1/2}}\right) \frac{h'(x)}{h(x)} \mathrm{d}y \\
&- \quad 2\int_S u_0 \frac{\partial u_0}{\partial y_2} \beta_\varepsilon^{1/2} \frac{\partial}{\partial y_1}\left(1/\beta_\varepsilon^{1/2}\right) \frac{y_2^2}{\varepsilon^{1/2}}(\tau+\alpha') \left(\frac{x}{\varepsilon^{1/2}}\right) \frac{h'(x)}{h(x)} \mathrm{d}y \\
&- \quad 2\int_S |u_0|^2 \beta_\varepsilon \frac{\partial}{\partial y_1}\left(1/\beta_\varepsilon^{1/2}\right) \frac{\partial}{\partial y_2}\left(1/\beta_\varepsilon^{1/2}\right) \frac{y_2^2}{\varepsilon^{1/2}}(\tau+\alpha') \left(\frac{x}{\varepsilon^{1/2}}\right) \frac{h'(x)}{h(x)} \mathrm{d}y \\
&+ \quad 2\int_S u_0 \frac{\partial u_0}{\partial y_2} \beta_\varepsilon^{1/2} \frac{\partial}{\partial y_2}\left(1/\beta_\varepsilon^{1/2}\right) \frac{y_1^2}{\varepsilon}(\tau+\alpha')^2 \left(\frac{x}{\varepsilon^{1/2}}\right) \mathrm{d}y
\end{aligned}
$$

$$
\begin{aligned}
&+ \quad 2\int_S \frac{\partial u_0}{\partial y_2}\frac{\partial u_0}{\partial y_1}\frac{y_1^2}{\varepsilon^{1/2}}(\tau+\alpha')\left(\frac{x}{\varepsilon^{1/2}}\right)\frac{h'(x)}{h(x)}\mathrm{d}y \\
&+ \quad 2\int_S u_0\frac{\partial u_0}{\partial y_2}\beta_\varepsilon^{1/2}\frac{\partial}{\partial y_1}\left(1/\beta_\varepsilon^{1/2}\right)\frac{y_1^2}{\varepsilon^{1/2}}(\tau+\alpha')\left(\frac{x}{\varepsilon^{1/2}}\right)\frac{h'(x)}{h(x)}\mathrm{d}y \\
&+ \quad 2\int_S \left[\frac{\partial u_0}{\partial y_2}\right]^2\frac{y_1y_2}{\varepsilon^{1/2}}(\tau+\alpha')\left(\frac{x}{\varepsilon^{1/2}}\right)\frac{h'(x)}{h(x)}\mathrm{d}y \\
&+ \quad 2\int_S u_0\frac{\partial u_0}{\partial y_2}\beta_\varepsilon^{1/2}\frac{\partial}{\partial y_2}\left(1/\beta_\varepsilon^{1/2}\right)\frac{y_1y_2}{\varepsilon^{1/2}}(\tau+\alpha')\left(\frac{x}{\varepsilon^{1/2}}\right)\frac{h'(x)}{h(x)}\mathrm{d}y \\
&+ \quad 2\int_S u_0\frac{\partial u_0}{\partial y_1}\beta_\varepsilon^{1/2}\frac{\partial}{\partial y_2}\left(1/\beta_\varepsilon^{1/2}\right)\frac{y_1^2}{\varepsilon^{1/2}}(\tau+\alpha')\left(\frac{x}{\varepsilon^{1/2}}\right)\frac{h'(x)}{h(x)}\mathrm{d}y \\
&+ \quad 2\int_S |u_0|^2\beta_\varepsilon\frac{\partial}{\partial y_1}\left(1/\beta_\varepsilon^{1/2}\right)\frac{\partial}{\partial y_2}\left(1/\beta_\varepsilon^{1/2}\right)\frac{y_1^2}{\varepsilon^{1/2}}(\tau+\alpha')\left(\frac{x}{\varepsilon^{1/2}}\right)\frac{h'(x)}{h(x)}\mathrm{d}y \\
&+ \quad 2\int_S u_0\frac{\partial u_0}{\partial y_2}\beta_\varepsilon^{1/2}\frac{\partial}{\partial y_2}\left(1/\beta_\varepsilon^{1/2}\right)\frac{y_1y_2}{\varepsilon^{1/2}}(\tau+\alpha')\left(\frac{x}{\varepsilon^{1/2}}\right)\frac{h'(x)}{h(x)}\mathrm{d}y \\
&+ \quad 2\int_S |u_0|^2\beta_\varepsilon\left[\frac{\partial}{\partial y_2}\left(1/\beta_\varepsilon^{1/2}\right)\right]^2\frac{y_1y_2}{\varepsilon^{1/2}}(\tau+\alpha')\left(\frac{x}{\varepsilon^{1/2}}\right)\frac{h'(x)}{h(x)}\mathrm{d}y \\
&+ \quad 2\int_S u_0\frac{\partial u_0}{\partial y_1}\beta_\varepsilon^{1/2}\frac{\partial}{\partial y_1}\left(1/\beta_\varepsilon^{1/2}\right)y_1^2\left[\frac{h'(x)}{h(x)}\right]^2\mathrm{d}y \\
&+ \quad 2\int_S \frac{\partial u_0}{\partial y_1}\frac{\partial u_0}{\partial y_2}y_1y_2\left[\frac{h'(x)}{h(x)}\right]^2\mathrm{d}y \\
&+ \quad 2\int_S u_0\frac{\partial u_0}{\partial y_1}\beta_\varepsilon^{1/2}\frac{\partial}{\partial y_2}\left(1/\beta_\varepsilon^{1/2}\right)y_1y_2\left[\frac{h'(x)}{h(x)}\right]^2\mathrm{d}y \\
&+ \quad 2\int_S u_0\frac{\partial u_0}{\partial y_2}\beta_\varepsilon^{1/2}\frac{\partial}{\partial y_1}\left(1/\beta_\varepsilon^{1/2}\right)y_1y_2\left[\frac{h'(x)}{h(x)}\right]^2\mathrm{d}y \\
&+ \quad 2\int_S |u_0|^2\beta_\varepsilon\frac{\partial}{\partial y_1}\left(1/\beta_\varepsilon^{1/2}\right)\frac{\partial}{\partial y_2}\left(1/\beta_\varepsilon^{1/2}\right)y_1y_2\left[\frac{h'(x)}{h(x)}\right]^2\mathrm{d}y \\
&+ \quad 2\int_S u_0\frac{\partial u_0}{\partial y_2}\beta_\varepsilon^{1/2}\frac{\partial}{\partial y_2}\left(1/\beta_\varepsilon^{1/2}\right)y_2^2\left[\frac{h'(x)}{h(x)}\right]^2\mathrm{d}y.
\end{aligned}
$$

We must call attention for some terms in the definition of $m_\varepsilon(x)$. First observe that

$$\int_S u_0\langle\nabla_y u_0, Ry\rangle\mathrm{d}y = 0.$$

We define

$$C(S) := \int_S |\langle\nabla_y u_0, Ry\rangle|^2\mathrm{d}y.$$

Thus, taking account (A.5) and (A.7), we have

$$-\int_S\left[u_0\frac{\partial u_0}{\partial y_1}\frac{y_2}{\varepsilon}(\tau+\alpha')\left(\frac{x}{\varepsilon^{1/2}}\right) - u_0\frac{\partial u_0}{\partial y_2}\frac{y_1}{\varepsilon}(\tau+\alpha')\left(\frac{x}{\varepsilon^{1/2}}\right)\right]\mathrm{d}y = 0$$

and, taking (A.1), (A.3) and (A.8), we have

$$\int_S \left[\left| \frac{\partial u_0}{\partial y_1} \right|^2 y_2^2(\tau_\varepsilon + \alpha_\varepsilon') - 2\frac{\partial u_0}{\partial y_1}\frac{\partial u_0}{\partial y_2} y_1 y_2 (\tau_\varepsilon + \alpha_\varepsilon') + \left| \frac{\partial u_0}{\partial y_2} \right|^2 y_1^2(\tau_\varepsilon + \alpha_\varepsilon') \right] \mathrm{d}y$$
$$= C(S)(\tau_\varepsilon + \alpha_\varepsilon')(x).$$

For the another terms we have the following situations. For example,

$$\frac{\partial}{\partial y_1}\left(1/\beta_\varepsilon^{1/2}\right) = \frac{\varepsilon^{1/2}}{2} h(x) k\left(\frac{x}{\varepsilon^{1/2}}\right) \cos\alpha_\varepsilon(x) \left(1/\beta_\varepsilon^{3/2}\right)$$

and then, see integral (A.4),

$$\int_S |u_0|^2 \beta_\varepsilon \left[\frac{\partial}{\partial y_1}\left(1/\beta_\varepsilon^{1/2}\right)\right]^2 y_1^2 \left[\frac{h'(x)}{h(x)}\right]^2 \mathrm{d}y \to 0$$

uniformly as $\varepsilon \to 0$. Now observe the integral (A.2). There exists $D_1' > 0$ so that

$$\left| \int_S |u_0|^2 \beta_\varepsilon \left[\frac{\partial}{\partial y_1}\left(1/\beta_\varepsilon^{1/2}\right)\right]^2 \frac{y_2^2}{\varepsilon} (\tau + \alpha')\left(\frac{x}{\varepsilon^{1/2}}\right) \mathrm{d}y \right| \le D_1',$$

for $\varepsilon > 0$ small enough (recall that we are assuming that conditions (I) and (II) in Section 3 hold true). On the other hand, if we consider the integral (A.6),

$$p_\varepsilon^1(x) := -\int_S |u_0|^2 \beta_\varepsilon^{1/2} \frac{\partial}{\partial y_1}\left(1/\beta_\varepsilon^{1/2}\right) \frac{y_2}{\varepsilon} (\tau' + \alpha'')\left(\frac{x}{\varepsilon^{1/2}}\right) \mathrm{d}y,$$

we have that there exists $D_1'' > 0$ so that

$$\left| \varepsilon^{1/2} p_\varepsilon^1(x) \right| \le D_1'',$$

for $\varepsilon > 0$ small enough.

In general, for all terms of $m_\varepsilon(x)$ occur one of the possibilities above. Thus, we can say that $m_\varepsilon(x)$ can be written as

$$m_\varepsilon(x) = g_\varepsilon(x)|w|^2 + b_\varepsilon(x)|w|^2 + p_\varepsilon(x)|w|^2 + C(S)(\tau_\varepsilon + \alpha_\varepsilon')^2|w|^2,$$

where the functions $g_\varepsilon, b_\varepsilon$ and p_ε satisfy

(1) $g_\varepsilon(x) \to 0$ uniformly as $\varepsilon \to 0$;

(2) $b_\varepsilon(x) \in \mathrm{L}^\infty(I)$ and there exists $D' > 0$ so that $\|b_\varepsilon\|_\infty \le D'$, for all $\varepsilon > 0$ small enough;

(3) we can write $p_\varepsilon(x) = 1/\varepsilon^{1/2} q_\varepsilon(x/\varepsilon^{1/2})$, with $q_\varepsilon \in \mathrm{L}^\infty(I)$ and there exists $D'' > 0$ so that $\|q_\varepsilon\|_\infty \le D''$, for all $\varepsilon > 0$ small enough.

B Proof of inequality (24)

We are going to prove inequality (24), which has appeared in the proof of Theorem 3. First observe that the bilinear form $t_\varepsilon(wu_0,\eta)$ has the following expression

$$\begin{aligned}
t_\varepsilon(wu_0,\eta) &= 2\int_{I\times S}\left(\frac{\partial w}{\partial x}u_0 - w\,u_0\,\beta_\varepsilon^{1/2}\frac{\partial}{\partial x}\left(1/\beta_\varepsilon^{1/2}\right) - w\,u_0\frac{h'(x)}{h(x)}\right.\\
&+ w\,\nabla_y u_0\cdot Ry\,(\tau_\varepsilon+\alpha'_\varepsilon) + w\,u_0\,\beta_\varepsilon^{1/2}\nabla_y\left(1/\beta_\varepsilon^{1/2}\right)\cdot Ry\,(\tau_\varepsilon+\alpha'_\varepsilon)\\
&- \left. w\,\nabla_y\,u_0\cdot y\,\frac{h'(x)}{h(x)} - w\,u_0\,\beta_\varepsilon^{1/2}\nabla_y\left(1/\beta_\varepsilon^{1/2}\right)\cdot y\frac{h'(x)}{h(x)}\right)\times\\
&\left(\frac{\partial\eta}{\partial x} - \eta\,\beta_\varepsilon^{1/2}\frac{\partial}{\partial x}\left(1/\beta_\varepsilon^{1/2}\right) - \eta\frac{h'(x)}{h(x)}\right.\\
&+ \nabla_y\eta\cdot Ry\,(\tau_\varepsilon+\alpha'_\varepsilon) + \eta\,\beta_\varepsilon^{1/2}\nabla_y\left(1/\beta_\varepsilon^{1/2}\right)\cdot Ry\,(\tau_\varepsilon+\alpha'_\varepsilon)\\
&- \left.\nabla_y\eta\cdot y\,\frac{h'(x)}{h(x)} - \eta\,\beta_\varepsilon^{1/2}\nabla_y\left(1/\beta_\varepsilon^{1/2}\right)\cdot y\frac{h'(x)}{h(x)}\right)\mathrm{d}x\mathrm{d}y\\
&+ 2\int_{I\times S} w\left(\frac{\nabla_y u_0\nabla_y\eta}{\varepsilon^2 h^2(x)} - \frac{\nu_0\,u_0\,\eta}{\varepsilon^2 h^2(x)}\right)\mathrm{d}x\mathrm{d}y - 2\int_{I\times S}\frac{k_\varepsilon^2(x)}{4}\,w\,u_0\,\eta\,\mathrm{d}x\mathrm{d}y\\
&+ 2\int_{I\times S}\frac{\nu_0}{\varepsilon^2}\left(\frac{1}{h^2(x)} - \frac{1}{M^2}\right)w\,u_0\,\eta\,\mathrm{d}x\mathrm{d}y + 2\int_{I\times S}\frac{c}{\varepsilon}\,w\,u_0\,\eta\,\mathrm{d}x\mathrm{d}y.
\end{aligned}$$

Since $\eta\in\mathcal{L}^\perp$ and u_0 is an eigenfunction of the Laplacian in $\mathcal{H}_0^1(S)$, we have

$$\int_S u_0\,\eta\,\mathrm{d}y = 0, \qquad \int_S u_0\,\frac{\partial\eta}{\partial x}\,\mathrm{d}y = 0 \quad \text{a.e.[x]}, \tag{B.1}$$

$$\int_S \nabla_y u_0\nabla_y\eta\,\mathrm{d}y = 0. \tag{B.2}$$

Thus, $t_\varepsilon(wu_0,\eta)$ is simply

$$\begin{aligned}
t_\varepsilon(wu_0,\eta) &= 2\int_{I\times S}\left(\frac{\partial w}{\partial x}u_0 - w\,u_0\,\beta_\varepsilon^{1/2}\frac{\partial}{\partial x}\left(1/\beta_\varepsilon^{1/2}\right) - w\,u_0\frac{h'(x)}{h(x)}\right.\\
&+ w\,\nabla_y u_0\cdot Ry\,(\tau_\varepsilon+\alpha'_\varepsilon) + w\,u_0\,\beta_\varepsilon^{1/2}\nabla_y\left(1/\beta_\varepsilon^{1/2}\right)\cdot Ry\,(\tau_\varepsilon+\alpha'_\varepsilon)\\
&- \left. w\,\nabla_y u_0\cdot y\,\frac{h'(x)}{h(x)} - w\,u_0\beta_\varepsilon^{1/2}\nabla_y\left(1/\beta_\varepsilon^{1/2}\right)\cdot y\frac{h'(x)}{h(x)}\right)\times\\
&\left(\frac{\partial\eta}{\partial x} - \eta\,\beta_\varepsilon^{1/2}\frac{\partial}{\partial x}\left(1/\beta_\varepsilon^{1/2}\right) - \eta\frac{h'(x)}{h(x)}\right.\\
&+ \nabla_y\eta\cdot Ry\,(\tau_\varepsilon+\alpha'_\varepsilon) + \eta\,\beta_\varepsilon^{1/2}\nabla_y\left(1/\beta_\varepsilon^{1/2}\right)\cdot Ry\,(\tau_\varepsilon+\alpha'_\varepsilon)\\
&- \left.\nabla_y\eta\cdot y\,\frac{h'(x)}{h(x)} - \eta\,\beta_\varepsilon^{1/2}\nabla_y\left(1/\beta_\varepsilon^{1/2}\right)\cdot y\frac{h'(x)}{h(x)}\right)\mathrm{d}x\mathrm{d}y.
\end{aligned}$$

We write $t_\varepsilon(wu_0,\eta) = \sum_{i=1}^{7} t_\varepsilon^i(wu_0,\eta)$ where

$$\begin{aligned}
t_\varepsilon^1(wu_0,\eta) &= 2\int_{I\times S} \frac{\partial w}{\partial x}\, u_0 \left(\frac{\partial \eta}{\partial x} - \eta\, \beta_\varepsilon^{1/2} \frac{\partial}{\partial x}\left(1/\beta_\varepsilon^{1/2}\right) - \eta \frac{h'(x)}{h(x)} \right.\\
&+ \nabla_y \eta \cdot Ry\,(\tau_\varepsilon + \alpha_\varepsilon') + \eta\, \beta_\varepsilon^{1/2} \nabla_y \left(1/\beta_\varepsilon^{1/2}\right) \cdot Ry\,(\tau_\varepsilon + \alpha_\varepsilon')\\
&- \left. \nabla_y \eta \cdot y\, \frac{h'(x)}{h(x)} - \eta\, \beta_\varepsilon^{1/2} \nabla_y \left(1/\beta_\varepsilon^{1/2}\right) \cdot y \frac{h'(x)}{h(x)} \right) \mathrm{d}x \mathrm{d}y\,,
\end{aligned}$$

$$\begin{aligned}
t_\varepsilon^2(wu_0,\eta) &= 2\int_{I\times S} w\, u_0\, \beta_\varepsilon^{1/2} \frac{\partial}{\partial x}\left(1/\beta_\varepsilon^{1/2}\right) \left(\frac{\partial \eta}{\partial x} - \eta\, \beta_\varepsilon^{1/2} \frac{\partial}{\partial x}\left(1/\beta_\varepsilon^{1/2}\right) \right.\\
&- \eta \frac{h'(x)}{h(x)} + \nabla_y \eta \cdot Ry\,(\tau_\varepsilon + \alpha_\varepsilon') + \eta\, \beta_\varepsilon^{1/2} \nabla_y \left(1/\beta_\varepsilon^{1/2}\right) \cdot Ry\,(\tau_\varepsilon + \alpha_\varepsilon')\\
&- \left. \nabla_y \eta \cdot y\, \frac{h'(x)}{h(x)} - \eta\, \beta_\varepsilon^{1/2} \nabla_y \left(1/\beta_\varepsilon^{1/2}\right) \cdot y \frac{h'(x)}{h(x)} \right) \mathrm{d}x \mathrm{d}y\,,
\end{aligned}$$

$$\begin{aligned}
t_\varepsilon^3(wu_0,\eta) &= -2\int_{I\times S} w\, u_0\, \frac{h'(x)}{h(x)} \left(\frac{\partial \eta}{\partial x} - \eta\, \beta_\varepsilon^{1/2} \frac{\partial}{\partial x}\left(1/\beta_\varepsilon^{1/2}\right) - \eta \frac{h'(x)}{h(x)} \right.\\
&+ \nabla_y \eta \cdot Ry\,(\tau_\varepsilon + \alpha_\varepsilon') + \eta\, \beta_\varepsilon^{1/2} \nabla_y \left(1/\beta_\varepsilon^{1/2}\right) \cdot Ry\,(\tau_\varepsilon + \alpha_\varepsilon')\\
&- \left. \nabla_y \eta \cdot y\, \frac{h'(x)}{h(x)} - \eta\, \beta_\varepsilon^{1/2} \nabla_y \left(1/\beta_\varepsilon^{1/2}\right) \cdot y \frac{h'(x)}{h(x)} \right) \mathrm{d}x \mathrm{d}y\,,
\end{aligned}$$

$$\begin{aligned}
t_\varepsilon^4(wu_0,\eta) &= 2\int_{I\times S} w\, \nabla_y u_0 \cdot Ry\,(\tau_\varepsilon + \alpha_\varepsilon') \left(\frac{\partial \eta}{\partial x} - \eta\, \beta_\varepsilon^{1/2} \frac{\partial}{\partial x}\left(1/\beta_\varepsilon^{1/2}\right) \right.\\
&- \eta \frac{h'(x)}{h(x)} + \nabla_y \eta \cdot Ry\,(\tau_\varepsilon + \alpha_\varepsilon') + \eta\, \beta_\varepsilon^{1/2} \nabla_y \left(1/\beta_\varepsilon^{1/2}\right) \cdot Ry\,(\tau_\varepsilon + \alpha_\varepsilon')\\
&- \left. \nabla_y \eta \cdot y\, \frac{h'(x)}{h(x)} - \eta\, \beta_\varepsilon^{1/2} \nabla_y \left(1/\beta_\varepsilon^{1/2}\right) \cdot y \frac{h'(x)}{h(x)} \right) \mathrm{d}x \mathrm{d}y\,,
\end{aligned}$$

$$\begin{aligned}
t_\varepsilon^5(wu_0,\eta) &= 2\int_{I\times S} w\, u_0\, \beta_\varepsilon^{1/2}\, \nabla_y \left(1/\beta_\varepsilon^{1/2}\right) \cdot Ry\,(\tau_\varepsilon + \alpha_\varepsilon') \times\\
&\left(\frac{\partial \eta}{\partial x} - \eta\, \beta_\varepsilon^{1/2} \frac{\partial}{\partial x}\left(1/\beta_\varepsilon^{1/2}\right) - \eta \frac{h'(x)}{h(x)} + \nabla_y \eta \cdot Ry\,(\tau_\varepsilon + \alpha_\varepsilon') \right.\\
&+ \eta\, \beta_\varepsilon^{1/2} \nabla_y \left(1/\beta_\varepsilon^{1/2}\right) \cdot Ry\,(\tau_\varepsilon + \alpha_\varepsilon') - \nabla_y \eta \cdot y\, \frac{h'(x)}{h(x)}\\
&- \left. \eta\, \beta_\varepsilon^{1/2} \nabla_y \left(1/\beta_\varepsilon^{1/2}\right) \cdot y \frac{h'(x)}{h(x)} \right) \mathrm{d}x \mathrm{d}y\,,
\end{aligned}$$

$$
\begin{aligned}
t_\varepsilon^6(wu_0,\eta) &= -2\int_{I\times S} w\,\nabla_y u_0\cdot y\,\frac{h'(x)}{h(x)}\left(\frac{\partial\eta}{\partial x} - \eta\,\beta_\varepsilon^{1/2}\frac{\partial}{\partial x}\left(1/\beta_\varepsilon^{1/2}\right)\right.\\
&\quad -\eta\,\frac{h'(x)}{h(x)} + \nabla_y\eta\cdot Ry\,(\tau_\varepsilon+\alpha'_\varepsilon) + \eta\,\beta_\varepsilon^{1/2}\nabla_y\left(1/\beta_\varepsilon^{1/2}\right)\cdot Ry\,(\tau_\varepsilon+\alpha'_\varepsilon)\\
&- \left.\nabla_y\eta\cdot y\,\frac{h'(x)}{h(x)} - \eta\,\beta_\varepsilon^{1/2}\nabla_y\left(1/\beta_\varepsilon^{1/2}\right)\cdot y\frac{h'(x)}{h(x)}\right)\mathrm{d}x\mathrm{d}y\,,
\end{aligned}
$$

$$
\begin{aligned}
t_\varepsilon^7(wu_0,\eta) &= -2\int_{I\times S} w\,u_0\,\beta_\varepsilon^{1/2}\,\nabla_y\left(1/\beta_\varepsilon^{1/2}\right)\cdot y\frac{h'(x)}{h(x)}\,\times\\
&\quad \left(\frac{\partial\eta}{\partial x} - \eta\,\beta_\varepsilon^{1/2}\frac{\partial}{\partial x}\left(1/\beta_\varepsilon^{1/2}\right) - \eta\frac{h'(x)}{h(x)} + \nabla_y\eta\cdot Ry\,(\tau_\varepsilon+\alpha'_\varepsilon)\right.\\
&+ \eta\,\beta_\varepsilon^{1/2}\nabla_y\left(1/\beta_\varepsilon^{1/2}\right)\cdot Ry\,(\tau_\varepsilon+\alpha'_\varepsilon)\\
&- \left.\nabla_y\eta\cdot y\,\frac{h'(x)}{h(x)} - \eta\,\beta_\varepsilon^{1/2}\nabla_y\left(1/\beta_\varepsilon^{1/2}\right)\cdot y\frac{h'(x)}{h(x)}\right)\mathrm{d}x\mathrm{d}y\,.
\end{aligned}
$$

We are going to analyze $t_\varepsilon^1(wu_0,\eta)$ in detail. Recall that $R = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and $\int_S u_0\,\eta\,\mathrm{d}y = 0$. Thus, we can rewrite $t_\varepsilon^1(wu_0,\eta)$ as

$$
\begin{aligned}
t_\varepsilon^1(wu_0,\eta) &= 2\int_{I\times S}\left(-\frac{\partial w}{\partial x}u_0\,\eta\,\beta_\varepsilon^{1/2}\frac{\partial}{\partial x}\left(1/\beta_\varepsilon^{1/2}\right)\right.\\
&+ \frac{\partial w}{\partial x}u_0\,\frac{\partial\eta}{\partial y_1}\,y_2\,(\tau_\varepsilon+\alpha'_\varepsilon) - \frac{\partial w}{\partial x}u_0\,\frac{\partial\eta}{\partial y_2}\,y_1\,(\tau_\varepsilon+\alpha'_\varepsilon)\\
&+ \frac{\partial w}{\partial x}u_0\,\eta\,\beta_\varepsilon^{1/2}\nabla_y\left(1/\beta_\varepsilon^{1/2}\right)\cdot Ry\,(\tau_\varepsilon+\alpha'_\varepsilon)\\
&- \frac{\partial w}{\partial x}u_0\,\frac{\partial\eta}{\partial y_1}\,y_1\,\frac{h'(x)}{h(x)} - \frac{\partial w}{\partial x}u_0\,\frac{\partial\eta}{\partial y_2}\,y_2\,\frac{h'(x)}{h(x)}\\
&- \left.\frac{\partial w}{\partial x}u_0\,\eta\,\beta_\varepsilon^{1/2}\nabla_y\left(1/\beta_\varepsilon^{1/2}\right)\cdot y\frac{h'(x)}{h(x)}\right)\mathrm{d}x\mathrm{d}y.
\end{aligned}
$$

Using integration by parts in the terms with $\partial\eta/\partial y_1$ and $\partial\eta/\partial y_2$, we obtain

$$
\begin{aligned}
t_\varepsilon^1(wu_0,\eta) &= 2\int_{I\times S}\left(-\frac{\partial w}{\partial x}u_0\,\eta\,\beta_\varepsilon^{1/2}\frac{\partial}{\partial x}\left(1/\beta_\varepsilon^{1/2}\right)\right.\\
&- \frac{\partial w}{\partial x}\frac{\partial u_0}{\partial y_1}\,\eta\,\frac{y_2}{\varepsilon^{1/2}}\,(\tau+\alpha')\left(\frac{x}{\varepsilon^{1/2}}\right) + \frac{\partial w}{\partial x}\frac{\partial u_0}{\partial y_2}\,\eta\,\frac{y_1}{\varepsilon^{1/2}}\,(\tau+\alpha')\left(\frac{x}{\varepsilon^{1/2}}\right)\\
&+ \frac{\partial w}{\partial x}u_0\,\eta\,\beta_\varepsilon^{1/2}\nabla_y\left(1/\beta_\varepsilon^{1/2}\right)\cdot Ry\,\frac{1}{\varepsilon^{1/2}}(\tau+\alpha')\left(\frac{x}{\varepsilon^{1/2}}\right)\\
&+ \frac{\partial w}{\partial x}\frac{\partial(u_0y_1)}{\partial y_1}\,\eta\,\frac{h'(x)}{h(x)} + \frac{\partial w}{\partial x}\frac{\partial(u_0y_2)}{\partial y_2}\,\eta\,\frac{h'(x)}{h(x)}\\
&- \left.\frac{\partial w}{\partial x}u_0\,\eta\,\beta_\varepsilon^{1/2}\nabla_y\left(1/\beta_\varepsilon^{1/2}\right)\cdot y\frac{h'(x)}{h(x)}\right)\mathrm{d}x\mathrm{d}y.
\end{aligned}
$$

Taking into account conditions (I) and (II), we see that there exists $L' > 0$ so that

$$\left|t_\varepsilon^1(wu_0,\eta)\right| \;\leq\; L'\left(1+\frac{1}{\varepsilon^{1/2}}\right)\left(\int_I\left|\frac{\partial w}{\partial x}\right|\mathrm{d}x\right)^{1/2}\left(\int_{I\times S}|\eta|^2\mathrm{d}x\mathrm{d}y\right)^{1/2}.$$

Now, note that, for $\varepsilon > 0$ small enough,

$$\int_I\left|\frac{\partial w}{\partial x}\right|^2\mathrm{d}x \leq t_\varepsilon(wu_0), \quad \forall w \in \mathcal{H}_0^1(I). \tag{B.3}$$

This inequality together the inequality (26) show that

$$\left|t_\varepsilon^1(wu_0,\eta)\right| \;\leq\; \frac{2L'}{\nu_1-\nu_0}\,\varepsilon^{1/2}\,\left(t_\varepsilon(wu_0)\right)^{1/2}\left(t_\varepsilon(\eta)\right)^{1/2}.$$

For $t_\varepsilon^2(wu_0,\eta)$ the analyze is even simpler. Observe that

$$\begin{aligned}
|t_\varepsilon^2(wu_0,\eta)| \;&\leq\; 2\left[\int_{I\times S}|w|^2\,|u_0|^2\,\beta_\varepsilon\left(\frac{\partial}{\partial x}\left(1/\beta_\varepsilon^{1/2}\right)\right)^2\mathrm{d}x\mathrm{d}y\right]^{1/2}\times\\
&\quad\left[\int_{I\times S}\left(\frac{\partial\eta}{\partial x}-\eta\,\beta_\varepsilon^{1/2}\frac{\partial}{\partial x}\left(1/\beta_\varepsilon^{1/2}\right)-\eta\frac{h'(x)}{h(x)}\right.\right.\\
&+\;\nabla_y\eta\cdot Ry\,(\tau_\varepsilon+\alpha'_\varepsilon)+\eta\,\beta_\varepsilon^{1/2}\nabla_y\left(1/\beta_\varepsilon^{1/2}\right)\cdot Ry\,(\tau_\varepsilon+\alpha'_\varepsilon)\\
&-\;\left.\left.\nabla_y\eta\cdot y\,\frac{h'(x)}{h(x)}-\eta\,\beta_\varepsilon^{1/2}\nabla_y\left(1/\beta_\varepsilon^{1/2}\right)\cdot y\frac{h'(x)}{h(x)}\right)^2\mathrm{d}x\mathrm{d}y\right]^{1/2},\\
&\leq\; L''\left(\int_I|w|^2\mathrm{d}x\right)^{1/2}\left(t_\varepsilon(\eta)\right)^{1/2}\\
&\leq\; L''\,F\,\varepsilon^{1/2}\,\left(t_\varepsilon(wu_0)\right)^{1/2}\left(t_\varepsilon(\eta)\right)^{1/2},
\end{aligned}$$

for some $L'' > 0$. Here we made use of conditions (I) and (II) and inequalities (25) and (26).

Similar analysis can be repeated for t_ε^i, $i = 3,\cdots,7$ by considering again conditions (I) and (II), equalities (B.1), (B.2), (25), (26) and (B.3). Hence, we can conclude that there exists $L > 0$ so that

$$|t_\varepsilon(wu_0,\eta)| \leq L\,\varepsilon^{1/2}\left(t_\varepsilon(wu_0)\right)^{1/2}\left(t_\varepsilon(\eta)\right)^{1/2},$$

for all $w \in \mathcal{H}_0^1(I)$ and for all $\eta \in \mathcal{L}^\perp$.

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