A REMARK ON MULTIPLICITY OF POSITIVE SOLUTIONS FOR A CLASS OF QUASILINEAR ELLIPTIC SYSTEMS

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Abstract. Using variational methods, we prove a nonexistence and multiplicity result of positive solutions for a class of elliptic systems involving a parameter.

1. Introduction

In this paper we study multiplicity of positive solutions for the the elliptic system

$$(P_\lambda)\qquad \begin{cases} -\Delta_p u = \lambda F_u(x,u,v), & \text{in } \Omega,\\ -\Delta_q v = \lambda F_v(x,u,v), & \text{in } \Omega,\\ u = v = 0, & \text{on } \partial\Omega,\end{cases}$$

where Ω is an open bounded domain with smooth boundary $\partial\Omega$, $1 < p, q < \infty$, and λ a positive real parameter. The nonlinearity $F : \Omega \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a C^1 - function, satisfying $F(x, 0, 0) = 0$ and

$$(1)\qquad \begin{array}{l} F(x,t,s) = F(x,0,s) \quad \forall\ x \in \Omega,\ \ s \in \mathbb{R} \ \text{ and } \ t \le 0;\\ F(x,t,s) = F(x,t,0) \quad \forall\ x \in \Omega,\ \ t \in \mathbb{R} \ \text{ and } \ s \le 0.\end{array}$$

It follows that $F_t(x, 0, 0) = F_s(x, 0, 0) = 0$, so that ($P_\lambda$) has the trivial solution $(0, 0)$. Moreover, assume that F satisfies the following condition

$$(2)\quad |F_t(x,t,s)| \le C(1+|t|^{p-1}+|s|^{q\frac{p-1}{p}}) \ \text{ and } \ |F_s(x,t,s)| \le C(1+|s|^{q-1}+|t|^{p\frac{q-1}{q}})$$

for all $(x, t, s) \in \Omega \times \mathbb{R} \times \mathbb{R}$. Under this assumptions the weak solutions of (P_λ) are critical points of the following C^1-functional, defined on $E = W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega)$,

$$(3)\qquad \Phi_\lambda(u,v) = \int_\Omega \left(\frac{1}{p}|\nabla u|^p + \frac{1}{q}|\nabla v|^q - \lambda F(x,u,v)\right) dx,$$

(see [2]). We remark that (2) implies that F satisfies the following growth condition:

$$(4)\qquad |F(x,t,s)| \le C(1+|t|^p+|s|^q), \quad \forall\,(x,t,s) \in \Omega \times \mathbb{R} \times \mathbb{R}.$$

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This condition was called "of resonant type" in [2].

It is known that the existence and multiplicity of solutions of (P_λ) rely strongly on the behavior of F at origin and at infinity. Our hypotheses are motivated by an eigenvalue problem considered in [2]. Let $G : R^2 \to [0, \infty)$ be a C^1 function such that

$$G(t,s) \leq k(|t|^p + |s|^q). \tag{5}$$

Now, we introduce the following assumptions:

$$\limsup_{||(t,s)|| \to 0} \frac{F(x,t,s)}{G(t,s)} \leq 0, \quad \text{uniformly in } \ x \in \Omega, \tag{6}$$

and

$$\limsup_{||(t,s)|| \to \infty} \frac{F(x,t,s)}{H(t,s)} \leq 0, \quad \text{uniformly in } \ x \in \Omega, \tag{7}$$

where G and H are functions satisfying (5).

Theorem 1.1. *Under the above hypotheses there exists a $\underline{\lambda}$ such that (P_λ) has no positive solution for $\lambda < \underline{\lambda}$. Moreover, if in addition we assume that there exist a ball $B \subset \Omega$ and t_0, $s_0 > 0$ such that $F(x, t_0, s_0) > 0$ for $x \in B$, then there is a $\overline{\lambda}$ such that (P_λ) has at least two positive solutions for $\lambda \geq \overline{\lambda}$.*

Remark 1. By a positive solution we mean a nonnegative pair $(u, v) \in E$ such that

$$\begin{aligned} 0 &= \int_\Omega |\nabla u|^{p-2} \nabla u \nabla \phi + \int_\Omega |\nabla v|^{q-2} \nabla v \nabla \psi \\ &\quad - \lambda \int_\Omega \Big(F_u(x,u,v)\phi + F_v(x,u,v)\psi \Big) dx. \end{aligned}$$

and either $u > 0$ in Ω or $v > 0$ in Ω.

Remark 2. Theorem 1.1 contains and extends the results of Fernández Bonder in [4] and Manouni & Perera in [6]. In [4, 6], the authors studied the system (P_λ), with similar conditions on F, but with particular classes of G, namely, $G(u,v) = |u|^p + |v|^q$ and $G(u,v) = u^{\alpha+1} v^{\beta+1}$ with $\frac{\alpha+1}{p} + \frac{\beta+1}{q} = 1$, respectively. Related results for the simple equation can be found in [5, 7]. Systems of the form (P_λ) are usually called gradient systems. See [2], for a comprehensive analysis of such systems.

Notation. In what follows, we will denote by $\lambda_{p,q} = \min\{\lambda_p, \lambda_q\}$, where λ_p and λ_q denote the main eigenvalues of $(-\Delta_p, W_0^{1,p}(\Omega))$ and $(-\Delta_q, W_0^{1,q}(\Omega))$, respectively. We shall often use the same letter C to denote different constants, and C_λ to denote different constants depending only on λ.

Remark 3. We have, for $(u, v) \in E = W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega)$,

$$\lambda_p \int_\Omega |u|^p \leq \int_\Omega |\nabla u|^p dx \ \text{ and } \ \lambda_q \int_\Omega |v|^q \leq \int_\Omega |\nabla v|^q dx,$$

from which it follows that

$$0 < \lambda_{p,q} \leq \frac{\int_\Omega [|\nabla u|^p + |\nabla v|^q] dx}{\int_\Omega [|u|^p + \int_\Omega |v|^q] dx} \ \text{ for } \ (u,v) \in E. \tag{8}$$

2. Proof of Theorem 1.1.

Proof of nonexistence part:

If (P_λ) has a positive solution (u, v), multiplying the first equation of (P_λ) by u, the second by v, and integrating by parts we get

$$\int_\Omega |\nabla u|^p + \int_\Omega |\nabla v|^q = \lambda \int_\Omega \Big(F_u(x, u, v)u + F_v(x, u, v)v \Big) dx.$$

Using (2) and Holder's inequality, we can show that

$$|F_u(x, u, v)u + F_v(x, u, v)v| \le C(u^p + v^q),$$

and so

$$\int_\Omega |\nabla u|^p + \int_\Omega |\nabla v|^q \le \lambda C \int_\Omega (|u|^p + |v|^q) dx.$$

It follows that $\lambda \ge \lambda_{p,q}/C$, by (8), which implies that there is no positive solution for λ small. □

Proof of multiplicity part:

First, observe that nontrivial critical points of Φ_λ are positive solutions of (P_λ). Indeed, if (u, v) is a critical point of Φ_λ, denoting by u^- and v^- the negative parts of u and v, respectively, we have

$$\begin{aligned} 0 &= [\Phi'_\lambda(u, v), (u^-, v^-)] \\ &= \int_\Omega \left[|\nabla u|^{p-2} \nabla u \nabla u^- + |\nabla v|^{q-2} \nabla v \nabla v^- \right] dx \\ &\quad -\lambda \int_\Omega \left[(F_u(x, u, v)u^- + F_v(x, u, v)v^-) \right] dx \\ &= -||u^-|| - ||v^-|| - \lambda \int_\Omega \left[(F_u(x, u, v)u^- + F_v(x, u, v)v^-) \right] dx. \end{aligned}$$

By (1), we have that $F_u(x, u, v)u^- = 0$ and $F_v(x, u, v)v^- = 0$. It follows that $||u^-|| + ||v^-|| = 0$, and so $u, v \ge 0$. Also, by [1, 3], we have that $u, v \in L^\infty(\Omega) \cap C^1(\Omega)$. Consequently, by the Harnack inequality, we get that either $u > 0$ in Ω or $v > 0$ in Ω, provided $(u, v) \neq (0, 0)$ (see [9]). Thus, nontrivial critical points of Φ_λ are positive solutions of (P_λ).

Consider k the constant in (5). Then by (4) and (7), there exists a constant C_λ such that

$$\lambda F(x, t, s) \le \frac{\lambda_{p,q}}{2k \max\{p, q\}} H(t, s) + C_\lambda, \quad \forall\, (x, t, s) \in \Omega \times \mathbb{R} \times \mathbb{R}. \tag{9}$$

Hence, using (5), we have

$$\begin{aligned} \Phi_\lambda(u, v) &\ge \int_\Omega \left[\frac{1}{p} |\nabla u|^p + \frac{1}{q} |\nabla v|^q - \frac{\lambda_{p,q}}{2k \max\{p, q\}} H(u, v) - C_\lambda \right] dx && (10) \\ &\ge \frac{1}{p} \int_\Omega |\nabla u|^p + \frac{1}{q} \int_\Omega |\nabla v|^q - \frac{\lambda_p}{2p} \int_\Omega |u|^p - \frac{\lambda_q}{2q} \int_\Omega |v|^q - \int_\Omega C_\lambda && (11) \\ &= \frac{1}{2} \Big(||u||^p + ||v||^q \Big) - C_\lambda |\Omega|. && (12) \end{aligned}$$

So, Φ_λ is bounded from below and coercive. This yields a global minimizer (u_1, v_1) since Φ_λ is weakly lower semi-continuous.

As a consequence of the next lemma, the solution (u_1, v_1) is nontrivial for λ large.

Lemma 2.1. *There is a* $\overline{\lambda}$ *such that* $\inf \Phi_\lambda < 0$ *for* $\lambda \geq \overline{\lambda}$.

Proof. Let us consider a sufficiently large compact subset B' of B, where B is a ball such that $F(x, t_0, s_0) > 0$ for $x \in B$ and some $t_0, s_0 > 0$. Consider u_0 and v_0, smooth functions with compact support in B, such that

$$u_0(x) = t_0, v_0(x) = s_0, \text{ in } B',$$
$$0 \leq u_0(x) \leq t_0, 0 \leq v_0(x) \leq s_0, \text{ in } B \setminus B'.$$

Then, we obtain

$$\begin{aligned}\int_\Omega F(x, u_0, v_0) &= \int_{B'} F(x, u_0, v_0) + \int_{B\setminus B'} F(x, u_0, v_0)\\ &\geq \int_{B'} F(x, t_0, s_0) - C\int_{B\setminus B'} (|u_0|^p + |v_0|^q)\\ &\geq \int_{B'} F(x, t_0, s_0) - C(|t_0|^p + |s_0|^q)|B \setminus B'| > 0,\end{aligned}$$

provided $|B \setminus B'|$ is small enough. Hence

$$\Phi_\lambda(u_0, v_0) = \frac{1}{p}\int_\Omega |\nabla u_0|^p + \frac{1}{q}\int_\Omega |\nabla v_0|^q - \lambda\int_\Omega F(x, u_0, v_0) < 0, \tag{13}$$

for λ large enough. So there is $\overline{\lambda}$ such that $\inf \Phi_\lambda < 0$, for $\lambda \geq \overline{\lambda}$. □

Now we fix $\lambda \geq \overline{\lambda}$. Then (u_1, v_1), the global minimum of Φ_λ, is a positive solution of (P_λ), provided $\Phi_\lambda(u_1, v_1) < 0$.

Lemma 2.2. *The origin is a strict local minimizer of* Φ_λ.

Proof. It follows from (4) and (6) that there is $C_\lambda > 0$ such that

$$\lambda F(x, t, s) \leq \frac{\lambda_{p,q}}{2k\max\{p, q\}} G(t, s) + C_\lambda(|t|^\alpha + |s|^\beta), \quad \forall\, (x, t, s) \in \Omega \times \mathbb{R} \times \mathbb{R}, \tag{14}$$

where $p < \alpha < p^*$ and $q < \beta < q^*$ (here: $p^* = pN/(N-p)$ if $N > p$ and $p^* = \infty$ if $N \leq p$; and $q^* = qN/(N-q)$ if $N > q$ and $q^* = \infty$ if $N \leq q$).

Hence

$$\begin{aligned}\Phi_\lambda(u, v) &= \frac{1}{p}\int_\Omega |\nabla u|^p + \frac{1}{q}\int_\Omega |\nabla v|^q - \frac{\lambda_{p,q}}{2k\max\{p, q\}}\int_\Omega G(u, v)\\ &\quad - C_\lambda\int_\Omega (|u|^\alpha + |v|^\beta)\\ &\geq \frac{1}{p}\int_\Omega |\nabla u|^p + \frac{1}{q}\int_\Omega |\nabla v|^q - \frac{\lambda_p}{2p}\int_\Omega |u|^p - \frac{\lambda_p}{2p}\int_\Omega |u|^p\\ &\quad - C_\lambda\int_\Omega (|u|^\alpha + |v|^\beta).\end{aligned}$$

By the Sobolev embeddings, we get

$$\Phi_\lambda(u, v) \geq \frac{1}{2p}||u||^p + \frac{1}{2q}||v||^q - C_\lambda||u||^\alpha - C_\lambda|||v|||^\beta,$$

which shows that there are positive numbers a and r such that $\Phi(u, v) > a$ for $||u|| + ||v|| \leq r$. □

Conclusion of the proof.

In order to prove the existence of the second solution, we will apply the mountain pass theorem. First, observe that Φ_λ is coercive, so we have that Φ_λ satisfies the Palais-Smale condition. From the above discussion, we know that $(0,0)$ is a local minimum of Φ_λ on E and $\Phi_\lambda(u_1,v_1)<0$. Now, consider the minimax level

$$c := \inf_{\gamma\in\Gamma} \max_{(u,v)\in\gamma([0,1])} \Phi_\lambda(u,v) > 0,$$

where $\Gamma = \{\gamma \in C([0,1], W_0^{1,p}\times W_0^{1,q}) : \gamma(0)=(0,0), \quad \gamma(1)=(u_1,v_1)\}$. The mountain pass theorem gives a critical point (u_2,v_2) of Φ_λ at level c (see [8]). Thus, (u_2,v_2) is a second positive solution of (P_λ). □

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