A REMARK ON MULTIPLICITY OF POSITIVE SOLUTIONS FOR A CLASS OF QUASILINEAR ELLIPTIC SYSTEMS

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Abstract. Using variational methods, we prove a nonexistence and multiplicity result of positive solutions for a class of elliptic systems involving a parameter.

1. Introduction

In this paper we study multiplicity of positive solutions for the elliptic system

\[
\begin{aligned}
-\Delta_p u &= \lambda F_u(x,u,v), & \text{in } \Omega, \\
-\Delta_q v &= \lambda F_v(x,u,v), & \text{in } \Omega, \\
u = v = 0, & \text{on } \partial \Omega,
\end{aligned}
\]

where \( \Omega \) is an open bounded domain with smooth boundary \( \partial \Omega \), \( 1 < p, q < \infty \), and \( \lambda \) a positive real parameter. The nonlinearity \( F : \Omega \times \mathbb{R} \times \mathbb{R} \to \mathbb{R} \) is a \( C^1 \) function, satisfying

\[
F(x,t,s) = F(x,0,s) \quad \forall \ x \in \Omega, \ s \in \mathbb{R} \text{ and } t \leq 0;
\]

\[
F(x,t,s) = F(x,t,0) \quad \forall \ x \in \Omega, \ t \in \mathbb{R} \text{ and } s \leq 0.
\]

It follows that \( F_t(x,0,0) = F_s(x,0,0) = 0 \), so that \( (P_\lambda) \) has the trivial solution \((0,0)\). Moreover, assume that \( F \) satisfies the following condition

\[
|F_t(x,t,s)| \leq C(1 + |t|^{p-1} + |s|^{q-1}) \quad \text{and} \quad |F_s(x,t,s)| \leq C(1 + |s|^{q-1} + |t|^{q-1})
\]

for all \((x,t,s) \in \Omega \times \mathbb{R} \times \mathbb{R}\). Under this assumptions the weak solutions of \((P_\lambda)\) are critical points of the following \( C^1 \)-functional, defined on \( E = W^{1,p}_0(\Omega) \times W^{1,q}_0(\Omega) \),

\[
\Phi_\lambda(u,v) = \int_\Omega \left( \frac{1}{p} |\nabla u|^p + \frac{1}{q} |\nabla v|^q - \lambda F(x,u,v) \right) dx,
\]

(see [2]). We remark that (2) implies that \( F \) satisfies the following growth condition:

\[
|F(x,t,s)| \leq C(1 + |t|^p + |s|^q), \quad \forall \ (x,t,s) \in \Omega \times \mathbb{R} \times \mathbb{R}.
\]

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This condition was called “of resonant type” in [2].

It is known that the existence and multiplicity of solutions of \((P_\lambda)\) rely strongly on the behavior of \(F\) at origin and at infinity. Our hypotheses are motivated by an eigenvalue problem considered in [2]. Let \(G : \mathbb{R}^2 \to [0, \infty)\) be a \(C^1\) function such that

\[ G(t, s) \leq k(|t|^p + |s|^q). \]

Now, we introduce the following assumptions:

\[ \limsup_{||(t, s)|| \to 0} \frac{F(x, t, s)}{G(t, s)} \leq 0, \quad \text{uniformly in } x \in \Omega, \]

and

\[ \limsup_{||(t, s)|| \to \infty} \frac{F(x, t, s)}{H(t, s)} \leq 0, \quad \text{uniformly in } x \in \Omega, \]

where \(G\) and \(H\) are functions satisfying (5).

**Theorem 1.1.** Under the above hypotheses there exists a \(\lambda\) such that \((P_\lambda)\) has no positive solution for \(\lambda < \lambda\). Moreover, if in addition we assume that there exist a ball \(B \subset \Omega\) and \(t_0, s_0 > 0\) such that \(F(x, t_0, s_0) > 0\) for \(x \in B\), then there is a \(\lambda\) such that \((P_\lambda)\) has at least two positive solutions for \(\lambda \geq \lambda\).

**Remark 1.** By a positive solution we mean a nonnegative pair \((u, v) \in E\) such that

\[ 0 = \int_\Omega \left| \nabla u \right|^{p-2} \nabla u \nabla \phi + \int_\Omega \left| \nabla v \right|^{q-2} \nabla v \nabla \psi - \lambda \int_\Omega \left( F_u(x, u, v) \phi + F_v(x, u, v) \psi \right) dx. \]

and either \(u > 0\) in \(\Omega\) or \(v > 0\) in \(\Omega\).

**Remark 2.** Theorem 1.1 contains and extends the results of Fernández Bonder in [4] and Manouni & Perera in [6]. In [4, 6], the authors studied the system \((P_\lambda)\), with similar conditions on \(F\), but with particular classes of \(G\), namely, \(G(u, v) = |u|^p + |v|^q\) and \(G(u, v) = u^{p+1} v^{q+1}\) with \(\frac{1}{p+1} + \frac{1}{q+1} = 1\), respectively. Related results for the simple equation can be found in [5, 7]. Systems of the form \((P_\lambda)\) are usually called gradient systems. See [2], for a comprehensive analysis of such systems.

**Notation.** In what follows, we will denote by \(\lambda_{p, q} = \min\{\lambda_p, \lambda_q\}\), where \(\lambda_p\) and \(\lambda_q\) denote the main eigenvalues of \((-\Delta_p, W_0^{1,p}(\Omega))\) and \((-\Delta_q, W_0^{1,q}(\Omega))\), respectively. We shall often use the same letter \(C\) to denote different constants, and \(C_\lambda\) to denote different constants depending only on \(\lambda\).

**Remark 3.** We have, for \((u, v) \in E = W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega)\),

\[ \lambda_p \int_\Omega |u|^p dx \leq \int_\Omega |\nabla u|^p dx \quad \text{and} \quad \lambda_q \int_\Omega |v|^q dx \leq \int_\Omega |\nabla v|^q dx, \]

from which it follows that

\[ 0 < \lambda_{p, q} \leq \frac{\int_\Omega |\nabla u|^p + |\nabla v|^q dx}{\int_\Omega |u|^p + \int_\Omega |v|^q dx} \quad \text{for } (u, v) \in E. \]
2. Proof of Theorem 1.1.

Proof of nonexistence part:
If \((P_\lambda)\) has a positive solution \((u, v)\), multiplying the first equation of \((P_\lambda)\) by \(u\), the second by \(v\), and integrating by parts we get
\[
\int_\Omega |\nabla u|^p + \int_\Omega |\nabla v|^q = \lambda \int_\Omega \left( F_u(x, u, v)u + F_v(x, u, v)v \right) dx.
\]
Using (2) and Holder’s inequality, we can show that
\[
|F_u(x, u, v)u + F_v(x, u, v)v| \leq C(u^p + v^q),
\]
and so
\[
\int_\Omega |\nabla u|^p + \int_\Omega |\nabla v|^q \leq \lambda C \int_\Omega (|u|^p + |v|^q) dx.
\]
It follows that \(\lambda \geq \lambda_{p,q}/C\), by (8), which implies that there is no positive solution for \(\lambda\) small. \(\square\)

Proof of multiplicity part:
First, observe that nontrivial critical points of \(\Phi_\lambda\) are positive solutions of \((P_\lambda)\). Indeed, if \((u, v)\) is a critical point of \(\Phi_\lambda\), denoting by \(u^-\) and \(v^-\) the negative parts of \(u\) and \(v\), respectively, we have
\[
0 = [\Phi_\lambda'(u, v), (u^-, v^-)]
\]
\[
= \int_\Omega \left[ |\nabla u|^{p-2} \nabla u \nabla u^- + |\nabla v|^{q-2} \nabla v \nabla v^- \right] dx
\]
\[
- \lambda \int_\Omega \left[ (F_u(x, u, v)u^- + F_v(x, u, v)v^-) \right] dx
\]
\[
= -||u^-|| - ||v^-|| - \lambda \int_\Omega \left[ (F_u(x, u, v)u^- + F_v(x, u, v)v^-) \right] dx.
\]
By (1), we have that \(F_u(x, u, v)u^- = 0\) and \(F_v(x, u, v)v^- = 0\). It follows that \(||u^-|| + ||v^-|| = 0\), and so \(u, v \geq 0\). Also, by [1, 3], we have that \(u, v \in L^\infty(\Omega) \cap C^1(\Omega)\). Consequently, by the Harnack inequality, we get that either \(u > 0\) in \(\Omega\) or \(v > 0\) in \(\Omega\), provided \((u, v) \neq (0, 0)\) (see [9]). Thus, nontrivial critical points of \(\Phi_\lambda\) are positive solutions of \((P_\lambda)\).

Consider \(k\) the constant in (5). Then by (4) and (7), there exists a constant \(C_\lambda\) such that
\[
\lambda F(x, t, s) \leq \frac{\lambda_{p,q}}{2k \max\{p, q\}} H(t, s) + C_\lambda, \quad \forall (x, t, s) \in \Omega \times \mathbb{R} \times \mathbb{R}.
\]
Hence, using (5), we have
\[
\Phi_\lambda(u, v) \geq \int_\Omega \left[ \frac{1}{p} |\nabla u|^p + \frac{1}{q} |\nabla v|^q - \frac{\lambda_{p,q}}{2k \max\{p, q\}} H(u, v) - C_\lambda \right] dx
\]
\[
\geq \frac{1}{p} \int_\Omega |\nabla u|^p + \frac{1}{q} \int_\Omega |\nabla v|^q - \frac{\lambda_{p,q}}{2k \max\{p, q\}} \int_\Omega |u|^p - \frac{\lambda_{p,q}}{2} \int_\Omega |v|^q - \int_\Omega C_\lambda
\]
\[
= \frac{1}{2} (||u||^p + ||v||^q) - C_\lambda ||\nabla||.
\]
So, \(\Phi_\lambda\) is bounded from below and coercive. This yields a global minimizer \((u_1, v_1)\) since \(\Phi_\lambda\) is weakly lower semi-continuous.

As a consequence of the next lemma, the solution \((u_1, v_1)\) is nontrivial for \(\lambda\) large.
Lemma 2.1. There is a $\overline{\lambda}$ such that $\inf \Phi_\lambda < 0$ for $\lambda \geq \overline{\lambda}$.

Proof. Let us consider a sufficiently large compact subset $B'$ of $B$, where $B$ is a ball such that $F(x,t_0,s_0) > 0$ for $x \in B$ and some $t_0,s_0 > 0$. Consider $u_0$ and $v_0$, smooth functions with compact support in $B$, such that

$u_0(x) = t_0, v_0(x) = s_0$, in $B'$,

$0 \leq u_0(x) \leq t_0, 0 \leq v_0(x) \leq s_0$, in $B \setminus B'$.

Then, we obtain

$$\int_{\Omega} F(x,u_0,v_0) = \int_{B'} F(x,u_0,v_0) + \int_{B \setminus B'} F(x,u_0,v_0) \geq \int_{B'} F(x,t_0,s_0) - C \int_{B \setminus B'} (|u_0|^p + |v_0|^q)$$

provided $|B \setminus B'|$ is small enough. Hence

$$\Phi_{\lambda}(u_0,v_0) = \frac{1}{p} \int_{\Omega} (\nabla u_0)^p + \frac{1}{q} \int_{\Omega} (\nabla v_0)^q - \lambda \int_{\Omega} F(x,u_0,v_0) < 0,$$

for $\lambda$ large enough. So there is $\overline{\lambda}$ such that $\inf \Phi_{\lambda} < 0$, for $\lambda \geq \overline{\lambda}$. □

Now we fix $\lambda \geq \overline{\lambda}$. Then $(u_1,v_1)$, the global minimum of $\Phi_\lambda$, is a positive solution of $(P_\lambda)$, provided $\Phi_{\lambda}(u_1,v_1) < 0$.

Lemma 2.2. The origin is a strict local minimizer of $\Phi_\lambda$.

Proof. It follows from (4) and (6) that there is $C_\lambda > 0$ such that

$$\lambda F(x,t,s) \leq \frac{\lambda p_{p,q}}{2k \max\{p,q\}} G(t,s) + C_\lambda (\alpha + |s|^{\beta}), \quad \forall (x,t,s) \in \Omega \times \mathbb{R} \times \mathbb{R},$$

where $p < \alpha < p^*$ and $q < \beta < q^*$ (here: $p^* = pN/(N-p)$ if $N > p$ and $p^* = \infty$ if $N \leq p$; and $q^* = qN/(N-q)$ if $N > q$ and $q^* = \infty$ if $N \leq q$).

Hence

$$\Phi_{\lambda}(u,v) = \frac{1}{p} \int_{\Omega} |\nabla u|^p + \frac{1}{q} \int_{\Omega} |\nabla v|^q - \frac{\lambda p_{p,q}}{2k \max\{p,q\}} \int_{\Omega} G(u,v) - C_\lambda \int_{\Omega} (|u|^{\alpha} + |v|^{\beta})$$

$$\geq \frac{1}{p} \int_{\Omega} |\nabla u|^p + \frac{1}{q} \int_{\Omega} |\nabla v|^q - \frac{\lambda p}{2p} \int_{\Omega} |u|^p - \frac{\lambda q}{2q} \int_{\Omega} |v|^q$$

$$- C_\lambda \int_{\Omega} (|u|^{\alpha} + |v|^{\beta}).$$

By the Sobolev embeddings, we get

$$\Phi_{\lambda}(u,v) \geq \frac{1}{2p} ||u||^p + \frac{1}{2q} ||v||^q - C_\lambda ||u||^\alpha - C_\lambda ||v||^\beta,$$

which shows that there are positive numbers $a$ and $r$ such that $\Phi(u,v) > a$ for $||u|| + ||v|| \leq r$. □

Conclusion of the proof.
In order to prove the existence of the second solution, we will apply the mountain pass theorem. First, observe that \( \Phi_\lambda \) is coercive, so we have that \( \Phi_\lambda \) satisfies the Palais-Smale condition. From the above discussion, we know that \((0, 0)\) is a local minimum of \( \Phi_\lambda \) on \( E \) and \( \Phi_\lambda(u, v_1) < 0 \). Now, consider the minimax level
\[
c := \inf_{\gamma \in \Gamma} \max_{(u, v) \in \gamma([0, 1])} \Phi_\lambda(u, v) > 0,
\]
where \( \Gamma = \{ \gamma \in C([0, 1], W^{1,p}_0 \times W^{1,q}_0) : \gamma(0) = (0, 0), \gamma(1) = (u_1, v_1) \} \). The mountain pass theorem gives a critical point \((u_2, v_2)\) of \( \Phi_\lambda \) at level \( c \) (see [8]). Thus, \((u_2, v_2)\) is a second positive solution of \((P_\lambda)\).

References


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