Bifurcation of stable equilibria and nonlinear flux boundary condition with indefinite weight *†

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Abstract

We study bifurcation and stability of positive equilibria of a parabolic problem under a nonlinear Neumann boundary condition having a parameter and an indefinite weight.

Local and global structures of the set of equilibria is given. While the stability of constant equilibria is analyzed, the exponential stability of the unique bifurcating nonconstant equilibrium solution is established. Diagrams exhibiting the bifurcation and stability structures are also furnished. Moreover the asymptotic behavior of such solutions on the boundary of the domain, as the positive parameter goes to infinity, is also provided.

The results are obtained via classical tools like the implicit function theorem, bifurcation from a simple eigenvalue theorem and the exchange of stability principle, in a combination with variational and dynamical arguments.

1 Introduction

Many problems appearing in several fields of knowledge are related to boundary value problems involving partial differential equations. For instance in Physics, Biology, Chemistry, and others. Particularly in Biology, an interesting problem comes from population genetics, namely, the problem of selection-migration for alleles in a given region of space. For a population confined in such a region, a model describing the changes of gene frequency considering natural selection effects only in the interior of the region and no flux throughout the boundary was introduced by Fisher [13] and generalized by Fleming [14]

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and Henry [17] in a new approach. It gives rise to a nonlinear parabolic equation supplied with a homogeneous Neumann boundary condition studied under various aspects by several authors, for instance [8, 14, 17, 18, 26] and in references therein. More specifically in [17] the following problem is considered

$$(\boldsymbol{N_0}) \qquad \begin{cases} \partial_t u = \Delta u + \lambda s(x) f(u) & \text{in} \quad \Omega \times \mathbb{R}^+, \\ \\ \dfrac{\partial u}{\partial \nu} = 0 \quad \text{on} \quad \partial\Omega \times \mathbb{R}^+. \end{cases}$$

This work is devoted to study a parabolic problem which is somehow related (N_0). More precisely, we consider the problem

$$\begin{cases} \partial_t u = \Delta u \quad \text{in} \quad \Omega \times \mathbb{R}^+ \\ \\ \dfrac{\partial u}{\partial \nu} = \lambda s(x) f(u) \quad \text{on} \quad \partial\Omega \times \mathbb{R}^+, \end{cases} \tag{1}$$

in a smooth bounded domain $\Omega \subset \mathbb{R}^n$, $n \geq 2$. The hypotheses are the same as those found in [17] for (N_0). Namely, λ is a positive parameter and $f : \mathbb{R} \longrightarrow \mathbb{R}$ is a smooth function, say of class C^4, satisfying

$$(\boldsymbol{H_1}) \qquad \begin{cases} f > 0 \quad \text{in} \quad (0,1), \qquad f(0) = 0 = f(1), \\ \\ f'(0) > 0, \quad f'(1) < 0. \end{cases}$$

The weight $s : \partial\Omega \longrightarrow \mathbb{R}$ is of class $C^{1,\theta}(\partial\Omega)$, $0 < \theta < 1$, and is a *sign-changing* function such that

$$(\boldsymbol{H_2}) \qquad \int_{\partial\Omega} s(x)\, d\mathcal{H}^{n-1} \neq 0.$$

Our primary goal is to show that the global bifurcation diagram of positive stationary solutions to (1), with respect to the parameter λ, as well as its stability properties are the same as those encountered in [17] for its counterpart (N_0). In addition, as will be seen, the asymptotic behavior of such solutions, as $\lambda \to \infty$, in $\partial\Omega$ is very much like the one presented in [17] for the positive stationary solutions to (N_0) in the entire domain Ω.

The interested reader is referred to [18, 19, 20, 8, 26], and the more recent works [25, 21], for generalizations in several directions or discussions on modeling aspects. We emphasize here we deal with (1) on arbitrary smooth domains of dimension greater than one.

A suitable phase space to treat parabolic problem (1) taking into account the physical motivation provided by (N_0) is

$$\mathfrak{X} \doteq \big\{ u \in H^1(\Omega) : 0 \leq u(x) \leq 1 \quad \text{a.e. } x \in \Omega \big\}.$$

Since problem (1) generates a dynamical system in $\mathfrak{X}$ which is a gradient system, see [22], the equilibrium solutions play a fundamental role in dynamics of (1) for large times. They are the solutions of

$$\begin{cases} \Delta u = 0 \quad \text{in} \quad \Omega \\ \\ \dfrac{\partial u}{\partial \nu} = \lambda s(x) f(u) \quad \text{on} \quad \partial\Omega \\ \\ u \in \mathfrak{X}. \end{cases} \tag{2}$$

Our main goal in this paper is to study the (local and global) bifurcation and (local) stability structures of equilibria to (1) for all $\lambda > 0$. The investigation developed herein is mainly based on classical results. Namely, the Implicit Function Theorem, the bifurcation from a simple eigenvalue theorem and the exchange of stability principle, due to Crandall and Rabinowitz [11, 12], in a combination of variational and dynamical arguments.

For other results on existence of nonconstant stable equilibrium solutions to parabolic problems under nonlinear boundary conditions see [5, 6, 10] and the references therein.

To employ the methods from bifurcation theory we introduce the nonlinear mapping $\mathscr{F} : W_p^2(\Omega) \longrightarrow L^p(\Omega) \times W_p^{1-1/p}(\partial\Omega)$, $p > n$, given by

$$\mathscr{F}(\lambda, u) = \left(\Delta u, \frac{\partial u}{\partial \nu} - \lambda s(\cdot) f(u)\right).$$

Such a mapping is of class C^3 in Fréchet sense and $u \in W_p^2(\Omega)$ is a solution to (2) corresponding to λ if, and only if, $(\lambda, u) \in \mathscr{F}^{-1}(0,0)$ and $u \in \mathfrak{X}$.

Concerning bifurcation, we will establish the complete diagrams related to the equilibria to (1), that is, to the solutions to elliptic problem (2) in $\mathfrak{X}$. Those diagrams depend on the sign of the average of $s(\cdot)$ over $\partial\Omega$; in fact, this and a necessary condition for bifurcation we prove will characterize from which trivial branch bifurcation occurs. The trivial branches are the curves

$$\Gamma_0 \doteq \{(\lambda, 0) : \lambda > 0\} \qquad \text{and} \qquad \Gamma_1 \doteq \{(\lambda, 1) : \lambda > 0\}$$

determined by the constant equilibria $u \equiv 0$ and $u \equiv 1$ of (1).

The analysis developed in this paper will be concentrated in the situation when the weight $s(\cdot)$ has negative average over $\partial\Omega$. In such case, bifurcation occurs only with respect to Γ_0 and that is the branch furnishing the main results related to (1), as will be seen. Actually, the change of variables $\sigma : \mathbb{R} \to \mathbb{R}$ given by $\sigma(u) = 1 - u$ allows one to transport all analysis made for Γ_0 to Γ_1, yielding perfectly symmetric results that can be read by interchanging the roles of $u \equiv 0$ and $u \equiv 1$. Thus, *along this paper we assume*

$$\int_{\partial\Omega} s(x)\, d\mathcal{H}^{n-1} < 0.$$

But it should be remarked that nonzero average of $s(\cdot)$ is essential for the approach adopted here and to assure the precise structure of the solution set of (2) we establish takes place. In fact, if $s(\cdot)$ has null average over $\partial\Omega$ the so-called Crandall-Rabinowitz transversality condition is no longer true for elliptic problem (2), precluding one to use the classical results of [11, 12]. The investigation of that situation will be carried out elsewhere.

Related to the structure of the solution set of (2), in Section 2 one encounters firstly that (1) does not have equilibrium solutions other than the constant ones for $\lambda > 0$ small. This is a consequence of a theorem of author's previous work [23] (cf. Theorem 2.1). Searching for bifurcation points with respect to the trivial branches, we prove in Subsection 2.1 a necessary condition for bifurcation related to the (positive) principal eigenvalue λ_0 of

$$\begin{cases} \Delta v = 0 & \text{in} \quad \Omega \\ \dfrac{\partial v}{\partial \nu} = \lambda\omega(x)v & \text{on} \quad \partial\Omega \end{cases}$$

with suitable weights $\omega(\cdot)$ (cf. Theorem 2.2). The role of the eigenvalue λ_0 in the analysis depends on whether it is positive or zero, what is connected with the sign of the average of $s(\cdot)$ over $\partial\Omega$ and the trivial branch under analysis (see Remark 2.1). Those information will give us condition to infer that bifurcation can only occur with respect to one trivial branch (see also Corollary 2.1).

Once the partial derivatives $D_u\mathscr{F}(\lambda_0, \bar{u})$ are not injective, where $\bar{u} \equiv 0$ or $\bar{u} \equiv 1$, they are not surjective since they are Fredholm operators of zero index. We are able to furnish in Theorem 2.3 a precise class for which the nonsurjectivity is verified. Equivalently, we prove a nonexistence result to a nonhomogeneous Neumann indefinite linear elliptic problem, carried over by using variational arguments and a Piccone's identity.

Next step, from local viewpoint, is to prove in Subsection 2.2 the existence of a smooth curve bifurcating from the appropriate trivial branch and to conclude the existence of only one bifurcation point from that branch (cf. Theorem 2.4 and Corollary 2.2). This is achieved as a consequence of Crandall and Rabinowitz theorem [11] and the necessary condition for bifurcation from trivial branches previously established in Theorem 2.2. Under the additional hypothesis of strict concavity of f on $[0, 1]$, we also prove in Theorem 2.5 the bifurcation occuring is transcritical.

For the global analysis of solution set of (2), we first prove in Subsection 2.3 a key tool for using the Implicit Function Theorem. Namely, the injectivity of the linearization around nonconstant equilibria to (1) (cf. Theorem 2.6). Some consequences of such a result are so derived like nonexistence of secondary bifurcation (Corollary 2.3) and the precise region of uniqueness of trivial equilibria for $\lambda > 0$ small (Theorem 2.7). Finally, in Subsection 2.4 we extend the local bifurcating curve to a smooth global curve containing all nonconstant solutions to (2) (Theorem 2.8) and prove the uniqueness of nontrivial equilibrium solutions to (1) (Theorem 2.9). The bifurcation analysis is concluded in Subsection 2.5 by drawing the complete bifurcation diagrams according to the sign of the average of $s(\cdot)$ over $\partial\Omega$.

Next issue is to analyze in Section 3 the stability properties (in the Lyapunov sense) of trivial and bifurcating equilibria to the parabolic problem (1). Initially, we treat in

Subsection 3.1 the stability of trivial equilibria. We prove (cf. Theorem 3.1) the solution $u \equiv 0$ is exponentially stable for $0 < \lambda < \lambda_0$ and unstable for $\lambda > \lambda_0$, while $u \equiv 1$ is unstable for all $\lambda > 0$. We recall those conclusions correspond to the case of negative average of $s(\cdot)$ over $\partial\Omega$; otherwise, the role of $u \equiv 0$ is played by $u \equiv 1$ vice-versa.

In the critical case of stability for $\lambda = \lambda_0$, we prove by a dynamical argument that $u \equiv 0$ is asymptotically stable. Indeed, this is possible since the Lyapunov functional to the dynamical system generated by (1) has a global minimum and, by Theorem 2.7, the only equilibria to (1) are the constant ones for $\lambda = \lambda_0$ (cf. Theorem 3.2).

In Subsection 3.2 we prove that the exponential stability is transferred to the global smooth curve bifurcating from Γ_0 through a detailed analysis based on the exchange of stability principle [12] (see Theorem 3.3). We then collect all stability results obtained and provide new diagrams having complete information on the structure of solutions to (2) from bifurcation/stability viewpoint.

Finally, in Section 4 we study the behavior of the solution to (2) for λ large. Under mild hypotheses, such as $s^{-1}(0)$ has zero Hausdorff measure and $\mathcal{M} = s^{-1}(0, +\infty)$ has finite capacity, we prove the trace of the solution to (2) tends to concentrate on $\mathcal{M}$. More precisely, we prove (see Theorem 4.1) the trace converges to the characteristic function $\chi_{\mathcal{M}}$, as λ goes to infinity, in $L^p(\partial\Omega)$ for all $1 < p < \infty$.

2 Bifurcation structure of equilibria

We study in this section bifurcation of equilibrium solutions to (1). More precisely, we completely describe the structure of the solution set of (2) and furnish the corresponding bifurcation diagrams.

2.1 Necessary condition for bifurcation from trivial branches and the nonsurjectivity of derivative

The first result related to the solution set of equilibria can be proved in the same manner as in [23], so we only state it.

Theorem 2.1 *Suppose* $(\boldsymbol{H_2})$ *holds true. Then for all* $\lambda > 0$ *sufficiently small the only equilibria to* (1) *are the constant ones.*

Later in Theorem 2.7 we will be able to precisely describe the uniqueness λ-region where Theorem 2.1 is valid.

It is well known that a necessary condition for bifurcation from the trivial branches Γ_0 and Γ_1 is the failure of injectivity of the operators

$$D_u\mathscr{F}(\lambda, \bar{u}) : W_p^2(\Omega) \longrightarrow L^p(\Omega) \times W_p^{1-1/p}(\partial\Omega)$$

given by

$$D_u\mathscr{F}(\lambda, \bar{u}) \cdot v = \Big(\Delta v,\ \frac{\partial v}{\partial \nu} - \lambda s(\cdot) f'(\bar{u}) v\Big)$$

for all $v \in W_p^2(\Omega)$, where $\bar{u} \equiv 0$ or $\bar{u} \equiv 1$, respectively, and $p > n$. But a more specific condition is given in the following

Theorem 2.2 *If $\lambda > 0$ is a bifurcation point with respect to the trivial branch Γ_0 (respect. Γ_1), then λ is a principal eigenvalue of*

$$\begin{cases} \Delta v = 0 \quad in \quad \Omega \\ \\ \dfrac{\partial v}{\partial \nu} = \lambda \omega(x) v \quad on \quad \partial\Omega \end{cases} \tag{3}$$

where $\omega(\cdot) = f'(0)s(\cdot)$ (respect. $\omega(\cdot) = f'(1)s(\cdot)$)

Proof: We prove the case related to Γ_0. By hypothesis, there exists a sequence $\{u_{\lambda_k}\}$ of solutions to (2) satisfying

$$\lambda_k \overset{k\to\infty}{\longrightarrow} \lambda \qquad \text{and} \qquad 0 \not\equiv u_{\lambda_k} \overset{k\to\infty}{\longrightarrow} 0 \quad \text{in} \quad H^1(\Omega).$$

Considering the sequence $v_k := \frac{u_{\lambda_k}}{||u_{\lambda_k}||_{L^2(\partial\Omega)}}$, by the weak formulation of (2) we have

$$\int_\Omega \nabla v_k \cdot \nabla \phi \, dx = \lambda_k \int_{\partial\Omega} s(x) f'(0) v_k \phi \, d\mathcal{H}^{n-1} + \lambda_k \int_{\partial\Omega} s(x) \rho_k v_k \phi \, d\mathcal{H}^{n-1} \tag{4}$$

for all $\phi \in H^1(\Omega)$, where $\rho_k = O(u_{\lambda_k})$ as $k \to \infty$. Taking $\phi = v_k$ as a test function in (4) and using Ma'zja's inequality (see [24])

$$\int_\Omega v_{\lambda_k}^2 \, dx \le C \left[\int_\Omega |\nabla v_{\lambda_k}|^2 \, dx + \int_{\partial\Omega} v_{\lambda_k}^2 \, d\mathcal{H}^{n-1} \right]$$

we can infer that $\{v_k\}$ is a bounded sequence in $H^1(\Omega)$. Thus, by passing to a subsequence if necessary, by compactness there exists $v_\lambda \in H^1(\Omega)$ such that as $k \to \infty$

$v_k \rightharpoonup v_\lambda$ in $H^1(\Omega)$,

$v_k \to v_\lambda$ in $L^2(\partial\Omega)$,

$v_k \to v_\lambda$ a.e. in $\partial\Omega$.

We are now in position to pass to the limit in (4) getting

$$\int_\Omega \nabla v_\lambda \cdot \nabla \phi \, dx = \lambda \int_{\partial\Omega} f'(0) s(x) v_\lambda \phi \, d\mathcal{H}^{n-1}$$

for all $\phi \in H^1(\Omega)$. Therefore, v_λ is a positive (weak) solution to (3) with $\omega(\cdot) = f'(0)s(\cdot)$ what implies that $\lambda > 0$ is a principal eigenvalue to (3). □

Remark 2.1 *As Theorem 2.2 indicates the possible bifurcation points with respect to the trivial branches Γ_0 and Γ_1 are in the set of principal eigenvalues to (3). It was proved in [27] that the number*

$$\lambda_0 \;=\; \inf \left\{ \frac{\int_\Omega |\nabla v|^2 \, dx}{\int_{\partial\Omega} \omega(x) v^2 \, d\mathcal{H}^{n-1}} \;:\; v \in H^1(\Omega) \quad and \quad \int_{\partial\Omega} \omega(x) v^2 \, d\mathcal{H}^{n-1} > 0 \right\} \tag{5}$$

is the only positive principal eigenvalue to (3) *if, and only if,* $\omega(\cdot)$ *changes sign and has a negative average over* $\partial\Omega$*. Further, if* $\omega(\cdot)$ *changes sign and has a nonnegative average over* $\partial\Omega$*, then* (3) *has no positive principal eigenvalues.*

Corollary 2.1 *There is no bifurcation with respect to* Γ_1*.*

Proof: Recalling $(\boldsymbol{H_1})$, this follows readily from Theorem 2.2 and Remark 2.1. □

Let us state a result from [3] known as Picone's identity that will be useful in next ones.

Lemma 2.1 *Let* $v > 0$ *and* $u \geq 0$ *be continuous functions in* Ω*, almost everywhere differentiable. Setting*

$$\begin{aligned} \mathcal{L}(u,v) &= |\nabla u|^2 + \frac{u^2}{v^2}|\nabla v|^2 - 2\frac{u}{v}\nabla v \cdot \nabla u \\ \mathcal{R}(u,v) &= |\nabla u|^2 - \nabla\left(\frac{u^2}{v}\right) \cdot \nabla v \end{aligned}$$

the following holds.

(i) $\mathcal{L}(u,v) = \mathcal{R}(u,v)$.

(ii) $\mathcal{L}(u,v) \geq 0$ *a.e. in* Ω.

(iii) $\mathcal{L}(u,v) = 0$ *a.e. in* Ω *if, and only if,* $u = kv$ *for some* $k \in \mathbb{R}$.

Remark 2.2 *Still related to* (3) *a fact needed later and not treated in [27] is that* λ_0 *has geometric multiplicity one. That is,* $\dim\ker(D_u\mathscr{F}(\lambda_0,\bar{u}) - \mu I) = 1$ *for* $\mu = 0$*, where* I *is the inclusion operator* $W_p^2(\Omega) \hookrightarrow L^p(\Omega) \times W_p^{1-1/p}(\partial\Omega)$*,* $p > n$*, and* $\bar{u} \equiv 0$ *or* $\bar{u} \equiv 1$*.*

To see that in a direct way, first note the assertion is trivial if $\omega(\cdot)$ *has a nonnegative average over* $\partial\Omega$ *since in this case* $\lambda_0 = 0$*. Otherwise, let* u, v *be eigenfunctions to* (3) *corresponding to* $\lambda_0 > 0$*, smooth by elliptic regularity and positive on* $\overline{\Omega}$ *from the variational characterization of* (3) *and the maximum principle. By Lemma 2.1, we have*

$$\begin{aligned} \int_\Omega \mathcal{R}(u,v)\,dx &= \int_\Omega |\nabla u|^2\,dx - \int_\Omega \nabla\left(\frac{u^2}{v}\right)\cdot\nabla v\,dx \\ &= \lambda_0 \int_{\partial\Omega} \omega(x) u^2\,d\mathcal{H}^{n-1} - \int_\Omega \nabla\left(\frac{u^2}{v}\right)\cdot\nabla v\,dx. \end{aligned}$$

Multiplying the first equation of (3) *by* u^2/v *when* v *is a solution, we are able to conclude that*

$$\int_\Omega \mathcal{R}(u,v)\,dx = 0.$$

Thus, Lemma 2.1 guarantees u *is a multiple of* v*.*

As pointed out in Corollary 2.1 we need to investigate bifurcation from the trivial branch Γ_0. It follows from previous remarks that the operator $D_u\mathscr{F}(\lambda_0, 0)$ is not injective and, since it is a Fredholm operator of zero index (see [15]), it is not surjective also. This is equivalent to the nonhomogeneous Neumann problem

$$\begin{cases} \Delta v = \varphi \quad \text{in} \quad \Omega \\ \dfrac{\partial v}{\partial \nu} = \lambda_0 s(x) f'(0) v + \psi \quad \text{on} \quad \partial\Omega \end{cases} \tag{6}$$

having no solution for some $(\varphi, \psi) \in L^p(\Omega) \times W_p^{1-1/p}(\partial\Omega)$, $p > n$, where λ_0 is given by (5) with $\omega(\cdot) = f'(0)s(\cdot)$. In next result, inspired in [7], we establish a class of data for which (6) can not be solved.

Theorem 2.3 *Problem* (6) *does not has a solution for all* $(\varphi, \psi) = (-\xi, \eta)$, *where* (ξ, η) *belongs to the set*

$$\mathfrak{C} \doteq \Big\{ (\xi, \eta) \in C^{0,\theta}(\overline{\Omega}) \times C^{1,\theta}(\partial\Omega) \, : \, 0 \not\equiv \xi, \eta \geq 0 \Big\}.$$

Proof: By contradiction, suppose that v is a weak solution to (6) with $(\varphi, \psi) = (-\xi, \eta) \in \mathfrak{C}$, that is,

$$\int_\Omega \nabla v \cdot \nabla \phi \, dx - \lambda_0 \int_{\partial\Omega} s(x) f'(0) v \phi \, d\mathcal{H}^{n-1} = \int_\Omega \xi \phi \, dx + \int_{\partial\Omega} \eta \phi \, d\mathcal{H}^{n-1}$$

for all $\phi \in H^1(\Omega)$. Taking $\phi = v^-$ as a test function, we get

$$\int_\Omega |\nabla v^-|^2 \, dx - \lambda_0 \int_{\partial\Omega} s(x) f'(0) (v^-)^2 \, d\mathcal{H}^{n-1} = -\int_\Omega \xi v^- \, dx - \int_{\partial\Omega} \eta v^- \, d\mathcal{H}^{n-1}$$

and thus

$$\int_\Omega |\nabla v^-|^2 \, dx \leq \lambda_0 \int_{\partial\Omega} s(x) f'(0) (v^-)^2 \, d\mathcal{H}^{n-1}.$$

If $v^- \not\equiv 0$, last inequality implies that v^- is an admissible function in the set of definition of λ_0 in (5) and that λ_0 is attained by v^-, which must be a solution to (3). By elliptic regularity and the maximum principle we have $v^- > 0$ in $\overline{\Omega}$, and then $v = v^-$ and $\xi \equiv 0$ in Ω, $\eta \equiv 0$ on $\partial\Omega$; a contradiction.
If $v^- \equiv 0$ we have $v = v^+$ and again by elliptic regularity and the maximum principle it follows that $v > 0$ in $\overline{\Omega}$ since $\eta \geq 0$ on $\partial\Omega$. Now, choosing the test function $\phi = v_0^2/v \in H^1(\Omega)$, where v_0 is an eigenfunction corresponding to λ_0, by Lemma 2.1 we have

$$\begin{aligned} 0 < \int_\Omega \xi \frac{v_0^2}{v} \, dx + \int_{\partial\Omega} \eta \frac{v_0^2}{v} \, d\mathcal{H}^{n-1} &= \int_\Omega \nabla v \cdot \nabla \Big(\frac{v_0^2}{v} \Big) \, dx - \int_\Omega |\nabla v_0|^2 \, dx \\ &= -\int_\Omega \mathcal{R}(v_0, v) \, dx \, \leq 0 \end{aligned}$$

a contradiction, proving the theorem. □

2.2 Local bifurcation from trivial branches

We prove now a local bifurcation theorem with respect to Γ_0 by applying Crandall-Rabinowitz's theorem [11] for simple eigenvalues. For the rest of this section we consider λ_0 given by (5) with $\omega(\cdot) = f'(0)s(\cdot)$, unless otherwise mentioned.

Theorem 2.4 *The principal eigenvalue λ_0 is a bifurcation point with respect to Γ_0. Precisely, in a neighborhood of $(\lambda_0, 0)$ in $\mathbb{R}^+ \times W_p^2(\Omega)$, $p > n$, the only nontrivial equilibria to* (1) *lie in a C^2-curve*

$$\mathscr{C} = \big\{(\lambda(r), u(r)) : r \in \tilde{\mathcal{I}} \subset \mathbb{R}\big\}.$$

Moreover, $u(r) = ru_0 + r\rho(r)$, where u_0 spans $\ker\big(D_u\mathscr{F}(\lambda_0, 0)\big)$, $\tilde{\mathcal{I}}$ is an open interval containing 0 *and $\lambda : \tilde{\mathcal{I}} \to \mathbb{R}$, $\rho : \tilde{\mathcal{I}} \to W_p^2(\Omega)$ are C^2-functions in Fréchet sense such that $\lambda(0) = \lambda_0$ and $\rho(0) = 0$.*

Proof: The nonlinear mapping $\mathscr{F} : W_p^2(\Omega) \longrightarrow L^p(\Omega) \times W_p^{1-1/p}(\partial\Omega)$, $p > n$, given by

$$\mathscr{F}(\lambda, u) = \left(\Delta u, \frac{\partial u}{\partial \nu} - \lambda s(\cdot) f(u)\right)$$

is of class C^3 in Fréchet sense and $\Gamma_0 \subset \mathscr{F}^{-1}(0,0)$. The partial derivative given by

$$D_u\mathscr{F}(\lambda_0, 0) \cdot v = \left(\Delta v, \frac{\partial v}{\partial \nu} - \lambda_0 s(\cdot) f'(0) v\right)$$

for all $v \in W_p^2(\Omega)$, is a Fredholm operator of zero index (see [15]) and λ_0 has geometric multiplicity one, according to Remark 2.2. To apply Theorem 1.7 of [11] we need to verify Crandall-Rabinowitz's transversality condition. Suppose it does not hold, that is, the condition $D_\lambda D_u\mathscr{F}(\lambda_0, 0) \cdot u_0 \notin R\big(D_u\mathscr{F}(\lambda_0, 0)\big)$, where u_0 spans $\ker\big(D_u\mathscr{F}(\lambda_0, 0)\big)$, fails. Then there exists a solution $v \in W_p^2(\Omega)$, $p > n$, of

$$\begin{cases} \Delta v = 0 \quad \text{in} \quad \Omega \\ \\ \dfrac{\partial v}{\partial \nu} - \lambda_0 s(x) f'(0) v = -s(x) f'(0) u_0 \quad \text{on} \quad \partial\Omega. \end{cases}$$

By Green's formula,

$$\begin{aligned} 0 = \int_\Omega u_0 \Delta v \, dx - \int_\Omega v \Delta u_0 \, dx &= \int_{\partial\Omega} u_0 \frac{\partial v}{\partial \nu} \, d\mathcal{H}^{n-1} - \int_{\partial\Omega} v \frac{\partial u_0}{\partial \nu} \, d\mathcal{H}^{n-1} \\ &= -\int_{\partial\Omega} s(x) f'(0) u_0^2 \, d\mathcal{H}^{n-1}. \end{aligned}$$

Since u_0 satisfies

$$\int_{\partial\Omega} s(x) f'(0) u_0^2 \, d\mathcal{H}^{n-1} = \frac{1}{\lambda_0} \int_\Omega |\nabla u_0|^2 \, dx$$

we get a contradiction. Therefore, by Crandall-Rabinowitz's theorem, the proof will be concluded by proving that $u(r) \in \mathfrak{X}$ for r small. But this follows by choosing an eigenfunction u_0 sufficiently small and using the embedding (see [1]) $W_p^2(\Omega) \hookrightarrow C^{1,\theta}(\overline{\Omega})$, where $0 < \theta < 1 - \frac{n}{p}$, for $p > n$. □

Corollary 2.2 $(\lambda_0, 0)$ *is the only bifurcation point with respect to* Γ_0.

Proof: This is a consequence of Theorem 2.2 and Remark 2.1. □

To precise the type of bifurcation occurring in $(\lambda_0, 0)$ and get more accurate results about equilibria to (1), *from now on* we assume f is strictly concave:

$$f''(u) < 0 \qquad \text{in} \quad [0, 1].$$

Next result shows how the local curve given by Theorem 2.4 crosses $(\lambda_0, 0)$.

Theorem 2.5 *The bifurcation occurring in* $(\lambda_0, 0)$ *is transcritical. More precisely, one has* $\dot{\lambda}(0) > 0$.

Proof: By Theorem 2.4 we have for all $r \in \tilde{\mathcal{I}}$

$$\begin{cases} \Delta u(r) = 0 \quad \text{in} \quad \Omega \\ \\ \dfrac{\partial u(r)}{\partial \nu} = \lambda(r) s(x) f(u(r)) \quad \text{on} \quad \partial\Omega \end{cases}$$

where $u(r) = ru_0 + r\rho(r)$. By chain rule one can compute d/dr $(= \dot{}\ )$ and taking into account that $\dot{u}(0) = u_0$ and $\ddot{u}(0) = 2\dot{\rho}(0)$, we get

$$\begin{cases} \Delta \dot{\rho}(0) = 0 \quad \text{in} \;\; \Omega \\ \\ \dfrac{\partial \dot{\rho}(0)}{\partial \nu} = \dot{\lambda}(0) s(x) f'(0) u_0 + \dfrac{\lambda_0}{2} s(x) f''(0) u_0^2 + \lambda_0 s(x) f'(0) \dot{\rho}(0) \;\; \text{on} \;\; \partial\Omega. \end{cases}$$

Now, by Green's formula we have

$$\begin{aligned} 0 &= \int_{\partial\Omega} u_0 \frac{\partial \dot{\rho}(0)}{\partial \nu} \, d\mathcal{H}^{n-1} - \int_{\partial\Omega} \dot{\rho}(0) \frac{\partial u_0}{\partial \nu} \, d\mathcal{H}^{n-1} \\ &= \dot{\lambda}(0) \int_{\partial\Omega} s(x) f'(0) u_0^2 \, d\mathcal{H}^{n-1} + \int_{\partial\Omega} \frac{\lambda_0}{2} s(x) f''(0) u_0^3 \, d\mathcal{H}^{n-1} \end{aligned}$$

and thus

$$\dot{\lambda}(0) \int_{\partial\Omega} s(x) f'(0) u_0^2 \, d\mathcal{H}^{n-1} = -\int_{\partial\Omega} \frac{\lambda_0}{2} s(x) f''(0) u_0^3 \, d\mathcal{H}^{n-1}.$$

Since its not difficul to see that

$$\int_{\partial\Omega} s(x) f'(0) u_0^2 \, d\mathcal{H}^{n-1} = \frac{1}{\lambda_0} \int_{\Omega} |\nabla u_0|^2 \, dx$$

and

$$\int_{\partial\Omega} s(x) u_0^3 \, d\mathcal{H}^{n-1} = \frac{2}{\lambda_0 f'(0)} \int_{\Omega} |\nabla u_0|^2 u_0 \, dx$$

the conclusion is

$$\dot{\lambda}(0) = -\lambda_0 \frac{f''(0)}{f'(0)} \frac{\displaystyle\int_{\Omega} |\nabla u_0|^2 u_0 \, dx}{\displaystyle\int_{\Omega} |\nabla u_0|^2 \, dx}$$

and the theorem is proved. □

2.3 Injectitivity of derivative at nonconstant equilibria and consequences

An important tool to help us getting better knowledge of the bifurcation and stability structures of equilibria to (1) is

Theorem 2.6 *Suppose that u_λ is a nontrivial equilibrium solution to* (1) *for* $\lambda > 0$. *Then the operator* $D_u\mathscr{F}(\lambda, u_\lambda) : W_p^2(\Omega) \longrightarrow L^p(\Omega) \times W_p^{1-1/p}(\partial\Omega)$, $p > n$, *given by*

$$D_u\mathscr{F}(\lambda, u_\lambda) \cdot v = \left(\Delta v, \frac{\partial v}{\partial \nu} - \lambda s(\cdot) f'(u_\lambda) v\right)$$

for all $v \in W_p^2(\Omega)$, *is an injective mapping.*

Proof: This theorem generalizes a result in [23] but with a parallel proof, so we only indicate the main points. By contradiction, suppose the operator $D_u\mathscr{F}(\lambda, u_\lambda)$ was not injective, that is, the problem

$$\begin{cases} \Delta\psi = 0 \quad \text{in} \quad \Omega \\ \dfrac{\partial \psi}{\partial \nu} = \lambda s(x) f'(u_\lambda)\psi \quad \text{on} \quad \partial\Omega \end{cases}$$

would have a (smooth) solution $\psi \not\equiv 0$. Note that by the maximum principle and Hopf's Lemma we have $u_\lambda > 0$, $1 - u_\lambda > 0$ in $\overline{\Omega}$ and so $f(u_\lambda) > 0$ in $\overline{\Omega}$.
The function $\Psi = \dfrac{\psi}{f(u_\lambda)}$ would be a solution to a elliptic problem

$$\begin{cases} \Delta\Psi + \displaystyle\sum_{i=1}^{n} b_i(x)\frac{\partial \Psi}{\partial x_i} + c(x)\Psi = 0 \quad \text{in} \quad \Omega \\ \dfrac{\partial \Psi}{\partial \nu} = 0 \quad \text{on} \quad \partial\Omega \end{cases}$$

where the coefficients b_i and c are smooth and $c \leq 0$ in Ω. But the maximum principle and Hopf's Lemma would force one to conclude $\Psi \equiv 0$, a contradiction. □

The first immediate consequence of Theorem 2.6 and the Implicit Function Theorem is

Corollary 2.3 *There is no secondary bifurcation of equilibrium solutions to* (1).

The next result shows that Theorem 2.1 can be sharpened, giving the precise range of uniqueness of constant equilibrium solutions to (1).

Theorem 2.7 *Theorem* 2.1 *holds true for all* $0 < \lambda \leq \lambda_0$.

Proof: It suffices to prove the result for $0 < \lambda < \lambda_0$. Thus, by contradiction, suppose that there exists $0 < \hat{\lambda} < \lambda_0$ such that $u_{\hat{\lambda}}$ is a nonconstant equilibrium solution to (1). Since $D_u\mathscr{F}(\hat{\lambda}, u_{\hat{\lambda}})$ is a Fredholm operator of zero index, it follows from Theorem 2.6 that $D_u\mathscr{F}(\hat{\lambda}, u_{\hat{\lambda}})$ is a bijection. Then, by the Implicit Function Theorem we get an interval $\hat{J} \doteq (\hat{\lambda}-\delta, \hat{\lambda}+\delta)$, $\delta > 0$, and solutions $u_{\hat{\lambda}}(\lambda) \in W^2_p(\Omega)$, $p > n$, of (2) for all $\lambda \in \hat{J}$ such that $u_{\hat{\lambda}}(\hat{\lambda}) = u_{\hat{\lambda}}$. It is not difficult to see that, reducing δ if necessary, one has $u_{\hat{\lambda}}(\lambda) \in \mathfrak{X}\backslash\{0,1\}$ are nonconstant solutions to (2) for all $\lambda \in \hat{J}$.

Taking a sequence $\{\lambda_k\} \subset \hat{J}$ which converges to $\hat{\lambda} - \delta$, and denoting $u_{\hat{\lambda}}(\lambda_k)$ by u_k, we have

$$\int_\Omega |\nabla u_k|^2\, dx = \lambda_k \int_{\partial\Omega} s(x) f(u_k) u_k \, d\mathcal{H}^{n-1} = O(\lambda_k).$$

Thus, $\{u_k\}$ is a bounded sequence in $H^1(\Omega)$ and, passing to a subsequence if necessary, there exists $\hat{u} \in H^1(\Omega)$ verifying

$$u_k \rightharpoonup \hat{u} \quad \text{in} \quad H^1(\Omega) \qquad \text{and} \qquad u_k \to \hat{u} \quad \text{a.e.} \quad \text{in} \quad \Omega \quad \text{and} \quad \partial\Omega.$$

But so $\hat{u}$ and is a weak solution to (2) with $\lambda = \hat{\lambda} - \delta$, which is nonconstant by Corollary 2.2. Therefore, all previous argumentation can be done from $\hat{u}$. By induction, we would construct a sequence $\{u_{\lambda_j}\}$ of nonconstant equilibria to (1), where $\lambda_1 = \hat{\lambda}$, such that $\lambda_j \to 0$ as $j \to \infty$. This is impossible by Theorem 2.1. □

2.4 Uniqueness and global bifurcation of equilibrium solutions

Previous theorem gives an uniqueness result for the constant equilibria to (1), i.e., the zeros of f in $[0, 1]$. The following one furnishes an uniqueness result related to nonconstant equilibria.

Theorem 2.8 *Problem* (1) *has an unique equilibrium solution for each* $\lambda > \lambda_0$.

Proof: The main idea is to extend the curve $\mathscr{C}$ given by Theorem 2.7 to a smooth curve defined on $(\lambda_0, +\infty)$ and containing all nonconstant equilibria to (1) for varying λ.

By Theorem 2.5, the function $\lambda(r)$ is increasing near $r = 0$; fixed a small interval $(0, \bar{r})$, $\bar{r} > 0$, take a sequence $r_j \to \bar{r}$ as $j \to \infty$. It can be proved as in Theorem 2.7 that the corresponding sequence $\{u(r_j)\}$ of nonconstant solutions to (2) converges weakly in $H^1(\Omega)$ and strongly in $L^2(\Omega)$ and in $L^2(\partial\Omega)$ to a nonconstant weak solution $\bar{u}$ to (2). Arguing as in the same theorem, it follows by Theorem 2.6 and the Implicit Function Theorem the existence of an open interval $\mathcal{I}$ containing $\lambda(\bar{r})$ and a C^3-function $\Theta : \mathcal{I} \longrightarrow W^2_p(\Omega)$, $p > n$, such that $\Theta(\lambda(\bar{r})) = \bar{u}$ and $\Theta(\lambda)$ is a nontrivial solution to (2) for all $\lambda \in \mathcal{I}$.

Now, the convergence of $\{u(r_j)\}$ can be improved since $\bar{u}$ is a classical solution to (2), [22], and by Amann's estimate (cf. [4]) there exists $C > 0$ such that, for $p > n$,

$$\begin{aligned} ||u(r_j) - \bar{u}||_{W^1_p(\Omega)} &\leq C\Big[||\Delta(u(r_j) - \bar{u})||_{L^p(\Omega)} + ||\lambda(r_j)s(\cdot)f(u(r_j)) + (u(r_j) - \bar{u})||_{L^p(\partial\Omega)}\Big] \\ &\leq C\Big[|\lambda(r_j)|\, ||s(\cdot)f(u(r_j))||_{L^p(\partial\Omega)} + ||u(r_j) - \bar{u}||_{L^p(\partial\Omega)}\Big] \overset{j\to\infty}{\longrightarrow} 0. \end{aligned}$$

Thus, $\{u(r_j)\}$ converges to $\bar{u}$ in $W_p^1(\Omega)$ and, by the standard L^p estimate from [2], in $W_p^2(\Omega)$, $p > n$, as $j \to \infty$. By uniqueness we have $\Theta(\lambda(r)) = u(r)$ for $r \sim \bar{r}$ but, as before, Θ can be extended in order to get

$$\Theta(\lambda(r)) = u(r), \qquad \forall r \in (0, \bar{r}). \tag{7}$$

By the same arguments Θ can be extended now to a function defined on $(\lambda_0, +\infty)$, providing an extension of $\mathscr{C}$.

The proof will be finished by proving that any nontrivial equilibrium solution to (1) belongs to $\mathscr{C}$. For $\lambda \sim \lambda_0$, this follows by (7) and Theorem 2.4. If there exists $\tilde{\lambda} \gg \lambda_0$ such that $u_{\tilde{\lambda}} \in \mathfrak{X} \setminus \mathscr{C}$ is a nontrivial solution to (2), arguing as above and recalling Corollary 2.3 we would get a function $\Upsilon : (\lambda_0, +\infty) \longrightarrow \left[W_p^2(\Omega) \cap \mathfrak{X}\right]$, $p > n$, such that $\Upsilon(\tilde{\lambda}) = u_{\tilde{\lambda}}$ and $\Upsilon(\lambda) \neq \Theta(\lambda)$ for all $\lambda \in (\lambda_0, +\infty)$. By (7) and the uniqueness of $\mathscr{C}$ near $(\lambda_0, 0)$ this is impossible. □

It follows from the proof of Theorem 2.8 that the bifurcation with respect to Γ_0 is not only a local phenomenon but a global one. Actually, it was proved that nontrivial equilibria to (1) form an unbounded smooth curve in $(\lambda_0, +\infty) \times \left[W_p^2(\Omega) \cap \mathfrak{X}\right]$, $p > n$. That is, the following holds.

Theorem 2.9 *The principal eigenvalue λ_0 is a global bifurcation point with respect to Γ_0. Furthermore, the mapping*

$$(\lambda_0, +\infty) \ni \lambda \longmapsto u_\lambda \in \left[W_p^2(\Omega) \cap \mathfrak{X}\right]$$

$p > n$, which associates $\lambda > \lambda_0$ to the unique nonconstant equilibrium solution to (1) is of class C^3 in Fréchet sense.

2.5 Bifurcation diagrams

Collecting the results obtained in this section we can draw the following schematic bifurcation diagrams.

3 Stability of equilibria

In this section we analyze stability of equilibria to (1), namely, of the trivial or constant equilibria and of those bifurcating from $(\lambda_0, 0)$. As before mentioned, we are supposing $\int_{\partial\Omega} s(x)\, d\mathcal{H}^{n-1} < 0$, which corresponds to the case of bifurcation from Γ_0. The case when previous integral is positive can be treated similarly and furnishes symmetric results related to Γ_1. The main tools used here are the linearized stability principle for evolution equations (see for example [17]) and the exchange of stability principle due to Crandall and Rabinowitz [12].

3.1 Stability of trivial equilibria

The stability properties of trivial equilibria to (1) are proved in next two results.

Theorem 3.1 *With respect to the trivial equilibria to* (1)*, the following holds.*

(i) *For* $0 < \lambda < \lambda_0$*, the solution* $u \equiv 0$ *is exponentially stable.*

(ii) *For* $\lambda > \lambda_0$*, the solution* $u \equiv 0$ *is unstable.*

(iii) *For all* $\lambda > 0$*, the solution* $u \equiv 1$ *is unstable.*

Proof: To infer stability/instability of the equilibria $u \equiv 0$ and $u \equiv 1$, we study the spectrum of the linearized eigenvalue problems corresponding to (2)

$$\begin{cases} \Delta\varphi = \mu\varphi & \text{in} \quad \Omega \\ \dfrac{\partial\varphi}{\partial\nu} = \lambda\omega(x)\varphi & \text{on} \quad \partial\Omega, \end{cases} \tag{8}$$

where $\omega(\cdot) = f'(0)s(\cdot)$ and $\omega(\cdot) = f'(1)s(\cdot)$, respectively. The first eigenvalue of (8) is given by (see [9])

$$\mu_1(\lambda) = \sup_{v \in H^1(\Omega)\setminus\{0\}} \left\{ \frac{-\displaystyle\int_\Omega |\nabla v|^2 \, dx + \lambda \int_{\partial\Omega} s(x)f'(0)v^2 \, d\mathcal{H}^{n-1}}{||v||^2_{L^2(\Omega)}} \right\} \tag{9}$$

and it is associated to a nonconstant smooth eigenfunction $\varphi > 0$ in $\overline{\Omega}$ satisfying (8) and such that $||\varphi||_{L^2(\Omega)} = 1$. Suppose that $\omega(\cdot) = f'(0)s(\cdot)$.

If $\int_{\partial\Omega} s(x)\varphi^2 \, d\mathcal{H}^{n-1} \leq 0$, we have

$$\begin{aligned} \mu_1(\lambda) = \int_\Omega \Delta\varphi \, \varphi \, dx &= -\int_\Omega |\nabla\varphi|^2 \, dx + \lambda \int_{\partial\Omega} s(x)f'(0)\varphi^2 \, d\mathcal{H}^{n-1} \\ &\leq -\int_\Omega |\nabla\varphi|^2 \, dx. \end{aligned}$$

On the other hand, if $\int_{\partial\Omega} s(x)\varphi^2 \, d\mathcal{H}^{n-1} > 0$ it follows that φ is admissible in the definition set of λ_0 in (5), so

$$-\int_\Omega |\nabla\varphi|^2 \, dx \leq -\lambda_0 f'(0) \int_{\partial\Omega} s(x)\varphi^2 \, d\mathcal{H}^{n-1}$$

and thus

$$\mu_1(\lambda) = -\int_\Omega |\nabla\varphi|^2 \, dx + \lambda \int_{\partial\Omega} s(x)f'(0)\varphi^2 \, d\mathcal{H}^{n-1} \leq (\lambda - \lambda_0)f'(0) \int_{\partial\Omega} s(x)\varphi^2 \, d\mathcal{H}^{n-1}.$$

Therefore, for $0 < \lambda < \lambda_0$ we have $\mu_1(\lambda) < 0$ and $u \equiv 0$ as an exponentially stable equilibrium solution to (1), proving **(i)**.

Now, as pointed out in Remark 2.1, λ_0 is attained by a smooth function, say ψ, such that $||\psi||_{L^2(\Omega)} = 1$ and

$$\int_\Omega |\nabla \psi|^2 \, dx = \lambda_0 \int_{\partial\Omega} s(x) f'(0) \psi^2 \, d\mathcal{H}^{n-1}.$$

Then for each $\lambda > \lambda_0$

$$\mu_1(\lambda) \geq -\int_\Omega |\nabla \psi|^2 \, dx + \lambda \int_{\partial\Omega} s(x) f'(0) \psi^2 \, d\mathcal{H}^{n-1} = (\lambda - \lambda_0) \int_{\partial\Omega} s(x) f'(0) \psi^2 \; d\mathcal{H}^{n-1}$$

and so $u \equiv 0$ is an unstable equilibrium solution to (1), proving **(ii)**.

The instability of $u \equiv 1$ for all $\lambda > 0$ follows since $f'(1) < 0$ and $\int_{\partial\Omega} s(x) \, d\mathcal{H}^{n-1} < 0$ imply $\mu_1(\lambda) > 0$ for all $\lambda > 0$, proving **(iii)**. □

Note that, for $\lambda = \lambda_0$, from the variational characterization (9) and Remark 2.1 we have $\mu_1(\lambda) \geq 0$. But since the map $\lambda \longmapsto \mu_1(\lambda)$ is continuous (see [9]) and $\mu_1(\lambda) < 0$ for $\lambda \in (0, \lambda_0)$, we get $\mu_1(\lambda_0) = 0$. Therefore we can not apply the linearized stability principle and another argument is needed, what is done in the following

Theorem 3.2 *The equilibrium solution $u \equiv 0$ of (1) is asymptotically stable for $\lambda = \lambda_0$.*

Proof: It was proved in [22] the problem (1) generates a dynamical system in the phase space $\mathfrak{X}$, which is a gradient system having

$$\mathcal{J}_\lambda(u) = \frac{1}{2} \int_\Omega |\nabla u|^2 \, dx - \int_{\partial\Omega} s(x) F(u) \, d\mathcal{H}^{n-1}, \qquad u \in \mathfrak{X}$$

(where $F' = f$) as a Lyapunov function for all $\lambda > 0$. Further, $\mathcal{J}_\lambda\big|_{\mathfrak{X}}$ has a global minimum for all $\lambda > 0$ which is a solution to (2) (see also the argumentation in [23]). Thus, by Theorem 2.7 the set of equilibrium solutions to (1) for $\lambda = \lambda_0$ is $\{0, 1\}$ and so $u \equiv 0$ globally minimizes $\mathcal{J}_{\lambda_0}\big|_{\mathfrak{X}}$ since $\mathcal{J}_{\lambda_0}(0) < \mathcal{J}_{\lambda_0}(1)$. Therefore, once the equilibria attract all orbits dissipating energy (see [17]), the theorem follows. □

3.2 Stability of bifurcating equilibria

The results given by Theorems 3.1 and 3.2 implies that the nonconstant equilibria to (1) in the global branch bifurcating from $(\lambda_0, 0)$ are asymptotically stable. But we prove a stronger result, namely, such equilibria are exponentially stable for $\lambda > \lambda_0$. Once we know the shape of that curve near $(\lambda_0, 0)$ by Theorem 2.5, a natural tool to establish such a result is the exchange of stability principle [12].

Let us recall that if $\mathcal{T}, \mathcal{K}$ are bounded linear operators between Banach spaces, we say that $\mu \in \mathbb{R}$ is a $\mathcal{K}$-simple eigenvalue of $\mathcal{T}$ if

(i) $\dim \ker \big(\mathcal{T} - \mu\mathcal{K}\big) = \dim \Big(Y / R\big(\mathcal{T} - \mu\mathcal{K}\big)\Big) = 1,$

(ii) $\mathcal{K}x_0 \notin R\big(\mathcal{T} - \mu\mathcal{K}\big)$, where x_0 spans $\ker\big(\mathcal{T} - \mu\mathcal{K}\big)$.

Having in mind the eigenvalue problems arising in the stability analysis, we consider the compact operator $\mathcal{K} : W^2_p(\Omega) \longrightarrow L^p(\Omega)\times W^{1-1/p}_p(\partial\Omega)$, $p > n$, given by $\mathcal{K}(u) := (u, 0)$.

Lemma 3.1 *Zero is a* $\mathcal{K}$*-simple eigenvalue of* $D_u\mathscr{F}(\lambda_0, 0)$.

The proof of Lemma 3.1 is similar to that of Lemma 3.2 below and will be omitted.

As a consequence of Lemma 3.1 it follows from [12], Corollary 1.13, the existence of $\epsilon, \delta > 0$ and of C^2-functions

$$\gamma : (\lambda_0 - \epsilon, \lambda_0 + \epsilon) \longrightarrow \mathbb{R}, \qquad z : (\lambda_0 - \epsilon, \lambda_0 + \epsilon) \longrightarrow W^2_p(\Omega),$$

and

$$\mu : (-\delta, \delta) \longrightarrow \mathbb{R}, \qquad w : (-\delta, \delta) \longrightarrow W^2_p(\Omega),$$

where $p > n$, such that

$$D_u\mathscr{F}(\lambda, 0)\cdot z(\lambda) = \gamma(\lambda)\,\mathcal{K}(z(\lambda)), \qquad \forall\lambda \in (\lambda_0 - \epsilon, \lambda_0 + \epsilon) \tag{10}$$

and

$$D_u\mathscr{F}(\lambda(r), u(r))\cdot w(r) = \mu(r)\,\mathcal{K}(w(r)), \qquad \forall r \in (-\delta, \delta). \tag{11}$$

One still has $\gamma(\lambda_0) = 0 = \mu(0)$ and $z(\lambda_0) = u_0 = w(0)$, where $\lambda(r), u(r)$ are as in Theorem 2.4. Moreover, by Theorem 1.16 of [12], the functions $\mu(r)$ and $-r\dot\lambda(r)\gamma'(\lambda_0)$ (where $d/dr = \dot{}$, $d/d\lambda = {}'$) have the same zeros and the same sign, and satisfy the relation

$$\lim_{\substack{r\to 0\\ \mu(r)\neq 0}} \frac{-r\dot\lambda(r)\gamma'(\lambda_0)}{\mu(r)} = 1. \tag{12}$$

Thus, to know the sign of $\mu(r)$ for r small we need to know that of $\gamma'(\lambda_0)$ because, by Theorem 2.5, we have $\dot\lambda(0) > 0$. Note that (10) is equilvalent for all $\lambda \in (\lambda_0 - \epsilon, \lambda_0 + \epsilon)$ to the problem

$$\begin{cases} \Delta z(\lambda) = \gamma(\lambda) z(\lambda) & \text{in} \quad \Omega \\[2mm] \dfrac{\partial z(\lambda)}{\partial\nu} = \lambda s(x) f'(0) z(\lambda) & \text{on} \quad \partial\Omega. \end{cases}$$

Calculating $d/d\lambda$ for $\lambda = \lambda_0$ we get

$$\begin{cases} \Delta z'(\lambda_0) = \gamma'(\lambda_0) u_0 & \text{in} \quad \Omega \\[2mm] \dfrac{\partial z'(\lambda_0)}{\partial\nu} = s(x) f'(0) u_0 + \lambda_0 s(x) f'(0) z'(\lambda_0) & \text{on} \quad \partial\Omega. \end{cases}$$

So, by one hand

$$\int_\Omega \Delta u_0\, z'(\lambda_0)\; dx - \int_\Omega \Delta z'(\lambda_0)\, u_0\; dx = -\gamma'(\lambda_0)\int_\Omega u_0^2\, dx$$

and by the other hand, from Green's formula, we have

$$\begin{aligned} -\gamma'(\lambda_0)\int_\Omega u_0^2\,dx &= \int_{\partial\Omega}\frac{\partial u_0}{\partial\nu}z'(\lambda_0)\,d\mathcal{H}^{n-1}-\int_{\partial\Omega}\frac{\partial z'(\lambda_0)}{\partial\nu}u_0\,d\mathcal{H}^{n-1} \\ &= -\int_{\partial\Omega}s(x)f'(0)u_0^2\,d\mathcal{H}^{n-1} \\ &= -\frac{1}{\lambda_0}\int_\Omega|\nabla u_0|^2\,dx. \end{aligned}$$

That is,

$$\gamma'(\lambda_0) = \frac{\displaystyle\int_\Omega|\nabla u_0|^2\,dx}{\lambda_0\displaystyle\int_\Omega u_0^2\,dx}.$$

Now, (11) is equivalent to the indefinite eigenvalue problem

$$\begin{cases} \Delta\phi=\mu\phi & \text{in}\quad \Omega \\ \dfrac{\partial\phi}{\partial\nu}=\lambda(r)s(x)f'(u(r))\phi & \text{on}\quad \partial\Omega, \end{cases} \tag{13}$$

where $\phi = w(r)$ e $\mu = \mu(r)$ for all $r\in(-\delta,\delta)$, which corresponds to the eigenvalue problem associated to the linearization of (2) around bifurcated equilibria to (1) near $(\lambda_0,0)$.

We want to know the sign of the first eigenvalue to the problem (13), given by

$$\mu_1(\lambda(r)) = \sup_{v\in H^1(\Omega)\setminus\{0\}}\left\{\frac{-\displaystyle\int_\Omega|\nabla v|^2\,dx+\lambda(r)\int_{\partial\Omega}s(x)f'(u(r))v^2\,d\mathcal{H}^{n-1}}{||v||^2_{L^2(\Omega)}}\right\}.$$

Lemma 3.2 *The first eigenvalue* $\mu_1(\lambda(r))$ *of problem* (13) *is a* $\mathcal{K}$*-simple eigenvalue of* $D_u\mathscr{F}(\lambda(r),u(r))$ *for all* $r\in(-\delta,\delta)$.

Proof: The operator

$$D_u\mathscr{F}(\lambda(r),u(r))-\mu_1(\lambda(r))\,\mathcal{K}\;:\;W^2_p(\Omega)\longrightarrow L^p(\Omega)\times W_p^{1-1/p}(\partial\Omega)$$

where $p>n$, given for $r\in(-\delta,\delta)$ by

$$\big(D_u\mathscr{F}(\lambda(r),u(r))-\mu_1(\lambda(r))\mathcal{K}\big)\cdot\phi=\left(\Delta\phi-\mu_1(\lambda(r))\phi,\;\frac{\partial\phi}{\partial\nu}-\lambda(r)s(x)f'(u(r))\phi\right)$$

for all $\phi\in W^2_p(\Omega)$, is a Fredholm operator of zero index (see [15]). Since $\mu_1(\lambda(r))$ is algebraically simple (see [9]), we have

$$\begin{aligned} 1 &= \dim\ker\big(D_u\mathscr{F}(\lambda(r),u(r))-\mu_1(\lambda(r))\,\mathcal{K}\big) \\ &= \dim\Big(L^p(\Omega)\times W_p^{1-1/p}(\partial\Omega)/R\big(D_u\mathscr{F}(\lambda(r),u(r))-\mu_1(\lambda(r))\,\mathcal{K}\big)\Big) \end{aligned}$$

for all $r \in (-\delta, \delta)$. Now, suppose

$$\mathcal{K}(u_0(r)) \in R\big(D_u\mathscr{F}(\lambda(r), u(r)) - \mu_1(\lambda(r))\mathcal{K}\big)$$

for some $r \in (-\delta, \delta)$, where $u_0(r) \in W_p^2(\Omega)$ spans $\ker\big(D_u\mathscr{F}(\lambda(r), u(r)) - \mu_1(\lambda(r))\mathcal{K}\big)$. This implies that the problem

$$\begin{cases} \Delta h(r) - \mu_1(\lambda(r))h(r) = u_0(r) & \text{in} \quad \Omega \\ \dfrac{\partial h(r)}{\partial \nu} = \lambda(r)s(x)f'(u(r))h(r) & \text{on} \quad \partial\Omega \end{cases}$$

has a solution $h(r) \in W_p^2(\Omega)$, $p > n$. By Green's formula and after some calculations, the conclusion is

$$\int_\Omega u_0^2(r)\, dx = 0$$

what is impossible. The lemma is proved. □

We are now in position to prove the main result of this section.

Theorem 3.3 *For all $\lambda > \lambda_0$ the nontrivial equilibrium solution to* (1) *bifurcating from* $(\lambda_0, 0)$ *is exponentially stable.*

Proof: It follows from Lemma 3.2 that $\mu(r) = \mu_1(\lambda(r))$ in (13) for all $r \in (-\delta, \delta)$, as a consequence of Lemma 1.3 and Corollary 1.13 of [12]. Thus $\mu(r) \neq 0$ for all $r > 0$ small by Corollary 2.3 and Crandall-Rabinowitz's theorem for simple eigenvalues. The relation (12) then implies $\mu_1(\lambda(r)) < 0$ for $r > 0$ small, that is, nontrivial equilibria in the beginning of the global branch emanating from $(\lambda_0, 0)$ are exponentially stable.

Now, the map $\lambda \longmapsto \mu_1(\lambda)$ is continuous (see [9]), where $\mu_1(\lambda)$ is the first eigenvalue of

$$\begin{cases} \Delta\phi = \mu\phi & \text{in} \quad \Omega \\ \dfrac{\partial\phi}{\partial\nu} = \lambda s(x)f'(u_\lambda)\phi & \text{on} \quad \partial\Omega \end{cases}$$

and u_λ stands for the unique nonconstant equilibrium solution to (1) given by Theorem 2.8. Therefore, since $\mu_1(\lambda) < 0$ for $\lambda \sim \lambda_0$ it follows from Corollary 2.3 that $\mu_1(\lambda) < 0$ for all $\lambda > \lambda_0$, and the theorem is proved. □

3.3 Bifurcation and stability diagrams

Summarizing the stability analysis above we can complete the bifurcation diagrams of previous section, getting the more elaborated following ones.

4 Trace convergence when the parameter is large

In this section we will establish the convergence of the trace on $\partial\Omega$ of the nontrivial equilibrium solution to (1) for large $\lambda > 0$. In fact, we will prove that such a trace concentrates, as $\lambda \to +\infty$ according to a suitable topology, in a subset of $\partial\Omega$ related to the indefinite weight function $s(\cdot)$.

Firstly, let us recall that if $\mathcal{M}$ is smooth connected Riemannian manifold and $K \subset \mathcal{M}$ is a compact set, the capacity of K is defined by

$$\operatorname{cap}(K) \doteq \inf\left\{\int_{\mathcal{M}} |\nabla\phi|^2\, d\mu \;:\; \phi \in C_0^\infty(\mathcal{M}) \quad \text{and} \quad \phi = 1 \;\text{ in a neighborhood of } K\right\}$$

where μ is the Riemannian volume of $\mathcal{M}$. If K is an open and precompact set, the capacity of K is defined by $\operatorname{cap}(K) = \operatorname{cap}(\overline{K})$. For more details, the reader is referred to [16] and references therein.

Consider the sets

$$\mathcal{Z} \doteq \{x \in \partial\Omega \;:\; s(x) = 0\}$$

and

$$\mathcal{M} \doteq \{x \in \partial\Omega \;:\; s(x) > 0\}.$$

Denoting by $u_\lambda\big|_{\partial\Omega}$ the trace on $\partial\Omega$ of the unique equilibrium solution to (1) and by χ_A the characteristic function of a set A, the result we prove in this section reads as

Theorem 4.1 *Suppose the $\mathcal{M}$ has finite capacity and $\mathcal{Z}$ has zero $(n-1)$-dimensional Hausdorff measure. Then, for all $1 < p < \infty$ one has*

$$u_\lambda\big|_{\partial\Omega} \overset{\lambda\to\infty}{\longrightarrow} \chi_{\mathcal{M}} \qquad \textit{in} \quad L^p(\partial\Omega).$$

Proof: It suffices to prove that

$$u_\lambda\big|_{\partial\Omega} \overset{\lambda\to\infty}{\longrightarrow} \chi_{\mathcal{M}} \qquad \text{in measure,}$$

that is, that for all $\varepsilon > 0$ the sets

$$N_\varepsilon^\lambda \doteq \{x \in \partial\Omega : |\chi_{\mathcal{M}}(x) - u_\lambda(x)| \geq \varepsilon\}$$

are such that $\mathcal{H}^{n-1}(N_\varepsilon^\lambda) \to 0$ as $\lambda \to \infty$. Since $\mathcal{M}$ has finite capacity there exists $\phi \in C_0^\infty(\partial\Omega)$ supported in an open neighborhood of $\partial\Omega$ containing $\mathcal{M}$ and satisfying $\phi\big|_{\overline{\mathcal{M}}} \equiv 1$. Moreover, one has $0 \leq \phi < 1$ in the complement of $\overline{\mathcal{M}}$ in that neighborhood.

Consider an extension $\Phi \in C_0^\infty(\overline{\Omega})$ of ϕ satisfying $0 \leq \Phi < 1$ over $\overline{\Omega}$ and the energy functional corresponding to (2), defined on $\mathfrak{X}$, given by

$$\mathcal{J}_\lambda(u) = \frac{1}{2}\int_\Omega |\nabla u|^2\, dx - \lambda \int_{\partial\Omega} s(x) F(u)\, d\mathcal{H}^{n-1}$$

where $F(u) = \int_0^u f(\tau)\,d\tau$. Note that

$$\frac{1}{\lambda}\mathcal{J}_\lambda\big(\Phi^{\sqrt{\lambda}}\big) = \frac{1}{2}\int_\Omega |\nabla\Phi|^2\Phi^{\,2(\sqrt{\lambda}-1)}dx - \int_{\mathcal{M}} s(x)F(1)\,d\mathcal{H}^{n-1} - \int_{\partial\Omega\setminus\mathcal{M}} s(x)F(\Phi^{\sqrt{\lambda}})\,d\mathcal{H}^{n-1}. \tag{14}$$

It follows that

$$\frac{1}{\lambda}\mathcal{J}_\lambda\big(\Phi^{\sqrt{\lambda}}\big) \overset{\lambda\to\infty}{\longrightarrow} -F(1)\int_{\mathcal{M}} s(x)\,d\mathcal{H}^{n-1}$$

by Lebegue's theorem. By the other hand, since u_λ globally minimizes $\mathcal{J}_\lambda\big|_{\mathfrak{X}}$ for each $\lambda > \lambda_0$ (see [22]) and $\Phi^{\sqrt{\lambda}} \in \mathfrak{X}$, we get

$$\begin{aligned}\frac{1}{\lambda}\mathcal{J}_\lambda\big(\Phi^{\sqrt{\lambda}}\big) \;\geq\; \frac{1}{\lambda}\mathcal{J}_\lambda(u_\lambda) \;&\geq\; -\int_{\mathcal{M}} s(x)F(u_\lambda)\,d\mathcal{H}^{n-1} - \int_{\partial\Omega\setminus\mathcal{M}} s(x)F(u_\lambda)\,d\mathcal{H}^{n-1}\\ &\geq\; \int_{\mathcal{M}} s(x)F(u_\lambda)\,d\mathcal{H}^{n-1} \;\geq\; -F(1)\int_{\mathcal{M}} s(x)\,d\mathcal{H}^{n-1}.\end{aligned}$$

Thus,

$$\lim_{\lambda\to\infty}\frac{1}{\lambda}\mathcal{J}_\lambda(u_\lambda) = -F(1)\int_{\mathcal{M}} s(x)\; d\mathcal{H}^{n-1}$$

what implies

$$\lim_{\lambda\to\infty}\int_{\partial\Omega}\big|s(x)\big[F(\chi_{\mathcal{M}}) - F(u_\lambda)\big]\big|\,d\mathcal{H}^{n-1} = 0. \tag{15}$$

Now, we have the following estimates

$$\begin{aligned}&\int_{\partial\Omega}\Big|s(x)\big[F(\chi_{\mathcal{M}}) - F(u_\lambda)\big]\Big|\,d\mathcal{H}^{n-1} \;\geq\; \int_{N_\varepsilon^\lambda}|s(x)|\left|\int_{u_\lambda}^{\chi_{\mathcal{M}}} f(\tau)d\tau\right|\,d\mathcal{H}^{n-1}\\ &= \int_{N_\varepsilon^\lambda\cap\mathcal{M}}|s(x)|\left[\int_{u_\lambda}^{1} f(\tau)d\tau\right]\,d\mathcal{H}^{n-1} + \int_{N_\varepsilon^\lambda\cap(\partial\Omega\setminus\mathcal{M})}|s(x)|\left[\int_0^{u_\lambda} f(\tau)d\tau\right]\,d\mathcal{H}^{n-1}\\ &\geq C\int_{N_\varepsilon^\lambda}|s(x)|\,d\mathcal{H}^{n-1}\end{aligned}$$

where $C > 0$ is a constant depending only on ε and f. Previous estimates and (15) then imply

$$\lim_{\lambda\to\infty}\int_{N_\varepsilon^\lambda}|s(x)|\,d\mathcal{H}^{n-1} = 0. \tag{16}$$

We claim that $\mathcal{H}^{n-1}(N_\varepsilon^\lambda) \longrightarrow 0$ as $\lambda \to \infty$.

In fact, if the contrary holds true, one gets $\varepsilon_0 > 0$ and a sequence $\{\lambda_j\}$, $\lambda_j \to \infty$, such that

$$\mathcal{H}^{n-1}(N_\varepsilon^{\lambda_j}) \;\geq\; \varepsilon_0, \qquad \forall j.$$

Consider the family of open sets containing $\mathcal{Z}$ given by

$$\mathcal{Z}_\delta \doteq \big\{x \in \partial\Omega \;:\; |s(x)| < \delta\big\}$$

where $\delta > 0$, satisfying $\mathcal{H}^{n-1}(\mathcal{Z}_\delta) \to 0$ as $\delta \to 0$. Choosing $\delta_0 > 0$ such that $\mathcal{H}^{n-1}(\mathcal{Z}_{\delta_0}) < \varepsilon_0$, we have

$$\begin{aligned}
\frac{1}{\delta_0}\int_{N_\varepsilon^{\lambda_j}} |s(x)|\, d\mathcal{H}^{n-1} &\geq \frac{1}{\delta_0}\int_{N_\varepsilon^{\lambda_j}\cap(\partial\Omega\setminus\mathcal{Z}_{\delta_0})} |s(x)|\, d\mathcal{H}^{n-1} \\
&\geq \mathcal{H}^{n-1}(N_\varepsilon^{\lambda_j} \cap (\partial\Omega \setminus \mathcal{Z}_{\delta_0})) \\
&= \mathcal{H}^{n-1}(N_\varepsilon^{\lambda_j}) - \mathcal{H}^{n-1}(N_\varepsilon^{\lambda_j} \cap \mathcal{Z}_{\delta_0}) \\
&\geq \mathcal{H}^{n-1}(N_\varepsilon^{\lambda_j}) - \mathcal{H}^{n-1}(\mathcal{Z}_{\delta_0}) \\
&\geq \varepsilon_0 - \mathcal{H}^{n-1}(\mathcal{Z}_{\delta_0}).
\end{aligned}$$

Therefore, as $j \to \infty$, thanks to (16) we get a contradiction, proving the theorem. □

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