# $L_p$-Solvability of a Full Superconductive Model

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## Contents



**Abstract.** In this article the mean-field vortex model arising from the II-type superconductivity is investigated. The vortex model is reduced to a nonlinear hyperbolic-elliptic system of PDEs in a bounded domain. Motivated by experiments, we consider physical boundary conditions, which describe a flux of superconducting vortices through the boundary of the domain. We prove the global solvability for the system. To show the solvability result we take a vanishing "viscosity" limit in an approximated parabolic-elliptic system. Since the approximated solutions do not have a compactness property, we justify this limit transition, using a kinetic formulation of our problem. The main trick is that instead of the nonlinear system, we have to investigate a linear transport equation.

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# 1 Introduction

This article is concerned with a 2-D reduction of the 3-D mean field model, describing the motion of magnetic vortices in a II-type superconductor [6]-[7]. More precisely, considering the particular case in which all the vortices are rectilinear, aligned and oriented with $x_3-$ direction along with the magnetic field $\mathbf{H}$, we have that $\mathbf{H} = (0, 0, h(t, \mathbf{x}))$ and $\mathbf{W} = (0, 0, \omega(t, \mathbf{x}))$, being $\mathbf{W}$ the 3-D vortex density. Then the evolution of non-zero components $h = h(t, \mathbf{x})$ and $\omega = \omega(t, \mathbf{x})$ is governed by the following system of differential equations

$$\omega_t + \mathrm{div}(|\omega|\mathbf{v}) = 0, \qquad (1.1)$$

$$\mathbf{v} = -\nabla h, \qquad (1.2)$$

$$-\Delta h + h = \omega, \qquad \text{in} \quad \Omega_T := (0, T) \times \Omega, \qquad (1.3)$$

where $\Omega$ is a bounded domain of $\mathbb{R}^2$. The system has to be closed by the condition for the magnetic field $h$ on the boundary $\Gamma$ of the superconducting sample $\Omega$

$$h = a \qquad \text{on} \quad \Gamma_T := (0, T) \times \Gamma \qquad (1.4)$$

and by the initial condition on the vorticity

$$\omega(\cdot, 0) = \omega_0 \qquad \text{in} \quad \Omega. \qquad (1.5)$$

The hyperbolic equation (1.1) needs an additional boundary condition for $\omega$, depending on whether the characteristics for (1.1) are directed into or out of $\Omega$ on the boundary $\Gamma$. Let $\mathbf{n} = \mathbf{n}(\mathbf{x})$ be the outward normal at $\mathbf{x} \in \Gamma$. Then if $(\mathbf{u} \cdot \mathbf{n})(t, \mathbf{x}) > 0$, vortices are leaving the sample and on this outflow section of the boundary no extra boundary conditions are required. However if $(\mathbf{u} \cdot \mathbf{n})(t, \mathbf{x}) < 0$, the vortices are moving into the sample and an extra boundary condition for the flux of vorticity has been suggested by Chapman [6]-[7], which can be generalize as

$$\omega = b(t, \mathbf{x}, \frac{\partial h}{\partial \mathbf{n}}) \qquad (1.6)$$

for $\quad (t, \mathbf{x}) \in \Gamma_T^- := \{(t, \mathbf{x}) \in \Gamma_T : \quad \mathrm{sign}(\omega)\, (\mathbf{v} \cdot \mathbf{n})\, (t, \mathbf{x}) < 0\}$.

Let us comment on relevant results to the system (1.1)-(1.3). In the previous works the positivity of the vortex function $\omega$ has been considered an important assumption, transforming the quasilinear hyperbolic equation (1.1) into a linear transport equation and as a consequence leading the system (1.1)-(1.3) to be similar to the classical incompressible Euler equations. The latter equations have been studied for the whole $\Omega = \mathbb{R}^2$ in many works (see a more complete bibliography in [15]); the solvability has been shown in the class of $L_p-$bounded vorticity with $1 \leqslant p \leqslant \infty$ and in the class of (positive) bounded measure $\omega$. In [9], [17], the methods developed for the Euler equations, have been used for the system (1.1)-(1.3). Under the same assumption of positivity for $\omega$ the solvability of the system (1.1)-(1.6) was shown in [2], [3], [23], but in a bounded domain $\Omega$ of $\mathbb{R}^2$. In [1], the system (1.1)-(1.5) was also studied and viewed as a gradient flow on the space of measures equipped with the Wasserstein distance. Unfortunately, without the boundary condition (1.6) the mentioned system has been stated incorrectly.

The system (1.1)-(1.6) without the assumption of the positivity of $\omega$ was considered in the mentioned work [17]. Assuming the $W_p^1$-regularity of the initial data $\omega_0$, the authors showed the existence and uniqueness of $W_p^1-$solutions of (1.1)-(1.3), (1.5), which is crucial, to obtain compactness properties for $\omega$. Since solutions of a quasilinear hyperbolic equation (1.1) admit jumps, it is more interesting to obtain the existence of the solutions in the class of $L_p-$bounded vorticity $\omega$. In the present article the solvability result for (1.1)-(1.6) is shown by applying a kinetic method [19], which reduces the equation (1.1) into a linear transport one for a "distributional" function that involves an additional kinetic variable. Since the hyperbolic-elliptic system (1.1)-(1.6) does not allow any type of $L_1-$ stability estimates, such as Kruzkov´s ones [11], we follow the kinetic technique of [20]-[22]. While in these works problems have been studied for the whole $\Omega = \mathbb{R}^2$, here we develop the kinetic technique to a bounded domain $\Omega$ of $\mathbb{R}^2$. Let us also note that the kinetic theory has been studied for $L_\infty-$bounded solutions of hyperbolic equations, while in our article, we expand the theory for $L_p-$bounded solutions with $2 < p < \infty$ of the hyperbolic-elliptic system (1.1)-(1.6).

The solvability of the system (1.1)-(1.6) will be obtained as the limit of solutions of the following viscous equations (with $\varepsilon > 0$)

$$\begin{cases} \omega_t + \operatorname{div}(|\omega|\mathbf{v}) = \varepsilon\Delta\omega, \qquad \mathbf{v} = -\nabla h \quad \text{in} \quad \Omega_T, \\ \varepsilon\frac{\partial\omega}{\partial\mathbf{n}} + M(t)(\omega - b(\cdot,\cdot,\frac{\partial h}{\partial \mathbf{n}})) = 0 \qquad \text{on} \quad \Gamma_T, \\ \omega|_{t=0} = \omega_0 \qquad \text{in} \quad \Omega, \end{cases} \tag{1.7}$$

with $M(t) := \|\mathbf{v}(t,\cdot)\|_{L_\infty(\Omega)}$ and

$$\begin{cases} -\Delta h + h = \omega & \text{in} \quad \Omega_T, \\ h = a & \text{on} \quad \Gamma_T. \end{cases} \tag{1.8}$$

The boundary layer theory (see [5], [16], [18] and references therein) says that the viscous limit $\varepsilon \to 0$ in the parabolic equation of the system (1.7) creates two regions inside of $\Omega$: a very thin layer in a neighborhood of the boundary $\Gamma$, where the viscosity $\varepsilon$ plays an essential role, and the remaining region outside of this layer, where the influence of the viscosity is negligible. In this boundary layer the solution of the parabolic equation undergoes a sharp increase, that brings a great difficulty for the viscous limit. Despite of a considerable quantity of publications a strict mathematical theory has not been developed for the problem of formation of the boundary layer. In this article we suggest an artificial boundary condition (see the system (1.7)) instead of the physical condition (1.6) and show that the boundary layer does not appear under the viscous limit in the system (1.7)-(1.8). We prove the $L_p-$boundedness of the viscous solutions independently on $\varepsilon$ (Lemma 1). As a consequence we obtain a strong convergence of the viscous solutions of (1.7)-(1.8) to the solution of the problem (1.1)-(1.6), that presents a very important result for the theory of boundary layers.

Let us assume the following regularity properties on the data of the problem (1.1)-(1.6): $\Omega$ *is a bounded domain of* $\mathbb{R}^2$ *having the boundary* $\Gamma$ *of class* $C^2$,

$$\omega_0 \in L_p(\Omega), \quad a \in L_\infty(0,T; W_p^{2-1/p}(\Gamma)) \quad \textit{for some} \quad p > 2 \tag{1.9}$$

*and the function* $b = b(\cdot,\cdot,z)$ *is a continuous function on* $z \in \mathbb{R}$, *such that*

$$|b(t,\mathbf{x},z)| \leqslant b_0(t,\mathbf{x}) + b_1|z|^{\varkappa} \quad \textit{for a.e.} \quad (t,\mathbf{x}) \in \Gamma_T \tag{1.10}$$

*for some constants* $\varkappa \leqslant \frac{p-1}{p}$, $b_1 \geqslant 0$ *and a positive function* $b_0 \in L_\infty(0,T;L_p(\Gamma))$.

The solution of the quasilinear hyperbolic equation (1.1) has to be considered as a weak entropy solution. Hence, taking into account Lemmas 7.24, 7.34 and Theorem 7.31 of the book [16], the solution of (1.1)-(1.6) is defined as follows.

**Definition 1** *We say that the pair of functions*

$$\omega \in L_\infty(0,T;L_p(\Omega)) \quad \textit{and} \quad h \in L_\infty(0,T;W_p^2(\Omega))$$

*is a weak solution of the problem* (1.1)-(1.6) *if the pair* $\{\omega, h\}$ *satisfies:*

*1) the inequality*

$$\int_{\Omega_T} \{|\omega-\xi|\varphi_t + \operatorname{sign}(\omega-\xi)\left[(|\omega|-|\xi|)\left(\mathbf{v}\cdot\nabla\varphi\right) + |\xi|\left(h-\omega\right)\varphi\right]\}\ dtd\mathbf{x}$$
$$+ \int_\Omega |\omega_0 - \xi|\varphi(0,\cdot)\,d\mathbf{x} + \int_{\Gamma_T} M(t)|b(\cdot,\cdot,\frac{\partial h}{\partial \mathbf{n}}) - \xi|\,\varphi\ dtd\mathbf{x} \geqslant 0 \qquad (1.11)$$

*for any* $\xi \in \mathbb{R}$ *and any non-negative* $\varphi \in C^\infty(\overline{\Omega}_T)$, *such that* $\varphi(\cdot,T) = 0$. *Here* $M(t) := ||\mathbf{v}(t,\cdot)||_{L_\infty(\Omega)}$;

*2) and the functions* $h$ *and* $\mathbf{v}$ *fulfill the relations* (1.2)-(1.3) *a.e. in* $\Omega_T$ *and the boundary condition* (1.4) *a.e. on* $\Gamma_T$.

Our main result is the following theorem.

**Theorem 1** *If the data* $a$, $b$, $\omega_0$ *satisfy (1.9)-(1.10), then there exists at least one weak solution* $\{\omega, h\}$ *of the problem* (1.1)-(1.6). *Moreover this solution is obtained from a strong convergent subsequence of the viscous solutions, i.e. there exists a subsequence of the viscous solutions* $\{\omega_\varepsilon, h_\varepsilon\}$ *for the problem* (1.7)-(1.8), *such that*

$$\begin{aligned} &h_\varepsilon \to h \quad \textit{strongly in } L_\infty(0,T;W_p^2(\Omega)),\\ &\omega_\varepsilon \to \omega \quad \textit{strongly in } L_\infty(0,T;L_p(\Omega)) \quad \textit{for } \varepsilon \to 0. \end{aligned}$$

# 2 Approximated Problem and Kinetic Formulation

Using Density's Theorem IV.12, p. 63 in [4], we can approximate our data $a$, $\omega_0$, $b_0$, $g$ and $b_z := b(\cdot,\cdot,z(\cdot,\cdot))$ for any $z = z(t,\mathbf{x})$ by smooth functions $a^\varepsilon$, $\omega_0^\varepsilon$, $b_0^\varepsilon$, $g^\varepsilon$ and $b_z^\varepsilon$. In the sequel we omit the parameter $\varepsilon$ and indicate the dependence of functions and constants on $\varepsilon$, where it is necessary.

**Lemma 1** *The problem* (1.7)-(1.8) *has a solution* $\omega \in W_2^{1,2}(\Omega_T) \cap L_\infty(0,T;L_p(\Omega))$, $h \in W_2^{1,2}(\Omega_T) \cap L_\infty(0,T;W_p^2(\Omega))$, *such that the estimates*

$$||\omega||_{L_\infty(0,T;L_p(\Omega))} \leqslant C, \qquad (2.1)$$
$$\|h\|_{L_\infty(0,T;W_p^2(\Omega))} \leqslant C, \qquad (2.2)$$
$$\sqrt{\varepsilon}||\nabla\omega||_{L_2(\Omega_T)} \leqslant C \qquad (2.3)$$

*hold. Moreover, denoting by $h_a$ the solution of the system*

$$\begin{cases} -\Delta h_a + h_a = 0 & \text{in} \quad \Omega_T, \\ h_a = a & \text{on} \quad \Gamma_T, \end{cases} \tag{2.4}$$

*and $\mathcal{P} := L_2\big(0,T;W_2^{\frac{1}{2}}(\Gamma)\big) \cap W_2^{\frac{1}{2}}\big(0,T;L_2(\Gamma)\big)$ we have*

$$||\partial_t \nabla\,(h-h_a)\,||_{L_\infty(0,T;L_2(\Omega))} \leqslant C, \tag{2.5}$$

$$||\frac{\partial\,(h-h_a)}{\partial \mathbf{n}}||_{\mathcal{P}} \leqslant C. \tag{2.6}$$

*All constants $C$ are independent of $\varepsilon$.*

**Proof.** *1$^{st}$ step.* The solvability of the problem (1.7)-(1.8) will be proved with the help of the fixed point theorem of Leray - Schauder: Theorem 10.3, p. 222, [10]. In order to do this, consider the one parameter family of problems depending on $\lambda \in [0,1]$ :

$$\begin{cases} \omega_t + \mathrm{div}(\lambda|\omega|\mathbf{v}) = \varepsilon\Delta\omega, \quad \mathbf{v} := -\nabla h & \text{in} \quad \Omega_T, \\ \varepsilon\frac{\partial\omega}{\partial\mathbf{n}} + \lambda M(t)(\omega - b(\cdot,\cdot,\frac{\partial h}{\partial \mathbf{n}})) = 0 & \text{on} \quad \Gamma_T, \\ \omega|_{t=0} = \lambda\omega_0 \quad \text{in} \quad \Omega \end{cases} \tag{2.7}$$

and

$$\begin{cases} -\Delta h + h = \omega & \text{in} \quad \Omega_T, \\ h = a & \text{on} \quad \Gamma_T. \end{cases} \tag{2.8}$$

Admitting the solvability of (2.7)-(2.8), we obtain the estimates (2.1)-(2.3), which are independent on $\lambda$ and $\varepsilon$. We consider $\lambda > 0$ (the case $\lambda = 0$ is trivial). Let $(\eta, q)$ be an entropy pair, i.e. $\eta = \eta(s) \in C^2(\mathbb{R})$ is a positive convex function and $q'(s) := \eta'(s)sign(s)$ satisfying the condition

$$|q(s)| \leqslant \eta(s). \tag{2.9}$$

From the first equation of (2.7), we have

$$\partial_t\eta(\omega) + \mathrm{div}(\lambda\mathbf{v}q(\omega)) + \lambda(\omega - h)(|\omega|\eta'(\omega) - q(\omega)) = \varepsilon\eta'(\omega)\Delta\omega,$$

which can be shown by a standard mollifying procedure (for instance, see [12]). Let us multiply the last equality by $\varphi \in C^\infty(\overline{\Omega}_T)$, such that $\varphi|_{t=T} = 0$ and integrate over $\Omega_T$. Using that $\eta(b) = \eta(\omega) + \eta'(\omega)(b-\omega) + \frac{\eta''(s)}{2}(b-\omega)^2$ for some $s = s(\mathbf{x},t)$, we get that $\omega$ fulfills the entropy type equality

$$\int_{\Omega_T} \{\eta(\omega)\varphi_t + q(\omega)\lambda\,(\mathbf{v}\cdot\nabla\varphi) + (|\omega|\eta'(\omega) - q(\omega))\,\lambda(h-\omega)\ \varphi\} dtd\mathbf{x} + \int_\Omega \eta(\omega_0)\varphi(0,\cdot)\ d\mathbf{x}$$
$$+\lambda\int_{\Gamma_T} M(t)\eta(b(\cdot,\cdot,\frac{\partial h}{\partial\mathbf{n}}))\varphi\ dtd\mathbf{x} - \varepsilon\int_{\Omega_T}\eta'(\omega)\nabla\omega\nabla\varphi\ dtd\mathbf{x} = m_{\varepsilon,\eta,\lambda}(\varphi) \tag{2.10}$$

with

$$m_{\varepsilon,\eta,\lambda}(\varphi) := \int_{\Omega_T} \varepsilon\eta''(\omega)|\nabla\omega|^2\varphi \; dtd\mathbf{x} + \lambda\int_{\Gamma_T}\{(\mathbf{v}\cdot\mathbf{n})q(\omega)+M(t)[\eta(\omega)+\frac{1}{2}\eta''(s)(b-\omega)^2]\}\varphi \; dtd\mathbf{x}.$$

Observe that $m_{\varepsilon,\eta,\lambda}(\varphi) \geqslant 0$ for $\varphi \geqslant 0$ as a consequence of the inequality (2.9).

Next we derive a priori estimate (2.1) for the solution $\omega$ of (2.7)-(2.8). We choose in (2.10) $\eta(\omega) := |\omega|^p$, $q(\omega) := \int\limits_0^{\omega} \eta'(s)sign(s)ds = |\omega|^p sign(\omega)$ and $\varphi(t,\mathbf{x}) := 1 - 1_\delta(t-t_0)$ with $t_0 \in (0,T)$ and

$$1_\delta(s) := \begin{cases} 0 \quad \text{for } s<0 \qquad \text{and} \qquad 1 \quad \text{for } s>\delta, \\ \frac{s}{\delta} \quad \text{for } 0\leqslant s\leqslant\delta. \end{cases} \tag{2.11}$$

Using (2.9) and passing to the limit on $\delta \to 0$ in (2.10), we derive

$$\begin{aligned} ||\omega||^p_{L_p(\Omega)}(t_0) + \int_0^{t_0}\int_\Omega \Big\{\varepsilon p(p-1)|\omega|^{p-2}\left|\nabla\omega\right|^2 + (p-1)\lambda|\omega|^p \mathrm{sign}(\omega)\,(\omega-h)\Big\} \; dtd\mathbf{x} \\ \leqslant ||\omega_0||^p_{L_p(\Omega)} + \lambda\int_0^{t_0} M(t)\int_\Gamma |b|^p \; dtd\mathbf{x}. \end{aligned} \tag{2.12}$$

.

Let $\overline{h} := h - h_a$ (see (2.4)). Multiplying the equation of (2.8) by $|\overline{h}|^p sign(\overline{h})$, we obtain

$$p\int_\Omega |\overline{h}|^{p-1}|\nabla\overline{h}|^2 d\mathbf{x} + ||\overline{h}||^{p+1}_{L_{p+1}(\Omega)} \leqslant ||\overline{h}||^p_{L_{p+1}(\Omega)} \left\|\omega\right\|_{L_{p+1}(\Omega)},$$

from which it follows that

$$\left\|\overline{h}\right\|_{L_{p+1}(\Omega)} \leqslant \left\|\omega\right\|_{L_{p+1}(\Omega)}. \tag{2.13}$$

By Holder's inequality, we have

$$\int_\Omega |h||\omega|^p \; d\mathbf{x} \leqslant \left\|\overline{h}\right\|_{L_{p+1}(\Omega)} \left\|\omega\right\|^p_{L_{p+1}(\Omega)} + C\left\|\omega\right\|^p_{L_p(\Omega)}.$$

Hence taking into account (2.12)-(2.13), we deduce

$$\left\|\omega\right\|^p_{L_p(\Omega)}(t_0) \leqslant C + \int_0^{t_0}\left\{C\left\|\omega\right\|^p_{L_p(\Omega)} + M(t)\int_\Gamma |b|^p \; d\mathbf{x}\right\}dt, \tag{2.14}$$

which by the assumptions (1.9)-(1.10) implies that

$$\int_\Gamma |b|^p \; d\mathbf{x} \leqslant C\left(1+\int_\Gamma |\nabla h|^{p\varkappa} \; d\mathbf{x}\right). \tag{2.15}$$

Finally, using the embedding theorem $W^1_p(\Omega) \hookrightarrow C(\overline{\Omega})$, (2.14) and (2.15), we obtain

$$\left\|\omega\right\|^p_{L_p(\Omega)}(t_0) \leqslant C + C\int_0^{t_0}\left\|\omega\right\|^p_{L_p(\Omega)} dt.$$

Therefore by Gronwall's inequality, (2.1) holds.

The estimate (2.2) follows from (1.9), (2.1) and the classical estimate for the elliptic problem (2.8) (see [10], [14])

$$||h||_{W^2_p(\Omega)} \leqslant C\ (||\omega||_{L_p(\Omega)} + \|a\|_{W_p^{2-1/p}(\Gamma)}). \tag{2.16}$$

Note that the estimate (2.12) is valid for the particular case $p = 2$, hence using the estimates (2.1)-(2.2), we get (2.3).

*2^nd step.* Now we can construct an operator, which fulfills Leray-Schauder's fixed point theorem. First we choose some function $\widetilde{\omega} \in \mathcal{F} := L_\infty(0,T;L_2(\Omega))$. The elliptic problem

$$\begin{cases} -\Delta\widetilde{h} + \widetilde{h} = \widetilde{\omega} & \text{in} \quad \Omega_T, \\ \widetilde{h} = a(t,\mathbf{x}) & \text{on} \quad \Gamma_T \end{cases} \tag{2.17}$$

has an unique solution $\widetilde{h} \in L_\infty(0,T;W_2^2(\Omega))$, satisfying (2.16). Secondly we consider the linear parabolic problem

$$\begin{cases} \omega_t = \varepsilon \triangle \omega - \lambda div\,(\mathbf{g}) \qquad \text{in} \quad \Omega_T, \quad \text{where} \quad \widetilde{\mathbf{v}} := -\nabla\widetilde{h}, \quad \mathbf{g} := |\widetilde{\omega}|\widetilde{\mathbf{v}}, \\ \varepsilon\frac{\partial\omega}{\partial\mathbf{n}} + \lambda M(t)(\omega - b(\cdot,\cdot,\frac{\partial\widetilde{h}}{\partial\mathbf{n}})) = 0 \qquad \text{on} \quad \Gamma_T, \\ \omega|_{t=0} = \lambda\omega_0 \qquad \text{in} \quad \Omega \end{cases} \tag{2.18}$$

with $M(t) := \|\widetilde{\mathbf{v}}(t,\cdot)\|_{L_\infty(\Omega)}$. In view of $\|\mathbf{g}\|_{\mathcal{F}} \leqslant C||\widetilde{\omega}||^2_{\mathcal{F}}$, Theorem 5.1, p. 170 (compare with Theorems 4.1-4.3, p. 152-164) of [13], the system (2.18) has an unique solution $\omega \in W_2^{\frac{1}{2},1}(\Omega_T) \cap \mathcal{F}$, such that

$$\|\omega\|_{W_2^{\frac{1}{2},1}(\Omega_T)\cap\mathcal{F}} \leqslant C(\varepsilon)\ (||b(\cdot,\cdot,\frac{\partial\widetilde{h}}{\partial\mathbf{n}})||_{C^{1,2}(\overline{\Omega}_T)} + \|\mathbf{g}\|_{\mathcal{F}}), \tag{2.19}$$

where the constant $C(\varepsilon)$ depends on $\varepsilon$. Hence using (2.19) the operator $B : \mathcal{F} \times [0,1] \to \mathcal{F}$, defined as

$$\omega := B\,[\widetilde{\omega},\lambda]\,, \tag{2.20}$$

is a *compact continuous operator* $B : \mathcal{F} \times [0,1] \to \mathcal{F}$, such that $B\,[\widetilde{\omega},0] = 0$ (see also Theorem 4.5, p. 165-166 of [13]). Moreover, a solution of the equation $\widetilde{\omega} = B\,[\widetilde{\omega},\lambda]$ (if any) corresponds to a solution of (2.7)-(2.8), fulfilling the estimate (2.1) for all $\lambda \in [0,1]$. Therefore $B$ satisfies all conditions of Leray-Schauder's fixed point theorem and as a consequence the problem (1.7)-(1.8) is solvable in the Banach space $W_2^{\frac{1}{2},1}(\Omega_T) \cap \mathcal{F}$. Observing *the Ladyzhenskaya inequality*

$$||\omega||_{L_4(\Omega)} \leqslant C||\omega||^{1/2}_{L_2(\Omega)}||\omega||^{1/2}_{W_2^1(\Omega)},$$

by Theorem 6.1, p. 178 of [13], it follows that $\omega \in W_2^{1,2}(\Omega_T)$.

*3^rd step.* Now we derive a priori estimates (2.5)-(2.6). In view of (1.8) the function $H := \frac{\partial(h-h_a)}{\partial t}$ solves the following elliptic problem

$$\begin{cases} -\Delta H + H = \operatorname{div}\mathbf{G} \ \text{ in } \Omega, \\ H\big|_\Gamma = 0, \end{cases}$$

with $\mathbf{G} := \varepsilon\nabla\omega - \mathbf{v}|\omega|$, such that $||\mathbf{G}||_{L_2(\Omega_T)} \leqslant C$. Applying the classical results of [14], we derive

$$\|\nabla H\|_{L_2(\Omega)} \leqslant C\,\|\mathbf{G}\|_{L_2(\Omega)} \leqslant C \quad \text{for a.e. } t \in (0,T),$$

being the estimate (2.5).

In view of (2.2) and (2.5), we have

$$\nabla(h-h_a) \in \Re\,[\Omega] := L_2\big(0,T;W_2^1(\Omega)\big) \cap W_2^1\big(0,T;L_2(\Omega)\big),$$

that, using Lemma 5 of [3], implies

$$||\frac{\partial(h-h_a)}{\partial\mathbf{n}}||_{\mathcal{P}} \leqslant C||\nabla(h-h_a)||_{\Re[\Omega]} \leqslant C$$

being the estimate (2.6). ■

The estimates of Lemma 1 do not imply a strong convergence of a subsequence of $\{\omega_\varepsilon\}$, that brings a difficulty for the limit transition in the nonlinear term of (1.7). Hence in the sequel the principal aim is to derive the strong convergence of $\{\omega_\varepsilon\}$. For it we use the kinetic technique.

Let us denote by $|s|_\pm := \max(\pm s, 0)$, $sign_+(s) := \begin{cases} 1, & \text{if } s>0, \\ 0, & \text{if } s \leqslant 0. \end{cases}$ and consider $\eta(s) := |s|_+$. Since this $\eta$ is not $C^2$, we first take a *smooth* convex function $\eta_\delta$, such that $\eta_\delta(s) \to \eta(s)$ strongly in $L_{\infty,loc}(\mathbb{R})$ for $\delta \to 0$. The entropy pair $(\eta_\delta(\omega_\varepsilon - \xi),\, q_\delta(\omega_\varepsilon - \xi))$ with $\xi \in \mathbb{R}$ satisfies the equality (2.10) with $\lambda = 1$. Since

$$(\eta_\delta(\omega_\varepsilon-\xi), \quad q_\delta(\omega_\varepsilon-\xi)) \to \big(|\omega_\varepsilon - \xi|_+, \quad \mathrm{sign}_+(\omega_\varepsilon-\xi)(|\omega_\varepsilon| - |\xi|)\big)$$

strongly in $L_{\infty,loc}(\mathbb{R}\times\Omega_T)$ as $\delta \to 0$, we infer that (2.10) is true for $(\eta, q) = (|\omega_\varepsilon - \xi|_+$, $sign_+(\omega_\varepsilon - \xi)(|\omega_\varepsilon| - |\xi|)$ ) and $\lambda = 1$. Therefore, the function $f_\varepsilon(\xi,t,\mathbf{x}) :=sign_+(\omega_\varepsilon - \xi)$ fulfills

$$\int_{\Omega_T} \left\{ \int_{\xi}^{+\infty} f_\varepsilon(s,t,\mathbf{x})\{\varphi_t + \mathrm{sign}(s)(\mathbf{v}\cdot\nabla\varphi)\}\,ds + |\xi| f_\varepsilon(\xi,t,\mathbf{x})(h_\varepsilon - \omega_\varepsilon)\varphi \right\} dtd\mathbf{x}$$
$$+ \int_{\Omega} |\omega_{0,\varepsilon} - \xi|_+ \varphi(0,\cdot)d\mathbf{x} + \int_{\Gamma_T} M_\varepsilon(t)|b_\varepsilon(t,\mathbf{x},\frac{\partial h_\varepsilon}{\partial n}) - \xi|_+\,\varphi\; dtd\mathbf{x} - \varepsilon\int_{\Omega_T} f_\varepsilon(\xi,t,\mathbf{x})\nabla\omega\nabla\varphi dtd\mathbf{x}$$
$$= m_\varepsilon(\varphi) \tag{2.21}$$

with

$$\begin{aligned} m_\varepsilon(\varphi) := \quad & \int_{\Omega_T} \frac{\varepsilon}{2}\delta(\xi = \omega_\varepsilon(t,\mathbf{x}))|\nabla\omega_\varepsilon|^2\varphi\; dtd\mathbf{x} \\ + \quad & \int_{\Gamma_T} \{(\mathbf{v}_\varepsilon\cdot\mathbf{n})f_\varepsilon(\xi,t,\mathbf{x})(|\omega_\varepsilon| - |\xi|) + \\ + \quad & M_\varepsilon(t)\left[|\omega_\varepsilon - \xi|_+ + \frac{1}{4}\delta(\xi = s_\varepsilon(t,\mathbf{x}))(b_\varepsilon - \omega_\varepsilon)^2\right]\}\varphi\; dtd\mathbf{x}, \end{aligned}$$

such that $m_\varepsilon(\varphi) \geqslant 0$ for $\varphi \geqslant 0$. Here, $\delta(s)$ is the Dirac function and $M_\varepsilon(t) = \|\mathbf{v}_\varepsilon(t,\cdot)\|_{L_\infty(\Omega)}$.

Taking $\varphi(t,\mathbf{x}) := 1 - 1_\delta(t-(T-\delta))$ in (2.21) and passing to the limit on $\delta \to 0$, we get

$$m_\varepsilon(1) \leqslant \int\limits_\Omega |\omega_{0,\varepsilon} - \xi|_+ \; d\mathbf{x} + \int\limits_{\Gamma_T} M_\varepsilon(t) |b_\varepsilon(t,\mathbf{x},\frac{\partial h_\varepsilon}{\partial n}) - \xi|_+ \; dtd\mathbf{x}.$$

Hence by the Riesz representation theorem, the measure $m_\varepsilon$ is well defined on $\mathbb{R}\times\overline{\Omega}_T$ and locally uniformly bounded on $\varepsilon$, since

$$\int\limits_{[-N,N]\times\overline{\Omega}_T} m_\varepsilon \; d\xi dtd\mathbf{x} < C(N) \quad \text{for any} \quad N > 0. \tag{2.22}$$

We have the following standard properties, which are used in the sequel

$$\begin{cases} 0 \leqslant f_\varepsilon \leqslant 1 & \text{a.e. in } \mathbb{R}\times\Omega_T \quad \text{and} \quad \frac{\partial f_\varepsilon}{\partial \xi} \leqslant 0 \qquad \text{in } \mathcal{D}'(\mathbb{R}\times\Omega_T), \\ \omega_\varepsilon(t,\mathbf{x}) = \int\limits_0^{+\infty} f_\varepsilon(s,t,\mathbf{x})\, ds - \int\limits_{-\infty}^{0} (1 - f_\varepsilon(s,t,\mathbf{x}))\, ds. \end{cases} \tag{2.23}$$

If we take $\eta := |\omega - \xi|_-$ for $\xi \in \mathbb{R}$ in (2.10) with $\lambda = 1$, by a similar way as (2.21) has been deduced, we can show

$$\begin{aligned} &\int\limits_{\Omega_T} \int\limits_{-\infty}^{\xi} (1 - f_\varepsilon(s,t,\mathbf{x}))\{\varphi_t + \operatorname{sign}(s)(\mathbf{v}\cdot\nabla\varphi)\} \; ds + |\xi|(f_\varepsilon(\xi,t,\mathbf{x}) - 1)(h_\varepsilon - \omega_\varepsilon)\varphi \\ &+ \int\limits_\Omega |\omega_{0,\varepsilon} - \xi|_- \varphi(0,\cdot) d\mathbf{x} + \int\limits_{\Gamma_T} M_\varepsilon(t)|b_\varepsilon(t,\mathbf{x},\frac{\partial h_\varepsilon}{\partial n}) - \xi|_- \, \varphi \;\; dtd\mathbf{x} \\ &- \varepsilon \int_{\Omega_T} (1 - f_\varepsilon(\xi,t,\mathbf{x}))\nabla\omega\nabla\varphi \; dtd\mathbf{x} \geqslant 0 \qquad \text{for} \qquad \varphi \geqslant 0. \end{aligned} \tag{2.24}$$

Now if we take $\varphi := \frac{\partial \psi}{\partial \xi}$ with $\psi(\xi,t,\mathbf{x}) \in C_0^\infty(\mathbb{R}\times\Omega_T)$ in (2.21), then after integration of the equality (2.21) by parts on the parameter $\xi$, we derive that $f_\varepsilon$ satisfies the following identity

$$\int\limits_{\mathbb{R}\times\Omega_T} f_\varepsilon \left\{\psi_t + \operatorname{sign}(\xi)(\mathbf{v}\cdot\nabla_\mathbf{x}\psi) + |\xi|(h_\varepsilon - \omega_\varepsilon)\psi'_\xi\right\} \; d\xi dtd\mathbf{x} = \int\limits_{\mathbb{R}\times\Omega_T} \psi'_\xi \, m_\varepsilon \, d\xi dtd\mathbf{x}.$$

# 3 Limit transition on the viscosity

By Lemma 1 and (2.22)-(2.23), there exists a subsequence of $\{\omega_\varepsilon, f_\varepsilon, h_\varepsilon, m_\varepsilon\}$, such that

$$\begin{aligned}
h_\varepsilon &\rightharpoonup h \quad \text{weakly} - * \text{ in } L_\infty(0,T;W_p^2(\Omega)),\\
\mathbf{v}_\varepsilon &\to \mathbf{v} \quad \text{strongly in } L_\infty(0,T;L_p(\Omega)),\\
\varepsilon\nabla\omega_\varepsilon &\to 0 \quad \text{strongly in } L_2(\Omega_T),\\
\frac{\partial h_\varepsilon}{\partial \mathbf{n}} &\to \frac{\partial h}{\partial \mathbf{n}} \quad \text{strongly in } L_2(\Gamma_T),\\
\omega_\varepsilon &\rightharpoonup \omega \quad \text{weakly} - * \text{ in } L_\infty(0,T;L_p(\Omega)),\\
f_\varepsilon,\ \omega_\varepsilon f_\varepsilon &\rightharpoonup\ f,\ R \quad \text{weakly} - * \text{ in } L_\infty(\mathbb{R}\times(0,T);L_p(\Omega)),\\
m_\varepsilon &\rightharpoonup m \quad \text{weakly in } \mathcal{M}^+(\mathbb{R}\times\overline{\Omega}_T).
\end{aligned}$$

In the following considerations the strong convergence of $\omega_\varepsilon$ to $\omega$ will be shown, which will imply the solvability of our problem (1.1)-(1.6). Obviously, $\omega, f, h, \mathbf{v}$,$m$ satisfy the relations (1.2)-(1.4), the estimates (2.1), (2.2) and the integral relations

$$\int\limits_{\Omega_T}\int\limits_{\xi}^{+\infty} f(s,t,\mathbf{x})\left\{\varphi_t + \operatorname{sign}(s)(\mathbf{v}\cdot\nabla\varphi)\right\}\, ds + |\xi|(fh-R)\varphi\, dt\, d\mathbf{x}$$
$$+\int\limits_{\Omega}|\omega_0-\xi|_+\varphi(0,\cdot)\, d\mathbf{x} + \int\limits_{\Gamma_T} M(t)|b(t,\mathbf{x},\frac{\partial h}{\partial n}) - \xi|_+\varphi\, dt\, d\mathbf{x} \geqslant 0, \tag{3.1}$$

$$\int\limits_{\Omega_T}\int\limits_{-\infty}^{\xi}(1-f(s,t,\mathbf{x}))\left\{\varphi_t + \operatorname{sign}(s)(\mathbf{v}\cdot\nabla\varphi)\right\}\, ds + |\xi|\left\{(f-1)(h-\omega)+(f\omega-R)\right\}\varphi\, dt\, d\mathbf{x}$$
$$+\int\limits_{\Omega}|\omega_0-\xi|_-\varphi(0,\cdot)\, d\mathbf{x} + \int\limits_{\Gamma_T} M(t)|b(t,\mathbf{x},\frac{\partial h}{\partial n}) - \xi|_-\varphi\, dt\, d\mathbf{x} \geqslant 0, \tag{3.2}$$

$$\int\limits_{\mathbb{R}\times\Omega_T} f\{\partial_t\psi + \operatorname{sign}(\xi)(\mathbf{v}\cdot\nabla_{\mathbf{x}}\psi)\} + |\xi|(fh-R)\psi'_\xi\, d\xi\, dt\, d\mathbf{x} = \int\limits_{\mathbb{R}\times\Omega_T}\psi'_\xi\, m\, d\xi dt d\mathbf{x} \tag{3.3}$$

with $M(t) = ||\mathbf{v}(t,\cdot)||_{L_\infty(\Omega)}$, $\varphi$ a positive smooth function and $\psi$ a smooth function having compact support on $\mathbb{R}\times\Omega_T$. Moreover

$$\begin{cases}
0\leqslant f\leqslant 1 \qquad \text{a.e. in } \mathbb{R}\times\Omega_T \quad \text{and} \quad \frac{\partial f}{\partial\xi}\leqslant 0 \qquad \text{in } \mathcal{D}'(\mathbb{R}\times\Omega_T),\\[2ex]
\omega(t,\mathbf{x}) = \int\limits_0^{+\infty} f(s,t,\mathbf{x})\, ds - \int\limits_{-\infty}^{0}(1-f(s,t,\mathbf{x}))\, ds,\\[2ex]
m\in BV(\mathbb{R}, w-\mathcal{M}^+(\overline{\Omega}_T)), \quad \text{such that} \quad \int\limits_{[-N,N]\times\overline{\Omega}_T} m\, d\xi dt d\mathbf{x} < C(N) \quad \forall N>0
\end{cases} \tag{3.4}$$

where $\mathcal{M}^+(\overline{\Omega}_T)$ is the Banach space of bounded non-negative Radon measures on $\overline{\Omega}_T$ and $w-\mathcal{M}^+(\overline{\Omega}_T)$ denotes $\mathcal{M}^+(\overline{\Omega}_T)$ equipped with the weak topology. The last line in (3.4) follows from (2.22) and the left part of (3.1).

**Lemma 2** *For almost all point in $\Omega_T$, we have*

$$\lim_{\xi\to+\infty} \xi^p f(\xi) = 0 \quad \text{and} \quad \lim_{\xi\to-\infty} |\xi|^p(1-f(\xi)) = 0, \tag{3.5}$$

$$R - \omega f \geqslant 0, \qquad \text{a.e. in } \mathbb{R}\times\Omega_T, \tag{3.6}$$

$$\lim_{\xi\to\pm\infty} |\xi| A(\xi) = 0, \tag{3.7}$$

$$\lim_{\xi\to\pm\infty} m(\xi) = 0, \tag{3.8}$$

*where $A(\xi) := R(\xi) - \omega f(\xi)$.*

**Proof.** Let us show (3.5). Since

$$\begin{aligned} C \geqslant \limsup_{\epsilon\to 0} \int_\Omega |\omega_\epsilon|^p dx &= p \lim_{\epsilon\to 0} \int_\Omega \left[ \int_0^{+\infty} f_\epsilon(\xi)\xi^{p-1} d\xi + \int_{-\infty}^0 (1-f_\epsilon(\xi))|\xi|^{p-1} ds \right] d\mathbf{x} \\ &= p \int_\Omega \left[ \int_0^{+\infty} f(\xi)\xi^{p-1} d\xi + \int_{-\infty}^0 (1-f(\xi))|\xi|^{p-1} ds \right] d\mathbf{x}, \end{aligned}$$

it follows that $\left[\int_0^{+\infty} f(\xi)\xi^{p-1}d\xi + \int_{-\infty}^0 (1-f(\xi))|\xi|^{p-1}d\xi\right] \leqslant C_{x,t} < \infty$ for a.e. $(x,t) \in \Omega_T$. Since $0 \leqslant f(\xi) \leqslant 1$, the result follows.

Now we show (3.6). Observe that

$$(\omega_\varepsilon - \xi) f_\varepsilon(\xi) = \int_\xi^{+\infty} f_\varepsilon(s) ds \quad \text{and} \quad (\xi - \omega_\varepsilon)(1-f_\varepsilon(\xi)) = \int_{-\infty}^{\xi} (1-f_\varepsilon(s)) ds.$$

Taking $\varepsilon \to 0$, we derive the following formulas for $R$

$$R(\xi) = \xi f(\xi) + \int_\xi^{+\infty} f(s) ds \quad \text{and} \quad R(\xi) = \omega - \xi(1-f(\xi)) + \int_{-\infty}^{\xi} (1-f(s)) ds \tag{3.9}$$

a.e. in $\Omega_T$. By the second line of (3.4) and the first formula of (3.9), we get

$$R(\xi) - \omega f(\xi) = f(\xi) \left[ \int_{-\infty}^{\xi} (1-f(s)) ds \right] + (1-f(\xi)) \left[ \int_\xi^{+\infty} f(s) ds \right] \geqslant 0.$$

Next we prove (3.7). The first limit in (3.5) and the first formula in (3.9) imply that $A = o(\xi^{-p+1})$ for $\xi \approx +\infty$, which gives $\lim_{\xi\to+\infty} |\xi| A(\xi) = 0$. Also, the second limit in (3.5) and the second formula in (3.9) imply that

$$A(\xi) = (R-\omega) + \omega(1-f) = o(|\xi|^{-p+1})$$

for $\xi \approx -\infty$ and so $\lim_{\xi\to-\infty} |\xi| A(\xi) = 0$. Finally, to show (3.8) we use the equalities (3.1) and (3.2), with the help of (3.5) and (3.7). ■

The equation (3.3) can be written as

$$\partial_t f + \mathrm{div}_{\mathbf{x}}(\mathrm{sign}(\xi)\,\mathbf{v}\,f) + \partial_\xi\,[|\xi|(h-\omega)f] = \partial_\xi\,[|\xi|(R-\omega f)+m] \quad \text{in } \mathcal{D}'(\mathbb{R}\times\Omega_T), \tag{3.10}$$

that yields the following result.

**Lemma 3** *The function $F := f(1-f)$ satisfies*

$$\partial_t F + \mathrm{div}_{\mathbf{x}}(\mathrm{sign}(\xi)\mathbf{v}F) + \partial_\xi\,[|\xi|(h-\omega)F] \leqslant \partial_\xi[|\xi|A(1-2f)] \quad \text{in } \mathcal{D}'(\mathbb{R}\times\Omega_T), \tag{3.11}$$

*with $A(\xi)$ defined in Lemma 2.*

**Proof.** Let $\rho_\delta = \rho_\delta(\xi)$ be the following non-symmetric even kernel on the variable $\xi\in\mathbb{R}$:

$$\text{if } \xi \geqslant 0, \text{ then } \quad \rho_\delta(\xi,s) := \begin{cases} c_\delta \exp_\delta[(s-\xi)-h] & \text{for } s\in[\xi,\xi+2\delta], \\ 0 & \text{otherwise,} \end{cases}$$

$$\text{if } \xi < 0, \text{ then } \quad \rho_\delta(\xi,s) := \begin{cases} c_\delta \exp_\delta[(s-\xi)+h] & \text{for } s\in[\xi-2\delta,\xi], \\ 0 & \text{otherwise,} \end{cases}$$

where $\exp_\delta(t) := e^{-\frac{\delta^2}{\delta^2-t^2}}$ and $c_\delta = \left(\int_{|t|\leqslant\delta}\exp_\delta(t)dt\right)^{-1}$. We denote by $z^\delta = z*\rho_\delta$ the convolution in the $\xi$ variable, that is $z^\delta(\xi) := \int_{\mathbb{R}} z(s)\rho_\delta(\xi,s)\,ds$. Using this special kernel we derive the property

$$[|\xi|f(\xi)]^\delta = |\xi|f^\delta(\xi) \quad \forall\xi\in\mathbb{R}. \tag{3.12}$$

Now we are able to prove (3.11). We first regularize in the $\xi$ variable by convoluting equation (3.10) with $\rho_\delta$ and use (3.12) to obtain

$$\partial_t f^\delta + \mathrm{sign}(\xi)\mathrm{div}_{\mathbf{x}}(\mathbf{v}\,f^\delta) + (h-\omega)\partial_\xi\left[|\xi|f^\delta\right] = \partial_\xi\,[|\xi|(R-\omega f)+m]^\delta + \epsilon^\delta \tag{3.13}$$

in $\mathcal{D}'(\mathbb{R}\times\Omega_T)$, where the remainder $\epsilon^\delta$ is

$$\epsilon^\delta := (h-\omega)\partial_\xi(|\xi|f^\delta) - \partial_\xi[|\xi|(h-\omega)f]*\rho_\delta.$$

Since $|\xi|\in W^1_\infty(\mathbb{R})$, $f\in L_\infty(\mathbb{R}\times\Omega_T)$, $h,\omega\in L_\infty(0,T;L_p(\Omega))$, acting as in Lemma 2.2, p. 972, [12], we have that $\lim_{\delta\to 0}\epsilon^\delta = 0$ in $L_{1,loc}(\mathbb{R})$.

Next we regularize in the $\mathbf{x}$ variable by convoluting (3.13) with a standard symmetric kernel $\rho_\theta$ on the variable $\mathbf{x}\in\Omega$ (see for example [12]) to obtain

$$\partial_t f^{\delta,\theta} + \mathrm{sign}(\xi)\mathrm{div}_{\mathbf{x}}(\mathbf{v}\,f^{\delta,\theta}) + (h-\omega)\partial_\xi(|\xi|f^{\delta,\theta}) = \partial_\xi\,[|\xi|(R-\omega f)+m]^\delta*\rho_\theta + \epsilon_1^{\delta,\theta} + \epsilon_2^{\delta,\theta} + \epsilon_3^{\delta,\theta}, \tag{3.14}$$

where $u^\theta := u*\rho_\theta$ and

$$\epsilon_1^{\delta,\theta} := \epsilon^\delta * \rho_\theta,$$

$$\epsilon_2^{\delta,\theta} := \mathrm{sign}(\xi)\left[\mathrm{div}_{\mathbf{x}}(\mathbf{v}f^{\delta,\theta}) - \mathrm{div}_{\mathbf{x}}(\left[\mathbf{v}f^\delta\right]*\rho_\theta)\right],$$

$$\epsilon_3^{\delta,\theta} := (h-\omega)\partial_\xi(|\xi|f^{\delta,\theta}) - [(h-\omega)\partial_\xi(|\xi|f^\delta)]*\rho_\theta.$$

By the principal property of the convolution for integrable functions and the fact that $f^\delta$ is regular on $\xi$, it follows that

$$\lim_{\delta\to 0}\lim_{\theta\to 0}\epsilon_1^{\delta,\theta} = \lim_{\delta\to 0}(\epsilon^\delta) = 0 \quad \text{and} \quad \lim_{\theta\to 0}\epsilon_3^{\delta,\theta} = 0.$$

Since $\mathbf{v}(t,\mathbf{x}) \in L_\infty\left(0,T;W_p^1(\Omega)\right)$, we have

$$\lim_{\delta\to 0}\lim_{\theta\to 0}\epsilon_2^{\delta,\theta} = 0.$$

Moreover, in the left part of the equation for $f^{\theta,\delta}$ we can apply the chain rule, since $f^{\theta,\delta}$ is smooth in $(\xi,\mathbf{x})$. So, multiplying (3.14) by $(1-2f^{\theta,\delta})$, and taking the limit first on $\theta\to 0$, then on $\delta\to 0$, we obtain that $F := f(1-f)$ satisfies (3.11) because of the relation

$$\int_{\mathbb{R}} \partial_\xi[|\xi|\,(R-\omega f)+m]^{\delta,\theta}\,\left(1-2f^{\delta,\theta}\right)\,d\xi = 2\int_{\mathbb{R}}[|\xi|(R-\omega f)+m]^{\delta,\theta}\,\frac{\partial f^{\theta,\delta}}{\partial\xi}\,d\xi \leqslant 0.$$

The latter inequality follows from the properties (3.4) and Lemma 2. ∎

The equality (3.3) and the inequalities (3.1), (3.2) yields also the existence of traces for the function $f$ at the time $t=0$ and on the boundary $\Gamma$, having the following properties.

**Lemma 4** *1) There exists the trace $f^0 = f^0(\xi,\mathbf{x})$ at the time moment $t=0$ for the function $f$, such that*

$$f^0 = \lim_{\delta\to 0^+}\frac{1}{\delta}\int_0^\delta f(\cdot,t,\cdot)\,dt$$

*a.e. on $\mathbb{R}\times\Omega$ and*

$$\int_{\mathbb{R}\times\Omega}\left(f^0 - \left(f^0\right)^2\right)d\xi d\mathbf{x} = 0. \tag{3.15}$$

*2) There exists the trace $f^\Gamma = f^\Gamma(\xi,t,\mathbf{x})$ on the boundary $\Gamma$ for the function $f$, such that*

$$f^\Gamma(\xi,t,\mathbf{x}) = \lim_{\delta\to 0^+}\frac{1}{\delta}\int_0^\delta f(\xi,t,\mathbf{x}-s\,\mathbf{n}(\mathbf{x}))\,ds$$

*for a.e. $(\xi,t,\mathbf{x})\in\mathbb{R}\times\Gamma_T$, where $sign(\xi)(\mathbf{v}\cdot\mathbf{n})(t,\mathbf{x})\neq 0$ and*

$$\int_{\mathbb{R}\times\Gamma_T}|\mathrm{sign}(\xi)(\mathbf{v}\cdot\mathbf{n})|_-\left(f^\Gamma-\left(f^\Gamma\right)^2\right)\,d\xi dt d\mathbf{x} = 0. \tag{3.16}$$

**Proof.** Let $\sigma\in C_0^\infty(\mathbb{R})$ be a fixed function. We define a vector function $\mathbf{F}_\sigma = \mathbf{F}_\sigma(t,\mathbf{x})$ as

$$\mathbf{F}_\sigma := \left(\int_{\mathbb{R}} f(\xi,\cdot,\cdot)\,\sigma(\xi)\,d\xi, \quad \mathbf{v}\int_{\mathbb{R}}\mathrm{sign}(\xi)\,f(\xi,\cdot,\cdot)\,\sigma(\xi)\,d\xi\right)$$

From (3.10) we have that in the distributional sense

$$\begin{aligned}\operatorname{div}_{t,\mathbf{x}}\mathbf{F}_\sigma &= \int_{\mathbb{R}} (\partial_t f + \operatorname{div}_{\mathbf{x}}(\operatorname{sign}(\xi)\mathbf{v}f))\sigma(\xi)\,d\xi \\ &= \int_{\mathbb{R}} \{-m(\xi,\cdot,\cdot) + |\xi|(fh - R)\}\,\sigma'(\xi)\,d\xi \in \mathcal{M}(\Omega_T).\end{aligned}$$

Let $\Sigma$ be the boundary of a closed domain $\overline{\Omega}_T$. Therefore denoting by $\mathbf{n}_{(t,\mathbf{x})}$ the external normal to $\Sigma$, it follows from Theorem 2.1 of [8] that $\mathbf{F}_\sigma \cdot \mathbf{n}_{(t,\mathbf{x})}$ is a continuous linear functional defined over $L_\infty(\Sigma)$.

*Proof of the part 1).* If we take an arbitrary $\psi \in C^\infty([0,T]\times\Omega)$ with a compact support on $\Omega$, then

$$\begin{aligned}\left\langle \mathbf{F}_\sigma \cdot \mathbf{n}_{(t,\mathbf{x})}\Big|_{t=0}, \psi \right\rangle &= \lim_{\delta\to 0} \frac{1}{\delta}\int_0^\delta \int_\Omega \left( \int_{\mathbb{R}} f(\xi,t,\mathbf{x})\sigma(\xi)\,d\xi \right) \psi(t,\mathbf{x})\;dt\,d\mathbf{x} \\ &= \lim_{\delta\to 0} \int_{\mathbb{R}\times\Omega} \left( \frac{1}{\delta}\int_0^\delta f(\xi,t,\mathbf{x})\,dt \right) \sigma(\xi)\psi(0,\mathbf{x})\;d\xi\,d\mathbf{x}.\end{aligned}$$

Since $0 \leqslant \frac{1}{\delta}\int_0^\delta f(\cdot,t,\cdot)\,dt \leqslant 1$ for $\delta \in (0,T)$, the dominated convergence theorem implies that $\mathbf{F}_\sigma \cdot \mathbf{n}_{(t,\mathbf{x})}|_{t=0}$ is a bounded function in $L^\infty(\mathbb{R}\times\Omega)$, which is equal to $f^0\sigma$. As a consequence of $0 \leqslant f \leqslant 1$, we have

$$0 \leqslant f^0 \leqslant 1 \qquad \text{a.e. in} \quad \mathbb{R}\times\Omega. \tag{3.17}$$

Next, let us choose $\varphi := O_\delta(t)\,\phi$ in (3.1), where $O_\delta := 1 - 1_\delta$ with $1_\delta$ given by (2.11) and $\phi \in C_0^\infty(\Omega)$ is a non-negative function. Taking $\delta \to 0$ in an obtained inequality, we derive

$$-\int_\Omega \int_\xi^{+\infty} f^0(s,\mathbf{x})\,\phi(\mathbf{x})\,ds\,d\mathbf{x} + \int_\Omega |\omega_0(\mathbf{x}) - \xi|_+\,\phi(\mathbf{x})\,d\mathbf{x} \geqslant 0,$$

that is

$$-\int_\xi^{+\infty} f^0(s,\mathbf{x})\,ds + |\omega_0(\mathbf{x}) - \xi|_+ \geqslant 0 \qquad \text{for a.e. } \mathbf{x}\in\Omega.$$

By (3.17) for any fixed $\mathbf{x}\in\Omega$ and $\xi > \omega_0(\mathbf{x})$, we deduce $f^0(\xi,\mathbf{x}) = 0$.

By the same way if we put the above chosen $\varphi$ in (3.2) and pass on $\delta\to 0$ in an obtained inequality, we get

$$-\int_{-\infty}^{\xi} (1 - f^0(s,\mathbf{x}))\,ds + |\omega_0(\mathbf{x}) - \xi|_- d\mathbf{x} \geqslant 0 \;\text{ a.e. } \Omega,$$

hence for the fixed $\mathbf{x}\in\Omega$ and $\xi < \omega_0(\mathbf{x})$, we derive $f^0(\xi,\mathbf{x}) = 1$. Therefore $f^0$ satisfies (3.15).

*Proof of the part 2).* Let

$$d(\mathbf{x}) := \inf_{\mathbf{y}\in\Gamma} |\mathbf{x}-\mathbf{y}| \tag{3.18}$$

be the distance between $\mathbf{x} \in \overline{\Omega}$ and the boundary $\Gamma$, having the property $\nabla d|_\Gamma = -\mathbf{n}$. If we take an arbitrary $\psi \in C^\infty((0,T)\times\overline{\Omega})$ with a compact support on $(0,T)$, then

$$\begin{aligned}\left\langle \mathbf{F}_\sigma \cdot \mathbf{n}_{(t,\mathbf{x})}\Big|_{\Gamma_T}, \psi \right\rangle &= \lim_{\delta\to 0}\frac{1}{\delta}\int_0^\delta\int_{\Gamma_T}\left[-\int_{\mathbb{R}} \mathrm{sign}(\xi)(\mathbf{v}\cdot\nabla d)(t,\widetilde{\mathbf{x}})f(\xi,t,\widetilde{\mathbf{x}})\sigma(\xi)\,d\xi\right]\psi(t,\mathbf{x})\,dtd\mathbf{x}ds \\ &= \int_{\mathbb{R}\times\Gamma_T}\mathrm{sign}(\xi)(\mathbf{v}\cdot\mathbf{n})(t,\mathbf{x})\lim_{\delta\to 0}\left[\frac{1}{\delta}\int_0^\delta f(\xi,t,\widetilde{\mathbf{x}})\,ds\right]\sigma\,\psi|_\Gamma\,d\xi dt\,d\mathbf{x},\end{aligned}$$

where $\widetilde{\mathbf{x}} := \mathbf{x} - s\,\mathbf{n}(\mathbf{x})$. Since $0 \leqslant \frac{1}{\delta}\int_0^\delta f(\cdot,\cdot,\widetilde{\mathbf{x}})\,ds \leqslant 1$ for any $\delta \in (0,\delta_0)$, then $\mathbf{F}_\sigma\cdot\mathbf{n}_{(t,\mathbf{x})}|_{\Gamma_T}$ is a bounded function in $L^\infty(\mathbb{R}\times\Gamma_T)$, which is equal to $sign(\xi)(\mathbf{v}\cdot\mathbf{n})\ f^\Gamma\sigma$. The function $f^\Gamma$ fulfills

$$0 \leqslant f^\Gamma \leqslant 1 \quad \text{a.e. on} \quad \mathbb{R}\times\Gamma_T \qquad \text{and} \qquad \frac{\partial f^\Gamma}{\partial\xi} \leqslant 0 \quad \text{in} \ \mathcal{D}'(\mathbb{R}\times\Gamma_T), \tag{3.19}$$

as a consequence of $0 \leqslant f \leqslant 1,\quad f'_\xi \leqslant 0$ (see (3.4)).

Let $\mathbf{0}_\delta : \overline{\Omega} \to \mathbb{R}$ defined by $\mathbf{0}_\delta(\mathbf{x}) := 1 - 1_\delta(d(\mathbf{x}))$ with the function $1_\delta$ is given by (2.11). Now, choose in (3.1) $\varphi := \mathbf{0}_\delta(\mathbf{x})\,\phi$ for a non-negative $\phi \in C^\infty((0,T)\times\overline{\Omega})$, having a compact support on $(0,T)$. Taking $\delta \to 0$ in an obtained inequality, we derive

$$\int_{\Gamma_T}\int_\xi^{+\infty}\mathrm{sign}(s)\,(\mathbf{v}\cdot\mathbf{n})\ f^\Gamma(s,t,\mathbf{x})\,ds\ \phi(t,\mathbf{x})\,d\mathbf{x} + \int_{\Gamma_T} M(t)|b-\xi|_+\,\phi(t,\mathbf{x})\,d\mathbf{x} \geqslant 0.$$

Hence

$$m_+^\Gamma(\xi,t,\mathbf{x}) := M(t)\ |b(t,\mathbf{x},\frac{\partial h}{\partial\mathbf{n}}) - \xi|_+ + \int_\xi^{+\infty}\mathrm{sign}(s)\,(\mathbf{v}\cdot\mathbf{n})\ f^\Gamma(s,t,\mathbf{x})\,ds \qquad \text{on} \ \mathbb{R}\times\Gamma_T$$

is a positive function, which fulfills

$$\begin{cases} m_+^\Gamma(\xi,t,\mathbf{x}) \in W^1_\infty(\mathbb{R}, L_\infty(\Gamma_T)) \qquad \text{and} \qquad \lim\limits_{\xi\to+\infty} m_+^\Gamma = 0 \qquad \text{a.e. on } \Gamma_T, \\ \mathrm{sign}(\xi)(\mathbf{v}\cdot\mathbf{n})f^\Gamma = -M(t)\mathrm{sign}_+(b(\cdot,\cdot,\frac{\partial h}{\partial\mathbf{n}}) - \xi) - \partial_\xi m_+^\Gamma. \end{cases} \tag{3.20}$$

By the same way if we put the above chosen $\varphi$ in (3.2) and pass on $\delta \to 0$ in an obtained inequality, we derive

$$\int_{-\infty}^{\xi}\mathrm{sign}(\xi)(\mathbf{v}\cdot\mathbf{n})(1 - f^\Gamma(s,t,\mathbf{x}))\,ds + M(t)|b-\xi|_- \geqslant 0.$$

Hence

$$m_-^\Gamma(\xi,t,\mathbf{x}) := M(t)|b(t,\mathbf{x},\frac{\partial h}{\partial \mathbf{n}}) - \xi|_- + \int\limits_{-\infty}^{\xi} \operatorname{sign}(s)(\mathbf{v}\cdot\mathbf{n})(1-f^\Gamma(s,t,\mathbf{x}))\,ds \qquad \text{on } \mathbb{R}\times\Gamma_T$$

is a positive function, which fulfills

$$\begin{cases} m_-^\Gamma(\xi,t,\mathbf{x}) \in W_\infty^1(\mathbb{R}, L_\infty(\Gamma_T)) \qquad \text{and} \qquad \lim\limits_{\xi\to-\infty} m_-^\Gamma = 0 \qquad \text{a.e. on } \Gamma_T, \\ \operatorname{sign}(\xi)(\mathbf{v}\cdot\mathbf{n})(f^\Gamma - 1) = M(t)\left[1 - \operatorname{sign}_+(b(\cdot,\cdot,\frac{\partial h}{\partial \mathbf{n}}) - \xi)\right] - \partial_\xi m_-^\Gamma. \end{cases} \tag{3.21}$$

Since $s = -|s|_-$ for $s<0$, then using (3.19)-(3.21) we have

$$0 \leqslant -\int\limits_{\mathbb{R}} |\operatorname{sign}(\xi)(\mathbf{v}\cdot\mathbf{n})|_-\, f^\Gamma(f^\Gamma-1)\,d\xi = \int\limits_{-\infty}^{b} f^\Gamma\{M\left[1-\operatorname{sign}_+(b-\xi)\right] - \partial_\xi m_-^\Gamma)\}\,d\xi$$

$$+ \int\limits_{b}^{+\infty} [-M\operatorname{sign}_+(b-\xi) - \partial_\xi m_+^\Gamma](f^\Gamma-1)\,d\xi$$

$$= -\left[ (f^\Gamma m_-^\Gamma)|_{\xi=b-0} - \int\limits_{-\infty}^{b} \partial_\xi f^\Gamma m_-^\Gamma\,d\xi \right.$$

$$\left. + m_+^\Gamma(1-f^\Gamma)|_{\xi=b+0} - \int\limits_{b}^{+\infty} m_+^\Gamma \partial_\xi f^\Gamma d\xi \right] \leqslant 0 \qquad \text{a.e. on } \Gamma_T.$$

A formal integration on $\xi$ by parts in the last identity can be justified by mollifying the function $f^\Gamma$ and taking the limit transition on a mollifying parameter. Therefore $f^\Gamma$ satisfies (3.16). ■

**Lemma 5** *We have*

$$\int_{\mathbb{R}\times\Omega_T} F\ d\xi dt d\mathbf{x} \leqslant 0. \tag{3.22}$$

**Proof.** To obtain (3.22) we use the simple inequality

$$-\frac{1}{\delta}\int_0^\delta z^2(s)ds \leqslant -\left(\frac{1}{\delta}\int_0^\delta z(s)ds\right)^2, \tag{3.23}$$

which is valid for any positive integrable function $z = z(s)$.

Let us introduce

$$\varphi := (1_\varepsilon(\xi+\varepsilon^{-1}) - 1_\varepsilon(\xi-\varepsilon^{-1}))\varphi_1^\delta(t)\varphi_2^\delta(\mathbf{x})$$

with $\varphi_1^\delta(t) := (1_\delta(t) - 1_\delta(t-t_0+\delta))$ for $t_0 \in (2\delta, T)$ and $\varphi_2^\delta(\mathbf{x}) := 1_\delta(d(\mathbf{x}))$ for $\mathbf{x}\in\Omega$. Here, $1_\delta$ and $d = d(\mathbf{x})$ are given by (2.11) and (3.18), respectively. Choosing $\varphi = \varphi(\xi,t,\mathbf{x})$ as a test function in the respective integral form of (3.11) and taking the limit transition on $\varepsilon \to 0$, with the help of (3.5) and (3.7), we get the inequality

$$\frac{1}{\delta}\int_{t_0-\delta}^{t_0}\int_{\mathbb{R}\times\Omega} F\ \varphi_2^\delta(\mathbf{x})\ dtd\xi d\mathbf{x} \leqslant \frac{1}{\delta}\int_0^\delta\int_{\mathbb{R}\times\Omega} F\ \varphi_2^\delta(\mathbf{x})\ dtd\xi d\mathbf{x}$$
$$+\frac{1}{\delta}\int_{0\leqslant d(\mathbf{x})\leqslant\delta}\int_{\mathbb{R}\times[0,T]} |\mathrm{sign}(\xi)(\mathbf{v}\cdot\nabla d)|_-\ F\ \varphi_1^\delta(t)\ dtd\xi d\mathbf{x} = C_1^\delta + C_2^\delta. \tag{3.24}$$

From (3.23) it follows that

$$C_1^\delta \leqslant \int_{\mathbb{R}\times\Omega}\left[\left(\frac{1}{\delta}\int_0^\delta f(t)dt\right) - \left(\frac{1}{\delta}\int_0^\delta f(t)dt\right)^2\right]\ \varphi_2^\delta(\mathbf{x})\ d\xi d\mathbf{x}.$$

Since $0 \leqslant \frac{1}{\delta}\int_0^\delta f(t)dt \leqslant 1$, in view of the dominated convergence theorem and Lemma 4, we derive

$$\limsup_{\delta\to 0} C_1^\delta \leqslant \int_{\mathbb{R}\times\Omega}(f^0 - {f^0}^2)\ d\xi d\mathbf{x} = 0.$$

Let us now consider the term $C_2^\delta$. Because of $\Gamma \in C^2$ there exists a small enough $\delta$, such that any point $\mathbf{x}\in\Omega:\ d(\mathbf{x}) < \delta$ has an unique projection $\mathbf{x}_0 = \mathbf{x}_0(\mathbf{x})$ on the boundary $\Gamma$. We can parametrize all points $\mathbf{x}_0 \in \Gamma$ by a length parameter $l \in [0, L]$ with $L$ being the length of $\Gamma$, considering $\mathbf{x}_0 = \mathbf{x}_0(l)$. We have 2 important properties: 1) $\mathbf{v}(t,\mathbf{x}) = \mathbf{v}(t,\mathbf{x}_0) + O(\delta^\alpha)$, since $\mathbf{v}(t,\cdot) \in C^\alpha(\overline{\Omega})$ with $\alpha = 1-\frac{2}{p}$; 2) $\nabla d(\mathbf{x}) = -\mathbf{n}(\mathbf{x}_0)+O(\delta)$ and the Jacobian $\frac{D(x_1,x_2)}{D(l,s)} = 1+O(\delta)$, since $(l,s)$ forms the orthogonal coordinate system at $s = 0$. Making in the term $C_2^\delta$ the change of variables $\mathbf{x} = (x_1,x_2) \leftrightarrow (l,s)$ with $s := d(\mathbf{x})$ and using (3.23), we obtain

$$\begin{aligned} C_2^\delta &\leqslant \frac{1}{\delta}\int_0^\delta\int_0^L\left[\int_{\mathbb{R}\times[0,T]}|\mathrm{sign}(\xi)(\mathbf{v}\cdot\mathbf{n})(t,\mathbf{x}_0)|_-\ F(\xi,t,\widetilde{\mathbf{x}})\ \varphi_1^\delta(t)\ dtd\xi\right] dsdl\ + O(\delta^\alpha) \\ &\leqslant \int_{\mathbb{R}\times\Gamma_T}|\mathrm{sign}(\xi)(\mathbf{v}\cdot\mathbf{n})(t,\mathbf{x}_0)|_-\ [f^\delta - \left(f^\delta\right)^2]\ d\xi dtd\mathbf{x}_0 + O(\delta^\alpha), \end{aligned}$$

considering $\widetilde{\mathbf{x}}(l,s) := \mathbf{x}_0(l) - s\mathbf{n}(\mathbf{x}_0(l))$ in the $1^{st}$ inequality and $f^\delta := \frac{1}{\delta}\int_0^\delta f(\cdot,\cdot,\widetilde{\mathbf{x}})ds$ in the $2^{nd}$ inequality. Since $0 \leqslant \frac{1}{\delta}\int_0^\delta f(\cdot,\cdot,\mathbf{x}_0 - s\mathbf{n}(\mathbf{x}_0))ds \leqslant 1$, in view of Lemma 4, we obtain

$$\limsup_{\delta\to 0} C_2^\delta \leqslant \int_{\mathbb{R}\times\Gamma_T}|\mathrm{sign}(\xi)(\mathbf{v}\cdot\mathbf{n})(t,\mathbf{x})|_-(f^\Gamma - \left(f^\Gamma\right)^2)\ d\xi dtd\mathbf{x} = 0.$$

Finally integrating (3.24) over $t_0 \in [2\delta, T]$, applying Fubini's theorem to the left part of the inequality and taking the limit on $\delta \to 0$, we get (3.22). ■

By Lemma 5 and the fact that $F = f(1-f) \geqslant 0$, we get $F = 0$ a.e. in $\mathbb{R}\times\Omega_T$, i.e. $f$ takes only the values 0 and 1. Since $f$ is monotone decreasing on $\xi$, there exists a function $u = u(t,\mathbf{x})$, such that

$$f(\xi,t,\mathbf{x}) = \mathrm{sign}_+(u(t,\mathbf{x}) - \xi).$$

Therefore

$$|\omega_\varepsilon|_+ = \int\limits_0^{+\infty} f_\varepsilon(\xi,\cdot,\cdot)\ d\xi, \quad |\omega_\varepsilon|_- = \int\limits_{-\infty}^{0} (1 - f_\varepsilon(\xi,\cdot,\cdot))\ d\xi \rightharpoonup |u|_+, \quad |u|_-$$

weakly $-*$ in $L_\infty(0,T;L_p(\Omega))$. This implies $u = \omega$ (see (3.4)) and, as a consequence of it we derive the strong convergence in $L_\infty(0,T;L_p(\Omega))$ of $\{\omega_\varepsilon\}$ to $\omega$. Therefore $R = sign_+(\omega - \xi)\ \omega$ and taking the sum of the inequalities (3.1), (3.2), we derive that $\omega$ satisfies (1.11). By the Calderon-Zygmund inequality (Theorem 9.9, [10]) we get also the strong convergence of $\{h_\varepsilon\}$ to $h$ in $L_\infty(0,T;W_p^2(\Omega))$, that ends the proof of Theorem 1.

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# References

[1] Ambrosio, L., Serfaty, S. *A gradient flow approach to an evolution problem arising in superconductivity*, Comm. Pure and Applied Math., **61**, 11 (2008), 1495-1539.

[2] Antontsev S.N., Chemetov N.V., *Flux of superconducting vortices through a domain*, SIAM J. Math. Anal., **39** (2007), 263–280.

[3] Antontsev S.N., Chemetov N.V., *Superconducting Vortices: Chapman Full Model,* Adv. Math. Fluid Mech., New Directions in Math. Fluid Mech., The Alexander V. Kazhikhov Memorial Volume, 41-55 (2009).

[4] Brezis H., *Analyse fonctionelle. Théorie et applications*, MASSON, Paris, 1983.

[5] Cebeci T., Cousteix J., *Modeling and computation of boundary-layer flows, Spriger,* Springer-Verlag, Berlin, 2005.

[6] Chapman S.J., *A hierarchy of models for type-II superconductors*, SIAM Review, **42** (2000), 555–598.

[7] Chapman S.J., *Macroscopic models of superconductivity,* in ICIAM 99 (Edinburgh), Oxford Univ. Press, Oxford, (2000), 23–34.

[8] Chen G.-Q., Frid H., *On the theory of divergence-measure fields and its applications.* Bol. Soc. Bras. Mat., **32**, 3 (2001), 1-33.

[9] Du Q., Zhang P., *Existence of weak solutions to some vortex density models,* SIAM J. Math. Anal. 34 (2003), pp. 1279–1299.

[10] Gilbarg D. and Trudinger, N.S., *Elliptic partial differential equations of second order.* Springer-Verlag, Berlin, 1983.

[11] Kruzkov S.N., *First order quasilinear equations in several independent variables,* Math. USSR Sb., **10,** 2 (1970), 127-243.

[12] De Lellis C., *Ordinary differential equations with rough coefficients and the renormalization theorem of Ambrosio,* Bourbaki Seminar, Preprint, (2007) 1-26.

[13] Ladyzhenskaya O.A., Solonnikov V.A., Ural'tseva N.N., *Linear and quasilinear equations of parabolic type.* American Mathematical Society, Providence RJ (1968).

[14] Ladyzhenskaya O.A., Uraltseva N.N., *Linear and quasilinear elliptic equations.* Academic Press, New York and London (1968).

[15] Majda A.J., Bertozzi A.L., *Vorticity and incompressible flow,* Cambridge University Press, 2002.

[16] Malek J., Necas J., Rokyta M., Ruzicka M., *Weak and measure-valued solutions to evolutionary PDEs.* Chapman&Hall, London (1996).

[17] Masmoudi N., Zhang P., *Global solutions to vortex density equations arising from supconductivity,* Ann. I. H. Poincaré - AN, **22** (2005), 441-458.

[18] Oleinik O.A., Samokhin V.N., *Mathematical models in boundary layer theory,* Chapman&Hall/CRC, 1999.

[19] Perthame B., *Kinetic formulation of conservation laws,* Oxford University Press, 2002.

[20] Perthame B., Dalibard A.-L., *Existence of solutions of the hyperbolic Keller-Segel model,* Trans. Amer. Math. Soc., **361**, (2009) 2319-2335.

[21] Plotnikov P.I., Sazhenkov S.A., *Kinetic formulation for the Graetz–Nusselt ultra-parabolic equation,* J. Math. Anal. Appl. **304**, (2005), 703–724.

[22] Plotnikov P.I., Sazhenkov S.A., *Cauchy Problem for the Graetz–Nusselt Ultra-parabolic Equation,* Doklady Mathematics, **71,** 2 (2005), 234–237.

[23] Schatzle R., Styles V., *Analysis of a mean field model of superconducting vortices,* European J. Appl. Math., 10 (1999), 319–352.