Existence of Gevrey approximate solutions for certain systems of linear vector fields applied to involutive systems of first-order nonlinear pdes

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Abstract

Given a G^s-*involutive structure*, $(M, \mathcal{V})$, a Gevrey submanifold $X \subset M$ which is maximally real and a Gevrey function u_0 on X we construct a Gevrey function u which extend u_0 and is a Gevrey approximate solution for $\mathcal{V}$. We then use our construction to study Gevrey microlocal regularity of solutions, $u \in C^2(\mathbb{R}^N)$, of a system of nonlinear pdes of the form

$$F_j(x, u, u_x) = 0,\ j = 1, \dots, n$$

where $F_j(x, \zeta_0, \zeta)$ are Gevrey functions of order $s > 1$ and holomorphic in $(\zeta_0, \zeta) \in \mathbb{C} \times \mathbb{C}^N$. The functions F_j satisfy an involutive condition and $d_\zeta F_1 \wedge \cdots \wedge d_\zeta F_n \neq 0$.

1 Introduction

In this paper we are motivated to analyze the existence of approximate solutions of an overdetermined system of linear vector fields with initial data given on an appropriate initial submanifold since, in general, it is not possible to construct homogeneous solutions and several authors have used it in different applications.

For instance, by using the existence of approximate solutions for a system of C^∞ linear vector fields, (see Treves [T]), Asano [A] has characterized the C^∞ wave-front set of C^2 solutions of first-order nonlinear pdes, giving another proof of the well-known Chemin's result (see [Ch]), while Adwan and Berhanu [AB] have used it to describe the C^∞ wave-front set of the boundary values of approximate solutions in wedges of involutive structures that are not necessarily locally integrable.

We now are going to describe what is known about this subject in the Gevrey spaces $G^s, s \geq 1$. When $s = 1$ we are in the analytic case and therefore we have at our disposal the Cauchy-Kovalewsky theorem and hence we do not need approximate

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solutions. Thus, we will restrict ourselves to the case $s > 1$. First we recall the Carleman's problem that says (cf. Bruna [Br] and Carleman [Ca]): *given a sequence of complex numbers, $\{m_n\}$, satisfying $|m_n| \leq B^{n+1}n^{ns}$, $n = 0, 1, \ldots,$ where B is a positive constant and $s > 1$, is there a Gevrey function $f(x)$ of order s, defined on $[-1, 1]$, such that $f^{(n)}(0) = m_n$, $n = 0, 1, \ldots$?* This question has an affirmative answer, as proved by Mityagin [Mi]. In Džanašija [Dz] an explicit construction of such a function f can be found.

In Barostichi and Petronilho [BP] we showed that Džanašija's construction can be extended in order to achieve the following result: given a G^s vector field L, a G^s hypersurface Σ which is non-characteristic with respect to L, and a function $u_0 \in G^s(\Sigma)$ it is possible to extend u_0 as a G^s-function u which is an approximate solution of the equation $Lv = 0$. We applied this result in order to fill the gap between Chemin's and Hanges-Treves' results (Hanges and Treves, [HT] have proved the Chemin's result in the analytic cathegory). We remark that, for a different application, Adwan and Hoepfner [AH] also studied the existence of Gevrey approximate solutions for the same class of vector fields. Neverthless, their result is weaker, in the sense that their extension of $u_0 \in G^s(\Sigma)$ is only of Gevrey class $s' > s + 1$, with s' arbitrary.

In order to describe our results we shall recall some concepts. We start by considering a pair $(M, \mathcal{V})$ in which M is real analytic manifold and $\mathcal{V}$ is a G^s subbundle of the complexified tangent bundle $\mathbb{C}TM$. The pair $(M, \mathcal{V})$ is called a *G^s-involutive structure of rank n* if $\mathcal{V}$ is a G^s-formally integrable structure of rank n, that is, the bracket of two sections of $\mathcal{V}$ is also a section of $\mathcal{V}$. Notice that the local generators for the bundle $\mathcal{V}$ are given by G^s vector fields. The G^s-involutive structure $(M, \mathcal{V})$ is called *locally integrable* if the orthogonal of $\mathcal{V}$, $\mathcal{V}^{\perp}$, in $\mathbb{C}T^*M$ is locally generated by exact forms. A function (or distribution) u is called a *solution* of the G^s-involutive structure $(M, \mathcal{V})$ if $Lu = 0$ for every G^s section L of $\mathcal{V}$. For more details about involutive structures we refer the reader to Berhanu, Cordaro and Hounie [BCH].

Our first result says that given a *G^s-involutive structure*, $(M, \mathcal{V})$, a Gevrey submanifold $X \subset M$ which is maximally real and a Gevrey function u_0 on X we are able to construct a Gevrey function u which extend u_0 and is a Gevrey approximate solution for $\mathcal{V}$, (see definition 2.4).

In the sequence, motivated by Berhanu [B2] and Barostichi and Petronilho [BP], we study Gevrey micro-regularity of solutions of certain overdetermined system of first-order pdes of the form

$$F_j(x, u, u_x) = 0,\ j = 1, \ldots, n \tag{1.1}$$

which is involutive. The nonlinear systems considered here are generalizations of the linear case described above (see Berhanu [B2]). In the linear case, when the system is locally integrable, local and micro-local regularity of solutions u has been studied extensively (see e.g., Treves [T], Journé and Trépreau [JT] and references therein). Micro-regularity of solutions for a single nonlinear equation in the analytic, C^∞ and Gevrey cases was investigated, e.g., by Hanges and Treves [HT], Berhanu [B1], Chemin [Ch], Asano [A], Barostichi and Petronilho [BP]. In Asano [A] and Barostichi and Petronilho [BP] the existence of approximate solutions was crucial in order to obtain their result. For results on the micro-local analytic regularity of solutions of higher order linear partial differential operators, we mention the works [H1] and [H2] written by Himonas.

We now will compare, locally, the linear case with our nonlinear system in study. When $(M, \mathcal{V})$ is a G^s-involutive structure, near a point $p \in M$, one can choose local coordinates (x, t), $x = (x_1, \ldots, x_m)$, $t = (t_1, \ldots, t_n)$, $m + n = N$ and G^s vector fields

$$L_j = \frac{\partial}{\partial t_j} + \sum_{k=1}^{m} a_{jk}(x, t) \frac{\partial}{\partial x_k}, \; j = 1, \ldots, n,$$

such that $L_1, \ldots, L_n$ generates $\mathcal{V}$ on some neighborhood $\mathcal{O}$ of p. A solution $u \in D'(\mathcal{O})$ is a solution of the G^s-involutive structure $(M, \mathcal{V})$ if

$$L_j u = 0, \; j = 1, \ldots, n.$$

Similarly, for the nonlinear system $F_j(x, u, u_x) = 0, \; j = 1, \ldots, n$, where the F_j are Gevrey in x and holomorphic in the last two variables, we can choose local coordinates $(x, t) \in \mathbb{R}^m \times \mathbb{R}^n$ such that the equations take the form

$$u_{t_j} = f_j(x, t, u, u_x), \; j = 1, \ldots, n$$

with the $f_j(x, t, \zeta_0, \zeta)$ Gevrey in (x, t) and holomorphic in the variables (ζ_0, ζ).

We prove that the G^s wave-front set of a C^2 solution, u, of the G^s-involutive system of first-order nonlinear pdes given by (1.1) is contained in the characteristic set of the system of the linearized operators $L_1^u, \ldots, L_n^u$ of $F_1(x, u(x), u_x(x)), \ldots, F_n(x, u(x), u_x(x))$ at u, respectively. The analytic case, has been recently investigated by Berhanu [B2]. In contrast to the analytic case, we do not have first integrals which is one of the key points in Berhanu [B2], but we are able to replace it by the existence of Gevrey approximate solutions that we have contructed here.

This article is organized as follows. We start section 2 by recalling some standard notations, definitions and results which are necessary to state our main result about the existence of Gevrey approximate solutions. After that, we present our method in order to construct an approximate solution of a G^s-involutive structure with the initial data given on a maximally real submanifold. In section 3 we present properties of an involutive system of first-order nonlinear pde, in the Gevrey frame. A simple proof of a holomorphic version of the implicit function theorem with dependence on parameters in the Gevrey class, which is necessary to express, locally, our system of pdes is also presented. Finally, in section 4 we take advantage of the existence of approximate solutions, proved in the previous section, in order to characterize the G^s wave-front set of a C^2 solution of a Gevrey involutive system of first-order nonlinear pdes.

2 Gevrey approximate solutions

In order to state precisely our main result of this section we shall introduce some notations, definitions and we also recall some useful results. For results on Gevrey functions mentioned below we refer the reader to Rodino [R].

Let $\mathcal{O} \subset \mathbb{R}^N$ be an open subset and let $s \geq 1$ be a fixed real number.

Definition 2.1 *We say that a function $f(x) \in C^\infty(\mathcal{O})$ is in the Gevrey class $G^s(\mathcal{O})$ if for every compact subset $K \subset \mathcal{O}$ there exists a constant $C > 0$ such that $|\partial_x^\alpha f(x)| \leq C^{|\alpha|+1}(\alpha!)^s$, for all $\alpha \in \mathbb{Z}_+^N$ and for all $x \in K$. In particular, $G^1(\mathcal{O})$ is the space of all analytic functions, denoted by $C^\omega(\mathcal{O})$.*

Definition 2.2 *Assume $s > 1$. We shall denote by $G_0^s(\mathcal{O})$ the vector space of all $f \in G^s(\mathcal{O})$ with compact support in $\mathcal{O}$.*

In order to give our definition of approximate solution to a G^s-involutive structure we shall need to recall the following definition

Definition 2.3 *Let $(M, \mathcal{V})$ be a G^s-involutive structure of rank n and let $X \subset M$ be a submanifold. We say that X is maximally real if for every $p \in X$ we have*

$$\mathbb{C}T_pM = \mathbb{C}T_pX \oplus \mathcal{V}_p.$$

If m is the dimension of X then we have, of course, $m + n = N$.

Definition 2.4 *Let $(M, \mathcal{V})$, $X \subset M$ and $p \in X$ be as above. Let Ω be an open set such that $p \in \Omega \subset M$. We say that a C^1 function (or distribution) in Ω is an s-approximate solution for $\mathcal{V}$ over Ω if for every section L of $\mathcal{V}$ over Ω there is a constant $C > 0$ such that*

$$|Lu(p)| \leq C^{k+1}(k!)^{s-1}(\text{dist}(p, X))^k, \ \ \forall\, p \in \Omega,\ k = 0, 1, 2, \dots \ . \tag{2.1}$$

We now state a result that is an easy generalization of Lemma 3.3 in [BP] to the situation where m and n are not necessarily equals and it will be useful in the sequence.

Lemma 2.5 *Let $s > 1$ be a real number and $\{v_\beta(x)\}$, $\beta \in \mathbb{Z}_+^n$, be a multi-sequence of C^∞ functions defined on an open neighborhood of the origin $\mathcal{U} \subset \mathbb{R}^m$, such that, given a compact subset $K \subset \mathcal{U}$, there exists $B > 1$ satisfying*

$$|\partial_x^\alpha v_\beta(x)| \leq B^{|\alpha|+|\beta|+1}(\alpha!)^s(\beta!)^s, \ \ \forall x \in K, \alpha \in \mathbb{Z}_+^m, \beta \in \mathbb{Z}_+^n. \tag{2.2}$$

Then, shrinking $\mathcal{U}$, there exists $F \in G^s(\mathcal{U} \times (-1,1)^n)$ such that $\partial_t^\gamma F(x, 0) = v_\gamma(x)$, for all $x \in \mathcal{U}, \gamma \in \mathbb{Z}_+^n$.

We would like to point out that this result is a generalization of the Carleman's problem mentioned in the introduction.

We also shall recall some notations. Let $s > 1$. If E is a locally convex vector space, we denote by $G^s(\mathcal{O}, E)$ the space of the C^∞ functions $f : \mathcal{O} \to E$ such that the following holds true: for every compact set $K \subset \mathcal{O}$ and for every continuous seminorm p defined in E, there is a constant $C > 0$ such that

$$\sup_{x \in K} p(\partial^\alpha f(x)) \leq C^{|\alpha|+1}(\alpha!)^s, \ \forall\ \alpha \in \mathbb{Z}_+^N.$$

We are interested in the case when $E = H(\mathcal{N})$, the space of the holomorphic functions on $\mathcal{N}$, where $\mathcal{N} \subset \mathbb{C}^N$ is an open set. Notice that $G^s(\mathcal{O}, H(\mathcal{N}))$ is the space of the functions $f = f(x, \zeta)$ such that f is G^s in $x \in \mathcal{O}$ and holomorphic in $\zeta \in \mathcal{N}$.

We now are ready to state and prove our main result of this section.

Theorem 2.6 *Let $(M, \mathcal{V})$ be a G^s-involutive structure of rank n and $X \subset M$ be a maximally real submanifold. If $p \in X$ and $f \in G^s(X)$ are given then there exist an open neighborhood $\Omega \subset M$ of p and $u \in G^s(\Omega)$ such that u is an s-approximate solution for $\mathcal{V}$ over Ω and $u|_{\Omega \cap X} = f$.*

Proof. Let $(M, \mathcal{V})$ and $p \in X$ as in the statement. We can find local coordinates $(x, t) = (x_1, \dots, x_m, t_1, \dots, t_n)$ defined on a neighborhood of p, such that $x(p) = 0$, $t(p) = 0$ and X is locally given, in these coordinates, by

$$\{(x, 0) : |x| < r\},$$

for some $r > 0$. Since our problem is local, we may suppose that

$$X = \{(x, 0) \in \mathbb{R}^m \times \mathbb{R}^n : x \in U\},$$

where $U \subset \mathbb{R}^m$ is a neighborhood of the origin.

Moreover, since X is maximally real, we can find local generators $L_1, \dots, L_n$ of $\mathcal{V}$ given by

$$L_j = \frac{\partial}{\partial t_j} + \sum_{k=1}^{m} a_{jk} \frac{\partial}{\partial x_k}, \; j = 1, \dots, n, \tag{2.3}$$

where $a_{jk} \in G^s(U \times V)$, for all $j = 1, ..., n$, $k = 1, ..., m$ and V is a neighborhood of the origin in $\mathbb{R}^n$.

It follows from the fact that $\mathcal{V}$ is involutive and from (2.3) that the complex vector fields L_j are pairwise commuting, i.e.,

$$[L_i, L_j] = 0, \qquad 1 \le i, j \le n. \tag{2.4}$$

Thus, in order to prove Theorem 2.6 we will construct $u(x, t) \in G^s(U \times V)$ such that $u(x, 0) = f(x) \in G^s(U)$ for a given $f(x)$ and we will show that there exists a positive constant C such that

$$|L_j u(x, t)| \le C^{\nu+1} (\nu!)^{s-1} |t|^\nu, \qquad (x, t) \in U \times V, \, j = 1, \dots, n, \, \nu \in \mathbb{Z}_+. \tag{2.5}$$

Let $f \in G^s(U)$ be given and let us start the construction of an s-approximate solution $u(x, t)$.

Let $\beta \in \mathbb{Z}_+^n$. If $\beta = (0, \dots, 0)$, we define $u_0(x) \doteq u_{(0,\dots,0)}(x) = f(x)$. Suppose that $\beta = (\beta_1, ..., \beta_n) \neq 0$ and set j the smallest integer such that $1 \le j \le n$ and $\beta_j = \max_{1 \le \nu \le n} \{\beta_\nu\}$. Then, we define

$$u_\beta(x) = -\frac{1}{\beta_j} \sum_{\gamma \le \beta - e_j} \frac{1}{(\beta - e_j - \gamma)!} \sum_{k=1}^{m} \partial_t^{\beta - e_j - \gamma} a_{jk}(x, 0) \frac{\partial u_\gamma}{\partial x_k}(x), \tag{2.6}$$

where $\{e_1, \dots, e_n\}$ is the canonical basis of $\mathbb{R}^n$.

We shall need the following

Lemma 2.7 *Given a compact set $K \subset U$ there exist constants $M, N > 1$ such that*

$$|\partial_x^\alpha u_\beta(x)| \le \frac{M^{|\beta|}}{|\beta|!} N^{|\alpha|+1}(|\alpha| + |\beta|)!^s, \quad \forall x \in K, \alpha \in \mathbb{Z}_+^m, \beta \in \mathbb{Z}_+^n. \tag{2.7}$$

Proof. Since $f \in G^s(U)$ and $a_{jk} \in G^s(U \times V)$, there exists $A > 1$ such that for $\alpha \in \mathbb{Z}_+^m$, $\gamma \in \mathbb{Z}_+^n$, $x \in K$, $j = 1, ..., n$ and $k = 1, ..., m$, we have

$$|\partial_x^\alpha f(x)| \le A^{|\alpha|+1}(\alpha!)^s \quad \text{and} \quad |\partial_x^\alpha \partial_t^\gamma a_{jk}(x, 0)| \le A^{|\alpha|+|\gamma|+1}(\alpha!)^s(\gamma!)^s. \tag{2.8}$$

We will prove (2.7) by induction on $|\beta|$. Set $M = AL$ and $N = 2A$, where L is a constant to be chosen later. It follows from (2.8) that for $x \in K$ and $\alpha \in \mathbb{Z}_+^m$ we have

$$|\partial_x^\alpha u_0(x)| = |\partial_x^\alpha f(x)| \le A^{|\alpha|+1}(\alpha!)^s \le \frac{M^0}{0!} N^{|\alpha|+1}(|\alpha| + 0)!^s$$

and therefore (2.7) is true for $|\beta| = 0$.

Given $\beta \in \mathbb{Z}_+^n$, $\beta \ne 0$ let us assume that (2.7) is satisfied for all $0 \le |\gamma| < |\beta|$ and let us prove it for β.

We have

$$|\partial_x^\alpha u_\beta(x)| \le \frac{1}{\beta_j} \sum_{\gamma \le \beta - e_j} \frac{1}{(\beta - e_j - \gamma)!} \sum_{k=1}^m \sum_{\delta \le \alpha} \binom{\alpha}{\delta} |\partial_x^{\delta+e_k} u_\gamma(x)||\partial_x^{\alpha-\delta} \partial_t^{\beta-e_j-\gamma} a_{jk}(x, 0)|. \tag{2.9}$$

By the induction hypothesis we have

$$|\partial_x^{\delta+e_k} u_\gamma(x)| \le \frac{M^{|\gamma|}}{|\gamma|!} N^{|\delta|+2}(|\delta| + |\gamma| + 1)!^s \tag{2.10}$$

and from (2.8) we have

$$|\partial_x^{\alpha-\delta} \partial_t^{\beta-e_j-\gamma} a_{jk}(x, 0)| \le A^{|\alpha-\delta|+|\beta-e_j-\gamma|+1}(\alpha - \delta)!^s(\beta - e_j - \gamma)!^s. \tag{2.11}$$

By using the inequalities $p!q! \le (p + q)!$ for $p, q \in \mathbb{Z}_+$, $\binom{\alpha}{\delta} \le \binom{|\alpha|}{|\delta|}$ for $\alpha, \delta \in \mathbb{Z}_+^m$, with $\delta \le \alpha$ and $\theta! \le |\theta|!$ for $\theta \in \mathbb{Z}_+^m$ we obtain

$$\begin{aligned}
&\binom{\alpha}{\delta} \frac{(|\delta| + |\gamma| + 1)!^s(\alpha - \delta)!^s(\beta - e_j - \gamma)!^s}{|\gamma|!(\beta - e_j - \gamma)!} \\
&\le \frac{|\alpha|!(|\delta| + |\gamma| + 1)!^s(|\alpha| - |\delta|)!^s(|\beta| - 1 - |\gamma|)!^{s-1}}{(|\alpha| - |\delta|)!|\delta|!|\gamma|!} \\
&\le \frac{|\alpha|!(|\delta| + |\gamma| + 1)!}{|\delta|!|\gamma|!}(|\alpha| + |\beta|)!^{s-1}.
\end{aligned} \tag{2.12}$$

By using the inequality

$$\frac{|\alpha|!(|\delta| + |\gamma| + 1)!}{|\delta|!|\gamma|!} \le \frac{(|\alpha| + |\beta|)!}{(|\beta| - 1)!}, \quad \text{for } \delta \le \alpha,\ |\gamma| < |\beta|,\ \beta \ne 0$$

it follows from (2.12) that

$$\binom{\alpha}{\delta}\frac{(|\delta|+|\gamma|+1)!^s(\alpha-\delta)!^s(\beta-e_j-\gamma)!^s}{|\gamma|!(\beta-e_j-\gamma)!}\le\frac{(|\alpha|+|\beta|)^s}{(|\beta|-1)!}. \tag{2.13}$$

We now point out that thanks to definition of β_j we obtain

$$\frac{1}{\beta_j(|\beta|-1)!}\le\frac{n}{|\beta|!}.$$

It follows from (2.9), (2.10), (2.11), (2.13) and from the last inequality that

$$\begin{aligned}
|\partial_x^\alpha u_\beta(x)| &\le \frac{1}{\beta_j}\sum_{\gamma\le\beta-e_j}\sum_{k=1}^m\sum_{\delta\le\alpha}M^{|\gamma|}N^{|\delta|+2}A^{|\alpha|-|\delta|+|\beta|-|\gamma|}\frac{(|\alpha|+|\beta|)!^s}{(|\beta|-1)!}\\
&\le \frac{nm}{|\beta|!}M^{|\beta|}N^{|\alpha|+1}(|\alpha|+|\beta|)!^s\sum_{\gamma\le\beta-e_j}\sum_{\delta\le\alpha}M^{-(|\beta|-|\gamma|)}N^{|\delta|-|\alpha|+1}A^{|\alpha|-|\delta|+|\beta|-|\gamma|}\\
&= \frac{nm}{|\beta|!}M^{|\beta|}N^{|\alpha|+1}(|\alpha|+|\beta|)!^s\sum_{\gamma\le\beta-e_j}M^{-|\beta-\gamma|}A^{|\beta-\gamma|}\sum_{\delta\le\alpha}N^{|\delta|-|\alpha|+1}A^{|\alpha|-|\delta|}\\
&= \frac{nm}{|\beta|!}M^{|\beta|}N^{|\alpha|+1}(|\alpha|+|\beta|)!^s\sum_{\gamma\le\beta-e_j}\left(\frac{1}{L}\right)^{|\beta-\gamma|}\sum_{\delta\le\alpha}A2^{|\delta|-|\alpha|+1}\\
&= \frac{nm}{|\beta|!}M^{|\beta|}N^{|\alpha|+1}(|\alpha|+|\beta|)!^s\frac{2A}{L}\sum_{\gamma\le\beta-e_j}\left(\frac{1}{L}\right)^{|\beta-\gamma|-1}\sum_{\delta\le\alpha}2^{-|\alpha-\delta|}\\
&\le \frac{nm}{|\beta|!}M^{|\beta|}N^{|\alpha|+1}(|\alpha|+|\beta|)!^s\frac{2A}{L}\left(\sum_{\nu=0}^{\infty}\left(\frac{1}{L}\right)^{\nu}\right)^n\left(\sum_{i=0}^{\infty}2^{-i}\right)^m\\
&= \frac{nm}{|\beta|!}M^{|\beta|}N^{|\alpha|+1}(|\alpha|+|\beta|)!^s\frac{2^{m+1}A}{L}\left(\frac{L}{L-1}\right)^n\le\frac{M^{|\beta|}}{|\beta|!}N^{|\alpha|+1}(|\alpha|+|\beta|)!^s,
\end{aligned}$$

for $L>1$ large enough. It completes the proof of the Lemma 2.7. ■

If we define $v_\beta(x)=\beta!u_\beta(x)$, then it is easy to see that the multi-sequence $\{v_\beta(x)\}$ satisfies the condition (2.2). Thus, by the Lemma 2.5, there exists $u\in G^s(U\times V)$, shrinking V if it is necessary, such that $\partial_t^\beta u(x,0)=v_\beta(x)$.

Therefore, there exists $u\in G^s(U\times V)$ such that

$$u_\beta(x)=\frac{1}{\beta!}\partial_t^\beta u(x,0),\quad \forall x\in U,\beta\in\mathbb{Z}_+^n. \tag{2.14}$$

In particular, we have that $u(x,0)=f(x)$, for all $x\in U$. We must prove now that u is an s-approximate solution to the system $L_iZ=0$, $i=1,\dots,n$. For this, it is enough to see that $(\partial_t^\beta L_iu)(x,0)=0$, for all $\beta\in\mathbb{Z}_+^n$, since u and all the coefficients of L_i are Gevrey functions of order s.

Fix $i \in \{1, ..., n\}$. We will prove that $(\partial_t^{\beta} L_i u)(x,0) = 0$ by induction on $|\beta|$. For $|\beta| = 0$, we have

$$\begin{aligned}(L_i u)(x,0) &= (\partial_t^{e_i} u)(x,0) + \sum_{l=1}^{m} a_{il}(x,0)\frac{\partial u(x,0)}{\partial x_l} \\ &= u_{e_i}(x) + \sum_{l=1}^{m} a_{il}(x,0)\frac{\partial u_0(x)}{\partial x_l} \\ &= 0, \end{aligned} \tag{2.15}$$

by the definition of $u_{e_i}(x)$.

Let $\beta \in \mathbb{Z}_+^n$, $\beta \neq 0$ and suppose that $(\partial_t^{\gamma} L_i u)(x,0) = 0$, for every $|\gamma| < |\beta|$. Let $1 \leq j \leq n$ be the integer that comes from the definition of $u_{\beta}(x)$. Observe that

$$\begin{aligned}(\partial_t^{\beta} L_i)u &= \partial_t^{\beta - e_j}\left(\frac{\partial}{\partial t_j} L_i u\right) \\ &= \partial_t^{\beta - e_j}\left(L_j(L_i u) - \sum_{k=1}^{m} a_{jk}\frac{\partial}{\partial x_k}(L_i u)\right) \\ &= \partial_t^{\beta - e_j}(L_j(L_i u)) - \sum_{k=1}^{m}\sum_{\gamma \leq \beta - e_j}\binom{\beta - e_j}{\gamma}\partial_t^{\beta - e_j - \gamma} a_{jk}\frac{\partial}{\partial x_k}\partial_t^{\gamma} L_i u.\end{aligned}$$

Then, by induction hypothesis we obtain

$$(\partial_t^{\beta} L_i u)(x,0) = (\partial_t^{\beta - e_j} L_j L_i u)(x,0). \tag{2.16}$$

Analogously we have

$$(\partial_t^{\beta - e_j + e_i} L_j u)(x,0) = (\partial_t^{\beta - e_j} L_i L_j u)(x,0). \tag{2.17}$$

It follows from (2.16), (2.17) and the condition (2.4) that

$$(\partial_t^{\beta} L_i u)(x,0) = (\partial_t^{\beta - e_j + e_i} L_j u)(x,0). \tag{2.18}$$

Set $\beta^{'} = \beta + e_i$. In order to write $u_{\beta'}$ we must know what is the smallest integer p such that $1 \leq p \leq n$ and $\beta'_p = \max_{1 \leq \nu \leq n}\{\beta'_\nu\}$. For this, we have two cases to analyze:

Case 1: $\beta_i + 1 < \beta_j$ or $\beta_i + 1 = \beta_j$ with $j < i$. In this case we have $p = j$ and $\beta'_p = \beta'_j = \beta_j$.

Case 2: $\beta_i + 1 > \beta_j$ or $\beta_i + 1 = \beta_j$ with $j > i$. Here, we have $p = i$ and $\beta'_p = \beta'_i = \beta_i + 1$.

We start by analyzing the first case. We can write

$$\begin{aligned}u_{\beta'}(x) &= -\frac{1}{\beta_j}\sum_{\gamma \leq \beta' - e_j}\frac{1}{(\beta' - e_j - \gamma)!}\sum_{k=1}^{m}\partial_t^{\beta^{'} - e_j - \gamma} a_{jk}(x,0)\frac{\partial u_\gamma}{\partial x_k}(x) \qquad (2.19) \\ &= -\frac{1}{\beta_j}\sum_{\gamma \leq \beta - e_j + e_i}\frac{1}{(\beta - e_j + e_i - \gamma)!}\sum_{k=1}^{m}\partial_t^{\beta - e_j + e_i - \gamma} a_{jk}(x,0)\frac{\partial u_\gamma}{\partial x_k}(x).\end{aligned}$$

Therefore,

$$
\begin{aligned}
(\partial_t^{\beta-e_j+e_i} L_j u)(x,0) &= \left[\partial_t^{\beta-e_j+e_i}\left(\frac{\partial u}{\partial t_j}+\sum_{k=1}^m a_{jk}\frac{\partial u}{\partial x_k}\right)\right]\Big|_{t=0} \qquad (2.20)\\
&= (\beta+e_i)! u_{\beta+e_i}(x)\\
&+ \sum_{\gamma\le\beta-e_j+e_i}\frac{(\beta-e_j+e_i)!}{(\beta-e_j+e_i-\gamma)!}\sum_{k=1}^m \partial_t^{\beta-e_j+e_i-\gamma} a_{jk}(x,0)\frac{\partial u_\gamma}{\partial x_k}(x).
\end{aligned}
$$

Noticing that $(\beta-e_j+e_i)! = \frac{(\beta+e_i)!}{\beta_j}$ for $j\neq i$, (2.19) and (2.20) gives

$$
\begin{aligned}
&\partial_t^{\beta-e_j+e_i}(L_j u)(x,0) = (\beta+e_i)! u_{\beta+e_i}(x)\\
&+\frac{(\beta+e_i)!}{\beta_j}\sum_{\gamma\le\beta-e_j+e_i}\frac{1}{(\beta-e_j+e_i-\gamma)!}\sum_{k=1}^m \partial_t^{\beta-e_j+e_i-\gamma}a_{jk}(x,0)\frac{\partial u_\gamma}{\partial x_k}(x)\\
&= 0.
\end{aligned}
$$

Then, from (2.18), we conclude that $(\partial_t^\beta L_i u)(x,0)=0$.

In the second case we may write

$$
\begin{aligned}
u_{\beta'}(x) &= -\frac{1}{\beta_i+1}\sum_{\gamma\le\beta'-e_i}\frac{1}{(\beta'-e_i-\gamma)!}\sum_{l=1}^m \partial_t^{\beta'-e_i-\gamma}a_{il}(x,0)\frac{\partial u_\gamma}{\partial x_l}(x)\\
&= -\frac{1}{\beta_i+1}\sum_{\gamma\le\beta}\frac{1}{(\beta-\gamma)!}\sum_{l=1}^m \partial_t^{\beta-\gamma}a_{il}(x,0)\frac{\partial u_\gamma}{\partial x_l}(x). \qquad (2.21)
\end{aligned}
$$

Similarly to the proof of the first case we have

$$
\begin{aligned}
(\partial_t^\beta L_i u)(x,0) &= (\beta+e_i)! u_{\beta+e_i}(x) +\\
&+\sum_{\gamma\le\beta}\frac{\beta!}{(\beta-\gamma)!}\sum_{l=1}^m \partial_t^{\beta-\gamma}a_{il}(x,0)\frac{\partial u_\gamma}{\partial x_l}(x),
\end{aligned}
$$

and therefore it follows from (2.21) that

$$(\partial_t^\beta L_i u)(x,0)=0.$$

Hence, $(\partial_t^\beta L_i u)(x,0)=0$, for every $\beta\in\mathbb{Z}_+$ and $i=1,...,n$, which completes the proof of Theorem 2.6. ■

Following the lines of the proof of Theorem 2.6, one can prove the next result.

Proposition 2.8 *Let $\mathcal{O}\subset\mathbb{R}^{m+n}$ and $\mathcal{N}\subset\mathbb{C}^{m+1}$ be open sets such that $(0,0)\in\mathcal{O}$. Consider the complex vector fields*

$$L_j=\frac{\partial}{\partial t_j}+\sum_{k=1}^m a_{kj}(x,t,\zeta)\frac{\partial}{\partial x_k}+\sum_{k=0}^m b_{kj}(x,t,\zeta)\frac{\partial}{\partial \zeta_k},\quad j=1,\dots,n$$

where the coefficients a_{kj} and b_{kj} belong to the class $G^s(\mathcal{O}, H(\mathcal{N}))$. Suppose that $[L_i, L_j] = 0$, for $1 \leq i, j \leq n$. Let $U \subset \mathbb{R}^m$ be an open neighborhood of the origin such that $U \times \{0\} \subset \mathcal{O}$ and let $f(x, \zeta) \in G^s(U, H(\mathcal{N}))$. Then, shrinking U, there exist an open neighborhood $V \subset \mathbb{R}^n$ of the origin and $u(x, t, \zeta) \in G^s(U \times V, H(\mathcal{N}))$ such that u is an s-approximate solution of $L_j w = 0$, $j = 1, \ldots, n$ and $u(x, 0, \zeta) = f(x, \zeta)$.

3 Gevrey involutive systems of first-order nonlinear PDE

In this section we introduce some concepts about involutive systems of first-order nonlinear pdes, in the Gevrey frame. For the analytic set-up we refer the reader to Treves [T] and Berhanu [B2].

Let M be a real analytic manifold of dimension N and $\mathbb{C}J^1(M)$ be the complex one-jet bundle of M, which is the set of the triples (x, a, w), where $x \in M$, $a \in \mathbb{C}$ and $w \in \mathbb{C}T^*_x M$. It is easy to see that we can identify $\mathbb{C}J^1(M)$ with $\mathbb{C} \times \mathbb{C}T^*M$. Consider (U, x) a local chart of M, with $x = (x_1, \ldots, x_N)$, and let $\zeta_1, \ldots, \zeta_N$ denote the corresponding complex coordinates in $\mathbb{C}T^*_x M$ at $x \in U$. We denote by $(\zeta_0, \zeta) = (\zeta_0, \zeta_1, \ldots, \zeta_N)$ the coordinates in $\mathbb{C}^{N+1}$ and we also denote by

$$\mathcal{O} = \mathbb{C} \times (\mathbb{C}T^*M|_U) \cong U \times \mathbb{C}^{N+1} \tag{3.1}$$

the open subset of the one-jet bundle that lies over U. If $F(x, \zeta_0, \zeta)$ and $G(x, \zeta_0, \zeta)$ are G^s function on $\mathcal{O}$ which are holomorphic in (ζ_0, ζ), we define the holomorphic Hamiltonian of F by

$$\begin{aligned} H_F &= \sum_{i=1}^{N} \frac{\partial F}{\partial \zeta_i} \frac{\partial}{\partial x_i} - \sum_{i=1}^{N} \left(\frac{\partial F}{\partial x_i} + \zeta_i \frac{\partial F}{\partial \zeta_0} \right) \frac{\partial}{\partial \zeta_i} \\ &\quad + \left(\sum_{i=1}^{N} \zeta_i \frac{\partial F}{\partial \zeta_i} - F \right) \frac{\partial}{\partial \zeta_0} + \frac{\partial F}{\partial \zeta_0} \end{aligned} \tag{3.2}$$

and the Poisson bracket $\{F, G\}$ by

$$\{F, G\} = H_F G = -H_G F. \tag{3.3}$$

Notice that for the class of functions being considered, the definition of the Poisson bracket is independent of the choice of the local coordinates.

We now will state a global definition of G^s-involutive systems.

Definition 3.1 *A G^s-involutive system of first-order partial differential equations of rank n on M is a closed G^s submanifold Σ of $\mathbb{C}J^1(M)$ satisfying the following properties:*

(a) *the projection $\pi : \mathbb{C}J^1(M) \to M$ maps Σ onto M;*

(b) *each point of* Σ *has a neighborhood* $\mathcal{O}$ *given as in* (3.1) *on which there are* n G^s *functions* $F_1(x,\zeta_0,\zeta),\dots,F_n(x,\zeta_0,\zeta)$*, which are holomorphic in the variables* (ζ_0,ζ) *such that*

$$\Sigma\cap\mathcal{O}=\{(x,\zeta_0,\zeta)\in\mathcal{O}:F_j(x,\zeta_0,\zeta)=0,1\le j\le n\},\tag{3.4}$$

(c) *at every point of* $\mathcal{O}$ *holds*

$$d_\zeta F_1\wedge\cdots\wedge d_\zeta F_n\neq 0,\tag{3.5}$$

(d) *on* $\Sigma\cap\mathcal{O}$ *holds*

$$\{F_j,F_k\}=0.\tag{3.6}$$

The condition (3.5) gives the linear independence of the system, while the condition (3.6) gives a formal integrability condition for the system of equations

$$F_j(x,u,u_x)=0,\ j=1,\dots,n \text{ (for more details see Berhanu [B2]).}$$

Definition 3.2 *Let* Σ *be as in Definition 3.1. A* C^1 *function* u *on* M *is called a solution of* Σ *if its first jet lies in* Σ*.*

Remark 3.3 *The definition* (3.1) *generalizes the linear case given by the* G^s*-involutive structures* $(M,\mathcal{V})$ *defined in the introduction. In order to see it the reader is referred to Berhanu [B2].*

Let Σ be a G^s-involutive system of first-order partial differential equations as above and fix $(x_0,\zeta_0',\zeta')\in\Sigma\cap\mathcal{O}$. Let $F_j(x,\zeta_0,\zeta)$, $j=1,\dots,n$ be as in definition 3.1. Consider a local chart (U,x) of M such that $x_0\in U$ and $x(x_0)=0$. We may assume that $\pi(\mathcal{O})=U$.

Let u be a C^2 solution of Σ on U such that $u(x_0)=\zeta_0'$ and $u_x(x_0)=\zeta'$. Consider the following complex vector fields

$$L_j^u=\sum_{k=1}^{N}\frac{\partial F_j}{\partial\zeta_k}(x,u(x),u_x(x))\frac{\partial}{\partial x_k},\quad 1\le j\le n.\tag{3.7}$$

We refer to these vector fields as the linearized operators of $F_j(x,u(x),u_x(x))=0$ at u. If $v\in C^1(U)$ and $F(x,\zeta_0,\zeta)$ is a C^1 function, holomorphic in (ζ_0,ζ), then we denote by F^v the function given by

$$F^v(x)=F(x,v(x),v_x(x)).$$

We recall that the principal part of the holomorphic Hamiltonian of F_j, which we denote by $H^0_{F_j}$, is the complex vector field obtained by omitting the term $\frac{\partial F_j}{\partial\zeta_0}$ in the expression of H_{F_j} given in (3.2).

If $\Psi(x,\zeta_0,\zeta)$ is a C^1 function, holomorphic in (ζ_0,ζ), a straightforward computation shows that

$$L_j^u(\Psi^u)=\left(H^0_{F_j}\Psi\right)^u,\quad\forall\ j=1,\dots,n.\tag{3.8}$$

Lemma 3.4 *If H_{F_j}, $j = 1, \dots, n$ are the holomorphic Hamiltonians of the functions F_j given above, then there exist G^s functions $a^\ell_{jk}(x, \zeta_0, \zeta)$, $j, k, \ell = 1, \dots, n$, holomorphic in (ζ_0, ζ), such that on the set $\Sigma \cap \mathcal{O}$,*

$$[H_{F_j}, H_{F_k}] = \sum_{\ell=1}^{n} a^\ell_{jk}(x, \zeta_0, \zeta) H_{F_\ell}.$$

Lemma 3.5 *Let $\mathcal{V}^u$ be the bundle generated by $L^u_1, \dots, L^u_n$. Then $\mathcal{V}^u$ is involutive.*

The proofs of Lemmas 3.4 and 3.5 can be found in Treves [T] and Berhanu [B2], respectively.

Lemma 3.6 *Given Σ and u as before and $p \in M$, there are real analytic coordinates (x, t), $x = (x_1, \dots, x_m)$, $t = (t_1, \dots, t_n)$ such that $x(p) = 0$, $t(p) = 0$ and in this new coordinates,*

$$u_{t_j} = f_j(x, t, u, u_x), \quad 1 \le j \le n,$$

where f_j are G^s in (x, t, ζ_0, ζ), and holomorphic in (ζ_0, ζ) for (x, t, ζ_0, ζ) varying in an open subset of $\mathbb{R}^m \times \mathbb{R}^n \times \mathbb{C} \times \mathbb{C}^m$.

For the proof of Lemma 3.6, we need the following version of the implicit function theorem:

Proposition 3.7 *Let $\Omega \subset \mathbb{R}^M$, $\mathcal{N} \subset \mathbb{C}^N$ be open sets and $(x_0, \zeta^0) \in \Omega \times \mathcal{N}$. Suppose that $F = (F_1, \dots, F_n)$ and $F_j(x, \zeta) \in G^s(\Omega, H(\mathcal{N}))$, for $j = 1, \dots, n$ and $n \le N$. If $F(x_0, \zeta^0) = 0$ and*

$$\det \left(\frac{\partial F_j}{\partial \zeta_i}(x_0, \zeta^0) \right)_{1 \le i, j \le n} \ne 0,$$

then there are open neighborhoods U of $(\zeta^0_{n+1}, \dots, \zeta^0_N)$ in $\mathbb{C}^{N-n}$ and V of x_0 in $\mathbb{R}^M$ and there is a unique function $f = (f_1, \dots, f_n)$ where $f_j = f_j(x, \zeta_{n+1}, \dots, \zeta_N) \in G^s(V, H(U))$, such that

$$F(x, f(x, \zeta_{n+1}, \dots, \zeta_N), \zeta_{n+1}, \dots, \zeta_N) = 0, \ \forall\ (x, \zeta_{n+1}, \dots, \zeta_N) \in V \times U.$$

Proof. Since each F_j is G^s in x and holomorphic in ζ, writing $F_j = u_j + iv_j$ we conclude that the functions u_j, v_j are G^s, for every $j = 1, \dots, n$.

By using the fact that F_j is holomorphic in ζ, if we write $\zeta = t + iy = (t_1 + iy_1, \dots, t_N + iy_N)$ then we obtain

$$Z \doteq \left(\frac{\partial F_j}{\partial \zeta_k} \right)_{n \times n} = \left(\frac{\partial u_j}{\partial t_k} \right)_{n \times n} - i \left(\frac{\partial u_j}{\partial y_k} \right)_{n \times n} \doteq A + iB.$$

It follows from the hypotheses that Z is invertible in an open neighborhood of (x_0, ζ^0). Then, we can consider $Z^{-1} = C + iD$, with C, D being matrices of order $n \times n$ with real entrances. Since $Z^{-1} Z = Id_n$, we have

$$\begin{cases} CA - DB = Id_n \\ CB + DA = 0. \end{cases}$$

Thus,

$$\begin{pmatrix} C & -D \\ D & C \end{pmatrix}\begin{pmatrix} A & -B \\ B & A \end{pmatrix} = \begin{pmatrix} CA-DB & -(CB+DA) \\ CB+DA & CA-DB \end{pmatrix} = Id_{2n}.$$

Hence, since $A = \left(\frac{\partial u_j}{\partial t_k}\right)_{n\times n}$ and $B = -\left(\frac{\partial u_j}{\partial y_k}\right)_{n\times n}$ it follows from the Cauchy-Riemann equations that the matrix

$$\begin{pmatrix} \left(\frac{\partial u_j}{\partial t_k}\right)_{n\times n} & \left(\frac{\partial u_j}{\partial y_k}\right)_{n\times n} \\ \left(\frac{\partial v_j}{\partial t_k}\right)_{n\times n} & \left(\frac{\partial v_j}{\partial y_k}\right)_{n\times n} \end{pmatrix}$$

is invertible in an open neighborhood of (x_0, ζ^0). Therefore, by the Gevrey version of the implicit function theorem (see Komatsu [K]), there exist neighborhoods V of x_0 in $\mathbb{R}^M$, U_1 of $(t^0_{n+1}, \dots, t^0_N)$, U_2 of $(y^0_{n+1}, \dots, y^0_N)$ in $\mathbb{R}^{N-n}$ and there exist real functions f^1_j, $f^2_j \in G^s(V \times U_1 \times U_2)$ where

$$f^1_j = f^1_j(x, t_{n+1}, y_{n+1}, \dots, t_N, y_N)$$

and

$$f^2_j = f^2_j(x_0, t_{n+1}, y_{n+1}, \dots, t_N, y_N),$$

such that if we set $f_j = f^1_j + i f^2_j$ and $f = (f_1, \dots, f_n)$, then we have

$$F_j(x, f(x, \zeta_{n+1}, \dots, \zeta_N), \zeta_{n+1}, \dots, \zeta_N) = 0, \tag{3.9}$$

for $(x, \zeta_{n+1}, \dots, \zeta_N) \in V \times U$ and $j = 1, \dots, n$, where $U = U_1 + iU_2$.

In order to complete the proof we will show that the functions f_j, $j = 1, \dots, n$, are holomorphic in $\zeta_{n+1}, \dots, \zeta_N$. For this, it suffices to prove that

$$\frac{\partial f_j}{\partial \bar{\zeta}_k}(x, \zeta_{n+1}, \dots, \zeta_N) = 0,$$

for all $(x, \zeta_{n+1}, \dots, \zeta_N) \in V \times U$ and $k = n+1, \dots, N$.

For $k = n+1, \dots, N$ and $j = 1, \dots, n$ it follows from (3.9) that

$$\frac{\partial}{\partial \bar{\zeta}_k} F_j(x, f(x, \zeta_{n+1}, \dots, \zeta_N), \zeta_{n+1}, \dots, \zeta_N) = 0.$$

Thus, since each F_j is holomorphic in ζ, it follows that

$$\sum_{\ell=1}^{n} \frac{\partial F_j}{\partial \zeta_\ell} \frac{\partial f_\ell}{\partial \bar{\zeta}_k} = 0,$$

or equivalently,

$$\begin{pmatrix} \frac{\partial F_1}{\partial \zeta_1} & \cdots & \frac{\partial F_1}{\partial \zeta_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial F_n}{\partial \zeta_1} & \cdots & \frac{\partial F_n}{\partial \zeta_n} \end{pmatrix}\begin{pmatrix} \frac{\partial f_1}{\partial \bar{\zeta}_{n+1}} & \cdots & \frac{\partial f_1}{\partial \bar{\zeta}_N} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial \bar{\zeta}_{n+1}} & \cdots & \frac{\partial f_n}{\partial \bar{\zeta}_N} \end{pmatrix} = 0.$$

Shrinking V and U if it is necessary it follows from the hypotheses that the matrix

$$\left(\frac{\partial F_j}{\partial \zeta_k}(x, f(x, \zeta_{n+1}, \dots, \zeta_N), \zeta_{n+1}, \dots, \zeta_N)\right)_{n\times n}$$

is invertible, and therefore we obtain

$$\frac{\partial f_j}{\partial \bar{\zeta}_k}(x, \zeta_{n+1}, \dots, \zeta_N) = 0,$$

for all $(x, \zeta_{n+1}, \dots, \zeta_N) \in V \times U$, $j = 1, \dots, n$ and $k = n+1, \dots, N$. Hence, we conclude that $f_j \in G^s(V, H(U))$, for all $j = 1, \dots, n$. ■

Proof of Lemma 3.6. Let $p \in M$ and (U, x) be a local chart such that $x = (x_1, \dots, x_N)$ and $x(p) = 0$. Also, let $F_1(x, \zeta_0, \zeta), \dots, F_n(x, \zeta_0, \zeta)$ be the functions that define the system Σ on a neighborhood $\mathcal{O}$ of $(p, u(p), u_x(p))$. Recall that F_j, $j = 1, \dots, n$ are G^s functions on $\mathcal{O}$ that are holomorphic in (ζ_0, ζ), the conditions (3.5) and (3.6) hold and

$$F_j(x, u(x), u_x(x)) = 0,\ \ j = 1, \dots, n.$$

Thanks to condition (3.5), after relabeling coordinates, we have

$$\det\left(\frac{\partial F_j}{\partial \zeta_i}\right)_{1\le i,j\le n} \neq 0.$$

We now apply Proposition 3.7 and we may assume that

$$F_j(x, \zeta_0, \zeta) = \zeta_j - f_j(x, \zeta_0, \zeta_{n+1}, \dots, \zeta_N),\ \ j = 1, \dots, n,$$

where the f_j are G^s functions that are holomorphic in $(\zeta_0, \zeta_{n+1}, \dots, \zeta_N)$.

Writing $t_j = x_j$ and $\tau_j = \zeta_j$ for $1 \le j \le n$ and writing x_i instead of x_{n+i} and ζ_i instead of ζ_{n+i} for $1 \le i \le m = N - n$, we have that $\Sigma \cap \mathcal{O}$ is given by the equations

$$\tau_j = f_j(x, t, \zeta_0, \zeta),\ 1 \le j \le n,$$

where $(x, t) = (x_1, \dots, x_m, t_1, \dots, t_n)$ and $\zeta = (\zeta_1, \dots, \zeta_m)$. Therefore, since u is a solution of Σ, u satisfies

$$u_{t_j} = f_j(x, t, u, u_x),\ \ 1 \le j \le n,$$

which completes the proof. ■

4 Gevrey micro-regularity for solutions to involutive systems of first-order nonlinear pdes

We start this section by recalling the concept of Gevrey wave-front set that can be found in Rodino [R].

Definition 4.1 *Let $\Omega \subset \mathbb{R}^N$ be an open neighborhood of a point x_0, and $s > 1$. The distribution u is said to belong to the Gevrey class G^s at x_0 if and only if there exists $\varphi \in G_0^s(\Omega)$, $\varphi(x) = 1$ in a neighborhood of x_0, such that for some $C > 0$ and $\epsilon > 0$*

$$|\widehat{\varphi u}(\xi)| \leq Ce^{-\epsilon|\xi|^{1/s}}, \forall\, \xi \in \mathbb{R}^N. \tag{4.1}$$

In the case that u is non-G^s at x_0 one can obtain more information about its singularities by studying the directions in which the above condition (4.1) does not hold. This leads to the following definition of Gevrey wave front set.

Definition 4.2 *Let $s > 1$. For fixed $x_0 \in \Omega$ and $\xi_0 \in \mathbb{R}^N$, $\xi_0 \neq 0$, we say that $u \in D'(\Omega)$ is s-micro-regular at (x_0, ξ_0) if there exist $\varphi \in G_0^s(\Omega)$, $\varphi(x) = 1$ in a neighborhood U of x_0, and a conic neighborhood Γ of ξ_0, such that for some positive constants C and ϵ the condition (4.1) holds for $\xi \in \Gamma$. The s-wave-front set of u, $WF_s(u)$, is then defined as the complement in $\Omega \times (\mathbb{R}^N \backslash \{0\})$ of the set of all (x_0, ξ_0) where u is s-micro-regular.*

We would like to point out that this definition is equivalent to the Definition 2.6 in Barostichi and Petronilho [BP] where FBI transform has been used.

We now are in the position to state and prove the following result.

Theorem 4.3 *Let $s > 1$ be a real number and Σ be a G^s-involutive system of first order nonlinear pde of rank n defined on a real analytic manifold M. Let u be a C^2 solution of Σ. Then*

$$WF_s(u) \subset (\mathcal{V}^u)^{\perp} \cap T^*M = T^0M.$$

Remark 4.4 *We would like to point out that this result, in the analytic case, was recently proved by Berhanu [B2]. The case $n = 1$ was proved by Barostichi and Petronilho [BP].*

Let us describe the technique that we will use to prove Theorem 4.3.

Let $\Omega \subset \mathbb{R}^{m'} \times \mathbb{R}^{n'}$ be a neighborhood of the origin and consider complex vector fields given by

$$L_j = \frac{\partial}{\partial t_j} + \sum_{k=1}^{m'} a_{kj}(x,t)\frac{\partial}{\partial x_k}, \quad j = 1, \ldots, n', \tag{4.2}$$

where $a = (a_{kj})_{m' \times n'} \in C^1(\Omega)$, $j = 1, \ldots, n'$ and $k = 1, \ldots, m'$.

Following the lines of the proof of Lemma 5.1 in Barostichi and Petronilho [BP] one can prove the next result.

Lemma 4.5 *Let $\Psi(x,t) = (\Psi_{kj}(x,t))_{m' \times n'} \in C^1(\Omega)$ such that $Z(x,t) = x + \Psi(x,t)t$ is an s-approximate solution of the system $L_j Z = 0$, $j = 1, \ldots, n'$. Let $\xi_0 \in \mathbb{R}^{m'} \backslash \{0\}$ and suppose that there exists $T \in \mathbb{R}^{n'}$ such that $\xi_0 \cdot \operatorname{Im}\Psi(0,0)T < 0$. If $h \in C^1(\Omega)$ is an s-approximate solution of $L_j h = 0$, $j = 1, \ldots, n'$, then $(0, \xi_0) \notin WF_s(h(\cdot, 0))$.*

By using the hypotheses of Theorem 4.3 we will construct special vector fields L_j for which we can apply our previous results in order to guarantee the existence of s-approximate solutions Z and h to the system $L_j w = 0$ satisfying the hypotheses of Lemma 4.5 in such way that the trace of h is precisely $u(x,t)$ with u being a C^2 solution of Σ.

Proof of Theorem 4.3. Let $s > 1$ be a real number and let Σ be a G^s-involutive system of first-order nonlinear pdes of rank n defined on a real analytic manifold M. Let u be a C^2 solution of Σ and fix $p \in \Sigma$. If $x_0 = \pi(p)$ then by Lemma 3.6, locally, the solution u satisfies the system

$$u_{t_j} = f_j(x,t,u,u_x),\ j = 1,\dots,n, \tag{4.3}$$

where $f_j(x,t,\zeta_0,\zeta) \in G^s(\Omega, H(\mathcal{N}))$, $(x,t) \in \Omega \subset \mathbb{R}^{m+n}$, $(\zeta_0,\zeta) \in \mathcal{N} \subset \mathbb{C} \times \mathbb{C}^m$ with Ω being an open neighborhood of the origin and $\mathcal{N}$ being an open set.

The linearized operators of $F_j(x,t,u(x,t),u_x(x,t),u_t(x,t))$, at u, where

$$F_j(x,t,\zeta_0,\zeta,\tau) = \tau_j - f_j(x,t,\zeta_0,\zeta)$$

are given by

$$L_j^u = \frac{\partial}{\partial t_j} - \sum_{k=1}^{m} \frac{\partial f_j}{\partial \zeta_k}(x,t,u(x,t),u_x(x,t))\frac{\partial}{\partial x_k}, \quad j = 1,\dots,n. \tag{4.4}$$

We notice that the coefficients of L_j^u are in $C^1(\Omega)$.

We now use a trick from Hanges and Treves [HT]. We set

$$\tilde{u}(x,t,r) = u(x,t).$$

Since u is a solution of the system $u_{t_j} = f_j(x,t,u,u_x)$, $j = 1,\dots,n$, it follows that $\tilde{u}$ is a solution of the new system

$$\tilde{u}_{r_j} = f_j^\theta(x,t,r,\tilde{u},\tilde{u}_x,\tilde{u}_t), \quad j = 1,\dots,n, \tag{4.5}$$

where $f_j^\theta(x,t,r,\zeta_0,\zeta,\tau) = e^{-i\theta}(\tau_j - f_j(x,t,\zeta_0,\zeta))$, and $\theta \in [0,2\pi)$.

Note that this last system of equations is of the same kind as (4.3) with r replacing t and (x,t) replacing x. The equations (4.5) also define a G^s-involutive system of first-order pdes. In fact, if

$$F_j^\theta(x,t,r,\zeta_0,\zeta,\tau,\xi) = \xi_j - f_j^\theta(x,t,r,\zeta_0,\zeta,\tau),\ j = 1,\dots,n,$$

then the holomorphic Hamiltonian of F_j^θ satisfies

$$H_{F_j^\theta} = \frac{\partial}{\partial r_j} - e^{-i\theta} H_{F_j} - e^{-i\theta}\sum_{i=1}^{n} \xi_i \frac{\partial f_j}{\partial \zeta_0}\frac{\partial}{\partial \xi_i}.$$

Thanks to the fact that system F_j is a G^s-involutive system of first-order pdes, it follows from the above formula that the system F_j^θ is also a G^s-involutive system of first-order pdes, which shows our claim.

We also notice that, if we denote by $(L_j^\theta)^{\tilde u}$, $j = 1, \ldots, n$, the linearized operators of $F_j^\theta(x, t, r, \tilde u(x, t, r), \tilde u_x(x, t, r), \tilde u_t(x, t, r), \tilde u_r(x, t, r))$ at $\tilde u$, we obtain

$$(L_j^\theta)^{\tilde u} = \frac{\partial}{\partial r_j} - e^{-i\theta} L_j^u, \ j = 1, \ldots, n.$$

If $G(x, t, r, \zeta_0, \zeta, \tau, \xi)$ is a C^1 function, holomorphic in $(\zeta_0, \zeta, \tau, \xi)$, a straightforward computation shows that

$$(L_j^\theta)^{\tilde u}(G^{\tilde u}) = \left(H^0_{F_j^\theta} G\right)^{\tilde u}, \quad \forall \ j = 1, \ldots, n, \tag{4.6}$$

where we have used the notation

$$G^{\tilde u}(x, t, r) = G(x, t, r, \tilde u(x, t, r), \tilde u_x(x, t, r), \tilde u_t(x, t, r), \tilde u_r(x, t, r)).$$

It follows from Lemma 3.4, with F_k replaced by F_k^θ that on the set $\Sigma \cap \mathcal{O}$ we have

$$[H^0_{F_i^\theta}, H^0_{F_j^\theta}] = 0, \ \ i, j = 1, \ldots, n. \tag{4.7}$$

Since the coefficients of $H^0_{F_j^\theta}$ are in $G^s(\Omega \times V, H(\mathcal{N} \times \mathcal{M}))$, where V is an open neighborhood of origin in $\mathbb{R}^n$ and $\mathcal{M}$ is an open set in $\mathbb{C}^n \times \mathbb{C}^n$ and (4.7) holds, it follows from Proposition 2.8, shrinking $\Omega \times V$, that there exist functions

$$\Lambda_k^\theta(x, t, r, \zeta_0, \zeta, \tau, \xi), \ k = 1, \cdots, m \text{ and } \Xi^\theta(x, t, r, \zeta_0, \zeta, \tau, \xi)$$

which belong to $G^s(\Omega \times V, H(\mathcal{N} \times \mathcal{M}))$ and satisfy $\Lambda_k^\theta(x, t, 0, \zeta_0, \zeta, \tau, \xi) = x_k$, and $\Xi^\theta(x, t, 0, \zeta_0, \zeta, \tau, \xi) = \zeta_0$. Furthermore, Λ_k^θ and Ξ^θ are s-approximate solutions of the system $H^0_{F_j^\theta} w = 0$, $j = 1, \ldots, n$.

We can write $\Lambda_k^\theta(x, t, r, \zeta_0, \zeta, \tau, \xi) = x_k + r \cdot \Phi_k^\theta(x, t, r, \zeta_0, \zeta, \tau, \xi)$, $k = 1, \cdots, m$, where $\Phi_k^\theta = (\Phi_{k1}^\theta, \ldots, \Phi_{kn}^\theta)$ and then we have

$$Z_k^\theta(x, t, r) \doteq (\Lambda_k^\theta)^{\tilde u}(x, t, r) = x_k + r \cdot (\Phi_k^\theta)^{\tilde u}(x, t, r) \doteq x_k + r \cdot \Psi_k^\theta(x, t, r),$$

where $\Psi_k^\theta \in C^1(\Omega \times V)$. It follows from (4.6) that $Z_k^\theta(x, t, r)$ is an s-approximate solution of $(L_j^\theta)^{\tilde u} w = 0$, $j = 1, \ldots, n$.

We also define, for $j = 1, \ldots, n$,

$$Z_{m+j}^\theta(x, t, r) = t_j + r \cdot \Psi_{m+j}^\theta(x, t, r), \tag{4.8}$$

where $\Psi_{m+j}^\theta = e^{-i\theta} e_j$ and $\{e_1, \ldots, e_n\}$ is the canonical basis of $\mathbb{R}^n$.

It is easily seen that

$$(L_i^\theta)^{\tilde u} Z_{m+j}^\theta = 0,$$

for all $1 \le i, j \le n$. Thus, if we set $\Psi^\theta = (\Psi_{kj}^\theta)_{(m+n) \times n}$ and $Z^\theta = (Z_1^\theta, \ldots, Z_{m+n}^\theta)$, then we have $\Psi^\theta, Z^\theta \in C^1(\Omega \times V)$,

$$Z^\theta(x, t, r) = (x, t) + \Psi^\theta(x, t, r) r, \tag{4.9}$$

and Z^θ is an s-approximate solution for the system $(L_j^\theta)^{\tilde{u}}w = 0$, $j = 1, \ldots, n$, with respect to $r \in \mathbb{R}^n$.

We also notice that if $(0, 0, \xi, \tau) \notin \text{Char}(\mathbb{L})$ then there exist $j \in \{1, \ldots, n\}$ and $\theta \in [0, 2\pi)$ such that

$$\begin{pmatrix} \xi \\ \tau \end{pmatrix} \cdot \operatorname{Im} \Psi^\theta(0, 0, 0)e_j < 0,$$

where we are using the notation $\mathbb{L} = (L_1, \ldots, L_n)$ to represent the system of vector fields L_j, $j = 1, \ldots, n$.

We now define $h(x, t, r) = \Xi^{\tilde{u}}(x, t, r)$. Likewise we have done before, we conclude that h is a C^1 function that is an s-approximate solution of $(L_j^\theta)^{\tilde{u}}w = 0$, $j = 1, \ldots, n$ and satisfies $h(x, t, 0) = u(x, t)$.

Applying Lemma 4.5, with $m' = m+n$, $n' = n$, $\Psi = \Psi^\theta$, $Z = Z^\theta$ and $L_j = (L_j^\theta)^{\tilde{u}}$, we can conclude that $(0, 0, \xi, \tau) \notin WF_s(u)$ and therefore $WF_s(u)\big|_0 \subset \text{Char}(\mathbb{L})\big|_0$.

Since the point $p \in \Sigma$ is arbitrary, we conclude that $WF_s(u) \subset T^0M$, which completes the proof of Theorem 4.3. ■

Remark 4.6 *Our techniques can be applied in order to prove a C^∞ version of Theorem 4.3. Notice that in this case we already have C^∞-approximate solutions to a smooth system of pairwise commuting vector fields, see Treves [T].*

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