Existence of trace for solutions of locally integrable systems of vector fields

J. Hounie and E. R. da Silva

Abstract. We give sufficient conditions for the existence of trace of homogeneous solutions defined on wedges of general locally integrable structures, extending previous results that considered locally integrable structures of a particular nature.

Introduction

A classical result states ([**H1**, Thms. 3.1.14 and 3.1.15]) that a holomorphic function in one complex variable, defined on domain with smooth boundary, that has tempered growth at the boundary possesses a well defined distributional boundary value. In the case of several complex variables, one considers the more general situation of holomorphic functions defined on wedges and studies their boundary values at the edges and an analogous result holds [**BER**, Ch. VII]. If we view holomorphic functions as homogeneous solutions of an overdetermined system of equations, it is natural to ask for which kind of overdetermined systems of vector fields their continuous homogeneous solutions defined on wedges behave similarly, that is, they have weak boundary values provided some growth restriction is assumed at the edge. Several works have dealt with this problem in particular situations, the case of a single vector field has been considered in [**BH1**], [**BH2**], [**BH3**] and [**BCH**, Thm. VI.1.3] while E. Bär studied in her thesis [**B**] the case of solutions defined in a wedge for a locally integrable system of vector fields of co-rank one.

Our main result applies to continuous solutions of a general overdetermined system of first order partial differential equations that arises from a locally integrable

2010 *Mathematics Subject Classification.* Primary 35F45, 35B30; Secondary 35F35.

Key words and phrases. Locally integrable structures, Boundary value of homogeneous solutions, Baouendi-Treves approximation formula,

Work supported in part by CNPq and FAPESP.

involutive structure and gives a sufficient condition for the existence of boundary values. Involutive structures arise in many geometric contexts including foliations, complex structures, and CR structures (see for example [EG1], [EG2], [HJ] and [Sz]). A smooth locally integrable involutive structure is a pair $(\mathcal{M}, \mathcal{L})$ where $\mathcal{M}$ is a smooth manifold and $\mathcal{L}$ is a smooth, involutive subbundle of $\mathbb{C}T\mathcal{M}$ such that $\mathcal{L}^\perp$, the subbundle of $\mathbb{C}T^*\mathcal{M}$ orthogonal to $\mathcal{L}$, is locally generated by exact one-forms. Similarly, a real analytic involutive structure is a pair $(\mathcal{M}, \mathcal{L})$ where $\mathcal{M}$ is a real analytic manifold and $\mathcal{L}$ is a real analytic, involutive subbundle of $\mathbb{C}T\mathcal{M}$. It follows from the Cauchy-Kowaleska theorem that a real analytic involutive structure is always locally integrable, in particular, our results apply to general real analytic involutive structures.

The paper is organized as follows. In section 1 we state a sufficient growth condition that guarantees the existence of trace for a homogeneous solution, defined on a wedge with maximally real edge, of a locally integrable involutive structure (Theorem 1.1) which is our main result. This condition is (in general) strictly weaker that the usual requirement of tempered growth at the edge. However, this condition is formulated in terms of a special first integral, so in section 2, we address the invariance problem and prove that our growth condition is actually independent of the choice of the first integral by attaching a local invariant to points $p \in (\mathcal{M}, \mathcal{L}, \Sigma)$, where $(\mathcal{M}, \mathcal{L})$ is a locally integrable structure and $\Sigma \subset \mathcal{M}$ is a maximally real submanifold. In section 3 we prove a slightly strengthened form of Theorem 1.1 (Theorem 3.1). In Section 4 we give an application of the invariant defined in Section 2, showing that it can be used to characterize CR structures among general locally integrable structures.

1. Statement of the main result

Suppose $(\mathcal{M}, \mathcal{L})$ is a smooth locally integrable structure, that is, $\mathcal{M}$ is a smooth manifold of dimension $N = m + n$, $\mathcal{L}$ is a smooth subbundle of $\mathbb{C}T\mathcal{M}$ of fiber dimension n over $\mathbb{C}$ and its orthogonal $\mathcal{L}^\perp$ has fiber dimension m and can be generated on some neighborhood of any given point by the differentials of m complex functions $Z_1, \ldots, Z_m$. To avoid trivial cases, we will always assume that n (called the rank of $\mathcal{L}$) and m (called the co-rank of $\mathcal{L}$) are ≥ 1. A system of m locally defined functions $Z_1, \ldots, Z_m$ whose differentials $dZ_1, \ldots, dZ_m$ span $\mathcal{L}^\perp$ is called a complete set of first integrals for $\mathcal{L}$ or, in short form, we may say that $Z = (Z_1, \ldots, Z_m)$ is a first integral of the system. On the subject of locally integrable structures we refer to [**T**] and [**BCH**]. We recall that

Definition 1.1. *Let $(\mathcal{M},\mathcal{L})$ be a smooth locally integrable structure. A submanifold Σ of $\mathcal{M}$ is called maximally real with respect to $\mathcal{L}$ if*

$$\mathbb{C}T_p\mathcal{M} = \mathbb{C}T_p\Sigma \oplus \mathcal{L}_p, \qquad p \in \Sigma.$$

Definition 1.2. *Let Σ be a submanifold of $\mathcal{M}$, dim $\mathcal{M} = r+s$, dim $\Sigma = r$, $r, s > 0$. We say a subset $\mathcal{W}$ is a wedge in $\mathcal{M}$ at $p \in \Sigma$ with edge Σ if the following holds: there exists a diffeomorphism φ of a neighborhood V of 0 in $\mathbb{R}^{r+s}$ onto a neighborhood U of p in $\mathcal{M}$ with $\varphi(0) = p$ and a set $B \times \Gamma \subseteq V$ with B a ball centered at $0 \in \mathbb{R}^r$ and Γ a truncated open convex cone in $\mathbb{R}^s$ with vertex at 0 such that $\varphi(B\times\Gamma) = \mathcal{W}$ and $\varphi(B \times \{0\}) = \Sigma \cap U$.*

If Σ, $\mathcal{M}$, $\mathcal{W}$ and $p \in \Sigma$ are as in the previous definition, the direction wedge $\Gamma_p(\mathcal{W}) \subseteq T_p(\mathcal{M})$ is defined as the interior of

$$\{c'(0) \mid c : [0,1] \to \mathcal{M} \text{ smooth}, c(0) = p, \quad c(t) \in \mathcal{W} \quad \forall t > 0\}.$$

Equivalently, $\Gamma_p(\mathcal{W}) = \{d\varphi(\mathbb{R}^r \times \{\lambda v \mid v \in \Gamma\ ,\ \ \lambda > 0\})\}$. Thus $\Gamma_p(\mathcal{W})$ is a linear wedge in $T_p\mathcal{M}$ with edge $d\varphi(\mathbb{R}^r \times \{0\}) = T_p\Sigma$. If Σ is a hypersurface in $\mathcal{M}$, then a wedge $\mathcal{W}$ with edge Σ is simply a side of Σ.

From now on, we will assume that Σ is a maximally real submanifold, $\mathcal{W}$is a wedge in $\mathcal{M}$ at p and consider the existence problem for the trace of a continuous null solution u of $\mathcal{L}$, i.e., a continuous function whose (weak) differential $du|_q \in \mathcal{L}_q^\perp$, $q \in \mathcal{M}$. Since this is a local problem, we may choose local coordinates $x_1,\dots,x_m,t_1,\dots,t_n$, such that $(x(p),t(p)) = (0,0)$ and assume we are in the following situation:

(1) Σ is given by the equations $t_j = 0$, $j = 1,\dots,n$, so we may set $\Sigma = \{(x,0) : |x| < r\}$ after its identification with an open subset of $\mathbb{R}^m \times \{0\} \subset \mathbb{R}^m \times \mathbb{R}^n$;
(2) $\mathcal{W} = B_r^x(0) \times \Gamma_T$, where $B_r^x(0) \subset \mathbb{R}^m$ denotes the open ball of radius $r > 0$ centered at the origin, $\Gamma_T = \Gamma \cap \{t \in \mathbb{R}^n; |t| \leq T\}$, $\Gamma \subset \mathbb{R}^n$ is a convex open cone with vertex at the origin, and $T > 0$;
(3) the functions $Z_1,\dots,Z_m$, whose differentials span $\mathcal{L}^\perp$ may be chosen to have the form

$$Z_k(x,t) = x_k + i\varphi_k(x,t), \quad k = 1,\dots,m, \tag{1.1}$$

where the functions $\varphi_k(x,t)$ are real, $\varphi_k(0,0) = 0$, $k = 1,\dots,m$, and $\partial\varphi_k/\partial x_\ell(0,0) = 0$, $1 \le k,\ell \le m$;

(4) $\mathcal{L}$ is generated by pairwise commuting vector fields $L_1,\dots,L_n$ of the form

$$L_j = \frac{\partial}{\partial t_j} + \sum_{k=1}^{m} \lambda_{jk}(x,t)\frac{\partial}{\partial x_k}, \quad j = 1,\dots,n$$

and u satisfies in the sense of distributions the overdetermined system

$$L_j u(x,t) = 0, \quad (x,t) \in B_r^x(0) \times \Gamma_T, \quad j = 1,\dots,n. \tag{1.2}$$

In view of (1.2) it is customary to write $\mathcal{L}u = 0$ rather than $du|_q \in \mathcal{L}_q^\perp$, $q \in \mathcal{M}$. We will set $\Gamma^0 = \Gamma \cap S^{n-1} = \{t \in \Gamma : \ |t| = 1\}$ so we may write $\Gamma_T = \{\tau t' : \ t' \in \Gamma^0, \ 0 < \tau < T\}$.

We will now state our main result. Consider the map

$$Z(x,t) = (Z_1(x,t),\dots,Z_m(x,t)) : B_r^x(0) \times B_T^t(0) \longrightarrow \mathbb{C}^m$$

and the function

$$d(x,\tau,t') \doteq \mathrm{dist}\,(Z(x,\tau t'), Z(\Sigma)) : B_r^x(0) \times (0,T) \times \Gamma^0 \longrightarrow \mathbb{R}_+.$$

Theorem 1.1. *Let u be a continuous solution of* (1.2) *and assume that there exists $\nu \in \mathbb{N}$ such that*

$$\sup_{t' \in \Gamma^0} \int_0^T \int_{B_r^x(0)} \mathrm{dist}\,\big(Z(x,\tau t'), Z(\Sigma)\big)^\nu \,|u(x,\tau t')|\, dx d\tau < \infty. \tag{1.3}$$

Then $u(x,t)$ has a distributional limit as $t \to 0$ in Γ_T. More precisely, for any test function $\phi(x) \in C_c^\infty(B_r^x(0))$, the limit

$$\langle bu, \phi \rangle \doteq \lim_{\substack{t\to 0\\ t \in \Gamma_T}} \int u(x,t)\phi(x)\, dx$$

exists and defines a distribution of order $\nu + 1$.

Remark 1.1: It is easy to check that the alternative condition

$$\sup_{t \in \Gamma_T} d(Z(x,t), Z(\Sigma))^\nu |\, u(x,t)| < \infty \tag{1.4}$$

is stronger than (1.3). Furthermore, (1.4) is implied by

$$\sup_{t \in \Gamma} |t|^\nu |\, u(x,t)| < \infty, \tag{1.5}$$

In particular, tempered growth of $u(x,t)$ as $t \to 0$ guarantees the existence of bu. Observe also that Theorem 1.1 extends all previous special results mentioned in the introduction concerning the existence of boundary values.

2. Invariance of the growth condition

Although condition (1.3) was formulated in terms of a special choice of coordinates, it is easy to see by changing variables in the integrals that it is coordinate-free. On the other hand, a specific first integral $Z(x,t)$ is present in (1.3), so it is of interest to show that, in fact, this condition does not depend on the choice of the first integral. This will be shown now. The basic tool is the Baouendi-Treves approximation theorem [**BT**] of which several variations and extensions are known. We now describe briefly the version we will use (see, e.g., the proof of [**BCH**, Thm. II.1.1]). Assume $(\mathcal{M},\mathcal{L})$ is a locally integrable structure, $\mathcal{L}$ has fiber dimension n and $\mathcal{M}$ has dimension $N = n+m$. Then, given $p \in \mathcal{M}$, there exist an open neighborhood U of p and smooth complex functions $Z_1,\dots,Z_m$, defined in U and satisfying $\mathcal{L}Z_j = 0$ on U, such that to every open set $U_2 \subset U$ that contains p, we may find another open set U_1 such that $p \in U_1 \subset U_2$ with the following property: every function $u \in C^k(U_2)$, $k = 0,1,\dots,\infty$, that satisfies $\mathcal{L}u = 0$ on U_2 can be approximated uniformly in $\overline{U}_1$ together with its derivatives up to order k, by a sequence of functions of the form $u_j = P_j(Z_1,\dots,Z_m)$, where P_j is a polynomial in m variables with complex coefficients. A standard consequence is that if we assume that $u \in C^0(U_2)$ and write $Z = (Z_1, Z_2,\dots,Z_m)$, there exists a continuous function $\widehat{u} : Z(\overline{U}_1) \to \mathbb{C}$, such that the factorization $u = \widehat{u}\circ Z$ holds on $\overline{U}_1$. The function $\widehat{u}$ is obtained as the limit of the polynomials $P_j(\zeta)$, $\zeta \in \mathbb{C}^m$, which converge uniformly for $\zeta \in Z(\overline{U}_1)$. We will need an improved version of this fact.

Lemma 2.1. *With the previous notation, if $u \in C^1(U_2)$, then*

$$\widehat{u} : Z(\overline{U}_1) \longrightarrow \mathbb{C}^m$$

is a Lipschitz function, i.e., there exists $K > 0$ such that

$$|\widehat{u}(\zeta_1) - \widehat{u}(\zeta_0)| \le K|\zeta_1 - \zeta_0|, \quad \zeta_1, \zeta_0 \in \overline{U}_1.$$

Proof: By the proof of Theorem II.1.1 in [**BCH**] we may assume that the functions Z_k, $1 \le k \le m$, are given by (1.1), and the choice of local coordinates (x,t) is such that $p = (0,0)$ and $\overline{U}_1$ is expressed by $|x| \le a$, $|t| \le b$. Let $\zeta_0 = Z(x^0,t^0)$, $\zeta_1 = Z(x^1,t^1)$ be two arbitrary points in $Z(\overline{U}_1)$ and set $p_0 = (x^0,t^0)$, $p_1 = (x^1,t^1)$, $q = (x^1,t^0)$, $\zeta_2 = Z(q)$. Notice that $\Re\zeta_0 = x^0$, $\Re\zeta_1 = \Re\zeta_2 = x^1$ and consider smooth curves γ_0 and γ_1 given by

$$\gamma_0 = \{(x,\varphi(x,t^0)) : \ x \in [x^0,x^1] \subset \mathbb{R}^m\}$$

and

$$\gamma_1 = [Z(p_1), Z(q)] \subset \{(x^1, \varphi(x^1, t)); t \in [t^0, t^1] \subset \mathbb{R}^n\}$$

where $[A, B]$ denotes the closed convex hull of the points A and B. Next consider the approximating sequence $u_j = P_j \circ Z$, $j = 1, 2, \ldots$, and write

$$P_j(\zeta_2) - P_j(\zeta_0) = \int_{\gamma_0} dP_j,$$
$$P_j(\zeta_1) - P_j(\zeta_2) = \int_{\gamma_1} dP_j.$$

From the fact that ∇u_j converges uniformly to ∇u on $\overline{U}_1$ we may derive in a standard way (invoking the fact that $(\partial P_j/\partial \zeta_k) \circ Z = M_k(P_j \circ Z)$, $k = 1, \ldots, m$, for vector fields M_k defined on U_2) that $|dP_j|$ is bounded on $Z(\overline{U}_1)$ by a constant independent of $j \in \mathbb{N}$. We refer to [**BCH**, p. 24] on the definition of the M_k's. Since the curves γ_0 and γ_1 are contained in $Z(\overline{U}_1)$, it follows that

$$|P_j(\zeta_1) - P_j(\zeta_0)| \le C_0(|\gamma_0| + |\gamma_1|). \tag{2.1}$$

We will next show that

$$|\gamma_0| + |\gamma_1| \le C_1|\zeta_1 - \zeta_0| \tag{2.2}$$

with C_1 independent of $\zeta_1, \zeta_0 \in Z(\overline{U}_1)$. Indeed, γ_0 is the image by Z of the segment $[x^0, x^1]$, so if C is a bound for $\sup_{\overline{U}_1} |Z_x|$ we have

$$|\gamma_0| \le C|x^1 - x^0| \le C|Z(x^1, t^1) - Z(x^0, t^0)| = C|\zeta_1 - \zeta_0|$$

Furthermore,

$$\begin{aligned} |\gamma_1| = |\zeta_1 - \zeta_2| &\le |\zeta_1 - \zeta_0| + |\zeta_0 - \zeta_2| \\ &\le |\zeta_1 - \zeta_0| + |x^1 - x^0| + |\varphi(x^0, t^0) - \varphi(x^1, t^0)| \\ &\le |\zeta_1 - \zeta_0| + |x^1 - x^0| + C|x^1 - x^0| \\ &\le C_2|\zeta_1 - \zeta_0|, \end{aligned}$$

so (2.2) holds true. Hence, (2.1) and (2.2) imply that

$$|P_j(\zeta_1) - P_j(\zeta_0)| \le K|\zeta_1 - \zeta_0|$$

and letting $j \to \infty$ we obtain

$$|\widehat{u}(\zeta_1) - \widehat{u}(\zeta_0)| \le K|\zeta_1 - \zeta_0|$$

as we wished to prove. $\square$

Consider now a second set $Z_1^{\#}, \ldots, Z_m^{\#}$ of smooth first integrals defined on a neighborhood $U^{\#}$ of p and set $U_2 \doteq U^{\#} \cap U \doteq U_2^{\#}$. Then, we may find open neighborhoods of p, $U_1 \subset U_2$, $U_1^{\#} \subset U_2^{\#}$, and continuous functions

$$\widehat{Z}^{\#} : Z(\overline{U}_1) \to \mathbb{C}^m, \quad \widehat{Z} : Z^{\#}\left(\overline{U_1^{\#}}\right) \to \mathbb{C}^m,$$

such that $Z^{\#} = \widehat{Z}^{\#} \circ Z$ and $Z = \widehat{Z} \circ Z^{\#}$ on $U_1 \cap U_1^{\#}$. Choose an open set $p \in U_0 \subset \overline{U}_0 \subset \overline{U}_1 \cap \overline{U_1^{\#}}$. It turns out that $\widehat{Z}^{\#}$ maps homeomorphically $W \doteq Z\left(\overline{U}_0\right)$ onto $W^{\#} \doteq Z^{\#}\left(\overline{U}_0\right)$ and the inverse of $\widehat{Z}^{\#} : W \to W^{\#}$ is given by $\widehat{Z} : W^{\#} \to W$. Furthermore, by Lemma 2.1, $\widehat{Z}$ and $\widehat{Z}^{\#}$ are Lipschitz functions.

We now apply these considerations to the setup of Theorem 1.1 and the role of Σ in condition (1.3). Using the factorizations $Z^{\#} = \widehat{Z}^{\#} \circ Z$ and $Z = \widehat{Z} \circ Z^{\#}$, we see that

$$\text{dist}\,(Z(p'), Z(\Sigma \cap U_0)) \simeq \text{dist}\,(Z^{\#}(p'), Z^{\#}(\Sigma \cap U_0)), \quad p' \in U_0. \tag{2.3}$$

This has the following interpretation. Let V be a neighborhood of $p \in \mathcal{M}$ and assume that two sets of first integrals $Z = (Z_1, \ldots, Z_m)$ and $Z^{\#} = (Z_1^{\#}, \ldots, Z_m^{\#})$ are defined on V. Consider the functions $d(q) \doteq \text{dist}\,(Z(q), Z(\Sigma \cap V))$ and $d^{\#}(q) \doteq \text{dist}\,(Z^{\#}(q), Z^{\#}(\Sigma \cap V))$, $q \in V$. If f is a continuous function defined in a neighborhood of p, denote by $\mathbf{f}$ its germ at p. If $\mathbf{f}$ and $\mathbf{g}$ are two such germs declare that $\mathbf{f} \sim \mathbf{g}$ if for some representatives $f \in \mathbf{f}$ and $g \in \mathbf{g}$ and some constants $c_1, c_2 > 0$, and some some neighborhood V' of p, the estimates

$$c_1|f(q)| \leq |g(q)| \leq c_2|f(q)|, \quad q \in V'.$$

hold. It is clear that $\mathbf{f} \sim \mathbf{g}$ is an equivalence relation and we denote by $[\mathbf{f}]$ the equivalence class of $\mathbf{f}$. If $f(p) = 0$, g is a representative of $\mathbf{g}$ and $\mathbf{f} \sim \mathbf{g}$, it follows that the zero sets Z_f and Z_g of f and g coincide in a neighborhood of p and the quotients f/g and g/f remain bounded where they are defined. Thus, the class $[\mathbf{f}]$ represents the way in which $f(q)$ approaches 0 as q approaches the zero set $Z_f \ni p$. Hence, (2.3) can be rephrased by saying that the germs at p, $\mathbf{d}$ and $\mathbf{d}^{\#}$, of the functions

$$d(q) = \text{dist}\,(Z(q), Z(\Sigma \cap V)) \text{ and } d^{\#}(q) = \text{dist}\,(Z^{\#}(q), Z^{\#}(\Sigma \cap V))$$

are equivalent and write

$$\mathbf{d} \sim \mathbf{d}^{\#}. \tag{2.4}$$

In other words, the equivalence class $[\mathbf{d}]$ of the germ at p of the function $d(q)$ is independent of the choice of the first integrals $Z_1, \ldots, Z_m$ and it is a local invariant at $p \in \Sigma$ that only depends on the maximally real submanifold $\Sigma \subset \mathcal{M}$ and the locally integrable structure $(\mathcal{M}, \mathcal{L})$.

We now express everything in terms of our local coordinates (x,t) (in which $Z(x,t)$ has the special form $Z(x,t) = x + i\varphi(x,t)$ but $Z^{\#}(x,t)$ might not). We may assume that U_0 is of the form $B_r^x(0) \times B_T^t(0)$ if $r > 0$ and $T > 0$ are sufficiently small (note that (1.3) still holds if we shink r and T) and, to simplify the notation, write Σ instead of $\Sigma \cap U_0$. Let $u(x,t)$ be the continuous solution in the statement of Theorem 1.1. If follows from (2.3) that

$$\begin{aligned}
\int_0^T \int_{B_r^x(0)} &\operatorname{dist}\left(Z^{\#}(x,\tau t'), Z^{\#}(\Sigma)\right)^{\nu} |u(x,\tau t')|\, dx d\tau \\
&\leq C \int_0^T \int_{B_r^x(0)} \operatorname{dist}\left(Z(x,\tau t'), Z(\Sigma)\right)^{\nu} |u(x,\tau t')|\, dx d\tau \\
&\leq C \sup_{t' \in \Gamma^0} \int_0^T \int_{B_r^x(0)} \operatorname{dist}\left(Z(x,\tau t'), Z(\Sigma)\right)^{\nu} |u(x,\tau t')|\, dx d\tau \\
&= C_1 < \infty,
\end{aligned}$$

so taking the sup in $t' \in \Gamma^0$ on the left hand side we see that u satisfies a growth restriction analogous to (1.3) with $Z^{\#}$ in the place of Z. This argument can be reversed to show that a growth condition in terms of $Z^{\#}$ implies a similar a growth condition in terms of Z, possibly after shrinking r and T.

Remark 2.1: In the special local coordinates (x,t) in which $Z = (Z_1, \dots, Z_m)$ is written as $Z(x,t) = x + i\varphi(x,t)$ and Σ is given by $\{t = 0\}$, it is easy to see that

$$\operatorname{dist}\left(Z(x,t), Z(\Sigma)\right) \simeq |\varphi(x,t) - \varphi(x,0)| = |Z(x,t) - Z(x,0)|, \qquad \text{for } |x| \leq r,\ |t| \leq T. \tag{2.5}$$

This fact will be used in the next section.

Remark 2.2: Since the rank of the map $Z : B^x \times B^t \to \mathbb{C}^m$ might not be constant, $Z(B^x \times B^t)$ is, in general, neither an open set nor a submanifold and may be rather irregular. Nevertheless, it is arc-connected by piecewise differentiable curves and this is the main fact we exploited in the proof of Lemma 2.1. If we define a distance between two points $\zeta_0, \zeta_1 \in Z(B^x \times B^t)$ as the infimum of the lengths of the piecewise differentiable curves contained in $Z(B^x \times B^t)$ that join ζ_0 to ζ_1, the arguments in the proof of Lemma 2.1 show that this distance is equivalent to the Euclidean distance restricted to $Z(B^x \times B^t)$.

3. Proof of the main result

Consider special coordinates (x,t) in which a set of first integrals $Z_1,\ldots,Z_m$ have the form (1.1), $\mathcal{L}$ is generated by vector fields of the form $L_j = \frac{\partial}{\partial t_j} + \sum_{k=1}^m \lambda_{jk}(x,t)\frac{\partial}{\partial x_k}$, $j=1,\ldots,n$, and $\Sigma=\{t=0\}$.

In fact (*cf.* [**BCH**, Chapter I]), there exist smooth vector fields

$$M_k = \sum_{\ell=1}^m \mu_{k\ell}(x,t)\frac{\partial}{\partial x_\ell}, \quad k=1\cdots m,$$

satisfying $M_kZ_j=\delta_{kj}$ (Kronecker's delta) such that

$$\begin{aligned} L_j &= \frac{\partial}{\partial t_j} - i\sum_{k=1}^m \frac{\partial \varphi_k}{\partial t_j}(x,t)M_k, \\ &= \frac{\partial}{\partial t_j} + \sum_{k=1}^m \lambda_{jk}(x,t)\frac{\partial}{\partial x_k} \quad j=1,\cdots,n. \end{aligned}$$

The vector fields $L_1,\ldots,L_n,M_1\ldots,M_m$ are pairwise commuting and span $\mathbb{C}T\mathcal{M}$ over the local patch where they are defined. Furthermore, if f is of class C^1 we have

$$df = \sum_{j=1}^n L_jf\,dt_j + \sum_{k=1}^m M_kf\,dZ_k. \tag{3.1}$$

In view of (2.5), Theorem 1.1 will be a consequence of

Theorem 3.1. *Let $f(x,t)$ be a continuous function on the wedge $Q=B_r^x(0)\times\Gamma_T$ with edge $\Sigma=B_r^x(0)\times\{0\}$ and assume that*

(1) $L_jf\in L^\infty(Q)$, $\quad j=1,\ldots,n$;

(2) *for some $\nu\in\mathbb{N}$*

$$\sup_{t'\in\Gamma^0}\int_0^T\int_{B_r^x(0)}|\varphi(x,\tau t')-\varphi(x,0)|^\nu|f(x,\tau t')|dxd\tau<\infty. \tag{3.2}$$

Then $f(x,t)$ has a distributional limit as $t\to 0$ in Γ_T. More precisely, for any test function $\psi(x)\in C_c^\infty(B_r^x(0))$, the limit

$$\langle bf,\psi\rangle \doteq \lim_{\substack{t\to 0\\ t\in\Gamma_T}}\int f(x,t)\psi(x)\,dx$$

exists and defines a distribution of order $\nu+1$.

The proof of Theorem 3.1 will be carried out in three steps. In the first step we will assume that f is of class C^1 and will show that the limit exists as $t\to 0$ in Γ_T along a fixed direction. In the second step we will assume that f is of class C^0 but we will still approach 0 along a fixed direction. In the final step we will deal with the general case.

3.1. Step 1. Assume that $f \in C^1(Q)$, fix a point $\dot{t} = (\dot{t}_1, \dots, \dot{t}_n) \in \Gamma^0$ and consider the complex vector field

$$L_{(\dot{t})} = \dot{t}_1 L_1 + \cdots + \dot{t}_n L_n$$

which is tangent to the $m+1$-submanifold

$$\Pi(\dot{t}) \doteq \{(x, \tau\dot{t}) : \quad x \in B^x_r(0),\ 0 < \tau < T\} \subset \mathbb{R}^m_x \times \mathbb{R}^n_t$$

which is an open subset of the linear space generated by $\mathbb{R}^m_x$ and $\dot{t}$. We may express the restriction of $f(x,t)$ to $\Pi(\dot{t})$ as

$$f_{(\dot{t})}(x, \tau) \doteq f(x, \tau\dot{t}_1, \dots, \tau\dot{t}_n), \quad 0 < \tau < T,$$

and regard it as a function in the variables $\tau, x_1, \dots, x_m$. It is clear that

$$L_{(\dot{t})} = \frac{\partial}{\partial \tau} + \sum_{k=1}^{m} \lambda_k(x, \tau, \dot{t}) \frac{\partial}{\partial x_k}$$

since differentiating $t = \tau\dot{t}$ with respect to τ gives

$$\frac{\partial}{\partial \tau} = \dot{t}_1 \frac{\partial}{\partial t_1} + \cdots + \dot{t}_n \frac{\partial}{\partial t_n}$$

and that $L_{(\dot{t})} f_{(\dot{t})} = \sum_k \dot{t}_k L_k f(x, \tau\dot{t}) \in L^\infty(Q \cap \Pi(\dot{t}))$. We regard $L_{(\dot{t})}$ as a single locally integrable vector field in the $m+1$ variables $\tau, x_1, \dots, x_m$ with first integrals

$$Z^{(\dot{t})}_j(\tau, x) \doteq Z_j(\tau\dot{t}, x), \quad j = 1, \dots, m,$$

that depend on $\dot{t}$ as a parameter. In particular, $L_{(\dot{t})}$ satisfies the hypothesis of [**BCH**, Thm. VI.1.3] uniformly in $\dot{t} \in \tilde{\Gamma}^0$, provided that $\tilde{\Gamma}^0$ is a compact subset of Γ^0. To see this we will briefly describe below the main steps in the proof of [**BCH**, Thm. VI.1.3] that lead us to conclude that the constants involved can be taken independently of $\dot{t} \in \tilde{\Gamma}^0$; a more detailed proof would be a straightforward but long and tedious repetition of the arguments in Thm.VI.1.3.

Lemma 3.1. *Let $\psi \in C^\infty_c(B^x_r(0))$ and fix a positive integer ν. There exists a smooth function $u(\xi + i\eta, \dot{t})$ defined on $\mathbb{C}^m \times \Gamma^0$ such that*

(1) $u(Z(x,0), \dot{t}) = \psi(x)$, $|x| < r$;
(2) $\left| \dfrac{\partial u}{\partial \bar{\zeta}_j}(\xi + i\eta, \dot{t}) \right| \leq C \operatorname{dist}(\xi + i\eta, Z(\Sigma))^\nu$, $j = 1, \dots, m$, $|\xi| \leq r$, $|\eta| \leq R$, $\dot{t} \in \tilde{\Gamma}^0$,

where $R > 0$ is a constant such that $Z(B^x_r(0) \times B^t_T(0)) \subset B_r(0) + iB_R(0)$, $C > 0$ is a constant and $\tilde{\Gamma}^0$ is a compact subset of Γ^0. Furthermore, the function u is obtained by applying to $\psi(x)$ a linear partial differential operator $P(x, t, D_x, D_t)$ of order ν with smooth coefficients.

Corollary 3.1. *Let $u(\zeta, \dot{t})$ be the function considered in the lemma above. Setting $\psi(x,\tau,\dot{t}) = u(Z(x,\tau\dot{t}),\dot{t})$ we have*

1. $\psi(x, 0, \dot{t}) = \psi(x)$*,* $|x| < r$*;*
2. $L_{(\dot{t})}\psi(x,\tau,\dot{t}) \le C\,|\varphi(x,\tau t') - \varphi(x,0)|^{\nu}$*.*

If we set $M^k_{(\dot{t})} = \sum_{\ell=1}^m \mu_{k\ell}(x,\tau\dot{t})\frac{\partial}{\partial x_\ell}$, $k = 1,\cdots,m$, it follows from (3.1) that for any $g(x,\tau)$ of class C^1,

$$dg = \sum_{k=1}^m M^k_{(\dot{t})} g_{(\dot{t})} dZ^k_{(\dot{t})} + L_{(\dot{t})} g_{(\dot{t})} d\tau.$$

Writing $dZ_{(\dot{t})} = dZ_1(x,\tau\dot{t}) \wedge \cdots \wedge dZ_m(x,\tau\dot{t})$, the exterior derivative of the $m-$form $g(x,t)\,dZ_{(\dot{t})}$ is given by

$$d(g\,dZ_{(\dot{t})}) = L_{(\dot{t})} g_{(\dot{t})} d\tau \wedge dZ_{(\dot{t})}.$$

We now call Corollary 3.1 and set $g(x,\tau) = f_{(\dot{t})}(x,\tau)\psi(x,\tau,\dot{t})$. Using the above formulas and Stokes' theorem we get

$$\begin{aligned}(3.3)\qquad &\int_{B_r^x(0)} f_{(\dot{t})}(x,\epsilon)\psi(x,\epsilon,\dot{t})\,d_x Z_{(\dot{t})}(x,\epsilon) = \\ &\quad\int_{B_r^x(0)} f_{(\dot{t})}(x,T)\psi(x,T,\dot{t})\,d_x Z_{(\dot{t})}(x,T) \\ &\qquad + \int_{B_r^x(0)}\int_\epsilon^T f_{(\dot{t})}(x,\tau) L_{(\dot{t})}\psi(x,\tau,\dot{t})\,d\tau \wedge dZ_{(\dot{t})} \\ &\qquad + \int_{B_r^x(0)}\int_\epsilon^T L_{(\dot{t})} f_{(\dot{t})}(x,\tau)\psi(x,\tau,\dot{t})\,d\tau \wedge dZ_{(\dot{t})}.\end{aligned}$$

By (2) of Corollary 3.1 we have

$$|f_{(\dot{t})}(x,\tau)L_{(\dot{t})}\psi(x,\tau,\dot{t})| \le C|f_{(\dot{t})}(x,\tau)||\varphi_{(\dot{t})}(x,\tau) - \varphi_{(\dot{t})}(x,0)|^{\nu}$$

which shows that $|f_{(\dot{t})}(x,\tau)L_{(\dot{t})}\psi(x,\tau)| \in L^1(B_r^x(0)\times[0,T])$ in view of (3.2). Thus, the second integral of the right hand side of (3.3) has a limit when $\epsilon \searrow 0$ and is bounded by a constant independent of the direction $\dot{t}$. The existence of the limit when $\epsilon \searrow 0$ of the other two integrals on the right hand side of (3.3) is clear. We conclude that the limit when $\epsilon \searrow 0$ of the left hand side of (3.3) exists and

$$\left|\int_{B_r^x(0)} f_{(\dot{t})}(x,\epsilon)\psi(x,\epsilon,\dot{t})\,d_x Z_{(\dot{t})}(x,\epsilon)\right| \le C,$$

with $C > 0$ independent of $\dot{t} \in \tilde{\Gamma}^0$ and $0 < \epsilon \le T$. We next concatenate $\psi(x,\tau,\dot{t})$ and $\psi(x) \in C_c^\infty(B_r^x(0))$, i.e., we find a finite sequence of smooth functions $\psi_\ell(x,\tau,\dot{t})$, $\ell = 0,\ldots,\nu-1$, whose x-support is contained in a fixed compact subset independent of t such that

(1) $\psi_\nu(x,\tau,\dot{t}) = \psi(x,\tau,\dot{t})$;

(2) $\psi_0(x,\tau,\dot{t}) = \psi(x)$;

(3) for $\ell = 1,\ldots,\nu$,

$$\text{if } \lim_{\tau\to 0}\int_{B_r^x(0)} f_{(\dot{t})}(x,\tau)\psi_\ell(x,\tau)\,d_xZ_{(\dot{t})}(x,\tau) \quad \text{exists, then}$$
$$\lim_{\tau\to 0}\int_{B_r^x(0)} f_{(\dot{t})}(x,\tau)\psi_{\ell-1}(x,\tau)\,d_xZ_{(\dot{t})}(x,\tau) \quad \text{exists}$$

and both limits are equal;

(4) for $\ell = 1,\ldots,\nu$, a bound

$$\left|\int_{B_r^x(0)} f_{(\dot{t})}(x,\tau)\psi_\ell(x,\tau,\dot{t})\,d_xZ_{(\dot{t})}(x,\tau)\right| \le C_\ell, \quad 0<\tau\le T,\ \dot{t}\in\tilde{\Gamma}^0,$$

implies a bound

$$\left|\int_{B_r^x(0)} f_{(\dot{t})}(x,\epsilon)\psi_{\ell-1}(x,\tau,\dot{t})\,d_xZ_{(\dot{t})}(x,\tau)\right| \le C_{\ell-1}, \quad 0<\tau\le T,\ \dot{t}\in\tilde{\Gamma}^0.$$

The construction of the functions ψ_ℓ in the concatenation is described in detail in the proof of [**BCH**, Theorem VI.1.3], where the case of a single vector field is treated (they are denoted as $T_\ell g(x,t)$). The only difference is that the single vector field we are dealing with in the present case depends on the parameter $\dot{t}$ and we must check that the bounds are uniform with respect to $\dot{t}$.

Hence, by descending induction, we obtain for $\ell = 0$ that

$$\lim_{\tau\to 0}\int_{B_r^x(0)} f_{(\dot{t})}(x,\tau)\psi(x)\,d_xZ_{(\dot{t})}(x,\tau) \quad \text{exists and is equal to}$$
$$\int_{B_r^x(0)} f_{(\dot{t})}(x,T)\psi_\nu(x,T,\dot{t})\,d_xZ_{(\dot{t})}(x,T) + \int_{B_r^x(0)}\int_0^T f_{(\dot{t})}(x,\tau)L_{(\dot{t})}\psi_\nu(x,\tau,\dot{t})\,d\tau\wedge dZ_{(\dot{t})} + \int_{B_r^x(0)}\int_0^T L_{(\dot{t})}f_{(\dot{t})}(x,\tau)\psi_\nu(x,\tau,\dot{t})\,d\tau\wedge dZ_{(\dot{t})} \tag{3.4}$$

(this corresponds to formula (VI.32) in [**BCH**, p. 281]) and

$$\left|\int_{B_r^x(0)} f_{(\dot{t})}(x,\tau)\psi(x)\,d_xZ_{(\dot{t})}(x,\tau)\right| \le C_0, \quad 0<\tau\le T,\ \dot{t}\in\tilde{\Gamma}^0. \tag{3.5}$$

Notice that the expression for the limit in (3.4) involves derivatives of order one of the function $\psi_\nu(x,\tau,\dot{t}) = \psi(x,\tau,\dot{t})$ which, by Lemma 3.1 and its corollary, is a linear combination with smooth coefficients of derivatives of $\psi(x)$ up to order ν, so it defines a distribution of order $\nu+1$. More generally, if $g(x,\tau)$ is smooth on

$B^x_r(0)\times[0,T]$ and the support of $x\mapsto g(x,\tau)$ is contained in a compact subset that does not depend on $t\in[0,T]$ we have

$$\lim_{\tau\to 0}\int_{B^x_r(0)} f_{(\dot t)}(x,\tau)g(x,\tau)\,d_xZ_{(\dot t)}(x,\tau)\quad\text{exists}\tag{3.6}$$

and

$$\left|\int_{B^x_r(0)} f_{(\dot t)}(x,\tau)g(x,\tau)\,d_xZ_{(\dot t)}(x,\tau)\right|\le C,\quad 0<\tau\le T,\ \dot t\in\tilde\Gamma^0.\tag{3.7}$$

3.2. Step 2. Assume now that $f\in C^0(Q)$. In fact, the proof of Step 1 still holds for a continuous f but the fact that the restriction $f_{(\dot t)}$ of f to $\Pi(\dot t)$ satisfies weakly the equation $L_{(\dot t)}f_{(\dot t)}\in L^\infty$ and that Stokes's formula (3.3) is valid requires some justification (it could be proved, for instance, with the help of Baouendi-Treves approximation formula). An alternative approach is to regularize f and apply Step 1 to the regularizations. Let $\phi\in C^\infty_c(B)$, with B the unit ball in $\mathbb{R}^{m+1}$, $\int\int\phi\,dxd\tau=1$ and $\phi_\delta(x,\tau)=\frac{1}{\delta^{m+1}}\phi(\frac{x}{\delta},\frac{\tau}{\delta})$, $\delta>0$. For $\epsilon>0$, set $f^\epsilon_{(\dot t)}(x,\tau)=f_{(\dot t)}(x,\tau+\epsilon)$. Then, for $\delta<\epsilon$, $f^\epsilon_{(\dot t)}*\phi_\delta(x,\tau)$ is smooth on $\{\tau>0\}$. Set $g^{\epsilon,\delta}_{(\dot t)}(x,\tau)=\big(f^\epsilon_{(\dot t)}*\phi_\delta(x,\tau)\big)\psi(x,\tau+\epsilon,\dot t)$, $Z^\epsilon_{(\dot t)}(x,\tau)=Z_{(\dot t)}(x,\tau+\epsilon)$ and

$$L_{(\dot t)}=\frac{\partial}{\partial\tau}+\sum_{k=1}^m\lambda_k(x,t+\epsilon,\dot t)\frac{\partial}{\partial x_k}.$$

As in Step 1, we have

$$d(g^{\epsilon,\delta}_{(\dot t)}dZ^\epsilon_{(\dot t)})=L^\epsilon_{(\dot t)}g^{\epsilon,\delta}_{(\dot t)}d\tau\wedge dZ^\epsilon_{(\dot t)}$$

which we use to obtain the analogue of (3.3) for $f^\epsilon_{(\dot t)}*\phi_\delta(x,\tau)$. Then, repetition of the arguments of Step 1 lead to the analogue of (3.4), (3.5), (3.6) and (3.7) for $f^\epsilon_{(\dot t)}*\phi_\delta(x,\tau)$. If we let $\delta\searrow 0$ and invoke Friedrichs' lemma we derive (3.4) and (3.5) for $f^\epsilon_{(\dot t)}$. Finally, we let $\epsilon\searrow 0$ to get (3.4), (3.5), (3.6) and (3.7) for $f_{(\dot t)}$ itself.

3.3. Step 3. Since $\dot t$ appears on the right hand side of (3.4) the directional limit seems to depend on the direction $\dot t$. To show that this is not so, consider for $\psi(x)\in C^\infty_c(B^x_r(0))$ the function

$$T(t)=\int_{B^x_r(0)} f(x,t)\psi(x)\,dx,\quad t\in\Gamma_T.$$

We will show that ∇T is bounded for $t\in\Gamma_T$ if $t/|t|\in\tilde\Gamma^0$. A standard computation shows that the derivatives $\partial T/\partial t_j$, $j=1,\dots,n$ in the sense of distributions are given by

$$\frac{\partial T}{\partial t_j}(t) =$$

$$\int_{B_r^x(0)} L_j f(x,t)\psi(x)\,dx + \int_{B_r^x(0)} f(x,t)\sum_{k=1}^{m}\frac{\partial}{\partial x_k}\big(\lambda_{jk}(x,t)\psi(x)\big)\,dx.$$

The first term on the right hand side is bounded because $L_j f$ is bounded. To bound the second term, write

$$\sum_{k=1}^{m}\frac{\partial}{\partial x_k}\big(\lambda_{jk}(x,t)\psi(x)\big) = g(x,t) = g(x,\tau\dot{t})$$

with $\tau = |t|$, $\dot{t} = t/|t|$ and apply (3.7). Hence, $T(t)$ is a Lipschitz function and has a limit as $t \to 0$ on proper subcones of Γ_T. Letting $t \to 0$ along a fixed direction $\dot{t}$ we see that the limit is given by (3.4). As we have already pointed out, this shows that $\det Z_x(x,0)\, bf(x)$ is a distribution of order $\nu+1$ and dividing by $\det Z_x(x,0)$ so is $bf(x)$. $\square$

4. Another application

Let $(\mathcal{M},\mathcal{L})$ be a smooth locally integrable structure, $\Sigma \subset \mathcal{M}$ a maximally real submanifold, $p \in \Sigma$. If $Z = (Z_1,\dots,Z_m)$ is a complete set of first integrals defined in a neighborhood U of p we write

$$d_{\Sigma,Z}(q) = \operatorname{dist}\left(Z(q), Z(\Sigma\cap U)\right), \quad q \in U,$$
$$\delta_\Sigma(q) = \operatorname{dist}\left(q, \Sigma\cap U\right), \quad q \in U,$$

and denote by $\mathbf{d}_{\Sigma,Z}$ and $\boldsymbol{\delta}_\Sigma$ their corresponding germs at p. We have already seen that the vanishing rate of $\mathbf{d}_{\Sigma,Z}$ is an invariant of the pair $(\Sigma,\mathcal{L})$. Since clearly $d_{\Sigma,Z}(q) \le C\operatorname{dist}(q,\Sigma\cap U)$ as $q \to p$, we always have $\mathbf{d}_{\Sigma,Z} \lesssim \boldsymbol{\delta}_\Sigma$. It is a natural question to ask for which structures the opposite relation also holds, i.e., when $\mathbf{d}_{\Sigma,Z} \sim \boldsymbol{\delta}_\Sigma$. This question has a simple answer: this property characterizes CR structures among locally integrable structures. We recall that $\mathcal{L}$ is CR at p if $\mathcal{L}_p \cap \overline{\mathcal{L}}_p = \{0\}$ and $\mathcal{L}$ is CR on $\mathcal{M}$ if $\mathcal{L}$ is CR at p for every point $p \in \mathcal{M}$. Before stating the precise characterization result, we will need some facts about local canonical forms for generators of $\mathcal{L}^\perp$ in appropriate local coordinates. As before, N will denote the dimension of $\mathcal{M}$.

Theorem 4.1. *Let $(\mathcal{M},\mathcal{L})$ be a smooth locally integrable structure of rank n and co-rank m. Let $p \in \Omega$ and d be the real dimension of $T_p^0 \doteq \mathcal{L}_p^\perp \cap T_p^*\mathcal{M}$. Then* *there is a coordinate system vanishing at* p

$$\{x_1, \dots, x_\nu, y_1, \dots, y_\nu, s_1, \dots, s_d, t_1, \dots, t_{n'}\}$$

and smooth, real-valued functions $\phi_1, \dots, \phi_d$ *defined in a neighborhood of the origin and satisfying*

$$\phi_k(0) = 0, \quad d\phi_k(0) = 0, \quad k = 1, \dots, d,$$

such that the differentials of the functions

$$Z_j(x, y) = z_j \doteq x_j + iy_j, \quad j = 1, \dots, \nu;$$
$$W_k(x, y, s, t) = s_k + i\phi_k(z, s, t), \quad k = 1, \dots, d,$$

span $\mathcal{L}^\perp$ *in a neighborhood of the origin. In particular we have* $\nu + d = m$, $\nu + n' = n$ *and also*

$$T_p^0 = span\,\{ds_1|_0, \dots, ds_d|_0\}.$$

Furthermore, $\mathcal{L}$ *is CR if and only if* $n' = 0$ *(i.e., there are no* t *variables). In this case, we have* $\nu = n$ *and* $m = n + d$.

The theorem above summarizes well known results, see for instance [**BCH**, Theorem I.10.1] and [**BCH**, Section I.15].

We state now the characterization theorem.

Theorem 4.2. *Let* $(\mathcal{M}, \mathcal{L})$ *be a smooth locally integrable structure of rank* n *and co-rank* m. *The following conditions are equivalent:*

(1) $\mathcal{L}$ *is CR on a neighborhood of* p*;*
(2) *For any maximally real submanifold* Σ *passing through* p *and any complete set* $Z = (Z_1, \dots, Z_m)$ *of local first integrals defined in a neighborhood of* p*, the functions* $d_{\Sigma,Z}$ *and* δ_Σ *are comparable in a neighborhood of* p*. In other words,* $\mathbf{d}_{\Sigma,Z} \sim \boldsymbol{\delta}_\Sigma$*;*
(3) *For some maximally real submanifold* Σ *passing through* p *and some complete set* $Z = (Z_1, \dots, Z_m)$ *of local first integrals defined in a neighborhood of* p*,* $\mathbf{d}_{\Sigma,Z} \sim \boldsymbol{\delta}_\Sigma$.

Proof: Since it is trivial that (2) implies (3) it is enough to show that (1) implies (2) and that (3) implies (1).

(1) $\implies$ (2). Since $\mathcal{L}$ is CR in a neighborhood of p we may find a local coordinate system vanishing at p,

$$(x_1, \dots, x_n, y_1, \dots, y_n, s_1, \dots, s_d)$$

with $2n + d = N$, such that the maximally real submanifold Σ is given by the equations $y_j = 0$, $1 \leq j \leq n$ and there exist smooth, real valued functions $\phi_1, \ldots, \phi_d$ defined in a neighborhood U of p satisfying

$$\phi_k(0) = 0, \quad d\phi_k(0) = 0, \quad k = 1, \ldots, d, \tag{4.1}$$

such that the differential of the functions

$$Z_j = x_j + iy_j, \quad j = 1, \ldots, n;$$
$$W_k = s_k + i\phi_k(x, y, s), \quad k = 1, \ldots, d,$$

form a complete set of first integrals of $\mathcal{L}$ in a neighborhood of the origin. Assume that U is the cube $|x| < 1$, $|y| < 1$, $|s| < 1$. Given a point $q = (x, y, s) \in U$ we have that $Z(q) = (x + iy, s + i\phi(x, y, s))$ and $Z(\Sigma)$ is given by $\{(x + i0, s + i\phi(x, 0, s))\}$. Hence, $\delta_\Sigma(q) = |y|$ and

$$d_{\Sigma,Z}(q) \simeq |y| + |\phi(x, y, s) - \phi(x, 0, s)| = |y| + O(|y|^2)$$

where we have used (4.1) in the second equality. This shows what we wanted for this special choice of Z and the case of a general first integral $Z^\#$ follows from (2.4)

(3) $\Longrightarrow$ (1). We will show that if (1) does not hold then (3) does not hold either. Let Σ be a maximally real submanifold that is not CR on any neighborhood of p. We may choose local coordinates defined on an open neighborhood U of p

$$\{x_1, \ldots, x_\nu, y_1, \ldots, y_\nu, s_1, \ldots, s_d, t_1, \ldots, t_{n'}\}$$

with the properties described in Theorem 4.1 such that Σ is given by the equations $y = 0$, $t = 0$. Notice that $n' \geq 1$ because Σ is not CR on U. For $q = (x, y, s, t)$, $\delta_\Sigma(q) = (|y|^2 + |t|^2)^{1/2}$ while

$$d_{\Sigma,Z}(q) \simeq |y| + |\phi(x, y, s, t) - \phi(x, 0, s, 0)| \simeq |y| + O(|y|^2 + |t|^2).$$

Taking a sequence of points $q_k = (0, 0, 0, t_k) \in U \setminus \Sigma$ with $t_k \to 0$, we see that $d_{\Sigma,Z}(q_k)/\delta_\Sigma(q_k) \to 0$ as $k \to \infty$ so $\mathbf{d}_{\Sigma,Z} \not\sim \boldsymbol{\delta}_\Sigma$. Invoking (2.4) we have as well that $\mathbf{d}_{\Sigma,Z^\#} \not\sim \boldsymbol{\delta}_\Sigma$ for any other first integral $Z^\#$. $\square$

References

[BER] M. S. Baouendi, P. Ebenfelt, and L. P. Rothschild, *Real Submanifolds in Complex Space and their Mappings,* Princeton Math. Ser. **47**(1999), Princeton Univ. Press.

[BT] M. S. Baouendi and F. Treves, *A property of the functions and distributions annihilated by a locally integrable system of complex vector fields,* Ann. of Math. **113** (1981), 387–421.

[B] E. Bär, *Um teorema de F. e M. Riesz para um sistema de campos vetoriais de co-posto um*, Tese de doutorado, UFScar, São Carlos, 2008 (in Portuguese).

[BCH] S. Berhanu, P. Cordaro and J. Hounie, *An Introduction to Involutive Structures*, Cambridge University Press, 2008.

[BH1] S. Berhanu and J. Hounie, *An F. and M. Riesz theorem for planar vector fields,* Math. Ann. **320** (2001), 463–485

[BH2] S. Berhanu and J. Hounie, *On boundary properties of solutions of complex vector fields*, J. Funct. Anal. **192** (2002), 446–490.

[BH3] S. Berhanu and J. Hounie,*Traces and F. and M. Riesz theorem for planar vector fields*, Annales de l'institut Fourier **53** (2003), 1425–1460.

[EG1] M. Eastwood and R. Graham, *Some involutive structures in analysis and geometry*, Trends Math., Birkhäuser, (1997), 25–38.

[EG2] M. Eastwood and R. Graham, *The involutive structure on the blow-up of* $\mathbb{R}^n$ *in* $\mathbb{C}^n$, Comm. Anal. Geom. **7**, (1999), 609–622.

[HJ] N. Hanges and H. Jacobowitz, *Involutive structures on compact manifolds,* Amer. Jour. Math. **177**, (1995),491–522.

[H1] L. Hörmander, *The Analysis of linear partial differential operators I,* Second edition, Springer-Verlag, (1990).

[S] R. Szoke, *Involutive structures on the tangent bundle of symmetric spaces*, Math. Ann. **319**, (2001), 319–348.

[T] F. Treves, *Hypo-analytic structures,* Princeton University Press, (1992).

Departamento de Matemática, Universidade Federal de São Carlos, São Carlos, SP, 13565-905, Brasil

E-mail address: `hounie@dm.ufscar.br`

ICMC, Universidade de São Paulo, São Carlos, SP, 13560-970, Brasil

E-mail address: `evandro@icmc.usp.br`