EQUISINGULARITY OF FAMILIES OF MAP GERMS BETWEEN CURVES

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Abstract. Given a finite map germ $f : (X, 0) \to (Y, 0)$ between complex analytic reduced space curves, we look at invariants which control the topological triviality and the Whitney equisingularity in families of this type of map germs. In the case that $(Y, 0)$ is smooth, the main invariant is the Milnor number of a function on a curve. As an application, we deduce some applications to the equisingularity of families of finitely determined map germs $f : (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$ and $f : (\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0)$.

1. Introduction

Given a holomorphic finite function germ $f : (X, 0) \to (\mathbb{C}, 0)$ on a complex analytic reduced space curve $(X, 0)$, we showed in [16] that

$$\mu(f) = \mu(X, 0) + \deg(f) - 1,$$

where $\mu(X, 0)$ is the Milnor number of the curve, as defined by Buchweitz and Greuel [3], $\deg(f)$ is the degree of f (that is, the number of inverse images of a generic value), and $\mu(f)$ is the Milnor number of f, introduced by Goryunov [8] for curves in $(\mathbb{C}^3, 0)$ and by Mond and van Straten [15] for the general case of curves in $(\mathbb{C}^n, 0)$. As a consequence, we deduce that a 1-parameter family $f_t : (X_t, 0) \to (\mathbb{C}, 0)$ is topologically trivial if and only if the Milnor number $\mu(f_t)$ is constant and the family is Whitney equisingular if and only if $\mu(f_t)$ and the Hilbert-Samuel multiplicity $m_0(X_t, 0)$ are both constant.

The aim of this paper is to generalize these results to the case of a holomorphic finite and surjective map germ $f : (X, 0) \to (Y, 0)$ where now both $(X, 0)$ and $(Y, 0)$ are space curves with isolated singularity. The main difficulty is that there is no good invariant which substitutes the Milnor number $\mu(f)$ in this case. Instead of this, we will consider separately the Milnor numbers $\mu(X, 0)$, $\mu(Y, 0)$ and the degree $\deg(f)$ (which is well defined on each branch of $(Y, 0)$). We will show that a 1-parameter family $f_t : (X_t, 0) \to (Y_t, 0)$ is topologically trivial if and only if these three invariants are constant. Moreover, the family is Whitney equisingular if and only if, in addition we have also the constancy of the multiplicities $m_0(X_t, 0)$ and $m_0(Y_t, 0)$. In some particular cases, these results can be improved, for instance, when $(X, 0)$ and $(Y, 0)$ are plane curves or when f has degree 1.

In the last part of the paper we give some applications for equisingularity of families of finitely determined map germs. We deduce a theorem of

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Gaffney [5] that a family $f_t : (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$ is Whitney equisingular if and only if $\mu(\Delta_t)$ is constant, where Δ_t is the germ of the discriminant of f_t (i.e., the image of the singular set). In the case of a family $f_t : (\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0)$, we give a simpler proof of a theorem due to Bobadilla and Pe-Pereira [4] (see also [1]): f_t is topologically trivial if and only if $\mu(D^2)$ is constant, where now D^2 is the germ of the double point curve of f_t in $\mathbb{C}^2$.

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2. Equisingularity of families of maps on space curves

Let $f : (X, 0) \to (Y, 0)$ be any holomorphic map germ between complex analytic set germs. We consider $\alpha : (\mathcal{X}, 0) \to (\mathbb{C}, 0)$ and $\beta : (\mathcal{Y}, 0) \to (\mathbb{C}, 0)$ (flat) deformations of $(X, 0)$ and $(Y, 0)$ respectively. We denote the fibres by $X_t = \alpha^{-1}(t)$ and $Y_t = \beta^{-1}(t)$.

Definition 2.1. An *unfolding* of f is a map germ $F : (\mathcal{X}, 0) \to (\mathcal{Y}, 0)$ such that $\beta \circ F = \alpha$ and $f_0 = f$, where f_t denotes the restriction $F|_{X_t} : X_t \to Y_t$.

In the case $X = \mathbb{C}^n$ and $Y = \mathbb{C}^p$, we consider the trivial deformations $\alpha : (\mathbb{C} \times \mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ and $\beta : (\mathbb{C} \times \mathbb{C}^p, 0) \to (\mathbb{C}, 0)$ given by $\alpha(t, x) = t$ and $\beta(t, y) = t$. Then, an unfolding of a map germ $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$ is a map germ $F : (\mathbb{C} \times \mathbb{C}^n, 0) \to (\mathbb{C} \times \mathbb{C}^p, 0)$ of the form $F(t, x) = (t, f_t(x))$ and such that $f_0 = f$. Hence, our definition of unfolding coincides with the classical notion of 1-parameter unfolding in the Thom-Mather sense.

We assume that the deformations $\alpha : (\mathcal{X}, 0) \to (\mathbb{C}, 0)$ and $\beta : (\mathcal{Y}, 0) \to (\mathbb{C}, 0)$ admit sections $\sigma : (\mathbb{C}, 0) \to (\mathcal{X}, 0)$ and $\tau : (\mathbb{C}, 0) \to (\mathcal{Y}, 0)$. By abuse of notation, we denote by $0 = \sigma(t) \in X_t$ and $0 = \tau(t) \in Y_t$ the selected base points of each space. We will say that F is origin preserving if $f_t(0) = 0$ for any t, so that we have an induced family of map germs $f_t : (X_t, 0) \to (Y_t, 0)$. We will assume that all the unfoldings are origin preserving, unless otherwise specified.

We recall now the notion of regular stratification. We refer to [7] for details. Let $f : \mathbb{C}^n \to \mathbb{C}^s$ be a complex analytic map and let $A \subset \mathbb{C}^n$, $A' \subset \mathbb{C}^s$ be subsets such that $f(A) \subset A'$. A *stratification* of $f : A \to A'$ is a pair $(\mathcal{A}, \mathcal{A}')$ of stratifications of A, A' respectively such that f maps strata submersively to strata. The stratification $(\mathcal{A}, \mathcal{A}')$ is said to be *regular* if $\mathcal{A}, \mathcal{A}'$ satisfy the Whitney regularity conditions and if any stratum $Y \in \mathcal{A}$ satisfies Thom's condition A_f over any other stratum $X \in \mathcal{A}$.

Along this section, we consider $(X, 0) \subset (\mathbb{C}^n, 0)$ a germ of reduced space curve and $f : (X, 0) \to (\mathbb{C}^s, 0)$ a finite map germ on it. Let $(Y, 0) \subset (\mathbb{C}^s, 0)$ be the curve given by the image of f, which we assume irreducible. We denote the restriction by $f : (X, 0) \to (Y, 0)$. The degree of f is denoted also by $\deg(f)$ and is defined as the number of inverse images of a generic value $y \in Y$. This is well defined when $(Y, 0)$ is irreducible.

The notions of topologically trivial, good and Whitney equisingular unfolding of a finitely determined map germ $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$ were introduced by Gaffney in [5]. Here we adapt such definitions to the case of a map germ $f : (X, 0) \to (Y, 0)$ between curves with isolated singularity.

Definition 2.2. Let $F : (\mathcal{X}, 0) \to (\mathcal{Y}, 0)$ be an unfolding of f.

(1) We say that F is *topologically trivial* if there are homeomorphism map germs $\Phi : (\mathcal{X}, 0) \to (\mathbb{C} \times X, 0)$ and $\Psi : (\mathcal{Y}, 0) \to (\mathbb{C} \times Y, 0)$ which are unfoldings of the identity in $(X, 0)$ and $(Y, 0)$ respectively and such that $\Psi \circ F \circ \Phi^{-1} = \mathrm{id} \times f$.
(2) We say that F is *good* if there are representatives $F : \mathcal{X} \to \mathcal{Y}$, $\alpha : \mathcal{X} \to D$ and $\beta : \mathcal{Y} \to D$, where $D \subset \mathbb{C}$ is an open neighbourhood of the origin such that $X_t \setminus \{0\}$, $Y_t \setminus \{0\}$ are smooth, f_t is regular on $X_t \setminus \{0\}$ and $f_t^{-1}(0) = \{0\}$, for any $t \in D$.
(3) If F is good, we can consider the stratification $(\mathcal{A}, \mathcal{A}')$ given by

$$\mathcal{A} = \{\mathcal{X} \setminus S, S\}, \quad \mathcal{A}' = \{F(\mathcal{X}) \setminus T, T\},$$

where $S = \sigma(D) \subset \mathcal{X}$ and $T = \tau(D) \subset \mathcal{Y}$ are the parameter axes. We say that F is *Whitney equisingular* if $(\mathcal{A}, \mathcal{A}')$ is regular.

Again, by using an appropriate version of Thom's second isotopy lemma for complex analytic maps, it follows that any Whitney equisingular unfolding is topologically trivial (see [5] for details).

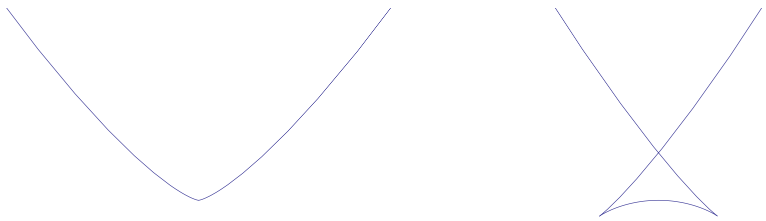
Example 2.3. Consider $(X, 0) = (\mathbb{C}, 0)$, $(Y, 0)$ the E_6 plane curve singularity in $\mathbb{C}^2$ given by $x^4 - y^3 = 0$ and $f : (\mathbb{C}, 0) \to (Y, 0)$ the map $f(s) = (s^3, s^4)$. For $(X, 0)$ we take the trivial deformation $\alpha : (\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$ given by $\alpha(t, s) = t$. The deformation of $(Y, 0)$ is $\beta : (\mathcal{Y}, 0) \to (\mathbb{C}, 0)$, where $(\mathcal{Y}, 0)$ is the swallowtail surface in $\mathbb{C}^3$ defined by

$$3t^4y - 2t^3x^2 - 18t^2y^2 + 54tx^2y - 27x^4 + 27y^3 = 0,$$

and $\beta(t, x, y) = t$. Finally, we take the unfolding $F : (\mathbb{C}^2, 0) \to (\mathcal{Y}, 0)$ given by

$$F(t, s) = (t, s^3 + ts, s^4 + \frac{2}{3}ts^2).$$

We have that F is not good since for any representative near the origin, Y_t has three singularities for any $t \neq 0$ (see figure 1).



$t = 0$ $\qquad$ $t \neq 0$

FIGURE 1

The following lemma is known as the universal property of the normalization. We present here an adapted version for analytic set germs.

Lemma 2.4. *Let $(Y, 0)$ be an irreducible germ and $n : (\bar{Y}, 0) \to (Y, 0)$ be a normalization. Let $(\bar{X}, \bar{0})$ be a normal multi-germ and $\varphi : (\bar{X}, \bar{0}) \to (Y, 0)$ be a map of multi-germs surjective on each irreducible component of $(\bar{X}, \bar{0})$.*

Then there exists a uniquely determined map germ $\bar{\varphi} : (\bar{X}, \bar{0}) \to (\bar{Y}, 0)$ *making the following diagram commutative.*

$$\begin{array}{ccc} & & (\bar{Y},0) \\ & \overset{\bar{\varphi}}{\nearrow} & \downarrow n \\ (\bar{X},\bar{0}) & \xrightarrow{\varphi} & (Y,0) \end{array}$$

Proof. Let us denote $(\bar{X}, \bar{0}) = (X_1, 0) \sqcup \cdots \sqcup (X_r, 0)$. Then it is enough to show the lemma on each irreducible component $(X_i, 0)$. Thus, we assume $r = 1$.

We denote by $R = \mathcal{O}_{Y,0}$, $S = \mathcal{O}_{X_1,0}$, and their corresponding total rings of fractions by $Q(R)$ and $Q(S)$ respectively. Since φ is surjective, the induced morphism $\varphi^* : R \to S$ is injective. Moreover, S is an integral domain because it is normal, local and Noetherian. Thus, φ^* extends in the obvious way to a monomorphism between the total ring of fractions $\widetilde{\varphi}^* : Q(R) \to Q(S)$.

On the other hand, $\bar{R} = \mathcal{O}_{\bar{Y},0}$ is the integral closure of R in $Q(R)$ and $\bar{S} = S$, since it is normal. Thus, $\widetilde{\varphi}^*(\bar{R}) \subset S$ and we can consider the restriction $\widetilde{\varphi}^*|_{\bar{R}} : \bar{R} \to S$. Finally, the map germ $\bar{\varphi} : (X_1, 0) \to (\bar{Y}, 0)$ is defined as the only map germ whose induced morphism of local rings is $\widetilde{\varphi}^*|_{\bar{R}} : \bar{R} \to S$. The unicity of $\bar{\varphi}$ is obvious. □

Theorem 2.5. *Let* $F : (\mathcal{X}, 0) \to (\mathcal{Y}, 0)$ *be a good unfolding of a finite map germ* $f : (X, 0) \to (Y, 0)$ *between curves with* $(Y, 0)$ *irreducible.*

1. *F is topologically trivial if and only if* $\mu(X_t, 0)$, $\mu(Y_t, 0)$ *and* $\deg(f_t, 0)$ *are constant.*
2. *F is Whitney equisingular if and only if* $\mu(X_t, 0)$, $\mu(Y_t, 0)$, $\deg(f_t, 0)$ *and the multiplicities* $m_0(X_t, 0)$, $m_0(Y_t, 0)$ *are constant.*

Proof. If F is topologically trivial, then $(\mathcal{X}, 0)$ and $(\mathcal{Y}, 0)$ are topologically trivial families of space curves and hence, $\mu(X_t, 0)$ and $\mu(Y_t, 0)$ are constant by [3]. Moreover, there are homeomorphism map germs $\Phi : (\mathcal{X}, 0) \to (\mathbb{C} \times X, 0)$ and $\Psi : (\mathcal{Y}, 0) \to (\mathbb{C} \times Y, 0)$ which are unfoldings of the identity in $(X, 0)$ and $(Y, 0)$ respectively and such that $\Psi \circ F \circ \Phi^{-1} = \mathrm{id} \times f$. For any t, this means that $f = \psi_t \circ f \circ \phi_t^{-1}$, which implies $\deg(f_t, 0) = \deg(f, 0)$.

Assume now that $\mu(X_t, 0)$, $\mu(Y_t, 0)$ and $\deg(f_t, 0)$ are constant. Therefore the families $(X_t, 0)$ and $(Y_t, 0)$ are also δ-constant and they admit a normalization in family (see [3] and [17]). This means that there are map germs $n : (\mathbb{C} \times \bar{X}, \bar{0}) \to (\mathcal{X}, 0)$ and $\eta : (\mathbb{C} \times \bar{Y}, \bar{0}) \to (\mathcal{Y}, 0)$ such that $n_t : (\bar{X}, \bar{0}) \to (X_t, 0)$ given by $n_t(x) = n(t, x)$ and $\eta_t : (\bar{Y}, \bar{0}) \to (Y_t, 0)$ given by $\eta_t(y) = \eta(t, y)$ are the normalizations of $(X_t, 0)$ and $(Y_t, 0)$ respectively. Since $(Y, 0)$ is irreducible, we can assume that $\bar{Y} = \mathbb{C}$.

By lemma 2.4 there exists a unique map $\bar{F} : (\mathbb{C} \times \bar{X}, \bar{0}) \to (\mathbb{C} \times \mathbb{C}, 0)$ with $\bar{F}(t, x) = (t, \bar{f}_t(x))$ such that $f_t \circ n_t = \eta_t \circ \bar{f}_t$. We have that $\bar{F}$ is an unfolding of a function multigerm $\bar{f}_0 : (\bar{X}, \bar{0}) \to (\mathbb{C}, 0)$, whose domain $(\bar{X}, \bar{0})$ is smooth and with constant degree on each component, since $\deg(\bar{f}_t) = \deg(f_t)$. It is well known that such a family is analytically trivial, that is, there is an analytical isomorphism multigerm $\bar{\Phi} : (\mathbb{C} \times \bar{X}, \bar{0}) \to (\mathbb{C} \times \bar{X}, \bar{0})$, defined by $\bar{\Phi}(t, x) = (t, \bar{\phi}_t(x))$ such that $\bar{f}_t \circ \bar{\phi}_t^{-1} = \bar{f}_0$. Then we have the following diagram

$$
\begin{array}{ccccccc}
 & & & & (\mathbb{C}\times\mathbb{C},0) & & \\
 & \nearrow^{\bar{F}} & & & \downarrow{\scriptstyle id\times\eta_t} & & \\
(\mathbb{C}\times\bar{X},\bar{0}) & \xrightarrow{n} & (\mathcal{X},0) & \xrightarrow{F} & (\mathcal{Y},0) & \xrightarrow{id\times\eta_t^{-1}} & (\mathbb{C}\times\mathbb{C},0) \\
\downarrow{\scriptstyle\bar{\Phi}} & & \downarrow{\scriptstyle\Phi} & & \downarrow{\scriptstyle\Psi} & \swarrow_{id\times\eta_0} & \\
(\mathbb{C}\times\bar{X},\bar{0}) & \xrightarrow{id\times n_0} & (\mathbb{C}\times X,0) & \xrightarrow{id\times f} & (\mathbb{C}\times Y,0) & &
\end{array}
$$

Now let us consider $\Phi : (\mathcal{X},0) \to (\mathbb{C}\times X,0)$, the unfolding of the identity given by $\phi_t = n_0 \circ \bar{\phi}_t \circ n_t^{-1}$ and $\Psi : (\mathcal{Y},0) \to (\mathbb{C}\times Y,0)$ the unfolding of the identity given by $\psi_t = \eta_0 \circ \eta_t^{-1}$. We have that Φ and Ψ are homeomorphisms germs which trivialize F. In fact,

$$
\begin{aligned}
\psi_t \circ f_t \circ \phi_t^{-1} &= \psi_t \circ f_t \circ n_t \circ \bar{\phi}_t^{-1} \circ n_0^{-1} \\
&= \eta_0 \circ \eta_t^{-1} \circ \eta_t \circ \bar{f}_t \circ \bar{\phi}_t^{-1} \circ n_0^{-1} \\
&= \eta_0 \circ \bar{f}_0 \circ n_0^{-1} = f_0 \circ n_0 \circ n_0^{-1} = f.
\end{aligned}
$$

We finish by showing part (2). If F is Whitney equisingular, then it is topologically trivial and hence $\mu(X_t,0)$, $\mu(Y_t,0)$ and $\deg(f_t,0)$ are constant, by (1). Moreover, $(\mathcal{X},0)$ and $(\mathcal{Y},0)$ become Whitney equisingular families of space curves, which imply that $m_0(X_t,0)$ and $m_0(Y_t,0)$ are also constant by [2].

Conversely, assume that $\mu(X_t,0)$, $\mu(Y_t,0)$, $\deg(f_t,0)$ and the multiplicities $m_0(X_t,0)$, $m_0(Y_t,0)$ are constant. Since that F is good there are representative $F : \mathcal{X} \to \mathcal{Y}$, $\alpha : \mathcal{X} \to D$ and $\beta : \mathcal{Y} \to D$, where $D \subset \mathbb{C}$ is an open neighbourhood of the origin, such that $X_t \setminus \{0\}$ and $Y_t \setminus \{0\}$ are smooth and f_t is regular on $X_t \setminus \{0\}$ and $f_t^{-1}(0) = \{0\}$, for any $t \in D$.

We have now a stratification $(\mathcal{A},\mathcal{A}')$ of F given by

$$\mathcal{A} = \{\mathcal{X} \setminus S, S\}, \quad \mathcal{A}' = \{F(\mathcal{X}) \setminus T, T\}.$$

Since $\mu(X_t,0)$ and $m_0(X_t,0)$ are constant, $\mathcal{A}$ is Whitney regular by [2]. The same argument applies to $\mathcal{A}'$. Moreover, F restricted to each stratum is a biholomorphism, so that the Thom A_F condition is also satisfied trivially. □

Remark 2.6. In the case that $(Y,0)$ is not irreducible, then we have to consider the degree of f on each irreducible component of $(Y,0)$. Then the statements of theorem 2.5 are still true if we assume that the degree of f_t is constant on each irreducible component.

It is well known that for families of plane curves the condition that the Milnor number is constant implies that the multiplicity is also constant [19]. Hence, we have the following immediate consequence of theorem 2.5.

Corollary 2.7. *Let $F : (\mathcal{X},0) \to (\mathcal{Y},0)$ be a good unfolding of a finite map germ $f : (X,0) \to (Y,0)$ between plane curves. Then, the following statements are equivalent:*

(1) *F is topologically trivial;*
(2) *F is Whitney equisingular.*

Another interesting particular case is when f has degree 1.

Corollary 2.8. *Let $F : (\mathcal{X}, 0) \to (\mathcal{Y}, 0)$ be a good unfolding of a finite map germ $f : (X, 0) \to (Y, 0)$ of degree 1 between curves with $(Y, 0)$ irreducible. Then:*

1. *F is topologically trivial if and only if $\mu(Y_t, 0)$ is constant.*
2. *F is Whitney equisingular if and only if $\mu(Y_t, 0)$, $m_0(Y_t, 0)$ and $m_0(X_t, 0)$ are constant.*

Proof. Since F is good, there is a representative $F : \mathcal{X} \to \mathcal{Y}$ which is finite, surjective and proper and such that $X_t \setminus \{0\}$ and $Y_t \setminus \{0\}$ are smooth, $f_t^{-1}(0) = \{0\}$ and f_t is regular on $X_t \setminus \{0\}$, for any $t \in D$.

On the other hand, f has degree 1, which implies that $f_t : X_t \to Y_t$ is generically 1-1. But $Y_t \setminus \{0\}$ is smooth, thus $f_t : X_t \to Y_t$ is in fact 1-1 and hence, a homeomorphism. By [3, 4.2.2], we have:

$$\mu(X, 0) - \mu(X_t, 0) = \dim_{\mathbb{C}} H^1(X_t, \mathbb{C}) = \dim_{\mathbb{C}} H^1(Y_t, \mathbb{C}) = \mu(Y, 0) - \mu(Y_t, 0).$$

We deduce that $\mu(X_t, 0)$ is constant if and only if $\mu(Y_t, 0)$ is constant. The results follow now from theorem 2.5. □

In the next theorem, we introduce a new invariant that we call $\delta(f)$, which measures the difference between the Milnor numbers of the two curves when f has degree 1.

Theorem 2.9. *Let $f : (X, 0) \to (Y, 0)$ be a finite map germ of degree 1 between irreducible curves. Then*

$$\mu(Y, 0) = \mu(X, 0) + 2\delta(f),$$

where $\delta(f) = \dim_{\mathbb{C}} \mathcal{O}_{X,0}/f^*(\mathcal{O}_{Y,0})$.

Proof. Let $n : (Z, 0) \to (X, 0)$ be a normalization of $(X, 0)$, then

$$f \circ n : (Z, 0) \to (Y, 0)$$

is a normalization of $(Y, 0)$, since f is generically 1-1 map. Then

$$\delta(X, 0) = \dim_{\mathbb{C}} \mathcal{O}_{Z,0}/n^*(\mathcal{O}_{X,0}),$$
$$\delta(Y, 0) = \dim_{\mathbb{C}} \mathcal{O}_{Z,0}/(f \circ n)^*(\mathcal{O}_{Y,0}).$$

Since n^* and f^* are both monomorphisms we have that

$$\delta(Y, 0) = \delta(X, 0) + \delta(f)$$

and by Milnor's formula $\mu(X, 0) = 2\delta(X, 0) - r + 1$ and $\mu(Y, 0) = 2\delta(Y, 0) - r + 1$, we obtain $\mu(Y, 0) = \mu(X, 0) + 2\delta(f)$. □

Mond and Pellikaan in [14, 3.6] give the following result, which applies in the particular case that $(Y, 0)$ is a plane curve.

Theorem 2.10. *Let $(X, 0)$ be a reduced curve germ and $f : (X, 0) \to (\mathbb{C}^2, 0)$ be finite and of degree 1 onto its image $(Y, 0)$. Then*

$$\mu(Y, 0) = \mu(X, 0) + 2 \dim_{\mathbb{C}} \mathcal{O}_2/\mathcal{F}_1(f),$$

where $\mathcal{F}_1(f)$ is the first Fitting ideal of f.

We observe that in fact the theorem above appears in [14] in the form

$$\delta(Y,0) = \delta(X,0) + \dim_{\mathbb{C}} \mathcal{O}_2/\mathcal{F}_1(f)$$

and we have to use again Milnor's formula $\mu(X,0) = 2\delta(X,0) - r + 1$, where r is the number of branches of $(X,0)$.

We recall that the Milnor number of a curve is upper semi-continuous [3] and the same is true for $\dim_{\mathbb{C}} \mathcal{O}_2/\mathcal{F}_1(f)$, when $(Y,0)$ is a plane curve. In fact, given $F : (\mathcal{X},0) \to (\mathcal{Y},0)$ an unfolding of f, we have that $(\mathcal{X},0)$ is Cohen-Macaulay by [9, 23.3] and hence $\dim_{\mathbb{C}} \mathcal{O}_3/\mathcal{F}_1(F)$ is also Cohen-Macaulay by [14, 3.4]. Hence, for $t \in \mathbb{C}$ near the origin,

$$\dim_{\mathbb{C}} \frac{\mathcal{O}_2}{\mathcal{F}_1(f)} = \sum_{y \in V(\mathcal{F}_1(f_t))} \frac{\mathcal{O}_{2,y}}{\mathcal{F}_1(f_t)} \geq \dim_{\mathbb{C}} \frac{\mathcal{O}_2}{\mathcal{F}_1(f_t)}.$$

Corollary 2.11. *Let $(X,0) \subset (\mathbb{C}^n,0)$ be a reduced curve germ and $(\mathcal{X},0)$ be a deformation of $(X,0)$. If there exists $p_2 : \mathbb{C}^n \to \mathbb{C}^2$ a generic linear projection such that $\mu(p_2(X_t),0)$ is constant, then $(\mathcal{X},0)$ is Whitney equisingular.*

Proof. The genericity condition on p_2 means that $m_0(p_2(X),0) = m_0(X,0)$ and $\mu(p_2(X),0) = \inf_p \mu(p(X),0)$, where $p : \mathbb{C}^n \to \mathbb{C}^2$ is any linear projection. If $\mu(p_2(X_t),0)$ is constant, by theorem 2.10 and by the upper semi-continuity of the invariants, then $\mu(X_t,0)$ and is also constant. On the other hand, $m_0(p_2(X_t),0) = m_0(X_t,0)$ is also constant, since $(p_2(X_t),0)$ is a family of plane curves with constant Milnor numbers. Hence, $(\mathcal{X},0)$ is Whitney equisingular. ☐

The following example shows that the converse of corollary 2.11 is not true in general.

Example 2.12. Let $(X,0) \subset (\mathbb{C}^4,0)$ be the space curve germ parametrized by

$$s \mapsto (s^4, s^7, s^9, s^{10})$$

and consider $(\mathcal{X},0)$ the deformation of $(X,0)$ parametrized by

$$(t,s) \mapsto (s^4, s^7 + ts^6, s^9, s^{10}).$$

We see that $\mu(X_t,0) = 12$, $m_0(X_t,0) = 4$ for all t, which implies that $(\mathcal{X},0)$ is Whitney equisingular. However, given a generic linear projection

$$\begin{aligned} & p_2(s^4, s^7 + ts^6, s^9, s^{10}) \\ &= (a_1 s^4 + a_2(s^7 + ts^6) + a_3 s^9 + a_4 s^{10}, b_1 s^4 + b_2(s^7 + ts^6) + b_3 s^9 + b_4 s^{10}), \end{aligned}$$

with $a_1 \neq 0$ or $b_1 \neq 0$ then $m_0(p_2(X_t),0) = m_0(X_t,0) = 4$. But if $t \neq 0$, then $\mu(p_2(X_t),0) = 16$ and $\mu(p_2(X),0) = 18$.

3. Equisingularity of map germs from $\mathbb{C}^2$ to $\mathbb{C}^2$

Consider a finite map germ $f : (\mathbb{C}^2,0) \to (\mathbb{C}^2,0)$. By the Mather-Gaffney criterion [18], f is finitely determined if and only if there is a finite representative $f : U \to V$, where $U, V \subset \mathbb{C}^2$ are open neighbourhoods of the origin, such that $f^{-1}(0) = \{0\}$ and the restriction $f : U \setminus \{0\} \to V \setminus \{0\}$ is stable. This means that the only singularities of f on $U \setminus \{0\}$ are folds, cusps and transverse double folds (i.e., transverse double points of the restriction of f to its singular set). In figure 2 we present the map $f : \mathbb{C}^2 \to \mathbb{C}^2$ defined by $f(x, y) = (x, xy + y^4 - y^2/2)$. This is a stable map with two cusps and one double fold and it is a stabilization of the unstable singularity $f_0(x, y) = (x, xy + y^4)$.

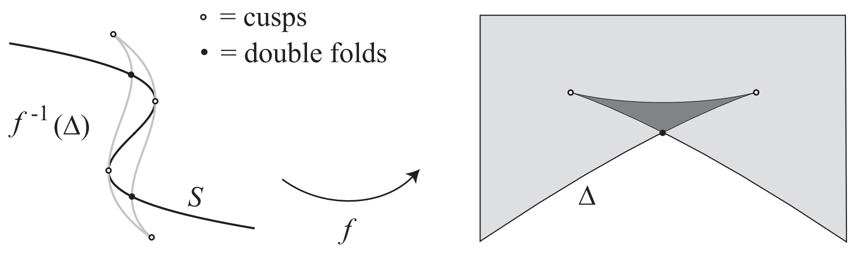


FIGURE 2

We denote by $S = V(J)$ the germ of the singular curve of f, which is defined by the vanishing of the jacobian determinant J of f. We also denote by $\Delta = f(S)$ the germ of the discriminant curve. The following criterion for finite determinacy is due to Gaffney-Mond [6]: *f is finitely determined if and only if S is reduced and the restriction $f : S \to \Delta$ is generically* $1-1$. Other important results of [6] which will be useful are:

Lemma 3.1. *Let $f : (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$ be a finitely determined map germ. Then:*

(1) *$\mu(\Delta) = \mu(S) + 2C + 2D$, where C, D are the numbers of cusps and transverse double folds respectively that appear in a stable deformation of f near the origin.*
(2) *$C = \mu(S) + d - 2$, where $d = \deg(f)$ is the degree of f.*

Notice that the three invariants C, D, d can be computed algebraically in terms of $f = (p, q)$ by means of the following formulas:

$$C = \dim_{\mathbb{C}} \frac{\mathcal{O}_2}{\langle J, p_x J_y - p_y J_x, q_x J_y - q_y J_x \rangle},$$
$$d = \dim_{\mathbb{C}} \frac{\mathcal{O}_2}{\langle p, q \rangle}, \quad C + D = \dim_{\mathbb{C}} \frac{\mathcal{O}_2}{\mathcal{F}_1(f_* \mathcal{O}_S)},$$

where the subscripts denote the partial derivatives and $\mathcal{F}_1(f_* \mathcal{O}_S)$ is the first Fitting ideal of the push-forward $f_* \mathcal{O}_S$ (see [6] for details). We just remark that, after comparing (1) of lemma 3.1 with our theorem 2.9, we deduce that $\delta(f|_S) = C + D$.

Since we are going to apply our theorem to the restriction $f : f^{-1}(\Delta) \to \Delta$, we need one more formula which relates the Milnor number of $f^{-1}(\Delta)$ to these invariants.

Lemma 3.2. *For a finitely determined map germ $f : (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$ of degree d, we have*

$$(d - 1)\mu(\Delta) = \mu(f^{-1}(\Delta)) + d - 2.$$

Proof. We consider $F : (\mathbb{C} \times \mathbb{C}^2, 0) \to (\mathbb{C} \times \mathbb{C}^2, 0)$ a 1-parameter stable unfolding of f, that is, $F(s, x) = (s, f_s(x))$ with $f_0 = f$ and such that f_s is stable for $s \neq 0$. We fix a small enough representative $F : B \times U \to V$ such that C, D are the numbers of cusps and transverse double folds respectively of f_s on U for any $s \in B \setminus \{0\}$.

On the other hand, we denote by Δ_F the discriminant of F and by Δ_s the discriminant of each f_s. Note that Δ_F and $F^{-1}(\Delta_F)$ are surfaces in $(\mathbb{C}^3, 0)$ which provide deformations of the plane curves Δ_0 and $f_0^{-1}(\Delta_0)$ respectively. We can also shrink the neighbourhoods, if necessary, so that we have the following relations between the Milnor numbers:

$$\mu(\Delta_0) - \mu(\Delta_s) = 1 - \chi(\Delta_s),$$
$$\mu(f_0^{-1}(\Delta_0)) - \mu(f_s^{-1}(\Delta_s)) = 1 - \chi(f_s^{-1}(\Delta_s)),$$

where $\mu(X)$ denotes now the sum of the Milnor numbers of all the singular points of X and $\chi(X)$ is the Euler characteristic of X (see [3]).

The singularities of Δ_s are simple cusps and nodes, corresponding to the cusps and transverse double points respectively of f_s. Since the simple cusp and the node have Milnor numbers 2 and 1 respectively, we get $\mu(\Delta_s) = 2C + D$. Analogously, the singularities of $f_s^{-1}(\Delta_s)$ can be either a simple cusp, if it is a regular inverse image of a cusp, or a singularity of type A_3 if it is a singular inverse image of a cusp, or a node, if it is an inverse image of a transverse double point (both regular or singular). The Milnor numbers of these type of singularities are 2, 3 or 1 respectively and each cusp has $d - 3$ regular inverse images and 1 singular inverse image, and each transverse double point has $d - 2$ inverse images. Hence, we have $\mu(f_s^{-1}(\Delta_s)) = 2(d-3)C + 3C + (d-2)D$.

We compute now the Euler characteristics $\chi(\Delta_s)$ and $\chi(f_s^{-1}(\Delta_s))$. We denote by $\Sigma_s \subset \Delta_s$ the set of cusps and transverse double folds of f_s and we put $\Delta_s^0 = \Delta_s \setminus \Sigma_s$. Then,

$$\chi(\Delta_s) = \chi(\Delta_s^0) + C + D,$$
$$\chi(f_s^{-1}(\Delta_s)) = \chi(f_s^{-1}(\Delta_s^0)) + (d-2)(C + D),$$

since each cusp or transverse double fold has $d-2$ inverse images. Moreover, the restriction $f_s : f_s^{-1}(\Delta_s^0) \to \Delta_s^0$ is now a covering of $d - 1$ sheets, which gives $\chi(f_s^{-1}(\Delta_s^0)) = (d-1)\chi(\Delta_s^0)$. Hence,

$$(d-1)\chi(\Delta_s) = \chi(f_s^{-1}(\Delta_s)) + C + D,$$

and we arrive to the desired relation between the Milnor numbers:

$$(d-1)\mu(\Delta) = \mu(f^{-1}(\Delta)) + d - 2.$$

□

We consider $F : (\mathbb{C} \times \mathbb{C}^2, 0) \to (\mathbb{C} \times \mathbb{C}^2, 0)$ a 1-parameter unfolding of f, that is, $F(t, x) = (t, f_t(x))$ with $f_0 = f$. We assume that F is origin preserving, that is, $f_t(0) = 0$, for any t. We denote by Δ_F the discriminant of F and by Δ_t the discriminant of each f_t. It is obvious that the restriction $F : F^{-1}(\Delta_F) \to \Delta_F$ is a 1-parameter unfolding of the restriction $f : f^{-1}(\Delta) \to \Delta$. We say that F is Whitney equisingular if there exists a representative $F : U \to V$ which admits a regular stratification and such that the parameter axes $S = U \cap (\mathbb{C} \times \{0\})$ and $T = V \cap (\mathbb{C} \times \{0\})$ are strata. By using our results on equisingularity of map germs between curves, we have a new proof of Gaffney's result [5, 9.9].

Theorem 3.3. *Let F be an unfolding of a finitely determined map germ $f : (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$. Then F is Whitney equisingular if and only if $\mu(\Delta_t)$ is constant.*

Proof. If F is Whitney equisingular, then it is topologically trivial by the second isotopy lemma. Since the homeomorphisms must preserve the discriminant, Δ_F is also topologically trivial. Hence, $\mu(\Delta_t)$ is constant.

Conversely, assume that $\mu(\Delta_t)$ is constant. All the invariants $\mu(S_t)$, C_t, D_t and d_t are upper semicontinuous (see [5]) and by lemma 3.1, d_t must be also constant. Then, by lemma 3.2 $\mu(f_t^{-1}(\Delta_t))$ is also constant and the restriction $F : F^{-1}(\Delta_F) \to \Delta_F$ is Whitney equisingular by corollary 2.7.

We fix a representative $F : U \to V$ where $U, V \subset \mathbb{C} \times \mathbb{C}^2$ are open neighbourhoods of the origin and such that $\mathcal{A} = \{F^{-1}(\Delta_F) \setminus S, S\}$ and $\mathcal{A}' = \{\Delta_F \setminus T, T\}$ provide a regular stratification of $F : F^{-1}(\Delta_F) \to \Delta_F$. We consider now

$$\mathcal{B} = \{U \setminus F^{-1}(\Delta_F), F^{-1}(\Delta_F) \setminus S, S\},$$
$$\mathcal{B}' = \{V \setminus \Delta_F, \Delta_F \setminus T, T\}.$$

Obviously, $(\mathcal{B}, \mathcal{B}')$ is a regular stratification of $F : U \to V$. □

We notice that the Whitney equisingularity of a family $f_t : (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$ can be also characterized by the constancy of the invariants C_t, D_t (see [6, 1.10]). In fact, by lemmas 3.1, 3.2 and by the upper semicontinuity of the invariants, if C_t, D_t are constant, then $\mu(\Delta_t)$ must be also constant.

4. Equisingularity of map germs from $\mathbb{C}^2$ to $\mathbb{C}^3$

Consider a finite map germ $f : (\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0)$. Again by the Mather-Gaffney criterion [18], f is finitely determined if and only if there is a finite representative $f : U \to V$, where $U \subset \mathbb{C}^2$, $V \subset \mathbb{C}^3$ are open neighbourhoods of the origin, such that $f^{-1}(0) = \{0\}$ and the restriction $f : U \setminus \{0\} \to V \setminus \{0\}$ is stable. This means that the only singularities of f on $U \setminus \{0\}$ are cross-caps or Whitney umbrellas, transverse double and triple points. We show in figure 3 a stabilization of the H_2 singularity which is a stable map from $\mathbb{C}^2$ to $\mathbb{C}^3$ with two cross-caps and one triple point [11].

The germ of the *double point curve* of f is defined as

$$D^2 = \{x \in \mathbb{C}^2 : f^{-1}(f(x)) \neq \{x\}\} \cup S,$$

where S is the germ of the singular set of f. If f is finite and generically 1-1, then D^2 is a closed analytic set germ of dimension 1 and it is possible to provide a convenient analytic structure on it (non-necessarily reduced) with the following property: *f is finitely determined if and only if D^2 is reduced* [10, 12]. The Milnor number $\mu(D^2)$ is called the *Mond number* of f. We also consider the germ of the lifting of the double point curve $\tilde{D}^2 \subset (\mathbb{C}^2 \times \mathbb{C}^2, 0)$ given by the pairs (x, y) such that either $f(x) = f(y)$ with $x \neq y$ or $x = y$, $x \in S$. We have the following relations between the Milnor numbers of the double point curves and the geometrical invariants [10, 12].

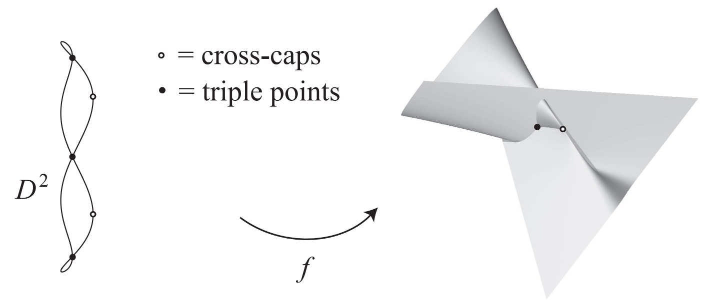


FIGURE 3

Lemma 4.1. *Let $f : (\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0)$ be a finitely determined map germ. Then:*

(1) $\mu(D^2) = \mu(\tilde{D}^2) + 6T$,
(2) $\mu(D^2) + 2T = 2\mu(f(D^2)) + C - 1$,

where C, T are the numbers of cross-caps and triple points respectively that appear in a stable deformation of f near the origin.

Again the geometrical invariants C, T can be computed algebraically in terms of f by means of the following formulas:

$$C = \dim_{\mathbb{C}} \frac{\mathcal{O}_2}{J}, \quad T = \dim_{\mathbb{C}} \frac{\mathcal{O}_3}{\mathcal{F}_2(f_*\mathcal{O}_2)},$$

where J is the ideal generated by the 2×2-minors of the jacobian matrix of f and $\mathcal{F}_2(f_*\mathcal{O}_2)$ is the second Fitting ideal of the push-forward $f_*\mathcal{O}_2$ (see [13, 14] for details).

We consider $F : (\mathbb{C} \times \mathbb{C}^2, 0) \to (\mathbb{C} \times \mathbb{C}^3, 0)$ a 1-parameter unfolding of f, that is, $F(t, x) = (t, f_t(x))$ with $f_0 = f$. We assume that F is origin preserving, that is, $f_t(0) = 0$, for any t. We denote by D^2_F the double point locus of F and by D^2_t the double point curve of each f_t. It is obvious that the restriction $F : D^2_F \to F(D^2_F)$ is a 1-parameter unfolding of the restriction $f : D^2 \to f(D^2)$. In this case, we deduce from our results on equisingularity of map germs between curves, a new proof of the following theorem due to Bobadilla and Pe-Pereira [4] (see also [1]).

Theorem 4.2. *Let F be an unfolding of a finitely determined map germ $f : (\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0)$. Then F is topologically trivial if and only if $\mu(D^2_t)$ is constant.*

Proof. Suppose F is topologically trivial. Since the homeomorphisms must preserve the double point curve, D^2_F is also a topologically trivial deformation of D^2. Hence, $\mu(D^2_t)$ is constant.

Conversely, assume that $\mu(D^2_t)$ is constant. All the invariants $\mu(f_t(D^2_t))$, C_t, and D_t are upper semicontinuous and by lemma 4.1, $\mu(f_t(D^2_t))$ must be also constant. Since the restriction $f_t : D^2_t \to f_t(D^2_t)$ has constant degree equal to 2, it is topologically trivial by theorem 2.5. In particular, there are homeomorphisms

$$\Phi : D^2_F \to \mathbb{C} \times D^2, \qquad \Phi(t,x) = (t, \phi_t(x)),$$
$$\Psi : F(D^2_F) \to \mathbb{C} \times f(D^2), \qquad \Psi(t,y) = (t, \psi_t(y)),$$

which are unfoldings of the identity and such that $\Psi \circ F \circ \Phi^{-1} = \mathrm{id} \times f$.

Note that D^2_F is a deformation of plane curves which is topologically trivial and hence, Whitney equisingular. Thus, the homeomorphism $\Phi : D^2_F \to \mathbb{C} \times D^2$ can be extended to an ambient homeomorphism $\tilde{\Phi} : (\mathbb{C} \times \mathbb{C}^2, 0) \to (\mathbb{C} \times \mathbb{C}^2, 0)$. Since F is a homeomorphism outside D^2_F, then $\tilde{\Phi}$ induces a homeomorphism $\hat{\Psi} : F(\mathbb{C}^3) \to \mathbb{C} \times f(\mathbb{C}^2)$ which is an extension of $\Psi : F(D^2_F) \to \mathbb{C} \times f(D^2)$. The last step is to extend $\hat{\Psi}$ to an ambient homeomorphism $\tilde{\Psi} : (\mathbb{C} \times \mathbb{C}^3, 0) \to (\mathbb{C} \times \mathbb{C}^3, 0)$. This can be done by using standard arguments of extension of stratified homeomorphisms by means of integration of vector fields [7]. □

References

[1] R. Calleja-Bedregal, K. Houston and M. A. S. Ruas, *Topological triviality of families of singular surfaces,* preprint, arXiv:math/0611699.

[2] J. Briançon, A. Galligo, M. Granger, *Déformations équisingulières des germes de courbes gauches réduites,* Mém. Soc. Math. France (N.S.) no. 1, 1980/81.

[3] R. O. Buchweitz, G. M. Greuel, *The Milnor number and deformations of complex curve singularities,* Invent. Math. 58 (1980), 241–281.

[4] J. Fernández de Bobadilla and M. Pe-Pereira, *Equisingularity at the normalisation*, J. Topol. 1 (2008), no. 4, 879–909.

[5] T. Gaffney, *Polar multiplicities and equisingularity of map germs*, Topology 32 (1993), no. 1, 185–223.

[6] T. Gaffney and D. Mond, *Cusps and double folds of germs of analytic maps* $\mathbb{C}^2 \to \mathbb{C}^2$, J. London Math. Soc. 42 (1991), 185–192.

[7] C. G. Gibson, K. Wirthmüller, A. A. du Plessis, E. J. N. Looijenga, *Topological stability of smooth mappings.* Lecture Notes in Mathematics, Vol. 552. Springer-Verlag, Berlin-New York, 1976.

[8] V. V. Goryunov, *Functions on space curves,* J. Lond. Math. Soc. (2) 61 (2000), no. 3, 807–822.

[9] H. Matsumura, *Commutative ring theory.* Cambridge Studies in Advanced Mathematics, 8. Cambridge University Press, Cambridge, 1986.

[10] W. L. Marar and D. Mond, *Multiple point schemes for corank* 1 *maps*, J. London Math. Soc. 39 (1989), 553–567.

[11] W. L. Marar and D. Mond, *Real map-germs with good perturbations*, Topology 35 (1996), no. 1, 157–165.

[12] W. L. Marar, J. J. Nuño-Ballesteros and G. Peñafort-Sanchis, *Double point curves for corank 2 map germs from* $\mathbb{C}^2$ *to* $\mathbb{C}^3$, preprint.

[13] D. Mond, *Some remarks on the geometry and classification of germs of maps from surfaces to 3-space*, Topology 26 (1987), 361–383.

[14] D. Mond, R. Pellikaan, *Fitting ideals and multiple points of analytic mappings.* Algebraic geometry and complex analysis, Pátzcuaro (1987), 107–161, Lecture Notes in Math., 1414.

[15] D. Mond, D. van Straten, *Milnor number equals Tjurina number for functions on space curves.* J. Lond. Math. Soc., II. Ser. 63, No.1, (2001) 177–187.

[16] J. J. Nuño-Ballesteros, J. N. Tomazella, *The Milnor number of a function on a space curve germ,* Bull. Lond. Math. Soc. 40 (2008), no.1, 129–138.

[17] B. Teissier, *The hunting of invariants in the geometry of discriminants.* Real and complex singularities, Oslo 1976, pp. 565–678. Sijthoff & Noordhoff International Publishers, 1977.

[18] C. T. C. Wall, *Finite determinacy of smooth map-germs,* Bull. London Math. Soc. 13 (1981), no. 6, 481–539.

[19] O. Zariski, *Studies in equisingularity. III. Saturation of local rings and equisingularity.* Amer. J. Math. 90 (1968), 961–1023.

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