Global analytic, Gevrey and C^{∞} hypoellipticity on the 3-torus

A. Alexandrou Himonas*
G. Petronilho and L. A. Carvalho dos Santos

Abstract

It is proved that a class of subLaplacians on the 3-dimensional torus is globally analytic, Gevrey and C^{∞} hypoelliptic if and only if either a Diophantine condition holds or there is a point of finite type for the vector fields defining the operator under consideration. This work is motivated by the desire of attaining a better understanding of necessary and sufficient conditions for the global Gevrey and C^{∞} hypoellipticity of a subLaplacian on a compact analytic manifold. This together with the more delicate local hypoellipticity are well known open problems in the theory of linear partial differential equations.

1 Introduction

Finding necessary and sufficient conditions for the local or global hypoellipticity of a subLaplacians is a well known open problem in the theory of linear partial differential equations. Here we shall focus our attention to the global problem. We shall find necessary and sufficient conditions for the global analytic, Gevrey and C^{∞} hypoellipticity of a class of subLaplacian on the three-dimensional torus. Next, we recall recall the terminology needed for stating our results precisely.

Recall that for $s \geq 1$ we say that a function $f(x) \in C^{\infty}(\mathbf{T}^N)$ is in the Gevrey class $G^s(\mathbf{T}^N)$ if there exists a contant $C > 0$ such that $|\partial_x^{\alpha} f(x)| \leq C^{|\alpha|+1}(\alpha!)^s$, for all $\alpha \in \mathbf{Z}_+^N$, $x \in \mathbf{T}^N$. In particular, $G^1(\mathbf{T}^N)$ is the space of all periodic analytic functions, denoted by $C^{\omega}(\mathbf{T}^N)$. One can prove that $u \in D'(\mathbf{T}^N)$ is in $G^s(\mathbf{T}^N)$ if and only if there exist positive constants ϵ and C such that

$$|\hat{u}(\xi)| \leq Ce^{-\epsilon|\xi|^{1/s}}, \ \ \forall\ \xi \in \mathbf{Z}^N \backslash \{0\}.$$

A linear partial differential operator P defined on $\mathbf{T}^N$ with coefficients in $C^{\omega}(\mathbf{T}^N)(C^{\infty}(\mathbf{T}^N))$ is said to be globally $G^s(C^{\infty})$ hypoelliptic in $\mathbf{T}^N$ if the conditions $u \in D'(\mathbf{T}^N)$ and $Pu \in G^s(\mathbf{T}^N)(C^{\infty}(\mathbf{T}^N))$ imply that $u \in G^s(\mathbf{T}^N)(C^{\infty}(\mathbf{T}^N))$. Global hypoellipticity should be contrasted with $G^s(C^{\infty})$ hypoellipticity, which means that for any open set $V \subset \mathbf{T}^N$, we have $u \in G^s(V)(C^{\infty}(V))$ for any $u \in D'(V)$ such that $Pu \in G^s(V)(C^{\infty}(V))$. We would like to point out that local hypoellipticity is a far more delicate subject than global hypoellipticity.

*Corresponding author: A. Alexandrou Himonas, Department of Mathematics, Notre Dame University, Notre Dame, IN 46556, USA, E-mail: himonas.1@nd.edu,
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Global hypoellipticity is implied by hypoellipticity, but the converse is not true in general. As a example, consider $L = \partial_{x_1} + \lambda \partial_{x_2}$ on $\mathbf{T}^2$ where λ is any real constant. Such a real vector field is never hypoelliptic but it is globally hypoelliptic for almost every λ (in the sense of Lebesgue measure). For the C^∞ case see e.g. [19] and for the Gevrey case see e.g. [18]. In particular, this example show that the situation on the torus is entirely different form the situation on a small open set in Euclidean space.

In the category of the operators given by a sum of squares of real vector fields we recall that the celebrate Hörmander's theorem [28] says that when the vector fields are C^∞ and all points are of finite type then P is C^∞ hypoelliptic and therefore is global C^∞ hypoelliptic (see also Kohn [30], Oleinik and Radkevich [36] and Rothchild and Stein [37]). However, there are operators P which are C^∞ hypoelliptic and the finite type condition does not hold (see, for example, Fedi [16], Kusuoka and Strook [32], Bell and Mohammed [3], Morimoto [35], Christ [11]).

The main motivation for this work is that global Gevrey (C^∞) hypoellipticity of sum of squares operators on a compact analytic manifold $\mathcal{M}$ is an open problem except for certain cases. From the complex analysis point of view, the most important situation is when $\mathcal{M}$ is the boundary of a domain in $\mathbf{C}^N$ and the operator is Kohn's Laplacian, $\Box_b$, for $\mathcal{M}$. More precisely, if $\mathcal{M}$ is defined by the equation Im $w = \varphi(z, \bar{z}, x)$, where $z = t_1 + it_2$, $w = x + iy$, and φ is a real valued function, then its Cauchy-Riemann operator $\bar{\partial}_b$ is given by the vector field

$$\bar{\partial}_b = \frac{\partial}{\partial \bar{z}} - i \frac{\varphi_{\bar{z}}}{1 + i\varphi_x} \frac{\partial}{\partial x} \doteq \bar{L}. \tag{1.1}$$

Separating real and imaginary parts gives

$$\bar{L} = \frac{1}{2}\Big[X_1 + iX_2\Big], \tag{1.2}$$

with

$$X_1 = \frac{\partial}{\partial t_1} + a_1(t_1, t_2, x)\frac{\partial}{\partial x}, \quad \text{and} \quad X_2 = \frac{\partial}{\partial t_2} + a_2(t_1, t_2, x)\frac{\partial}{\partial x}, \tag{1.3}$$

where a_1 and a_2 are real valued functions expressed in terms of the defining function φ and its derivatives. The Kohn Laplacian is the operator $\Box_b \doteq \bar{\partial}_b \bar{\partial}_b^* + \bar{\partial}_b^* \bar{\partial}_b$, where $\bar{\partial}_b^*$ is the adjoint of $\bar{\partial}_b$. Its principal part is given by

$$P = -\bar{L}L - L\bar{L}, \tag{1.4}$$

which, written it terms of the real vector fields X_1 and X_2, is the subLaplacian

$$P = -\frac{1}{2}X_1^2 - \frac{1}{2}X_2^2 = \frac{1}{2}\Big[- \Big(\frac{\partial}{\partial t_1} + a_1(t_1, t_2, x)\frac{\partial}{\partial x}\Big)^2 - \Big(\frac{\partial}{\partial t_2} + a_2(t_1, t_2, x)\frac{\partial}{\partial x}\Big)^2\Big]. \tag{1.5}$$

Notice that when the hypersurface $\mathcal{M} \subset \mathbf{C}^2$ is defined by an equation of the form Im $w = \varphi(z, \bar{z})$ then the coefficients a_1 and a_2 of the vector fields X_1 and X_2 defining the operator P are independent of the variable x.

In the symplectic case the global analytic hypoellipticity of $\Box_b$ follows from the fundamental result of Tartakoff [39], and Treves [41]. In fact, in this case $\Box_b$ is analytic hypoelliptic (see also Metivier [34] and Sjöstrand [38]). Some other, very interesting, partial results on the problem of global analytic hypoellipticity can be found in Chen [6], Derridj [13], Derridj and Tartakoff [15] and Komatsu [31]. It follows from, e.g., Chen [6], [7] that $\Box_b$ can be globally analytic hypoelliptic but fail to be analytic hypoelliptic. Also, the finite type condition and symplecticity are not sufficient for analytic hypoellipticity (see Hanges and Himonas [22]).

Finding necessary and sufficient conditions for the global analytic, Gevrey ($s > 1$) and C^∞ hypoellipticity of operators like (1.5) remains a challenging open problem.

In this work we focus on the case that the manifold $\mathcal{M}$ is the 3-dimensional torus and the coefficients of the principal part of Kohn Laplacian operator are independent of x and we study necessary and sufficient conditions for the global Gevrey and C^∞ hypoellipticity. This work is also motivated by the results in [27].

Next, we state our first result, which is about the global analytic and Gevrey regularity.

Theorem 1.1. *Let P be given by*

$$P = -\Big(\partial_{t_1} + a_1(t_1,t_2)\partial_x\Big)^2 - \Big(\partial_{t_2} + a_2(t_1,t_2)\partial_x\Big)^2, \tag{1.6}$$

where $(t_1,t_2,x) \in \mathbf{T}^3$ and $a_1, a_2 \in C^\omega(\mathbf{T}^2)$ are real-valued. We set

$$a_{10}(t_2) = \frac{1}{2\pi}\int_{\mathbf{T}} a_1(s,t_2)ds, \quad a_{20}(t_1) = \frac{1}{2\pi}\int_{\mathbf{T}} a_2(t_1,s)ds.$$

Then, P is globally G^s hypoelliptic in $\mathbf{T}^3$ if and only if either the range of $a_{10}(t_2)$ and $a_{20}(t_1)$ contain real numbers α_1 and α_2 respectively, which are not simultaneously approximable with exponent $s \geq 1$, or there exists a point $p \in \mathbf{T}^3$ of finite type for the vector fields $X_1 = \partial_{t_1} + a_1(t_1,t_2)\partial_x$ and $X_2 = \partial_{t_2} + a_2(t_1,t_2)\partial_x$.

We recall that two real numbers α_1, α_2 are said to be not simultaneously approximable with exponent $s \geq 1$ if for any $\epsilon > 0$ there exists a constant $C_\epsilon > 0$ such that for each $\eta = (\eta_1,\eta_2) \in \mathbf{Z}^2$ and each $\xi \in \mathbf{Z}\backslash\{0\}$ we have

$$|\eta_j - \alpha_j\xi| \geq C_\epsilon e^{-\epsilon|\xi|^{1/s}}, \quad \text{for some}\, j \in \{1,2\}.$$

We also recall that a point $p \in \mathbf{T}^3$ is of finite type for the vector fields X_1, X_2 if the Lie algebra $\mathcal{L}(X_1,X_2)$ spans the tangent space $T_p(\mathbf{T}^3)$.

When the coefficients a_1 and a_2 of the vector fields X_1 and X_2 defining the operator (1.6) are smooth instead of analytic we prove the following result, whose proof is similar to that of Theorem 1.1.

Theorem 1.2. *Let P be given by (1.6) with a_1 and a_2 belonging in $C^\infty(\mathbf{T}^2)$. Then, P is globally C^∞ hypoelliptic in $\mathbf{T}^3$ if and only if either the range of $a_{10}(t_2)$ and $a_{20}(t_1)$ contain real numbers α_1 and α_2 respectively, which are not simultaneously approximable numbers or there exists a point $p \in \mathbf{T}^3$ of finite type for the vector fields $X_1 = \partial_{t_1} + a_1(t_1,t_2)\partial_x$ and $X_2 = \partial_{t_2} + a_2(t_1,t_2)\partial_x$.*

Also, recall that two real numbers α_1 and α_2 are said to be not simultaneously approximable if there exist constants $C > 0$ and $K > 0$ such that for any $\eta = (\eta_1,\eta_2) \in \mathbf{Z}^2$ and $\xi \in \mathbf{Z}\backslash\{0\}$ we have

$$|\eta_j - \alpha_j\xi| \geq \frac{C}{|\xi|^K}, \quad \text{for some}\, j \in \{1,2\}.$$

For more results on the problem of global C^∞ hypoellipticity we refer the reader to the works by Amano [1], Fujiwara and Omori [17], Greenfield and Wallach [19], Himonas [23], Himonas and Petronilho [25], [26], and the references therein. Concerning global analytic and Gevrey hypoellipticity we refer the reader to Christ [10], Cordaro and Himonas [12], Derridj and Tartakoff [14], Himonas [24], Tartakoff [40], and the references therein. For more results on analytic hypoelliptic we refer the reader to Baouendi and Goulaouic [2], Metivier [33], Grigis and Sjöstrand [20], Hanges and Himonas [21], Christ [9], Bernardi, Bove and Tartakoff [4], Chanillo, Helffer and Laptev [5], and the references therein.

This paper is structured as follows. In section 2 we present a concatenation which reduces the study of the global hypoellipticity for P to that of a more convenient operator with the same regularity properties. In section 3 we prove the necessity of the conditions in Theorem 1.1 for global Gevrey hypoellipticity. We begin section 4 by stating the main L^2-estimate (4.3) needed for the proof of the sufficiency. Then we proceed by proving this estimate in the most important case. That is, when we are under the hypothesis that the range of the averages $a_{10}(t_2)$ and $a_{20}(t_1)$ contain numbers that are not simultaneously approximable with exponent s. Next, we provide the proof of estimate (4.3) in the case that we have simultaneous approximability and there exists a point of finite type. Finally, in section 5 we show that when the x-variable enters into the picture then the situation becomes much more complicated. First we consider a globally C^∞ hypoelliptic real vector field L that does not vanish on $\mathbf{T}^2$ and we prove that any real and C^∞ perturbation of zero order of it is global C^∞ hypoelliptic as well. We use this fact in order to characterize the global C^∞ hypoellipticity of the model operator $P = -\partial_{t_1}^2 - (\partial_{t_2} + a(t_2, x)\partial_x)^2$. We would like to point out that this model appears when one considers the operator $P = -\partial_{t_1}^2 - (\partial_{t_2} + a(t_1, t_2, x)\partial_x)^2$ and assumes that there are no points in $\mathbf{T}^3$ of finite type for the vector fields $X_1 = \partial_{t_1}$ and $X_2 = \partial_{t_2} + a(t_1, t_2, x)\partial_x$. Also we present a necessary condition for the operator $P = -\partial_{t_1}^2 - (\partial_{t_2} + a(t_1, t_2, x)\partial_x)^2$ to be globally C^∞ hypoelliptic in $\mathbf{T}^3$.

2 Concatenation

We begin with the following important lemma.

Lemma 2.1. *Let P, $a_{10}(t_2)$ and $a_{20}(t_1)$ be as in Theorem 1.1. For $j = 1, 2$, let $\tilde{P}_j$ be given by*

$$\tilde{P}_1 = - [\partial_{t_1} + a_{10}(t_2)\partial_x]^2 - [\partial_{t_2} + (a_{20}(t_1) + \partial_{t_2}(h_2 - h_1)(t))\partial_x]^2, \quad (2.1)$$

$$\tilde{P}_2 = - [\partial_{t_1} + (a_{10}(t_2) + \partial_{t_1}(h_1 - h_2)(t))\partial_x]^2 - [\partial_{t_2} + a_{20}(t_1)\partial_x]^2 \quad (2.2)$$

where $h_1(t) = \int_0^{t_1} a_1(r, t_2)dr - a_{10}(t_2)t_1$ and $h_2(t) = \int_0^{t_2} a_2(t_1, r)dr - a_{20}(t_1)t_2$. Then, for each $j \in \{1, 2\}$ and $s \geq 1$ fixed, there exists an isomorphism, S_j, from $D'(\mathbf{T}^3)$ onto itself which is also an isomorphism from $G^s(\mathbf{T}^3)$ onto $G^s(\mathbf{T}^3)$ such that

$$S_j P S_j^{-1} = \tilde{P}_j$$

and therefore the operator P is globally G^s hypoelliptic in $\mathbf{T}^3$ if and only if $\tilde{P}_j$ is globally G^s hypoelliptic in $\mathbf{T}^3$.

Proof. For each $j \in \{1, 2\}$ we define,

$$S_j : D'(\mathbf{T}^3) \to D'(\mathbf{T}^3)$$

by

$$S_j u = v_j = \sum_{\xi \in \mathbf{Z}} \hat{v}_j(t,\xi) e^{ix\xi} \doteq \sum_{\xi \in \mathbf{Z}} \hat{u}(t,\xi) e^{i\xi h_j(t)} e^{ix\xi}, \text{ for all } u \in D'(\mathbf{T}^3). \tag{2.3}$$

We also define

$$S_j^{-1} : D'(\mathbf{T}^3) \to D'(\mathbf{T}^3)$$

by

$$S_j^{-1} v = u_j = \sum_{\xi \in \mathbf{Z}} \hat{u}_j(t,\xi) e^{ix\xi} \doteq \sum_{\xi \in \mathbf{Z}} \hat{v}(t,\xi) e^{-i\xi h_j(t)} e^{ix\xi}, \text{ for all } v \in D'(\mathbf{T}^3).$$

It follows from a straightforward calculation that S_j is an isomorphism from $D'(\mathbf{T}^3)$ onto itself which is also an isomorphism from $G^s(\mathbf{T}^3)$ onto itself.

The rest of the proof is similar to that of Lemma 2.1 in [27] and therefore it will be omitted. □

3 Proof of Theorem 1.1 - Necessity

We begin by proving the necessity of the conditions for global G^s hypoellipticity. We do this by contradiction. If the condition in Theorem 1.1 does not hold then $a_{10}(t_2) \equiv \alpha_1$ and $a_{20}(t_1) \equiv \alpha_2$, where α_1, α_2 are simultaneously approximable numbers with exponent s, and all points in $\mathbf{T}^3$ are of infinite type for the vector fields $X_1 = \partial_{t_1} + a_1(t)\partial_x$ and $X_2 = \partial_{t_2} + a_2(t)\partial_x$. Since $[X_1, X_2] = (\partial_{t_1} a_2(t) - \partial_{t_2} a_1(t))\partial_x$ the statement that all points of $\mathbf{T}^3$ are of infinite type reads as $\partial_{t_1} a_2(t) = \partial_{t_2} a_1(t)$, $\forall\, t \in \mathbf{T}^2$. In this case we have

$$\begin{aligned} \partial_{t_2}(h_2 - h_1)(t) &= a_2(t) - \alpha_2 - \int_0^{t_1} \frac{\partial a_1}{\partial t_2}(r, t_2)dr = a_2(t) - \alpha_2 - \int_0^{t_1} \frac{\partial a_2}{\partial r}(r, t_2)dr \\ &= -\alpha_2 + a_2(0, t_2) \end{aligned}$$

and the operator, for example, $\tilde{P}_1$ becomes

$$\tilde{P}_1 = -(\partial_{t_1} + \alpha_1 \partial_x)^2 - (\partial_{t_2} + a_2(0, t_2)\partial_x)^2. \tag{3.1}$$

By introducing new variables $s_j = t_j$, $j = 1, 2$, $y = x - \int_0^{t_2} a_2(0, r)dr + a_2^0 t_2$, where $a_2^0 = \frac{1}{2\pi}\int_0^{2\pi} a_2(0, r)dr = a_{20}(0) = \alpha_2$, the operator $\tilde{P}_1$ becomes

$$Q_1 = -(\partial_{s_1} + \alpha_1 \partial_y)^2 - (\partial_{s_2} + \alpha_2 \partial_y)^2.$$

Now, it is easy to show that Q_1 is not globally G^s hypoelliptic in $\mathbf{T}^3$. In fact, since α_1, α_2 are simultaneously approximable numbers with exponent $s \geq 1$, there exist $\epsilon_0 > 0$, a sequence $\{p_n, \tilde{p}_n\} \subset \mathbf{Z}^2$ and another sequence $\{q_n\} \subset \mathbf{Z}\backslash\{0\}$ such that $|q_n| \geq 2$, $|q_n| \to \infty$ as $n \to \infty$ and

$$|q_n \alpha_1 - \tilde{p}_n| < e^{-\epsilon_0 |q_n|^{1/s}}, \quad |q_n \alpha_2 - p_n| < e^{-\epsilon_0 |q_n|^{1/s}}, \; n = 1, 2, \cdots. \tag{3.2}$$

We define $u \in \mathcal{D}'(\mathbf{T}^3) \setminus G^s(\mathbf{T}^3)$ by

$$u(s_1, s_2, y) = \sum_{n=1}^{\infty} e^{i(-\tilde{p}_n s_1 - p_n s_2 + q_n y)}.$$

It is easily seen that $Q_1 u \in G^s(\mathbf{T}^3)$. Hence, Q_1 and therefore P is not globally G^s hypoelliptic in $\mathbf{T}^3$. This completes the proof of the necessity.

4 Proof of Theorem 1.1 - Sufficiency

Assume that $u \in D'(\mathbf{T}^3)$ is such that

$$Pu = f \in G^s(\mathbf{T}^3). \tag{4.1}$$

We must prove that $u \in G^s(\mathbf{T}^3)$.

Taking Fourier transform with respect to x in the equation (4.1) and after multiplying it by $\overline{\hat{u}}(t,\xi)$ and integrating with respect to $t \in \mathbf{T}^2$ we obtain

$$\int_{\mathbf{T}^2} |(\partial_{t_1} + i\xi a_1(t))\hat{u}(t,\xi)|^2 dt + \int_{\mathbf{T}^2} |(\partial_{t_2} + i\xi a_2(t))\hat{u}(t,\xi)|^2 dt = \int_{\mathbf{T}^2} \hat{f}(t,\xi)\overline{\hat{u}}(t,\xi)dt. \tag{4.2}$$

In order to show that P is globally G^s hypoelliptic in $\mathbf{T}^3$ it suffices, by using a standard microlocal analysis argument, to prove the following lemma.

Lemma 4.1. *If either the range of $a_{10}(t_2)$ and $a_{20}(t_1)$ contain real numbers α_1, α_2 respectively which are not simultaneously approximable numbers with exponent $s \geq 1$ or there exists a point $p \in \mathbf{T}^3$ of finite type for the vector fields $X_1 = \partial_{t_1} + a_1(t_1,t_2)\partial_x$ and $X_2 = \partial_{t_2} + a_2(t_1,t_2)\partial_x$, then for any $\epsilon > 0$ there exist positive constants $C_\epsilon > 0$ and C'_ϵ such that*

$$||\hat{u}(\cdot,\xi)||_{L^2(\mathbf{T}^2)} \leq C_\epsilon e^{3\epsilon|\xi|^{1/s}} ||\hat{f}(\cdot,\xi)||_{L^2(\mathbf{T}^2)}, \quad \text{for } |\xi| > C'_\epsilon. \tag{4.3}$$

Setting $v_j = S_j u \in D'(\mathbf{T}^3)$ and $g_j = S_j f \in G^s(\mathbf{T}^3)$ and using the concatenations $S_j P S_j^{-1} = \tilde{P}_j$, $j = 1, 2$ one can prove Lemma 4.1 by showing that for any $\epsilon > 0$ there exist constants $C_\epsilon > 0$ and $C'_\epsilon > 0$ such that

$$||\hat{v}_j(\cdot,\xi)||_{L^2(\mathbf{T}^2)} \leq C_\epsilon e^{\epsilon|\xi|^{1/s}} ||\hat{g}_j(\cdot,\xi)||_{L^2(\mathbf{T}^2)} \text{ for } |\xi| > C'_\epsilon.$$

The proof of Lemma 4.1 can be done splitting it in cases. Under the hypothesis that the range of $a_{10}(t_2)$ and $a_{20}(t_1)$ contain real numbers α_1 and α_2 respectively, which are not simultaneously approximable with exponent $s \geq 1$ one can split the proof in four cases, namely,

Case R-1: $a_{10}(t_2) \equiv \alpha_1$ and $a_{20}(t_1) \equiv \alpha_2$, where α_1, α_2 are not simultaneously approximable numbers with exponent $s \geq 1$.

Case R-2: $a_{10} \equiv \kappa \in \mathbf{R} \setminus \mathbf{Q}$ and $a_{20}(t_1)$ is non-constant.

Case R-3: $\kappa = p/q \in \mathbf{Q}$, where $p, q \in \mathbf{Z}, q \neq 0$ and p and q do not have a common factor which is larger than one and $a_{20}(t_1)$ is non-constant.

Case R-4: $a_{10}(t_2)$ and $a_{20}(t_1)$ are non-constants.

On the other hand if we assume that there exists a point of finite type for the vector fields $X_1 = \partial_{t_1} + a_1(t_1,t_2)\partial_x$ and $X_2 = \partial_{t_2} + a_2(t_1,t_2)\partial_x$ and that the averages $a_{10}(t_2)$ and $a_{20}(t_1)$ are constants, then the proof can be done proving the following cases.

Case FT-1: $a_{10} = \frac{p_1}{q_1}$ and $a_{20} = \frac{p_2}{q_2}$, where for $j \in \{1,2\}$, $p_j, q_j \in \mathbf{Z}, q_j \neq 0$, and p_j and q_j do not have a common factor that is larger than one.

Case FT-2: $a_{10} \equiv \frac{m}{n}$ and $a_{20} \equiv \kappa \in \mathbf{R} \backslash Q$, where $m, n \in \mathbf{Z}, n \neq 0$ and m and n do not have a common factor which is larger than one.

Case FT-3: $a_{10} \equiv \kappa_1$ and $a_{20} \equiv \kappa_2$, where κ_1 and κ_2 are irrational numbers.

Every case mentioned above, except the **Case R-4**, can be proved by doing a clever adaptation of the methods used in [27]. As an example, here we present the proof of one of these cases, namely **FT-2**. Next, we begin with the most interesting case R-4.

4.1 Proof of estimate (4.3) when $a_{10}(t_2)$ and $a_{20}(t_1)$ are non-constants.

We shall need the following key lemma.

Lemma 4.2. *Let $y_0, y_1 \in \mathbf{T}$, $\varphi, \chi \in C^\infty(\mathbf{T}^2)$ real-valued functions. Let $\psi \in C^\infty(\bar{I})$, real-valued, i.e., $\psi \in C^\infty(I)$ and extends continously to $\bar{I}$, where I is an interval contained in $[0, 2\pi]$. Then, for $\eta \in \mathbf{Z}\backslash\{0\}$, we have*

$$\begin{aligned} &||(\partial_{t_1} + i\eta y_0)\varphi(t)||^2_{L^2(\mathbf{T}\times I)} \leq ||(\partial_{t_1} + i\eta\psi(t_2))\varphi(t)||^2_{L^2(\mathbf{T}\times I)} \\ &+2|\eta| \sup_{t_2\in\bar{I}}\{|\psi(t_2) - y_0|\}||(\partial_{t_1} + i\eta\psi(t_2))\varphi(t)||_{L^2(\mathbf{T}\times I)}\, ||\varphi||_{L^2(\mathbf{T}\times I)} \\ &+|\eta|^2 \sup_{t_2\in\bar{I}}\{|\psi(t_2) - y_0|^2\}||\varphi||^2_{L^2(\mathbf{T}\times I)}. \end{aligned} \tag{4.4}$$

Furtheremore, if $\psi(t_2) \in (y_0 - r, y_0 + r)$ for any $t_2 \in I$, with $r > 0$ then we have

$$\|(\partial_{t_1} + i\eta y_0)\varphi\|^2_{L^2(\mathbf{T}\times I)} \leq 2\|(\partial_{t_1} + i\eta\psi(t_2))\varphi\|^2_{L^2(\mathbf{T}\times I)} + 2|\eta|^2 r^2 \|\varphi\|^2_{L^2(\mathbf{T}\times I)}. \tag{4.5}$$

We also have

$$\begin{aligned} &||(\partial_{t_2} + i\eta(y_0 + \chi(t)))\varphi||^2_{L^2(\mathbf{T}^2)} \leq ||(\partial_{t_2} + i\eta(y_1 + \chi(t)))\varphi||^2_{L^2(\mathbf{T}^2)} \\ &+2|\eta|\, |y_1 - y_0|||(\partial_{t_2} + i\eta(y_1 + \chi(t)))\varphi||^2_{L^2(\mathbf{T}^2)}||\varphi||_{L^2(\mathbf{T}^2)} \\ &+|\eta|^2|y_1 - y_0|^2||\varphi||^2_{L^2(\mathbf{T}^2)}. \end{aligned} \tag{4.6}$$

Proof. Let $y_0 \in \mathbf{T}, \varphi$ and ψ be given as in the statement of Lemma 4.2. Since for any $\eta \in \mathbf{Z}\backslash\{0\}$ and for any $t_2 \in I$ we have

$$\begin{aligned} (\partial_{t_1} + i\eta y_0)^2\varphi(t) &= (\partial_{t_1} + i\eta\psi(t_2))^2\varphi(t) - 2i\eta(\psi(t_2) - y_0)(\partial_{t_1} + i\eta\psi(t_2))\varphi(t) \\ &- \eta^2(\psi(t_2) - y_0)^2\varphi(t), \end{aligned}$$

one can complete the proof easily. □

We also shall introduce some notations and results.

Assume that $a_{10}(t_2)$ and $a_{20}(t_1)$ are non-constant functions. Then, setting

$$m = \min_{t_2\in[0,2\pi]} a_{10}(t_2) \quad \text{and} \quad M = \max_{t_2\in[0,2\pi]} a_{10}(t_2)$$

we see that $M - m > 0$.

For $\xi \in \mathbf{Z}\backslash\{0\}$ we now define the set

$$\mathcal{Q}(\xi; m, M) = \left\{\frac{p}{q} \in \mathbf{Q};\ \xi\frac{p}{q} \in \mathbf{Z},\ \text{and } m \leq \frac{p}{q} \leq M\right\}$$

where we are assuming that p and q do not have a common factor which is larger than one.

We need the following result.

Lemma 4.3. *The following conditions hold true for the set $\mathcal{Q}(\xi; m, M)$:*

1) For $\xi \in \mathbf{Z}\backslash\{0\}$ we can write

$$\mathcal{Q}(\xi; m, M) \;=\; \{\frac{\ell}{\xi} : \ell \in \mathbf{Z} \text{ and } m\xi \leq \ell \leq M\xi, \text{ if } \xi > 0 \\ \text{or } M\xi \leq \ell \leq m\xi \text{ if } \xi < 0\}. \qquad (4.7)$$

2) For all $\xi \in \mathbf{Z}$ such that $(M-m)|\xi| \geq 1$ the set $\mathcal{Q}(\xi; m, M)$ is not empty.

Proof. 1) If $\xi \in \mathbf{Z}\backslash\{0\}$ and $w \in \mathcal{Q}(\xi; m, M)$ we can write $w = \frac{p}{q} = \frac{1}{\xi}(\xi\frac{p}{q})$, where $\ell \doteq (\xi\frac{p}{q}) \in \mathbf{Z}$. Furthermore, since $m \leq \frac{p}{q} \leq M$ we have $m\xi \leq \ell \leq M\xi$, if $\xi > 0$ or $M\xi \leq \ell \leq m\xi$ if $\xi < 0$. On the other hand, if $w = \frac{\ell}{\xi}$ with $\ell \in \mathbf{Z}$ and $m\xi \leq \ell \leq M\xi$, if $\xi > 0$ or $M\xi \leq \ell \leq m\xi$ if $\xi < 0$ then $w = \frac{\ell}{\xi} \in \mathbf{Q}$, $\xi w \in \mathbf{Z}$ and $m \leq w = \frac{\ell}{\xi} \leq M$ for any $\xi \in \mathbf{Z}\backslash\{0\}$.

2) We will prove 2) only for $\xi > 0$ since for $\xi < 0$ the proof is analougous. Let $\xi \in \mathbf{Z}^+$ be such that $(M-m)\xi \geq 1$. Thus, we can guarantee that there exists $\eta \in \mathbf{Z}$ such that $m\xi \leq \eta \leq M\xi$. In fact, if either $m\xi \in \mathbf{Z}$ or $M\xi \in \mathbf{Z}$ then we take $\eta = m\xi$ or $\eta = M\xi$. We now assume that $m\xi \notin \mathbf{Z}$ and $M\xi \notin \mathbf{Z}$ and we shall prove that there exists $\eta \in \mathbf{Z}$ such that $m\xi \leq \eta \leq M\xi$. We now suppose that for all $\eta \in \mathbf{Z}$ we have either $\eta > M\xi$ or $\eta < m\xi$. It follows from our hypotheses that $[m\xi] \leq m\xi < M\xi < [m\xi] + 1$ which implies that

$$M\xi - m\xi < [m\xi] + 1 - [m\xi] = 1$$

which is a contradiction with our hypothesis that $(M-m)\xi \geq 1$. Hence we may conclude that there exists $\eta \in \mathbf{Z}$ such that $m\xi \leq \eta \leq M\xi$. It follows from 1) that $\frac{\eta}{\xi} \in \mathbf{Q}(\xi; m, M)$, which completes the proof of item 2). □

It follows from item 1) of Lemma 4.3 that for each fixed $\xi \in \mathbf{Z}\backslash\{0\}$ the set $\mathcal{Q}(\xi; m, M)$ is a finite set. Thus, we can list its elements as

$$\{\frac{p_1^\xi}{q_1^\xi} < \frac{p_2^\xi}{q_2^\xi} < \cdots < \frac{p_{n(\xi)}^\xi}{q_{n(\xi)}^\xi}\} \text{ and we have } \left|\frac{p_i^\xi}{q_i^\xi} - \frac{p_j^\xi}{q_j^\xi}\right| = \frac{1}{|\xi|}, \text{ whenever } |i-j| = 1. \qquad (4.8)$$

Now, we define $\frac{p_0^\xi}{q_0^\xi} = \frac{p_1^\xi}{q_1^\xi} - \frac{1}{|\xi|}$ and $\frac{p_{n(\xi)+1}^\xi}{q_{n(\xi)+1}^\xi} = \frac{p_{n(\xi)}^\xi}{q_{n(\xi)}^\xi} + \frac{1}{|\xi|}$. Thus $\frac{p_0^\xi}{q_0^\xi} \notin \mathbf{Q}(\xi; m, M)$, $\frac{p_0^\xi}{q_0^\xi} < m$ and $\frac{p_{n(\xi)+1}^\xi}{q_{n(\xi)+1}^\xi} \notin \mathbf{Q}(\xi; m, M)$, $\frac{p_{n(\xi)+1}^\xi}{q_{n(\xi)+1}^\xi} > M$. Next, for $0 \leq j \leq n(\xi)+1$ we set

$$B_j(\xi) = B\left(\frac{p_j^\xi}{q_j^\xi}, \frac{1}{|\xi|^2}\right) \doteq \left\{x \in \mathbf{R} : \left|x - \frac{p_j^\xi}{q_j^\xi}\right| < \frac{1}{|\xi|^2}\right\} \text{ and } I_j(\xi) = a_{10}^{-1}(B_j).$$

Let $i, j \in \{0, 1, \cdots, n(\xi)+1\}$, $j \neq i$ and $\xi \in \mathbf{Z}\backslash\{-1, 0, 1\}$. Suppose that there exists $x \in B_i(\xi) \cap B_j(\xi)$. Then we have $|x - \frac{p_i^\xi}{q_i^\xi}| < \frac{1}{|\xi|^2}$ and $|x - \frac{p_j^\xi}{q_j^\xi}| < \frac{1}{|\xi|^2}$ which implies that

$$\frac{1}{|\xi|} \leq \left|\frac{p_i^\xi}{q_i^\xi} - \frac{p_j^\xi}{q_j^\xi}\right| < \frac{2}{|\xi|^2},$$

which is a contradiction since we have $|\xi| \geq 2$. Thus we can conclude that

$$B_i(\xi) \cap B_j(\xi) = \emptyset, \; \forall \; i, j \in \{0, 1, \cdots, n(\xi)+1\}, \, j \neq i, \text{ whenever } |\xi| \geq 2.$$

Hence, we have

$$I_i(\xi) \cap I_j(\xi) = \emptyset,\ \forall\ i, j \in \{0, 1, \cdots, n(\xi)+1\},\ j \neq i, \text{ whenever } |\xi| \geq 2.$$

Observe that the complement set in $[0, 2\pi]$ of $\cup_{j=0}^{n(\xi)+1} I_j(\xi)$ can be an empty set. For example, this is the case if we take $m = \frac{19}{200}$ and $M = \frac{21}{200}$. Then, for $\xi = 10$ we have $\mathbf{Q}(10, \frac{19}{200}, \frac{21}{200}) = \{\frac{1}{10}\}$. Thus, $\frac{1}{10} + \frac{1}{100} = \frac{11}{100} > \frac{21}{200}$ and $\frac{1}{10} - \frac{1}{100} = \frac{9}{100} < \frac{19}{200}$ and we can conclude that $[\frac{19}{200}, \frac{21}{200}] \subset B_1 = B(\frac{1}{10}, \frac{1}{100})$. Hence $[0, 2\pi] = I_1(10)$.

Next, we will analyze the complement set in $[0, 2\pi]$ of $\cup_{j=0}^{n(\xi)+1} I_j(\xi)$ when it is not an empty set. For this we shall need the following result.

Lemma 4.4. *Assume that* $[0, 2\pi] \setminus \{\cup_{j=0}^{n(\xi)+1} I_j(\xi)\} \neq \emptyset$. *Then, for* $|\xi| \geq 3$ *we have*

$$|e^{i2\pi\xi a_{10}(t_2)} - 1| \geq \frac{1}{|\xi|}, \quad \textit{for all } t_2 \in [0, 2\pi] \setminus \{\cup_{j=0}^{n(\xi)+1} I_j(\xi)\}. \tag{4.9}$$

Proof. Let $t_2 \in [0, 2\pi] \setminus \{\cup_{j=0}^{n(\xi)+1} I_j(\xi)\}$. We will split the proof in two cases.

Case 1. $\mathbf{Q}(\xi; m, M) = \{\frac{p_1^\xi}{q_1^\xi}\}$. Then, in this case, we are dealing with the sets $I_0(\xi), I_1(\xi)$ and $I_2(\xi)$.

Case 1.1 $B_1(\xi) \subset [m, M]$.

Case 1.2 $m \in B_1(\xi)$ and $M \notin B_1(\xi)$.

Case 1.3 $M \in B_1(\xi)$ and $m \notin B_1(\xi)$.

We begin by analyzing case **1.1.** If $t_2 \in [0, 2\pi] \setminus \{\cup_{j=0}^{2} I_j(\xi)\}$ and $B_1(\xi) \subset [m, M]$ then one of the following conditions occurs:

$$\textbf{a)}\, a_{10}(t_2) \in [m, \frac{p_1^\xi}{q_1^\xi} - \frac{1}{|\xi|^2}], \quad \textbf{b)}\, a_{10}(t_2) \in [\frac{p_1^\xi}{q_1^\xi} + \frac{1}{|\xi|^2}, M].$$

Here we will analyze only case **a)** since case **b)** is analogous. In case **a)** we have two subcases:

Case a-1) $\frac{p_0^\xi}{q_0^\xi} + \frac{1}{|\xi|^2} \leq m \leq \frac{p_1^\xi}{q_1^\xi} - \frac{1}{|\xi|^2}$.

Case a-2) $\frac{p_0^\xi}{q_0^\xi} < m < \frac{p_0^\xi}{q_0^\xi} + \frac{1}{|\xi|^2}$.

In case **a)** we have $a_{10}(t_2) \in [m, \frac{p_1^\xi}{q_1^\xi} - \frac{1}{|\xi|^2}]$ and we define $J_0(\xi) = a_{10}^{-1}\left(\left[\frac{p_0^\xi}{q_0^\xi} + \frac{1}{|\xi|^2}, \frac{p_1^\xi}{q_1^\xi} - \frac{1}{|\xi|^2}\right]\right)$. Thus, in case **a-1)** we have $t_2 \in J_0(\xi)$ and therefore

$$\frac{p_0^\xi}{q_0^\xi} + \frac{1}{|\xi|^2} \leq a_{10}(t_2) \leq \frac{p_1^\xi}{q_1^\xi} - \frac{1}{|\xi|^2}. \tag{4.10}$$

Assuming that $\xi > 0$ and multiplying the last inequality by $\pi\xi$ we obtain

$$\pi\xi\frac{p_0^\xi}{q_0^\xi} + \frac{\pi}{\xi} \leq \pi\xi a_{10}(t_2) \leq \pi\xi\frac{p_1^\xi}{q_1^\xi} - \frac{\pi}{\xi}.$$

Since $\frac{p_1^\xi}{q_1^\xi} \in \mathcal{Q}(\xi; m, M)$ we have $\xi\frac{p_1^\xi}{q_1^\xi} \doteq \ell \in \mathbf{Z}$. Then, using the definition of $\frac{p_0^\xi}{q_0^\xi}$ we have

$$\ell - 1 = \xi\frac{p_1^\xi}{q_1^\xi} - 1 = \xi(\frac{p_1^\xi}{q_1^\xi} - \frac{1}{\xi}) = \xi(\frac{p_1^\xi}{q_1^\xi} - (\frac{p_1^\xi}{q_1^\xi} - \frac{p_0^\xi}{q_0^\xi})) = \xi\frac{p_0^\xi}{q_0^\xi}.$$

Thus the last inequality can be written as $(\ell-1)\pi+\frac{\pi}{\xi}\le\pi\xi a_{10}(t_2)\le\ell\pi-\frac{\pi}{\xi}$ with $\ell\in\mathbf{Z}$. It follows from this that

$$|\sin(\pi\xi a_{10}(t_2))|\ge|\sin(\ell\pi-\frac{\pi}{\xi})|=|\sin(\frac{\pi}{\xi})|. \tag{4.11}$$

We now recall that

$$\frac{\sin x}{x}\ge\frac{\sin\pi/2}{\pi/2}=\frac{2}{\pi},\ \text{ if } 0<|x|\le\frac{\pi}{2}. \tag{4.12}$$

Since, by hypothesis, $\xi\ge 3$ we have $0<\frac{\pi}{\xi}\le\frac{\pi}{3}$. It follows from (4.11) and (4.12) that

$$|e^{i2\pi\xi a_{10}(t_2)}-1|=2|\sin(\pi\xi a_{10}(t_2))|\ge 2|\sin(\frac{\pi}{\xi})|\ge\frac{4}{\xi}>\frac{1}{\xi},$$

which proves the Lemma 4.4 in this case.

We now assume that $\xi<0$. Multiplying the inequality (4.10) by $\pi\xi$ we obtain

$$\pi\xi\frac{p_1^\xi}{q_1^\xi}-\frac{\pi}{\xi}\le\pi\xi a_{10}(t_2)\le\pi\xi\frac{p_0^\xi}{q_0^\xi}+\frac{\pi}{\xi}.$$

Since $\frac{p_1^\xi}{q_1^\xi}\in\mathcal{Q}(\xi;m,M)$ and recalling that $\xi\frac{p_1^\xi}{q_1^\xi}=\ell\in\mathbf{Z}$, we have

$$\ell+1=\xi\frac{p_1^\xi}{q_1^\xi}+1=\xi(\frac{p_1^\xi}{q_1^\xi}+\frac{1}{\xi})=\xi(\frac{p_1^\xi}{q_1^\xi}+(\frac{p_0^\xi}{q_0^\xi}-\frac{p_1^\xi}{q_1^\xi}))=\xi\frac{p_0^\xi}{q_0^\xi}.$$

The last inequality can be written as $\ell\pi-\frac{\pi}{\xi}\le\pi\xi a_{10}(t_2)\le(\ell+1)\pi+\frac{\pi}{\xi}$ with $\ell\in\mathbf{Z}$. It follows from this that

$$|\sin(\pi\xi a_{10}(t_2))|\ge|\sin(\ell\pi-\frac{\pi}{\xi})|=|\sin(\frac{\pi}{\xi})|. \tag{4.13}$$

Since, by hypotheses, $\xi\le -3$ we have $0>\frac{\pi}{\xi}\ge-\frac{\pi}{3}$. It follows from (4.13) and (4.12) that

$$|e^{i2\pi\xi a_{10}(t_2)}-1|=2|\sin(\pi\xi a_{10}(t_2))|\ge 2|\sin(\frac{\pi}{\xi})|\ge\frac{4}{|\xi|}>\frac{1}{|\xi|},$$

which proves the Lemma 4.4 in case **a-1)**.

In case **a-2)** we must have $a_{10}(t_2)\in[\frac{p_0^\xi}{q_0^\xi}+\frac{1}{|\xi|^2},\frac{p_1^\xi}{q_1^\xi}-\frac{1}{|\xi|^2}]$ since otherwise we will have $t_2\in I_0$. Thus

$$\frac{p_0^\xi}{q_0^\xi}+\frac{1}{|\xi|^2}\le a_{10}(t_2)\le\frac{p_1^\xi}{q_1^\xi}-\frac{1}{|\xi|^2}$$

and therefore, as before, we have the desired inequality. The proof of Lemma 4.4 is now complete in case **1.1.**

The proof of case **1.2** is similar to that of case **1.1-b)**. In case **1.3** we must have $a_{10}(t_2)\in[m,\frac{p_1^\xi}{q_1^\xi}-\frac{1}{|\xi|^2}]$ and the proof is the same as in case **a)**. Therefore the proof of Lemma 4.4 is complete in case **1.**

Case 2. $\mathcal{Q}(\xi;m,M)=\{\frac{p_1^\xi}{q_1^\xi}<\frac{p_2^\xi}{q_2^\xi}<\cdots<\frac{p_{n(\xi)}^\xi}{q_{n(\xi)}^\xi}\}$ with $n(\xi)>1$.

We begin by defining

$$J_j(\xi) = a_{10}^{-1}\left(\left[\frac{p_j^\xi}{q_j^\xi} + \frac{1}{|\xi|^2}, \frac{p_{j+1}^\xi}{q_{j+1}^\xi} - \frac{1}{|\xi|^2}\right]\right), \qquad \text{for } \ 1 \leq j \leq n(\xi).$$

Therefore, if $t_2 \in [0, 2\pi] \setminus \{\cup_{j=0}^{n(\xi)+1} I_j(\xi)\}$ then one of the following condition occurs: **a)** $a_{10}(t_2) \in [m, \frac{p_1^\xi}{q_1^\xi} - \frac{1}{|\xi|^2}]$, **b)** $t_2 \in J_j(\xi)$ for some $j \in \{1, \cdots, n(\xi) - 1\}$ **c)** $a_{10}(t_2) \in [\frac{p_{n(\xi)}^\xi}{q_{n(\xi)}^\xi} + \frac{1}{|\xi|^2}, M]$.

Here we will analyze only case **b)** since case **a)** has been analyzed in the case **1**, and case **c)** is analogous to case **a)**.

In case **b)** we have that $t_2 \in J_j(\xi)$ for some $j \in \{1, \cdots, n(\xi) - 1\}$. Therefore

$$\frac{p_j^\xi}{q_j^\xi} + \frac{1}{|\xi|^2} \leq a_{10}(t_2) \leq \frac{p_{j+1}^\xi}{q_{j+1}^\xi} - \frac{1}{|\xi|^2},$$

which, as before, implies the desired inequality. This completes the proof of Lemma 4.4. □

Remark 4.5 We set $K_j(\xi) = \left[\frac{p_j^\xi}{q_j^\xi} + \frac{1}{|\xi|^2}, \frac{p_{j+1}^\xi}{q_{j+1}^\xi} - \frac{1}{|\xi|^2}\right]$ for $j = 0, 1, \cdots, n(\xi)$. It is easy to see that the sets $K_j(\xi)$ are disjoint and therefore we conclude that the sets $J_i(\xi)$, $i \in \{0, \cdots, n(\xi)\}$ are disjoint too. We also notice that for each $t_2 \in [0, 2\pi]$ either $a_{10}(t_2) \in B_j(\xi)$ for some $j \in \{0, \cdots, n(\xi) + 1\}$ or $a_{10}(t_2) \in K_j(\xi)$ for some $j \in \{0, \cdots, n(\xi)\}$. Thus

$$[0, 2\pi] = \{\cup_{j=0}^{n(\xi)+1} I_j(\xi)\} \cup \{\cup_{j=1}^{n(\xi)+1} J_j(\xi)\}.$$

Also, it is easy to see that $\{\cup_{j=0}^{n(\xi)+1} I_j(\xi)\} \cap \{\cup_{j=1}^{n(\xi)+1} J_j(\xi)\} = \emptyset$.

From now on we shall use the following notations

$$\mathbf{A}_1(\xi) \doteq \{\cup_{j=0}^{n(\xi)+1} I_j(\xi)\} \ \text{ and } \ \mathbf{A}_2(\xi) \doteq \{\cup_{j=0}^{n(\xi)} J_j(\xi)\}.$$

Now, we are ready to prove Lemma 4.1 in case **R-4**.

Arguing along the lines of the proof of Lemma 3.2 in [27] we can prove that

$$|\hat{v}_1(t, \xi)| \leq \frac{1}{|e^{i2\pi\xi a_{10}(t_2)} - 1|} \int_{\mathbf{T}} |\partial_{t_1} \hat{v}_1(t, \xi) + i\xi a_{10}(t_2) \hat{v}_1(t, \xi)| dt_1,$$

provided that $e^{i2\pi\xi a_{10}(t_2)} - 1 \neq 0$. Using this inequality and Lemma 4.4 we obtain that

$$|\hat{v}_1(t, \xi)| \leq |\xi| \int_{\mathbf{T}} |\partial_{t_1} \hat{v}_1(t, \xi) + i\xi a_{10}(t_2) \hat{v}_1(t, \xi)| dt_1 \tag{4.14}$$

for $|\xi| \geq 3$, $t_1 \in \mathbf{T}$ and for all $t_2 \in \mathbf{A}_2(\xi)$. In (4.14), applying Cauchy-Schwartz inequality first, and then integrating with respect to $t = (t_1, t_2) \in \mathbf{T} \times \mathbf{A}_2(\xi)$ give

$$||\hat{v}_1(\cdot, \xi)||^2_{L^2(\mathbf{T}\times\mathbf{A}_2(\xi))} \leq 4\pi^2 |\xi|^2 \int_{\mathbf{T}\times\mathbf{A}_2(\xi)} |(\partial_{t_1} + i\xi a_{10}(t_2)) \hat{v}_1(t, \xi)|^2 dt.$$

From the last inequality and the Cauchy-Schwartz inequality we obtain that

$$\|\hat{v}_1(\cdot,\xi)\|^2_{L^2(\mathbf{T}\times\mathbf{A}_2(\xi))} \leq 4\pi^2|\xi|^2\|\hat{g}_1(\cdot,\xi)\|_{L^2(\mathbf{T}^2)}\|\hat{v}_1(\cdot,\xi)\|_{L^2(\mathbf{T}^2)}. \tag{4.15}$$

In the last inequality we have used (4.2) replacing $a_1(t)$ by $a_{10}(t_2)$, $a_2(t)$ by $a_{20}(t_1)+\partial_{t_2}(h_2-h_1)$, $\hat{u}(t,\xi)$ by $\hat{v}_1(t,\xi)$, $\hat{f}(t,\xi)$ by $\hat{g}_1(t,\xi)$.

Next, we analyze the norm $\|\hat{v}_1(\cdot,\xi)\|_{L^2(\mathbf{T}\times\mathbf{A}_1(\xi))}$. For this we shall need the following easy lemma.

Lemma 4.6. *For any intervals I_1, I_2 in $[0,2\pi]$ and $\varphi\in C^\infty(\mathbf{T}^2)$ we have*

$$||\varphi||^2_{L^2(\mathbf{T}\times I_2)} \leq 8\pi^2\bigg(\frac{1}{|I_1|}\int_{I_2}\int_{I_1}|\varphi(t_1,t_2)|^2dt_1dt_2 + ||\partial_{t_1}\varphi||^2_{L^2(\mathbf{T}\times I_2)}\bigg).$$

For $j\in\{0,\cdots,n(\xi)+1\}$ we apply Lemma 4.6 with $\varphi(t)=e^{i\xi\frac{p_j^\xi}{q_j^\xi}t_1}\hat{v}_1(t,\xi)$, $I_2=I_j(\xi)$ and we obtain

$$||\hat{v}_1(\cdot,\xi)||^2_{L^2(\mathbf{T}\times I_j(\xi))} \leq \frac{8\pi^2}{|I_1|}\int_{I_j(\xi)}\int_{I_1}|\hat{v}_1(t_1,t_2,\xi)|^2dt_1dt_2+8\pi^2||(\partial_{t_1}+i\xi\frac{p_j^\xi}{q_j^\xi})\hat{v}_1(t,\xi)||^2_{L^2(\mathbf{T}^1\times I_j(\xi))},$$

where $\xi\frac{p_j^\xi}{q_j^\xi}\in\mathbf{Z}$ since for $j\in\{1,\cdots,n(\xi)\}$ we have $\frac{p_j^\xi}{q_j^\xi}\in\mathbf{Q}(\xi;m,M)$, while for for $j=0$ and $j(\xi)+1$ it is true by the definition of $\frac{p_0^\xi}{q_0^\xi}$ and $\frac{p^\xi_{n(\xi)+1}}{q^\xi_{n(\xi)+1}}$. From the last inequality and from Lemma 4.2 - formula (4.5), applied with $y_0=\frac{p_j}{q_j}$, $\eta=\xi$, $I=I_j(\xi)$, therefore $r=\frac{2}{|\xi|^2}$, $\varphi(t)=\hat{v}_1(t,\xi)$ and $\psi(t_2)=a_{10}(t_2)$ it follows that

$$\begin{aligned}||\hat{v}_1(\cdot,\xi)||^2_{L^2(\mathbf{T}\times I_j(\xi))} &\leq 8\pi^2\bigg(\frac{1}{|I_1|}\int_{I_j(\xi)}\int_{I_1}|\hat{v}_1(t_1,t_2,\xi)|^2dt_1dt_2 \\ &+ 2\|(\partial_{t_1}+i\xi a_{10}(t_2))\hat{v}_1(\cdot,\xi)\|^2_{L^2(\mathbf{T}\times I_j(\xi))}+\frac{4}{|\xi|^2}\|\hat{v}_1(\cdot,\xi)\|^2_{L^2(\mathbf{T}\times I_j(\xi))}\bigg).\end{aligned} \tag{4.16}$$

Next, we will estimate the term $\int_{I_j(\xi)}\int_{I_1}|\hat{v}_1(t_1,t_2,\xi)|^2dt_1dt_2$. For this we shall introduce some results. Since non-Liouville numbers are dense in $\mathbf{R}$ and $a_{20}(t_1)$ is non-constant we can find a $t_1^0\in(0,2\pi)$ such that $a_{20}(t_1^0)=\alpha$ with α being a non-Liouville number. It follows from Lemma 3.3 in [27] that there exist constants $L=L(t_1^0)>0$ and $K=K(t_1^0)\geq 0$ such that

$$|\sin(\pi\eta a_{20}(t_1^0))| = |\sin(\pi\eta\alpha)| \geq L|\eta|^{-K},\ \ \forall\ \eta\in\mathbf{Z}\backslash\{0\}. \tag{4.17}$$

We shall need the following result whose proof follows the lines that of Lemma 3.4 in [27].

Lemma 4.7. *Suppose that $a_{20}(t_1)$ is non-constant. For each $\eta\in\mathbf{Z}\backslash\{0\}$ there exist positive constants L_1, L_2, which depend on t_1^0, and there exists an interval $I_\eta\subset(0,2\pi)$ such that $t_1^0\in I_\eta$ and*

$$\big|e^{i2\pi\eta a_{20}(t_1)}-1\big| \geq L_1|\eta|^{-K},\ \ \forall\ t_1\in I_\eta,$$

with $|I_\eta|\leq\frac{1}{\pi\beta|\eta|^{K+1}}$ and $|I_\eta|\geq\frac{L_2}{2\pi\beta|\eta|^{K+1}}$, where $\beta=||a'_{20}||_\infty+1$ and K is given in (4.17).

Using a simple variation of Lemma 3.2 in [27] and Lemma 4.7 we can see that

$$|\hat{v}_1(t,\xi)| \leq \frac{|\xi|^K}{L_1} \int_{\mathbf{T}} |(\partial_{t_2} + i\xi(a_{20}(t_1) + \partial_{t_2}(h_2 - h_1)(t))\hat{v}_1(\cdot,\xi)|dt_2$$

for $t_1 \in I_\xi$ and $t_2 \in \mathbf{T}$, where I_ξ, L_1 and K are given in Lemma 4.7. Integrating the last inequality with respect to $t_1 \in I_\xi$ and $t_2 \in I_j(\xi)$ we obtain

$$\begin{aligned} &\int_{I_j(\xi)} \int_{I_\xi} |\hat{v}_1(t_1,t_2,\xi)|dt_1 dt_2 \qquad (4.18) \\ &\leq \frac{\sqrt{2\pi}}{L_1}|\xi|^K |I_j(\xi)| \int_{\mathbf{T}} \int_{I_\xi} |(\partial_{t_2} + i\xi(a_{20}(t_1) + \partial_{t_2}(h_2 - h_1)(t))\hat{v}_1(t_1,t_2,\xi)|^2 dt_1 dt_2. \end{aligned}$$

It follows from (4.18), from Lemma 4.7 and from (4.16), with $I_1 = I_\xi$, that

$$\begin{aligned} &||\hat{v}_1(\cdot,\xi)||^2_{L^2(\mathbf{T}\times I_j(\xi))} \leq 8\pi^2 \frac{2\pi\beta|\xi|^{K+1}}{L_2} \frac{\sqrt{2\pi}}{L_1} |\xi|^K |I_j(\xi)| \\ &\times \int_{\mathbf{T}} \int_{I_\xi} |(\partial_{t_2} + i\xi(a_{20}(t_1) + \partial_{t_2}(h_2 - h_1)(t)))\hat{v}_1(t_1,t_2,\xi)|^2 dt_1 dt_2 \\ &+16\pi^2 \|(\partial_{t_1} + i\xi a_{10}(t_2))\hat{v}_1(\cdot,\xi)\|^2_{L^2(\mathbf{T}\times I_j(\xi))} + \frac{32\pi^2}{|\xi|^2} \|\hat{v}_1(\cdot,\xi)\|^2_{L^2(\mathbf{T}\times I_j(\xi))}. \end{aligned}$$

Thus, we can conclude that there exists a positive constant C such that for $|\xi| > 8\pi$ we have

$$\begin{aligned} ||\hat{v}_1(\cdot,\xi)||^2_{L^2(\mathbf{T}\times I_j(\xi))} &\leq C|\xi|^{2K+1}|I_j(\xi)| \int_{\mathbf{T}} \int_{I_\xi} |(\partial_{t_2} + i\xi(a_{20}(t_1) \\ &+ \partial_{t_2}(h_2 - h_1)(t))\hat{v}_1(t_1,t_2,\xi)|^2 dt_1 dt_2 \qquad (4.19) \\ &+ C\|(\partial_{t_1} + i\xi a_{10}(t_2))\hat{v}_1(\cdot,\xi)\|^2_{L^2(\mathbf{T}\times I_j(\xi))}. \end{aligned}$$

Finally, thanks to the fact that $I_i(\xi) \cap I_j(\xi) = \emptyset$ for all $i \neq j$ and (4.19) we have

$$
\begin{aligned}
&\|\hat{v}_1(\cdot,\xi)\|^2_{L^2(\mathbf{T}\times \mathbf{A}_1(\xi))} = \|\hat{v}_1(\cdot,\xi)\|^2_{L^2(\mathbf{T}\times\{\cup_{j=0}^{n(\xi)+1} I_j(\xi)\})} \\
&= \sum_{j=0}^{n(\xi)+1} \int_{I_j(\xi)} \int_{\mathbf{T}} |\hat{v}_1(t_1,t_2,\xi)|^2 dt_1 dt_2 = \sum_{j=0}^{n(\xi)+1} \|\hat{v}_1(\cdot,\xi)\|^2_{L^2(\mathbf{T}\times I_j(\xi))} \\
&\leq \sum_{j=0}^{n(\xi)+1} \Big(C|\xi|^{2K+1} |I_j(\xi)| \\
&\times \int_{\mathbf{T}} \int_{I_\xi} |(\partial_{t_2} + i\xi(a_{20}(t_1) + \partial_{t_2}(h_2-h_1)(t)))\hat{v}_1(t_1,t_2,\xi)|^2 dt_1 dt_2 \\
&+ C\|(\partial_{t_1} + i\xi a_{10}(t_2))\hat{v}_1(\cdot,\xi)\|^2_{L^2(\mathbf{T}\times I_j(\xi))} \Big) \\
&\leq C|\xi|^{2K+1} \left(\sum_{j=0}^{n(\xi)+1} |I_j(\xi)| \right) \\
&\times \int_{\mathbf{T}} \int_{I_\xi} |(\partial_{t_2} + i\xi(a_{20}(t_1) + \partial_{t_2}(h_2-h_1)(t_1,t_2)))\hat{v}_1(t_1,t_2,\xi)|^2 dt_1 dt_2 \\
&+ C||(\partial_{t_1} + i\xi a_{10}(t_2))\hat{v_1}(t,\xi)||^2_{L^2(\mathbf{T}^1\times \mathbf{A}_1(\xi))} \\
&\leq C 2\pi |\xi|^{2K+1} \\
&\times \int_{\mathbf{T}} \int_{I_\xi} |(\partial_{t_2} + i\xi(a_{20}(t_1) + \partial_{t_2}(h_2-h_1)(t_1,t_2)))\hat{v}_1(t_1,t_2,\xi)|^2 dt_1 dt_2 \\
&+ C||(\partial_{t_1} + i\xi a_{10}(t_2))\hat{v}_1(\cdot,\xi)||^2_{L^2(\mathbf{T}^1\times \mathbf{A}_1(\xi))} \\
&\leq C 2\pi |\xi|^{2K+1} ||(\partial_{t_2} + i\xi(a_{20}(t_1) + \partial_{t_2}(h_2-h_1)(t_1,t_2))\hat{v}_1(\cdot,\xi)||^2_{L^2(\mathbf{T}^2)} \\
&+ C||(\partial_{t_1} + i\xi a_{10}(t_2))\hat{v_1}(t,\xi)||^2_{L^2(\mathbf{T}^2)},
\end{aligned}
$$

since

$$\sum_{j=0}^{n(\xi)+1} |I_j(\xi)| = \sum_{j=0}^{n(\xi)+1} \int_{I_j(\xi)} ds = \int_{\{\cup_{j=0}^{n(\xi)+1} I_j(\xi)\}} ds \leq \int_{\mathbf{T}} ds = 2\pi.$$

Thanks to Cauchy-Schwartz inequality, as before, it follows from the last inequality that

$$||\hat{v}_1(\cdot,\xi)||^2_{L^2(\mathbf{T}\times\mathbf{A}_1(\xi))} \leq C|\xi|^{2K+1} ||\hat{g}_1(\cdot,\xi)\|_{L^2(\mathbf{T}^2)} ||\hat{v}_1(\cdot,\xi)\|_{L^2(\mathbf{T}^2)}. \tag{4.20}$$

Combining the inequalities (4.15) and (4.20) gives

$$
\begin{aligned}
\|\hat{v}_1(\cdot,\xi)\|^2_{L^2(\mathbf{T}^2)} &= ||\hat{v}_1(\cdot,\xi)||^2_{L^2(\mathbf{T}\times\mathbf{A}_1(\xi))} + ||\hat{v}_1(\cdot,\xi)||^2_{L^2(\mathbf{T}\times\mathbf{A}_2(\xi))} \\
&\leq C(|\xi|^{2K+1} + |\xi|^2) \|\hat{g}_1(\cdot,\xi)\|_{L^2(\mathbf{T}^2)} ||\hat{v}_1(\cdot,\xi)\|_{L^2(\mathbf{T}^2)},
\end{aligned}
$$

which proves Lemma 4.1 in this case if $|\xi| \geq \max\{8\pi, \frac{1}{M-m}\}$.

4.2 Proof of estimate (4.3) in Case FT-2

Now we assume that $a_{10} \equiv \frac{m}{n}$ and $a_{20} \equiv \kappa$, where $m, n \in \mathbf{Z}, n \neq 0$ and m and n do not have a common factor which is larger than one, and κ is an irrational number. Also, we assume that there exists a point of finite type for the vector fields $X_1 = \partial_{t_1} + a_1(t_1,t_2)\partial_x$ and

$X_2 = \partial_{t_2} + a_2(t_1, t_2)\partial_x$. Next, we will prove that Lemma 4.1 is true under these assumptions, i.e., we will present the proof of the **Case FT-2.**

For this we consider the j^{th}-order convergent $\frac{p_j}{q_j}$ of κ. We know that

$$q_j \to \infty \text{ as } j \to \infty \ \text{ and } \left|\kappa - \frac{p_j}{q_j}\right| \leq \frac{1}{q_j q_{j+1}} < \frac{1}{q_j^2}, \quad \forall j \in \mathbf{N}. \tag{4.21}$$

Next, let $M > 0$ be a fixed constant to be chosen later. Thanks to (4.21) there is $n_0 \in \mathbf{N}$ such that $q_{n_0} > 8M + 1$. Let $\xi \in \mathbf{Z}\backslash\{0\}$ be such that $|\xi| \geq \sqrt{q_{n_0}}$. Since $\{q_j\}$ increases as j goes to ∞ we have that $\{\sqrt{q_j}\}$ increases too. Thus, there exists some $j_0 \geq n_0$ such that

$$\sqrt{q_{j_0}} \leq |\xi| \leq \sqrt{q_{j_0+1}}. \tag{4.22}$$

Now, we set $q = \min\{\ell \in \mathbf{Z} : \ell\frac{p_{j_0}}{q_{j_0}} \in \mathbf{Z}, \ell\frac{m}{n} \in \mathbf{Z}\}$, define the sets

$$\mathcal{A} = q\mathbf{Z},\ \mathcal{B} = \mathbf{Z}\backslash\{\mathcal{A}\} \quad \text{ and } \quad \mathcal{C} = \{\xi \in \mathbf{Z} : \sqrt{q_{j_0}} \leq |\xi| \leq \sqrt{q_{j_0+1}}\},$$

and split the proof of Lemma 4.1 in two subcases.

First one: $\xi \in \mathcal{A} \cap \mathcal{C}$.

We shall need the following result, whose proof is a straightforward generalization of the proof of inequality (4.9) in [27].

Lemma 4.8. *Suppose that there exists a point $(t_1^\circ, t_2^\circ, x^\circ) \in \mathbf{T}^3$ of finite type for the vector fields $X_1 = \partial_{t_1} + a_1(t)\partial_x$ and $X_2 = \partial_{t_2} + a_2(t)\partial_x$. Then there exist $\delta > 0$ and $C > 0$ such that for $\xi \in \mathbf{Z}\backslash\{0\}$ and $\varphi_\xi \in C^\infty(\mathbf{T}^2)$ we have*

$$\int_{[-\frac{\delta}{2},\frac{\delta}{2}]^2} |\varphi_\xi(t)|^2 dt \leq C\epsilon^2 ||\varphi_\xi||^2_{L^2(\mathbf{T}^2)} + C(1 + \frac{1}{2\epsilon^2}) \sum_{j=1}^{2} ||Y_j\varphi_\xi||^2_{L^2(\mathbf{T}^2)} \tag{4.23}$$

for any $\epsilon > 0$, where $Y_j = \partial_{t_j} + i\xi a_j(t),\ j = 1, 2$.

Since $\xi \in \mathcal{A} \cap \mathcal{C}$ there exists $\ell \in \mathbf{Z}$ such that $\xi = \ell q$ with $\sqrt{q_{j_0}} \leq |\xi = \ell q| \leq \sqrt{q_{j_0+1}}$.

Recalling that $\hat{v}_1(t, \ell q) = e^{i\ell q h_1(t)}\hat{u}(t, \ell q)$, $\hat{g}_1(t, \ell q) = e^{i\ell q h_1(t)}\hat{f}(t, \ell q)$ and

$$\sum_{j=1}^{2} ||Y_j\hat{u}(\cdot, \ell q)||^2_{L^2(\mathbf{T}^2)} = \int_{\mathbf{T}^2} \hat{f}(t, \ell q)\bar{\hat{u}}(t, \ell q)dt,$$

it follows from the inequality (4.23), with $\varphi_{\ell q}(t) = \hat{u}(t, \ell q)$, and the Cauchy-Schwarz inequality that

$$\int_{[-\frac{\delta}{2},\frac{\delta}{2}]^2} |\hat{v}_1(t, \ell q)|^2 dt \leq \left(C\epsilon^2 + C(1 + \frac{1}{2\epsilon^2})\frac{\epsilon_1^2}{2}\right) ||\hat{v}_1(\cdot, \ell q)||^2_{L^2(\mathbf{T}^2)} \tag{4.24}$$
$$+C(1 + \frac{1}{2\epsilon^2})\frac{1}{2\epsilon_1^2}||\hat{g}_1(\cdot, \ell q)||^2_{L^2(\mathbf{T}^2)}.$$

Using inequalities (4.1) and (4.2) in [27] and choosing ϵ and ϵ_1 such that $C\epsilon^2 + C(1 + \frac{1}{2\epsilon^2})\frac{\epsilon_1^2}{2} = \frac{1}{2}$ we obtain

$$\begin{aligned} ||\hat{v}_1(\cdot, \ell q)||^2_{L^2(\mathbf{T}^2)} &\leq C||(\partial_{t_1} + i\ell q\frac{m}{n})\hat{v}_1(\cdot, \ell q)||^2_{L^2(\mathbf{T}^2)} \\ &+ C||[\partial_{t_2} + i\ell q(\frac{p_{j_0}}{q_{j_0}} + \partial_{t_2}(h_2 - h_1)(t))]\hat{v}_1(\cdot, \ell q)||^2_{L^2(\mathbf{T}^2)} \\ &+ C||\hat{g}_1(\cdot, \ell q)||_{L^2(\mathbf{T}^2)}. \end{aligned} \tag{4.25}$$

Thus, using Lemma 4.2 - formula (4.6), (4.21) and (4.22) it follows from the last inequality that

$$\begin{aligned} ||\hat{v}_1(\cdot,\ell q)||^2_{L^2(\mathbf{T}^2)} &\leq C||(\partial_{t_1}+i\ell q\frac{m}{n})\hat{v}_1(t,\ell q)||^2_{L^2(\mathbf{T}^2)} \\ &+ 2C\|(\partial_{t_2}+i\ell q(\kappa+\partial_{t_2}(h_2-h_1)(t))\hat{v}_1(t,\ell q)\|^2_{L^2(\mathbf{T}^2)} \\ &+ \frac{3C}{8Mq^2_{j_0}}\|\hat{v}_1(\cdot,\ell q)\|^2_{L^2(\mathbf{T}^2)}+C||\hat{g}_1(\cdot,\ell q)||_{L^2(\mathbf{T}^2)}. \end{aligned}$$

Finally, taking $M>\frac{3C}{2}$, recalling that $q_{j_0}\geq 1$, using the Cauchy-Schwartz inequality, from the last inequality we obtain

$$\|\hat{v}_1(\cdot,\ell q)\|^2_{L^2(\mathbf{T}^2)}\leq C\|\hat{g}_1(\cdot,\ell q)\|_{L^2(\mathbf{T}^2)}\|\hat{v}_1(\cdot,\ell q)\|_{L^2(\mathbf{T}^2)} \tag{4.26}$$

which implies the proof of Lemma 4.1 in this situation.

Second one: $\xi\in\mathcal{B}\cap\mathcal{C}$.

Since $\xi\in\mathcal{C}$ we have $\sqrt{q_{j_0}}\leq|\xi|\leq\sqrt{q_{j_0+1}}$. Since $\xi\in\mathcal{B}$ it follows that ξ can not be simultaneously a multiple of q_{j_0} and n.

To prove Lemma 4.1 in this case we need the following result, whose proof will be omitted.

Lemma 4.9. *Suppose that $\xi\in\mathcal{B}$. Then, we have either*

$$\left|e^{i2\pi\xi\frac{p_{j_0}}{q_{j_0}}}-1\right|\geq\frac{1}{q_{j_0}}\quad or\quad \left|e^{i2\pi\xi\frac{m}{n}}-1\right|\geq\frac{1}{n}.$$

Next, we set

$$\tilde{B}_1=\{\xi\in\mathcal{B}\cap\mathcal{C}:\left|e^{i2\pi\xi\frac{m}{n}}-1\right|\geq\frac{1}{n}\}\ \text{and}\ \tilde{B}_2=\{\xi\in\mathcal{B}\cap\mathcal{C}:\left|e^{i2\pi\xi\frac{p_{j_0}}{q_{j_0}}}-1\right|\geq\frac{1}{q_{j_0}}\}.$$

We know that

$$|\hat{v}_1(t,\xi)|\leq\frac{1}{|e^{i2\pi\xi\frac{p_{j_0}}{q_{j_0}}}-1|}\int_{\mathbf{T}}|(\partial_{t_2}+i\xi(\frac{p_{j_0}}{q_{j_0}}+\partial_{t_2}(h_2-h_1)(t))\hat{v}_1(\cdot,\xi)|dt_2, \tag{4.27}$$

provided $e^{i2\pi\xi\frac{p_{j_0}}{q_{j_0}}}-1\neq 0$. Then, thanks to last inequality and the definition of $\tilde{B}_2$ we have

$$||\hat{v}_1(\cdot,\xi)||^2_{L^2(\mathbf{T}^2)}\leq 4\pi^2q^2_{j_0}\int_{\mathbf{T}^2}|\partial_{t_2}\hat{v}_1(t,\xi)+i\xi(\frac{p_{j_0}}{q_{j_0}}+\partial_{t_2}(h_2-h_1))\hat{v}_1(t,\xi)|^2dt.$$

Thus, using Lemma 4.2 - formula (4.6), (4.21), (4.22) and choosing $M>\max\{\frac{3C}{2},8\pi^2\}$ it follows from the last inequality that there exists $C>0$ such that

$$\|\hat{v}_1(\cdot,\xi)\|^2_{L^2(\mathbf{T}^2)}\leq Cq^2_{j_0}\|(\partial_{t_2}+i\xi(\kappa+\partial_{t_2}(h_2-h_1))\hat{v}_1(\cdot,\xi))\|^2_{L^2(\mathbf{T}^2)}.$$

Using the fact that $\xi\in\mathcal{C}$ gives $q^2_{j_0}\leq|\xi|^4$ and therefore the last inequality implies that

$$\begin{aligned} \|\hat{v}_1(\cdot,\xi)\|^2_{L^2(\mathbf{T}^2)} &\leq C|\xi|^4\|(\partial_{t_2}+i\xi(\kappa+\partial_{t_2}(h_2-h_1))\hat{v}_1(\cdot,\xi))\|^2_{L^2(\mathbf{T}^2)} \\ &\leq C|\xi|^4\|(\partial_{t_2}+i\xi(\kappa+\partial_{t_2}(h_2-h_1))\hat{v}_1(\cdot,\xi))\|^2_{L^2(\mathbf{T}^2)} \\ &+ C|\xi|^4\|(\partial_{t_1}+i\xi\frac{m}{n})\hat{v}_1(t,\xi)\|^2_{L^2(\mathbf{T}^2)}. \end{aligned}$$

Then, as before, there exists a positive constant C such that

$$\|\hat{v}_1(\cdot,\xi)\|^2_{L^2(\mathbf{T}^2)}\leq C|\xi|^4\|\hat{g}_1(\cdot,\xi)\|_{L^2(\mathbf{T}^2)}\|\hat{v}_1(\cdot,\xi)\|_{L^2(\mathbf{T}^2)}, \tag{4.28}$$

which implies the proof of Lemma 4.1 when $\xi\in\tilde{B}_2$.

The proof of the case when $\xi\in\tilde{B}_1$ is similar. Taking $F=\{\xi\in\mathbf{Z}:|\xi|\leq\sqrt{q_{j_0}}\}$ where $q_{j_0}\geq 8K+1$ with $M>\max\{\frac{3C}{2},8\pi^2\}$ completes the proof of Lemma 4.1.

5 Concluding Remarks

We conclude with a few remarks concerning the global C^∞ hypoellipticity of the operator (1.5) when $a_1 = 0$ but a_2 depends on all variables. We would like to point out that it is an interesting and difficult open problem. More precisely, we start by considering the operator

$$P = -\partial_{t_1}^2 - \left(\partial_{t_2} + a(t_1, t_2, x)\partial_x\right)^2 \tag{5.1}$$

and analyze the case when there are no points of finite type (see Theorem 5.2 below). We notice that the condition that there are no points of finite type for the vector fields $X_1 = \partial_{t_1}$ and $X_2 = \partial_{t_2} + a(t_1, t_2, x)\partial_x$ reads as $a = a(t_2, x)$.

We begin by considering two related questions about a real vector field $L = \sum_{j=1}^N a_j(x)\partial_{x_j}$, where $a_j \in C^\infty(\mathbf{T}^N)$, $j = 1, \dots, N$, which does not vanish on $\mathbf{T}^N$.

Q1: Is it true that L is globally hypoelliptic in $\mathbf{T}^N$ iff its adjoint is globally hypoelliptic?

Q2: For $c \in C^\infty(\mathbf{T}^N; \mathbf{R})$ is it possible that L is globally hypoelliptic in $\mathbf{T}^N$ but $L + c$ is **not**?

Concerning question Q1 we recall that if its adjoint L^* is globally hypoelliptic in $\mathbf{T}^N$ then it follows from Theorem 2.2 in Chen and Chi [8] that L is globally hypoelliptic in $\mathbf{T}^N$. We do not know if the converse of Theorem 2.2 in [8] is true. However, here we will prove it when $N = 2$. Also, we will show that the answer to question Q2 is negative in this case.

Lemma 5.1. *Let $L = a(t,x)\partial_t + b(t,x)\partial_x$, where $a, b \in C^\infty(\mathbf{T}^2, \mathbf{R})$, and define*

$$M = L + c(t,x),\ c \in C^\infty(\mathbf{T}^2, \mathbf{R}).$$

We assume that L does not vanish on $\mathbf{T}^2$. If L is globally hypoelliptic in $\mathbf{T}^2$ then M is globally hypoelliptic in $\mathbf{T}^2$.

Note. $L^* = -a(t,x)\partial_t - b(t,x)\partial_x - \frac{\partial a}{\partial t} - \frac{\partial b}{\partial x}$ is a particular M.

Proof. Since L is a real vector field such that L is globally hypoelliptic in $\mathbf{T}^2$ and does not vanish it follows from Theorem A in Hounie [29] that there exist a differmorphism

$$(s, y) = \tau(t, x), \quad \tau : \mathbf{T}^2 \longrightarrow \mathbf{T}^2,$$

such that L becomes

$$\tilde{L} = f(s,y)(\partial_s + \alpha\partial_y),$$

where f is a never vanishing $C^\infty(\mathbf{T}^2, \mathbf{R})$ function and α is a non-Liouville number. Thus M becomes

$$\tilde{M} = f(s,y)(\partial_s + \alpha\partial_y) + \tilde{c}(s,y),$$

where $\tilde{c} \in C^\infty(\mathbf{T}^2, \mathbf{R})$. Then we have

$$\frac{1}{f(s,y)}\tilde{M} = \partial_s + \alpha\partial_y + g(s,y), \qquad \text{where } g \in C^\infty(\mathbf{T}^2; \mathbf{R}).$$

It is easy to conclude that M is globally hypoelliptic in $\mathbf{T}^2$ if and only if there exists $c > 0$ such that

$$|k + \alpha m - ig_{00}| \geq \frac{1}{(|k| + |m|)^c}, \quad \text{if } |k| + |m| \geq c,$$

where g_{00} is the avarage of g with respect to (s, y) and $(k, m) \in \mathbf{Z}^2$. But the last condition is easy to prove since

$$|k + \alpha m - ig_{00}| \geq |g_{00}| \geq \frac{1}{(|k| + |m|)}, \quad \text{if } |k| + |m| \text{ is large enough}$$

when $g_{00} \neq 0$. On the other hand, if $g_{00} = 0$ we use the fact that α is a non-Liouville number in order to guarantee that there exists $c > 0$ such that

$$|k + \alpha m| = |m|| \frac{k}{m} + \alpha| \geq |m| \frac{1}{|m|^c} \geq \frac{1}{|m|^c} \geq \frac{1}{(|k| + |m|)^c}, \quad \text{if } |k| + |m| \text{ is large enough.}$$

This complete the proof of the lemma. □

Next we use Lemma 5.1 in order to prove the following partial result about the global hypoellipticity of the operator (5.1).

Theorem 5.2. *Assume that every point in $\mathbf{T}^3$ is of infinite type for the vector fields $X_1 = \partial_{t_1}$ and $X_2 = \partial_{t_2} + a(t_1, t_2, x)\partial_x$. Then the following conditions are equivalents.*
1) The operator P in (5.1) is globally hypoelliptic in $\mathbf{T}^3$.
2) $L = \partial_{t_2} + a(t_2, x)\partial_x$ is globally hypoelliptic in $\mathbf{T}^2$.
3) There is a diffeomorphism in $\mathbf{T}^2$ such that $L = \alpha\partial_s + \beta\partial_y$, with the real numbers α, β satisfying the following Diophantine condition: There exist $K > 0$, $C > 0$ such that

$$|k\alpha + m\beta| \geq \frac{C}{(1 + |(k, m)|)^K}, \quad \forall\ (k, m) \in \mathbf{Z}^2 \backslash \{0\}. \tag{5.2}$$

Proof. **1) $\Longrightarrow$ 2):** Let $u \in \mathcal{D}'(\mathbf{T}^2_{t_2,x})$ such that $Lu = f$, $f \in C^\infty(\mathbf{T}^2_{t_2,x})$. Then $Pu = -L^2u = -Lf \in C^\infty \Rightarrow u \in C^\infty(\mathbf{T}^2_{t_2,x})$.

2) $\Longrightarrow$ 3): Since $L = \partial_{t_2} + a(t_2, x)\partial_x$ is globally hypoelliptic in $\mathbf{T}^2$ it follows from Lemma 5.1 that L^* is globally hypoelliptic in $\mathbf{T}^2$ and by using Theorem 1.3 in [8] it implies that condition 3) holds true.

3) $\Longrightarrow$ 1): The condition 3) implies that in the new variables (t_1, s, y) we can write

$$P = -\partial_{t_1}^2 - (\alpha\partial_s + \beta\partial_y)^2$$

where the real numbers α, β satisfy the Diophantine condition (5.2). Thus, P is globally hypoelliptic in $\mathbf{T}^3$. This completes the proof of the theorem. □

Finally we present a necessary condition for the global hypoellipticity for the operator (5.1). It is easily seen that the following result holds true.

Theorem 5.3. *Let P be given by $P = -\partial_{t_1}^2 - \left(\partial_{t_2} + a(t_1, t_2, x)\partial_x\right)^2$ where $a \in C^\infty(\mathbf{T}^3)$ is real-valued. We assume that P is globally hypoelliptic in $\mathbf{T}^3$. Then, for each $x \in \mathbf{T}$ fixed, the function*

$$t \in \mathbf{T}^2 \mapsto a(t, x) \in \mathbf{R}$$

does not vanish identically.

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