The Ambrosetti-Prodi type result to a quasilinear Neumann problem

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ABSTRACT. We study the problem $-\Delta_p u = f(x, u) + t$ in $\Omega$ with Neumann boundary condition $|\nabla u|^{p-2} \partial u/\partial \nu = 0$ on $\partial \Omega$. There exists at $t_0 \in \mathbb{R}$ such that for $t > t_0$, there is no solution. If $t \leq t_0$, there is at least a minimal solution, and for $t < t_0$, there are at least two distinct solutions. We use the sub-super-solution method, a priori estimates and degree theory.

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1 Introduction

We study the problem

\[(P_t) \begin{cases} -\Delta_p u = f(x, u) + t & \text{in } \Omega, \\
|\nabla u|^{p-2} \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial \Omega, \end{cases}\]

where \(\Omega \subset \mathbb{R}^n\) is an open bounded domain with smooth boundary \(\partial \Omega\), \(t \in \mathbb{R}\) is a parameter and \(f : \Omega \times \mathbb{R} \to \mathbb{R}\) is a Carathéodory function. Here \(\Delta_p u = \text{div}(|\nabla u|^{p-2} \nabla u)\) denotes the \(p\)-Laplace operator for \(1 < p < \infty\). We also assume

\[(1) \liminf_{s \to \infty} \frac{f(x, s)}{|s|^{p-2}s} > 0\]

and

\[(2) \limsup_{s \to -\infty} \frac{f(x, s)}{|s|^{p-2}s} < 0,\]

where the limits are uniform in \(x \in \Omega\). Conditions (1) and (2) are a sort of eigenvalue crossing of \(f\). These assumptions imply, respectively,

\[(3) f(x, s) \geq \mu |s|^{p-2}s - C, \ s > 0,\]

and

\[(4) f(x, s) \geq -\mu |s|^{p-2}s - C, \ s < 0,\]

for some constants \(C > 0\) and \(\mu > 0\). The next hypothesis is standard in order to apply the sub-super-solution method. We will assume that for all \(M > 0\) there is \(\lambda > 0\) such that

\[(5) f(x, u) + \lambda |u|^{p-2}u \text{ is nondecreasing in } u \text{ on } [-M, M].\]

Hypothesis (5) is in fact not needed when one assumes \(p = 2\) and \(f\) to be \(C^1\). Since in this case the derivative is \(f_u + \lambda > 0\) for \(u\) belonging to some interval \([-M, M]\) and \(\lambda\) large.

A function \(u \in W^{1,p}(\Omega)\) is called a (weak) solution of \((P_t)\) if

\[
\int_{\Omega} |\nabla u|^{p-2} \nabla u \nabla \phi dx = \int_{\Omega} f(x, u) \phi dx - t \int_{\Omega} u \phi dx, \text{ for all } \phi \in C^1(\bar{\Omega}),
\]

where \(F(x, t) = \int_0^t f(x, s) ds\).
Theorem 1.1  Suppose (1), (2), (5) and that there exists a constant $c$ such that

\[ f(x, s) \leq c(1 + |s|^{p-1}) \quad \text{for all } s \in \mathbb{R} \text{ and uniformly for } x \in \Omega. \]

Then there exists $t_0 \in \mathbb{R}$ such that

(i) if $t > t_0$, then $(P_t)$ has no solution; and  
(ii) if $t \leq t_0$, then $(P_t)$ has at least a minimal solution.

Moreover, assume $f$ is locally Lipschitz continuous in $s$ uniformly a.e. in $x \in \Omega$, then

(iii) there exists $t_1 \leq t_0$ such that for $t < t_1$ problem $(P_t)$ has at least two distinct solutions; and

(iv) if moreover $f \in C(\overline{\Omega} \times \mathbb{R})$, then $t_1 = t_0$.

Problems of this nature fit into a general framework devised in the pioneering paper [1]. They studied the problem

\[ \begin{cases}
-\Delta u = g(u) + h(x) & \text{in } \Omega \\
 u = 0 & \text{on } \partial\Omega,
\end{cases} \]

where $g$ interacts with the spectrum $0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \lambda_4 \ldots$ of $-\Delta$ in $H_0^1(\Omega)$ in such a way that $g'(-\infty) < \lambda_1 < g'(+\infty) < \lambda_2$. Assuming that $g'' > 0$, they proved that the singular set $S$ of the mapping $\Phi(u) = -\Delta u - g(u)$ from $U = \{u \in C^{2,\alpha}(\Omega) : u = 0 \text{ on } \partial\Omega\}$ to $V = \{u \in C^{0,\alpha}(\Omega) : u = 0 \text{ on } \partial\Omega\}$, $0 < \alpha < 1$, consists of a codimension one manifold parametrized over $\{u \in U : \int_{\Omega} u \varphi_1 = 0\}$, where $\varphi_1$ denotes the first eigenfunction corresponding to $\lambda_1$. Moreover, $\Phi(S)$ is a smooth codimension one manifold and $V - \Phi(S)$ has exactly two components $V_0$ and $V_2$. If $h \in V_2$, then (7) has 2 solutions. If $h \in V_0$, then (7) has no solution. If $h \in \Phi(S)$, then (7), has one solution. Subsequently in [5] these results have been generalized by parametrizing $\Phi(S)$ in $H_0^1$. A complete characterization showing that $\Phi$ is globally diffeomorphic to the fold map is in [4]. A rich structure is of the mapping $\Phi$ is described in [6], [7] and [8] in space dimension 1. In the present paper we do not give such a detailed description to $(P_t)$, since we do not have the Hilbert space structure. Also, $-\Delta_p$ does not possesses the same regularizing properties of $-\Delta$. 

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Equation (7) with Neumann condition $\partial u/\partial \nu = 0$ on $\partial \Omega$, was studied in [3] and [14]. A result similar to ours for Dirichlet boundary condition $u = 0$ on $\partial \Omega$, has been addressed in [2] and [10] with different techniques.

In Section 2 we show some a priori estimates for solutions of $(P_t)$. We use these lemmas to define adequate sets to apply degree theory to prove Theorem 1.1 in Section 3.

## 2 Preliminary

Throughout this section we assume (1), (2) and (6). We begin establishing an a priori bound for solutions of $(P_t)$.

**Lemma 2.1** Let $u$ be a weak solution of $(P_t)$ in $W^{1,p}_0(\Omega)$. If $t$ belongs to a bounded interval, then $\|u\|_{L^\infty} \leq c$, where $c > 0$ is a constant depending on $t$, but not on $u$.

**Proof.** First we will prove that $\|u^-\|_{L^\infty}$ is bounded. Indeed, multiply equation $(P_t)$ by $\varphi = \max(u^- - k, 0) \in W^{1,p}(\Omega)$ where $k > 0$. Denoting by $A_k = \{x \in \Omega : u^- > k\}$ and using (4), we obtain

$$\int_{A_k} |\nabla (u^- - k)|^p = -\int_{A_k} (f(x, u) + t)\varphi \leq \int_{A_k} (c(u^-)^{p-1} + c + |t|)(u^- - k)$$

$$= \int_{A_k} c(u^-)^{p-1}(u^- - k) + (c + |t|) \int_{A_k} (u^- - k)$$

$$\leq \int_{A_k} C((u^- - k)^p + k^{p-1}(u^- - k)) + (c + |t|) \int_{A_k} (u^- - k)$$

$$= C \int_{A_k} (u^- - k)^p + (Ck^{p-1} + c + |t|) \int_{A_k} (u^- - k).$$

(8)

In the course of this proof, constant $C > 0$ may vary from line to line. By Hölder and Sobolev inequality,

$$\int_{A_k} (u^- - k)^p \leq |A_k|^\frac{p}{n} \left( \int_{A_k} (u^- - k)^{\frac{np}{n-p}} \right)^{\frac{n-p}{p}}$$

(9)

$$\leq |A_k|^\frac{p}{n} C \left( \int_{A_k} |\nabla (u^- - k)|^p + \int_{A_k} (u^- - k)^p \right),$$

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Thus, (8) and (9) imply

(10) \((|A_k|^{-\frac{\pi}{n}} - C) \int_{A_k} (u^- - k)^p \leq C \int_{A_k} (u^- - k)^p + C(k^{p-1} + 1 + |t|) \int_{A_k} (u^- - k),\)

hence

\(\left(|A_k|^{-\frac{\pi}{n}} - C\right) \int_{A_k} (u^- - k)^p \leq C(k^{p-1} + 1 + |t|) \int_{A_k} (u^- - k).\)

Notice that \(|A_k| \to 0\) as \(k \to \infty\), indeed by the proof of Lemma 2.2 below one obtains

\(|A_k| = \int_{u^- > k} dx \leq \int_{A_k} \frac{(u^-)^{p-1}}{k^{p-1}} \leq C k^{1-p}.

Therefore, \(|A_k|^{-\frac{n}{p}} - C > 0\) for every \(k \geq k_0\) where \(k_0\) is fixed, large enough and does not depend on \(u\).

By Hölder inequality and (10),

\[\int_{A_k} (u^- - k) \leq |A_k|^\frac{p-1}{p} \left( \int_{A_k} (u^- - k)^p \right) \frac{1}{p} \leq C|A_k|^\frac{p-1}{p} \left( k^{p-1} + 1 + |t| \int_{A_k} (u^- - k) \right)^\frac{1}{p}.\]

Thus

\[\left( \int_{A_k} (u^- - k) \right)^{\frac{p-1}{p}} \leq C|A_k|^\frac{p-1}{p} \left( k^{p-1} + 1 + |t| \right)^\frac{1}{p} \left( |A_k|^{-\frac{\pi}{n}} - C \right).\]

Consequently

\[\int_{A_k} (u^- - k) \leq C|A_k| \left( \frac{k^{p-1} + 1 + |t|}{|A_k|^{-\frac{\pi}{n}} - C} \right)^\frac{1}{p-1} = C|A_k|^{1 + \frac{p-1}{n(p-1)}} \left( \frac{k^{p-1} + 1 + |t|}{1 - |A_k|^{-\frac{\pi}{n}}C} \right)^\frac{1}{p-1}.\]
We can assume that $1 - |A_k|^{\frac{p}{n}} C \geq 1/2$ for $k \geq k_0$, and then
\[
\int_{A_k} (u^- - k) \leq C |A_k|^{1 + \frac{p(p-1)}{n}} (k^{p-1} + 1 + |t|)^{p-1} \\
= C |A_k|^{1 + \frac{p(p-1)}{n}} k \left( 1 + \frac{1 + |t|}{k^{p-1}} \right)^{p-1} \\
\leq C |A_k|^{1 + \frac{p(p-1)}{n}} k \left( 1 + \frac{1 + |t|}{k_0^{p-1}} \right)^{p-1} \\
\leq C |A_k|^{1 + \frac{p(p-1)}{n}} k.
\]

We are in position to apply [11, Lemma 5.1, p. 71], to conclude that $||u^-||_{L^\infty}$ is bounded by a constant that depends only on $t, k_0, \mu, p, n$ and $||u^-||_{L^1(A_{k_0})}$.

As we said before, constant $k_0$ does not depend on $u$. We shall bound $||u^-||_{L^1(A_{k_0})}$ more accurately, this is done below.

Multiplying the equation $(P_t)$ by $-u^-$ and using (4),
\[
0 \leq \int_{\Omega} |\nabla u^-|^p = \int_{\Omega} - f(x, u^-) u^- - t \int_{\Omega} u^- \\
\leq -\mu \int_{\Omega} (u^-)^p + C \int_{\Omega} u^- - t \int_{\Omega} u^-.
\]

Hence
\[
\int_{\Omega} |\nabla u^-|^p + \mu \int_{\Omega} (u^-)^p \leq (C + |t|) \int_{\Omega} u^- \leq \frac{\mu}{2} \int_{\Omega} (u^-)^p + C(1 + |t|).
\]

It follows that $||u^-||_{W^{1,p}}$ is bounded. Thus,
\[
\int_{A_{k_0}} |u^-| \leq \int_{\Omega} |u^-| \leq C \left( \int_{\Omega} |u^-|^p \right)^{\frac{1}{p}} \leq C(1 + |t|).
\]

Therefore $||u^-||_{L^\infty}$ is bounded by a constant depending only on $t, \mu, p, n$.

Now we prove that $||u^+||_{L^\infty}$ is bounded. We only need to prove that $||u^+||_{L^p}$ is bounded, since the computations above to prove the boundedness of $||u^-||_{L^\infty}$ can be performed in a similar manner to conclude that $||u^+||_{L^\infty}$ is bounded.

Assume by contradiction that there are $a, b \in \mathbb{R}$ such that $||u^+_{t_n}||_{L^p} \to \infty$ with $t_n \in [a, b]$. Define $w_n = \frac{u^+_{t_n}}{||u^+_{t_n}||_{L^p}}$. Notice that $w_n$ is bounded in $W^{1,p}$. 

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Indeed, using (6),
\[
\int_{\Omega} |\nabla u_n|^p = \int_{\Omega} f(x, u_n)u_n + t \int_{\Omega} u_n \\
\leq c \int_{\Omega} |u_n|^p + c(1 + |t|) \int_{\Omega} |u_n| \\
\leq c(1 + |t|) \int_{\Omega} |u_n|^p.
\]

Thus \(w_n\) is bounded in \(W^{1,p}\). So we can assume that \(w_n \rightharpoonup w\) in \(W^{1,p}\), \(w_n \to w\) in \(L^p\) and \(w_n \to w\) a.e. \(x \in \Omega\). Moreover, \(||w||_{L^p} = 1\) and \(w \geq 0\), since \(||u_n^+||_{L^\infty}\) is bounded. Notice that \(w_n^- \to 0\) a.e. \(x \in \Omega\).

Let \(\varphi \in W^{1,p}(\Omega)\) and \(\varphi \geq 0\), then
\[
\int |\nabla w_n|^{p-2} \nabla w_n \nabla \varphi = \int \frac{f(x, u_n)}{|u_n^+|} \varphi + t \int \frac{\varphi}{|u_n^+|} \\
\geq \mu \int |w_n|^{p-1} \varphi + o(n).
\]
Letting \(n \to \infty\) we get
\[
\int |\nabla w|^{p-2} \nabla w \nabla \varphi \geq \mu \int w^{p-1} \varphi.
\]
Taking \(\varphi \equiv 1\), we obtain \(\mu \int w^{p-1} \leq 0\), a contradiction to the fact that \(w \geq 0\) and \(w \not\equiv 0\). The proof is complete. \(\square\)

Since weak solutions of \((P_t)\) are bounded, by a result from [11], these solutions belong to \(C(\Omega)\). The \(C^{1,\alpha}(\Omega)\) regularity follows from [13].

**Lemma 2.2** Problem \((P_t)\) has no solution for a large enough \(t > 0\).

**Proof.** By (3) and (4) we have that \(f(x, s) \geq \mu |s|^{p-1} - C\) for every \(s \in \mathbb{R}\). Integrating \((P_t)\), we obtain
\[
0 = \int_{\Omega} f(x, u_t) + t|\Omega| \\
\geq \mu \int_{\Omega} |u_t|^{p-1} - C|\Omega| + t|\Omega|.
\]
Then
\[ \mu \int_{\Omega} |u_t|^{p-1} + t|\Omega| \leq C|\Omega|, \]
which gives a contradiction for \( t > 0 \) large enough. \qed

**Lemma 2.3** Problem \((P_t)\) has a sub-solution for all \( t \).

**Proof.** There is a constant \( z_t \) satisfying
\[
\begin{cases}
-\Delta_p z \leq f(x, z) + t & \text{in } \Omega \\
z \leq 0 & \text{in } \partial \Omega \\
|\nabla z|^{p-2} \frac{\partial z}{\partial \nu} = 0 & \text{on } \partial \Omega,
\end{cases}
\]
In fact, since \((2)\) reads as \( f(x, u) \geq -\mu |u|^{p-2}u - C \) for some constants \( \mu, C > 0 \), then clearly
\[ z_t = - \left( \frac{|t| + C}{\mu} \right)^{\frac{1}{p-1}} \]
satisfies the requirements we need. \qed

**Remark 2.4** It is easy to see that every constant \( z < z_t \) is a strict sub-solution.

**Lemma 2.5** If \( u_t \) is a solution of problem \((P_t)\), then \( u_t \geq z_t \).

**Proof.** It is enough to show that \((z_t - \varepsilon - u_t)^+ = 0\) for all \( \varepsilon > 0 \). Suppose by contradiction that there is \( \varepsilon_0 > 0 \) such that \((z_t - \varepsilon_0 - u_t)^+ \neq 0\) is nontrivial, then \( \Omega_\varepsilon = \{ x \in \Omega : u_t(x) < z_t - \varepsilon \} \neq \emptyset \) for all \( 0 < \varepsilon < \varepsilon_0 \). Multiplying the equation \((P_t)\) by \((z_t - \varepsilon - u_t)^+ \neq 0\) and using \((4)\),
\[
\int_{\Omega_\varepsilon} |\nabla u_t|^{p-2} \nabla u_t \nabla (z_t - \varepsilon - u_t)^+ = \int_{\Omega_\varepsilon} f(x, u_t)(z_t - \varepsilon - u_t)^+
\]
\[
+ t \int_{\Omega_\varepsilon} (z_t - \varepsilon - u_t)^+
\]
\[
\geq -\mu \int_{\Omega_\varepsilon} |u_t|^{p-2} u_t (z_t - \varepsilon - u_t)^+
\]
\[
- (C - t) \int_{\Omega_\varepsilon} (z_t - \varepsilon - u_t)^+.
\]
Note that for $x \in \Omega$ we have $\nabla(z_t - \varepsilon - u_t)^+ = -\nabla u_t$, then one gets
\[
\int_{\Omega} |\nabla u_t|^p \leq \mu \int_{\Omega} |u_t|^{p-2} u_t (z_t - \varepsilon - u_t)^+ + (C - t) \int_{\Omega} (z_t - \varepsilon - u_t)^+
\]
\[
= -\mu \int_{\Omega} |u_t|^{p-1} (z_t - \varepsilon - u_t)^+ + (C - t) \int_{\Omega} (z_t - \varepsilon - u_t)^+
\]
\[
\leq -\mu \int_{\Omega} |z_t - \varepsilon|^{p-1} (z_t - \varepsilon - u_t)^+ + (C - t) \int_{\Omega} (z_t - \varepsilon - u_t)^+
\]
\[
= (-\mu|z_t - \varepsilon|^{p-1} + C - t) \int_{\Omega} (z_t - \varepsilon - u_t)^+.
\]
It is a contradiction, since $-\mu|z_t - \varepsilon|^{p-1} + C - t < 0$ for $\varepsilon$ small enough. \(\square\)

3 Proof of Theorem 1.1

The proof is divided in three parts:

Step 1. there is $t_0$ such that for $t \leq t_0$ there is a solution and no solution exists for $t > t_0$;

Step 2. there is a minimal solution for $t \leq t_0$;

Step 3. there is $t_1$ such that for $t < t_1$ there are two distinct solutions.

1. First, we will prove that there exists a $t'$ such that problem $(P_t)$ has a solution for all $t \leq t'$. Actually, take $t' = \inf\{-f(x,0) : x \in \Omega\}$, so $0$ is a super-solution for the problem $(P_t)$ for all $t \leq t'$. In fact, $-\Delta u = 0 \geq f(x,0) + t'$ if and only if $t' \leq -f(x,0)$ for all $x$. Since, by Lemma 2.3, for each $t$ there is a negative constant sub-solution $z_t$, the claim follows by the method of sub-super-solution, see [12].

Now, we will see that if $(P_t)$ has a solution for some $t$, then so does for all $s \leq t$. Indeed, let $u$ be a solution of $(P_t)$ corresponding to $t$, then clearly $u$ is a super-solution of $(P_t)$ corresponding to $s$ for every $s < t$, since
\[
-\Delta u = f(x,u) + t \geq f(x,u) + s.
\]
Again the assertion follows by the method of sub-super-solution.

Hence the set $S = \{t : (P_t) has a solution\}$ is nonempty and bounded from above by Lemma 2.2. Actually, we have $(-\infty, t_0) \subset S$, where $t_0$ is the supremum of $S$. Let $\{t_n\}$ be such that $t_n \nearrow t_0$. By virtue of the a
priori estimates for solutions $u_t$ corresponding to $t$ of Lemma 2.1, there is a subsequence $t_{n_k}$ such that $u_{t_{n_k}} \rightarrow u_{t_0}$ in $C^1(\Omega)$, and so $u_{t_0}$ is a solution of $(P_t)$. Thus $S = (-\infty, t_0]$, and Step 1 is proved. This proves $(i)$.

2. Let $t \leq t_0$, then $(P_t)$ has a minimal solution if $t \leq t_0$. Indeed, by Lemma 2.3, $(P_t)$ has a solution and that any solution satisfy

$$u \geq -\frac{1}{\mu}(|t| + C).$$

But

$$z = -\frac{1}{\mu}(|t| + C)$$

is a sub-solution of $(P_t)$ and $z \leq u$. Thus $(P_t)$ has a minimal solution in the set of functions satisfying $w \geq z$ in $\Omega$, since all solutions satisfy the property $w \geq z$ in $\Omega$. The proof of $(ii)$ is complete.

3. Define

$$t_1 = \sup \inf_{\mathbb{R}} \Omega \{ -f(x, s) \}.$$

By (3) and (4), $f(x, s)$ is bounded from below. Hence $t_1$ is well defined. If $t < t_1$, then there is an $\sigma$ such that $t < \inf_{\mathbb{R}} \Omega \{ -f(x, \sigma) : x \in \Omega \}$. Thus $w_t = \sigma$ is a super-solution for $(P_t)$ corresponding to $t$. Since for all $t$ the problem $(P_t)$ has a constant sub-solution $z_t$, which can be taken $z_t < w_t$, we can solve $(P_t)$ for $t$. Thus there is a solution $u_t$ for (1), corresponding to $t$, obtained by sub-super-solution method, so $z_t \leq u_t \leq w_t$. In particular, we have $t_1 \leq t_0$. We will apply degree theory to find a second solution for $t < t_1$.

First assume that $f(x, s)$ is locally Hölder continuous in $s$, uniformly in $x \in \Omega$. Then we can choose a constant $w$ such that $w > \sigma$ and $t < \inf_{\Omega} \{ -f(x, w) : x \in \Omega \}$. Thus $w > \sigma$ is a also a super-solution. Moreover, for a fixed constant $z$ with $z < z_t$ one concludes that $z$ is also a sub-solution. Define the open set

$$\Lambda = \{ v \in C(\overline{\Omega}) : z < v < w \text{ in } \overline{\Omega} \},$$

thus $u_t \in \Lambda$. There is a $\lambda > 0$ such that $f(x, u) + \lambda|u|^{p-2}u$ is nondecreasing in $u$ on $[z, w]$ for $x \in \overline{\Omega}$, see (5). Clearly, $u$ is a solution of $(P_t)$ if and only
if $u$ is a fixed point of the compact operator $K_t : C(\overline{\Omega}) \to C(\overline{\Omega})$ defined by

$$K_t v = u$$

where $u$ is a solution of

$$
\begin{cases}
-\Delta_p u + \lambda |u|^{p-2}u = f(x, v) + \lambda |v|^{p-2}v + t & \text{in } \Omega \\
|\nabla u|^{p-2} \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial \Omega.
\end{cases}
$$

Since $u_t \in \Lambda$, we can suppose that $\deg(I - K_t, \Lambda, 0)$ is well defined, i.e. $0 \notin (I - K_t)(\partial \Lambda)$. Otherwise the proof is complete. We claim that

$$\deg(I - K_t, \Lambda, 0) = 1.$$

In fact, take $\varphi \in \Lambda$ and define $T_\eta : C(\overline{\Omega}) \to C(\overline{\Omega})$ by $T_\eta v = \eta K_t v + (1 - \eta)\varphi$ for $\eta \in (0, 1]$. Hence for $v \in \Lambda$ we have that

$$-\Delta_p z + \lambda |z|^{p-2}z \leq -\Delta_p (K_t v) + \lambda |K_t v|^{p-2}K_t v \leq -\Delta_p w + \lambda |w|^{p-2}w.$$

It follows by a Weak Comparison Principle that $K_t v \in \overline{\Lambda}$, i.e. $z \leq K_t v \leq w$. Since $\Lambda$ is convex and $\varphi \in \Lambda$ we obtain $T_\eta : \Lambda \to \Lambda$ for all $\eta \in (0, 1]$. Thus $0 \notin (I - T_\eta)(\partial \Lambda)$ for all $\eta \in [0, 1]$. In this way, $\deg(I - T_\eta, \Lambda, 0)$ is well defined and independent of $\eta$. Hence

$$\deg(I - K_t, \Lambda, 0) = \deg(I - T_\eta, \Lambda, 0) = \deg(I - T_0, \Lambda, 0).$$

The map $T_0 u = \varphi$ for every $u \in \Lambda$ and $\varphi \in \Lambda$, then

$$\deg(I - T_0, \Lambda, 0) = 1.$$

The claim is proved.

We also claim that for large enough $M > 0$ we have

$$\deg(I - K_t, B_M, 0) = 0,$$

where $B_M = \{ u \in C(\overline{\Omega}) : \|u\|_{C(\overline{\Omega})} < M \}$.

By Lemma 2.1, one concludes that all fixed points $u_s$ of $K_s$, that is, $K_s u_s = u_s$ satisfy $\|u\|_{C(\overline{\Omega})} < M$ independently of $s \in [t, t_0 + 1]$. Notice that $t$ is kept fixed and $t < t_1 \leq t_0$. We can assume that $\Lambda \subset B_M$. Thus

$$\deg(I - K_t, B_M, 0) = \deg(I - K_{t_0 + 1}, B_M, 0) = 0,$$

since $K_{t_0 + 1}$ does not have fixed points. The claim is proved.
In conclusion

\[ \text{deg}(I - K, B_M - \Lambda, 0) = -1. \]

Therefore \((P_t)\) has a solution which is not in \(\Lambda\).

Finally, assume that \(f\) is continuous on \(\overline{\Omega} \times \mathbb{R}\). For \(t < t_0\) we have that \(u_{t_0}\), the minimal solution corresponding to \(t_0\), is a super-solution for \((P_t)\) corresponding to \(t\). Moreover, \(z_t \leq u_{t_0}\) and we have a solution \(u_t\) such that \(z_t \leq u_t \leq u_{t_0}\). In order to apply the ideas from the previous case (where \(f\) is only Hölder continuous), we need a sub-solution \(z\) and a super-solution \(w\) such that \(z < u_t < w\) in \(\overline{\Omega}\). We can choose \(z\) as a fixed number less than \(z_t\). We claim that \(w = u_{t_0} + \theta\) is a super-solution if \(\theta > 0\) is small enough. Actually, we have

\[ -\Delta_p w = f(x, u_{t_0}) + t_0 = f(x, u_{t_0} + \theta) + t + (f(x, u_{t_0}) - f(x, u_{t_0} + \theta) + t - t_0). \]

It is a consequence of the continuity of \(f\) that \(f(x, u_{t_0}) - f(x, u_{t_0} + \theta) + t - t_0 \geq 0\) for \(\theta > 0\) small enough. Thus \(-\Delta_p w \geq f(x, u_{t_0} + \theta) + t\), and the claim is proved.

The rest of the proof follows as in the previous case.

\[ \Box \]

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