The Ambrosetti-Prodi type result to a quasilinear Neumann problem

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ABSTRACT. We study the problem $-\Delta_p u = f(x,u) + t$ in Ω with Neumann boundary condition $|\nabla u|^{p-2}\partial u/\partial\nu = 0$ on $\partial\Omega$. There exists at $t_0 \in \mathbb{R}$ such that for $t > t_0$, there is no solution. If $t \le t_0$, there is at least a minimal solution, and for $t < t_0$, there are at least two distinct solutions. We use the sub-super-solution method, a priori estimates and degree theory.

AMS MATHEMATICS SUBJECT CLASSIFICATION 2000. Primary 35J65; Secondary 35J25, 58E07.

KEY WORDS. A priori estimates, degree theory, sub-super-solutions.

To appear in Proceedings of the Edinburgh Mathematical Society

1 Introduction

We study the problem

$$(P_t) \qquad \begin{cases} -\Delta_p u = f(x,u) + t & \text{in } \Omega \\ |\nabla u|^{p-2}\dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^n$ is an open bounded domain with smooth boundary $\partial\Omega$, $t \in \mathbb{R}$ is a parameter and $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function. Here $\Delta_p u = \mathrm{div}(|\nabla u|^{p-2}\nabla u)$ denotes the p-Laplace operator for $1 < p < \infty$. We also assume

$$(1) \qquad \liminf_{s\to\infty} \frac{f(x,s)}{|s|^{p-2}s} > 0$$

and

$$(2) \qquad \limsup_{s\to-\infty} \frac{f(x,s)}{|s|^{p-2}s} < 0,$$

where the limits are uniform in $x \in \Omega$. Conditions (1) and (2) are a sort of eigenvalue crossing of f. These assumptions imply, respectively,

$$(3) \qquad f(x,s) \geq \mu|s|^{p-2}s - C, \ \ s > 0,$$

and

$$(4) \qquad f(x,s) \geq -\mu|s|^{p-2}s - C, \ \ s < 0,$$

for some constants $C > 0$ and $\mu > 0$. The next hypothesis is standard in order to apply the sub-super-solution method. We will assume that for all $M > 0$ there is $\lambda > 0$ such that

$$(5) \qquad f(x,u) + \lambda|u|^{p-2}u \text{ is nondecreasing in } u \text{ on } [-M, M].$$

Hypothesis (5) is in fact not needed when one assumes $p = 2$ and f to be C^1. Since in this case the derivative is $f_u + \lambda > 0$ for u belonging to some interval $[-M, M]$ and λ large.

A function $u \in W^{1,p}(\Omega)$ is called a (weak) solution of (P_t) if

$$\int_\Omega |\nabla u|^{p-2}\nabla u \nabla \phi dx = \int_\Omega f(x,u)\phi dx - t\int_\Omega u\phi dx, \text{ for all } \phi \in C^1(\overline{\Omega}),$$

where $F(x,t) = \int_0^t f(x,s)ds$.

Theorem 1.1 *Suppose* (1), (2), (5) *and that there exists a constant c such that*

(6) $\quad f(x,s) \le c(1+|s|^{p-1})$ *for all* $s \in \mathbb{R}$ *and uniformly for* $x \in \Omega$.

Then there exists $t_0 \in \mathbb{R}$ *such that*

(i) if $t > t_0$*, then* (P_t) *has no solution; and*

(ii) if $t \le t_0$*, then* (P_t) *has at least a minimal solution.*

Moreover, assume f is locally Lipschitz continuous in s uniformly a.e. in $x \in \Omega$*, then*

(iii) there exists $t_1 \le t_0$ *such that for* $t < t_1$ *problem* (P_t) *has at least two distinct solutions; and*

(iv) if moreover $f \in C(\overline{\Omega} \times \mathbb{R})$*, then* $t_1 = t_0$.

Problems of this nature fit into a general framework devised in the pioneering paper [1]. They studied the problem

$$(7) \qquad \begin{cases} -\Delta u = g(u) + h(x) & \text{in } \Omega \\ \quad u = 0 & \text{on } \partial\Omega, \end{cases}$$

where g interacts with the spectrum $0 < \lambda_1 < \lambda_2 \le \lambda_3 \le \lambda_4 ...$ of $-\Delta$ in $H_0^1(\Omega)$ in such a way that $g'(-\infty) < \lambda_1 < g'(+\infty) < \lambda_2$. Assuming that $g'' > 0$, they proved that the singular set S of the mapping $\Phi(u) = -\Delta u - g(u)$ from $U = \{u \in C^{2,\alpha}(\Omega) : u = 0 \text{ on } \partial\Omega\}$ to $V = \{u \in C^{0,\alpha}(\Omega) : u = 0 \text{ on } \partial\Omega\}$, $0 < \alpha < 1$, consists of a codimension one manifold parametrized over $\{u \in U : \int_\Omega u\varphi_1 = 0\}$, where φ_1 denotes the first eigenfunction corresponding to λ_1. Moreover, $\Phi(S)$ is a smooth codimension one manifold and $V - \Phi(S)$ has exactly two components V_0 and V_2. If $h \in V_2$, then (7) has 2 solutions. If $h \in V_0$, then (7) has no solution. If $h \in \Phi(S)$, then (7), has one solution. Subsequently in [5] these results have been generalized by parametrizing $\Phi(S)$ in H_0^1. A complete characterization showing that Φ is globally diffeomorphic to the fold map is in [4]. A rich structure is of the mapping Φ is described in [6], [7] and [8] in space dimension 1. In the present paper we do not give such a detailed description to (P_t), since we do not have the Hilbert space structure. Also, $-\Delta_p$ does not possesses the same regularizing properties of $-\Delta$.

Equation (7) with Neumann condition $\partial u/\partial\nu = 0$ on $\partial\Omega$, was studied in [3] and [14]. A result similar to ours for Dirichlet boundary condition $u = 0$ on $\partial\Omega$, has been addressed in [2] and [10] with different techniques.

In Section 2 we show some a priori estimates for solutions of (P_t). We use these lemmas to define adequate sets to apply degree theory to prove Theorem 1.1 in Section 3.

2 Preliminary

Throughout this section we assume (1), (2) and (6). We begin establishing an a priori bound for solutions of (P_t).

Lemma 2.1 *Let u be a weak solution of (P_t) in $W_0^{1,p}(\Omega)$. If t belongs to a bounded interval, then $\|u\|_{L^\infty} \le c$, where $c > 0$ is a constant depending on t, but not on u.*

Proof. First we will prove that $\|u^-\|_{L^\infty}$ is bounded. Indeed, multiply equation (P_t) by $\varphi = \max(u^- - k, 0) \in W^{1,p}(\Omega)$ where $k > 0$. Denoting by $A_k = \{x \in \Omega : u^- > k\}$ and using (4), we obtain

$$\begin{aligned}
\int_{A_k} |\nabla(u^- - k)|^p &= -\int_{A_k} (f(x,u) + t)\varphi \\
&\le \int_{A_k} (c(u^-)^{p-1} + c + |t|)(u^- - k) \\
&= \int_{A_k} c(u^-)^{p-1}(u^- - k) + (c + |t|)\int_{A_k} (u^- - k) \\
&\le \int_{A_k} C((u^- - k)^p + k^{p-1}(u^- - k)) + (c + |t|)\int_{A_k} (u^- - k) \\
&= C\int_{A_k} (u^- - k)^p + (Ck^{p-1} + c + |t|)\int_{A_k} (u^- - k). \qquad (8)
\end{aligned}$$

In the course of this proof, constant $C > 0$ may vary from line to line. By Hölder and Sobolev inequality,

$$\begin{aligned}
\int_{A_k} (u^- - k)^p &\le |A_k|^{\frac{p}{n}} \left(\int_{A_k} (u^- - k)^{\frac{np}{n-p}}\right)^{\frac{n-p}{n}} \\
&\le |A_k|^{\frac{p}{n}} C \left(\int_{A_k} |\nabla(u^- - k)|^p + \int_{A_k} (u^- - k)^p\right), \qquad (9)
\end{aligned}$$

Thus, (8) and (9) imply

(10)

$$(|A_k|^{-\frac{p}{n}} - C) \int_{A_k} (u^- - k)^p \le C \int_{A_k} (u^- - k)^p + C(k^{p-1} + 1 + |t|) \int_{A_k} (u^- - k),$$

hence

$$(|A_k|^{-\frac{p}{n}} - C) \int_{A_k} (u^- - k)^p \le C(k^{p-1} + 1 + |t|) \int_{A_k} (u^- - k).$$

Notice that $|A_k| \to 0$ as $k \to \infty$, indeed by the proof of Lemma 2.2 below one obtains

$$|A_k| = \int_{u^- > k} dx \le \int_{A_k} \frac{(u^-)^{p-1}}{k^{p-1}} \le C k^{1-p}.$$

Therefore, $|A_k|^{-p/n} - C > 0$ for every $k \ge k_0$ where k_0 is fixed, large enough and does not depend on u.

By Hölder inequality and (10),

$$\begin{aligned} \int_{A_k} (u^- - k) &\le |A_k|^{\frac{p-1}{p}} \left(\int_{A_k} (u^- - k)^p \right)^{\frac{1}{p}} \\ &\le C|A_k|^{\frac{p-1}{p}} \left(\frac{k^{p-1} + 1 + |t|}{|A_k|^{-\frac{p}{n}} - C} \int_{A_k} (u^- - k) \right)^{\frac{1}{p}}. \end{aligned}$$

Thus

$$\left(\int_{A_k} (u^- - k) \right)^{\frac{p-1}{p}} \le C|A_k|^{\frac{p-1}{p}} \left(\frac{k^{p-1} + 1 + |t|}{|A_k|^{-\frac{p}{n}} - C} \right)^{\frac{1}{p}}.$$

Consequently

$$\begin{aligned} \int_{A_k} (u^- - k) &\le C|A_k| \left(\frac{k^{p-1} + 1 + |t|}{|A_k|^{-\frac{p}{n}} - C} \right)^{\frac{1}{p-1}} \\ &= C|A_k|^{1+\frac{p}{n(p-1)}} \left(\frac{k^{p-1} + 1 + |t|}{1 - |A_k|^{\frac{p}{n}} C} \right)^{\frac{1}{p-1}}. \end{aligned}$$

We can assume that $1-|A_k|^{\frac{p}{n}}C \geq 1/2$ for $k \geq k_0$, and then

$$\begin{aligned}\int_{A_k}(u^- - k) &\leq C|A_k|^{1+\frac{p(p-1)}{n}}(k^{p-1}+1+|t|)^{p-1}\\ &= C|A_k|^{1+\frac{p(p-1)}{n}}k\left(1+\frac{1+|t|}{k^{p-1}}\right)^{p-1}\\ &\leq C|A_k|^{1+\frac{p(p-1)}{n}}k\left(1+\frac{1+|t|}{k_0^{p-1}}\right)^{p-1}\\ &\leq C|A_k|^{1+\frac{p(p-1)}{n}}k.\end{aligned}$$

We are in position to apply [11, Lemma 5.1, p. 71], to conclude that $||u^-||_{L^\infty}$ is bounded by a constant that depends only on $t,\ k_0, \mu,\ p,\ n$ and $||u^-||_{L^1(A_{k_0})}$. As we said before, constant k_0 does not depend on u. We shall bound $||u^-||_{L^1(A_{k_0})}$ more accurately, this is done below.

Multiplying the equation (P_t) by $-u^-$ and using (4),

$$\begin{aligned}0 \leq \int_\Omega |\nabla u^-|^p &= \int_\Omega -f(x,u)u^- - t\int_\Omega u^-\\ &\leq -\mu\int_\Omega (u^-)^p + C\int_\Omega u^- - t\int_\Omega u^-.\end{aligned}$$

Hence

$$\int_\Omega |\nabla u^-|^p + \mu\int_\Omega (u^-)^p \leq (C+|t|)\int_\Omega u^- \leq \frac{\mu}{2}\int_\Omega (u^-)^p + C(1+|t|).$$

It follows that $\|u^-\|_{W^{1,p}}$ is bounded. Thus,

$$\int_{A_{k_0}} |u^-| \leq \int_\Omega |u^-| \leq C\left(\int_\Omega |u^-|^p\right)^{\frac{1}{p}} \leq C(1+|t|).$$

Therefore $\|u^-\|_{L^\infty}$ is bounded by a constant depending only on t, μ, p, n.

Now we prove that $\|u^+\|_{L^\infty}$ is bounded. We only need to prove that $\|u^+\|_{L^p}$ is bounded, since the computations above to prove the boundedness of $\|u^-\|_{L^\infty}$ can be performed in a similar manner to conclude that $\|u^+\|_{L^\infty}$ is bounded.

Assume by contradiction that there are $a, b \in \mathbb{R}$ such that $\|u_{t_n}^+\|_{L^p} \to \infty$ with $t_n \in [a,b]$. Define $w_n = \frac{u_{t_n}^+}{\|u_{t_n}^+\|_{L^p}}$. Notice that w_n is bounded in $W^{1,p}$.

Indeed, using (6),

$$\begin{aligned}\int_\Omega |\nabla u_{t_n}|^p &= \int_\Omega f(x,u_{t_n})u_{t_n} + t\int_\Omega u_{t_n}\\ &\leq c\int_\Omega |u_{t_n}|^p + c(1+|t|)\int_\Omega |u_{t_n}|\\ &\leq c(1+|t|)\int_\Omega |u_{t_n}|^p.\end{aligned}$$

Thus w_n is bounded in $W^{1,p}$. So we can assume that $w_n \rightharpoonup w$ in $W^{1,p}$, $w_n \to w$ in L^p and $w_n \to w$ a.e. $x \in \Omega$. Moreover, $||w||_{L^p} = 1$ and $w \geq 0$, since $||u_{t_n}^+||_{L^\infty}$ is bounded. Notice that $w_n^- \to 0$ a.e. $x \in \Omega$.

Let $\varphi \in W^{1,p}(\Omega)$ and $\varphi \geq 0$, then

$$\begin{aligned}\int |\nabla w_n|^{p-2}\nabla w_n \nabla\varphi &= \int \frac{f(x,u_{t_n})}{\|u_{t_n}^+\|}\varphi + t_n\int \frac{\varphi}{\|u_{t_n}^+\|}\\ &\geq \mu\int |w_n|^{p-1}\varphi + o(n).\end{aligned}$$

Letting $n \to \infty$ we get

$$\int |\nabla w|^{p-2}\nabla w\nabla\varphi \geq \mu\int w^{p-1}\varphi.$$

Taking $\varphi \equiv 1$, we obtain $\mu\int w^{p-1} \leq 0$, a contradiction to the fact that $w \geq 0$ and $w \not\equiv 0$. The proof is complete. □

Since weak solutions of (P_t) are bounded, by a result from [11], these solutions belong to $C(\overline{\Omega})$. The $C^{1,\alpha}(\overline{\Omega})$ regularity follows from [13].

Lemma 2.2 *Problem (P_t) has no solution for a large enough $t > 0$.*

Proof. By (3) and (4) we have that $f(x,s) \geq \mu|s|^{p-1} - C$ for every $s \in \mathbb{R}$. Integrating (P_t), we obtain

$$\begin{aligned}0 &= \int_\Omega f(x,u_t) + t|\Omega|\\ &\geq \mu\int_\Omega |u_t|^{p-1} - C|\Omega| + t|\Omega|.\end{aligned}$$

Then

$$\mu\int_\Omega |u_t|^{p-1} + t|\Omega| \leq C|\Omega|,$$

which gives a contradiction for $t > 0$ large enough. □

Lemma 2.3 *Problem (P_t) has a sub-solution for all t.*

Proof. There is a constant z_t satisfying

$$\text{(11)} \qquad \begin{cases} -\Delta_p z \leq f(x,z) + t & \text{in } \Omega \\ z \leq 0 & \text{in } \overline{\Omega} \\ |\nabla z|^{p-2}\dfrac{\partial z}{\partial \nu} = 0 & \text{on } \partial\Omega, \end{cases}$$

In fact, since (2) reads as $f(x,u) \geq -\mu|u|^{p-2}u - C$ for some constants $\mu, C > 0$, then clearly

$$z_t = -\left(\frac{|t| + C}{\mu}\right)^{\frac{1}{p-1}}$$

satisfies the requirements we need. □

Remark 2.4 *It is easy to see that every constant $z < z_t$ is a strict sub-solution.*

Lemma 2.5 *If u_t is a solution of problem (P_t), then $u_t \geq z_t$.*

Proof. It is enough to show that $(z_t - \varepsilon - u_t)^+ = 0$ for all $\varepsilon > 0$. Suppose by contradiction that there is $\varepsilon_0 > 0$ such that $(z_t - \varepsilon_0 - u_t)^+ 0$ is nontrivial, then $\Omega_\varepsilon = \{x \in \Omega : u_t(x) < z_t - \varepsilon\} \neq \emptyset$ for all $0 < \varepsilon < \varepsilon_0$. Multiplying the equation (P_t) by $(z_t - \varepsilon - u_t)^+ \not\equiv 0$ and using (4),

$$\begin{aligned} \int_{\Omega\varepsilon} |\nabla u_t|^{p-2}\nabla u_t \nabla(z_t - \varepsilon - u_t)^+ &= \int_{\Omega\varepsilon} f(x,u_t)(z_t - \varepsilon - u_t)^+ \\ &\quad + t\int_{\Omega\varepsilon} (z_t - \varepsilon - u_t)^+ \\ &\geq -\mu\int_{\Omega\varepsilon} |u_t|^{p-2}u_t(z_t - \varepsilon - u_t)^+ \\ &\quad -(C - t)\int_{\Omega\varepsilon} (z_t - \varepsilon - u_t)^+. \end{aligned}$$

Note that for $x \in \Omega_\varepsilon$ we have $\nabla(z_t - \varepsilon - u_t)^+ = -\nabla u_t$, then one gets

$$\begin{aligned}
\int_{\Omega_\varepsilon} |\nabla u_t|^p &\leq \mu \int_{\Omega_\varepsilon} |u_t|^{p-2} u_t (z_t - \varepsilon - u_t)^+ + (C-t)\int_{\Omega\varepsilon} (z_t - \varepsilon - u_t)^+ \\
&= -\mu \int_{\Omega_\varepsilon} |u_t|^{p-1} (z_t - \varepsilon - u_t)^+ + (C-t)\int_{\Omega\varepsilon} (z_t - \varepsilon - u_t)^+ \\
&\leq -\mu \int_{\Omega_\varepsilon} |z_t - \varepsilon|^{p-1} (z_t - \varepsilon - u_t)^+ + (C-t)\int_{\Omega\varepsilon} (z_t - \varepsilon - u_t)^+ \\
&= (-\mu |z_t - \varepsilon|^{p-1} + C - t)\int_{\Omega\varepsilon} (z_t - \varepsilon - u_t)^+ .
\end{aligned}$$

It is a contradiction, since $-\mu|z_t - \varepsilon|^{p-1} + C - t < 0$ for ε small enough. □

3 Proof of Theorem 1.1

The proof is divided in three parts:

Step 1. there is t_0 such that for $t \leq t_0$ there is a solution and no solution exists for $t > t_0$;

Step 2. there is a minimal solution for $t \leq t_0$;

Step 3. there is t_1 such that for $t < t_1$ there are two distinct solutions.

1. First, we will prove that there exists a t' such that problem (P_t) has a solution for all $t \leq t'$. Actually, take $t' = \inf\{-f(x,0) : x \in \Omega\}$, so 0 is a super-solution for the problem (P_t) for all $t \leq t'$. In fact, $-\Delta_p 0 = 0 \geq f(x,0) + t'$ if and only if $t' \leq -f(x,0)$ for all x. Since, by Lemma 2.3, for each t there is a negative constant sub-solution z_t, the claim follows by the method of sub-super-solution, see [12].

Now, we will see that if (P_t) has a solution for some t, then so does for all $s \leq t$. Indeed, let u be a solution of (P_t) corresponding to t, then clearly u is a super-solution of (P_t) corresponding to s for every $s < t$, since

$$-\Delta_p u = f(x,u) + t \geq f(x,u) + s.$$

Again the assertion follows by the method of sub-super-solution.

Hence the set $S = \{t : (P_t) \text{ has a solution}\}$ is nonempty and bounded from above by Lemma 2.2. Actually, we have $(-\infty, t_0) \subset S$, where t_0 is the supremum of S. Let $\{t_n\}$ be such that $t_n \nearrow t_0$. By virtue of the a priori estimates for solutions u_t corresponding to t of Lemma 2.1, there is a subsequence t_{n_k} such that $u_{t_{n_k}} \to u_{t_0}$ in $C^1(\overline{\Omega})$, and so u_{t_0} is a solution of (P_t). Thus $S = (-\infty, t_0]$, and Step 1 is proved. This proves (i).

2. Let $t \leq t_0$, then (P_t) has a minimal solution if $t \leq t_0$. Indeed, by Lemma 2.3, (P_t) has a solution and that any solution satisfy

$$u \geq -\frac{1}{\mu}(|t| + C).$$

But

$$z = -\frac{1}{\mu}(|t| + C)$$

is a sub-solution of (P_t) and $z \leq u$. Thus (P_t) has a minimal solution in the set of functions satisfying $w \geq z$ in $\overline{\Omega}$, since all solutions satisfy the property $w \geq z$ in $\overline{\Omega}$. The proof of (ii) is complete.

3. Define

$$t_1 = \sup_{\mathbb{R}} \inf_{\Omega} \{-f(x, s)\}.$$

By (3) and (4), $f(x, s)$ is bounded from below. Hence t_1 is well defined. If $t < t_1$, then there is an σ such that $t < \inf\{-f(x, \sigma) : x \in \Omega\}$. Thus $w_t = \sigma$ is a super-solution for (P_t) corresponding to t. Since for all t the problem (P_t) has a constant sub-solution z_t, which can be taken $z_t < w_t$, we can solve (P_t) for t. Thus there is a solution u_t for (1), corresponding to t, obtained by sub-super-solution method, so $z_t \leq u_t \leq w_t$. In particular, we have $t_1 \leq t_0$. We will apply degree theory to find a second solution for $t < t_1$.

First assume that $f(x, s)$ is locally Hölder continuous in s, uniformly in $x \in \Omega$. Then we can choose a constant w such that $w > \sigma$ and $t < \inf\{-f(x, w) : x \in \Omega\}$. Thus $w > \sigma$ is a also a super-solution. Moreover, for a fixed constant z with $z < z_t$ one concludes that z is also a sub-solution. Define the open set

$$\Lambda = \{v \in C(\overline{\Omega}) : z < v < w \text{ in } \overline{\Omega}\},$$

thus $u_t \in \Lambda$. There is a $\lambda > 0$ such that $f(x, u) + \lambda |u|^{p-2} u$ is nondecreasing in u on $[z, w]$ for $x \in \overline{\Omega}$, see (5). Clearly, u is a solution of (P_t) if and only if u is a fixed point of the compact operator $K_t : C(\overline{\Omega}) \to C(\overline{\Omega})$ defined by $K_t v = u$ where u is a solution of

$$
(12) \qquad \begin{cases} -\Delta_p u + \lambda |u|^{p-2}u = f(x,v) + \lambda |v|^{p-2}v + t & \text{in } \Omega \\ \qquad |\nabla u|^{p-2}\dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega. \end{cases}
$$

Since $u_t \in \Lambda$, we can suppose that $\deg(I - K_t, \Lambda, 0)$ is well defined, i.e. $0 \notin (I - K_t)(\partial\Lambda)$. Otherwise the proof is complete. We claim that

$$
\deg(I - K_t, \Lambda, 0) = 1.
$$

In fact, take $\varphi \in \Lambda$ and define $T_\eta : C(\overline{\Omega}) \to C(\overline{\Omega})$ by $T_\eta v = \eta K_t v + (1-\eta)\varphi$ for $\eta \in [0,1]$. Hence for $v \in \Lambda$ we have that

$$
-\Delta_p z + \lambda |z|^{p-2}z \leq -\Delta_p(K_t v) + \lambda |K_t v|^{p-2}K_t v \leq -\Delta_p w + \lambda |w|^{p-2}w.
$$

It follows by a Weak Comparison Principle that $K_t v \in \overline{\Lambda}$, i.e. $z \leq K_t v \leq w$. Since Λ is convex and $\varphi \in \Lambda$ we obtain $T_\eta : \Lambda \to \Lambda$ for all $\eta \in (0,1]$. Thus $0 \notin (I - T_\eta)(\partial\Lambda)$ for all $\eta \in [0,1]$. In this way, $\deg(I - T_\eta, \Lambda, 0)$ is well defined and independent of η. Hence

$$
\deg(I - K_t, \Lambda, 0) = \deg(I - T_\eta, \Lambda, 0) = \deg(I - T_0, \Lambda, 0).
$$

The map $T_0 u = \varphi$ for every $u \in \Lambda$ and $\varphi \in \Lambda$, then

$$
\deg(I - T_0, \Lambda, 0) = 1.
$$

The claim is proved.

We also claim that for large enough $M > 0$ we have

$$
\deg(I - K_t, B_M, 0) = 0,
$$

where $B_M = \{u \in C(\overline{\Omega}) : \|u\|_{C(\overline{\Omega})} < M\}$.

By Lemma 2.1, one concludes that all fixed points u_s of K_s, that is, $K_s u_s = u_s$ satisfy $\|u\|_{C(\overline{\Omega})} < M$ independently of $s \in [t, t_0 + 1]$. Notice that t is kept fixed and $t < t_1 \leq t_0$. We can assume that $\Lambda \subset B_M$. Thus

$$
\deg(I - K_t, B_M, 0) = \deg(I - K_{t_0+1}, B_M, 0) = 0,
$$

since K_{t_0+1} does not have fixed points. The claim is proved.

In conclusion

$$\deg(I - K, B_M - \Lambda, 0) = -1.$$

Therefore (P_t) has a solution which is not in Λ.

Finally, assume that f is continuous on $\overline{\Omega} \times \mathbb{R}$. For $t < t_0$ we have that u_{t_0}, the minimal solution corresponding to t_0, is a super-solution for (P_t) corresponding to t. Moreover, $z_t \leq u_{t_0}$ and we have a solution u_t such that $z_t \leq u_t \leq u_{t_0}$. In order to apply the ideas from the previous case (where f is only Hölder continuous), we need a sub-solution z and a super-solution w such that $z < u_t < w$ in $\overline{\Omega}$. We can choose z as a fixed number less than z_t. We claim that $w = u_{t_0} + \theta$ is a super-solution if $\theta > 0$ is small enough. Actually, we have

$$-\Delta_p w = f(x, u_{t_0}) + t_0 = f(x, u_{t_0} + \theta) + t + (f(x, u_{t_0}) - f(x, u_{t_0} + \theta) + t - t_0).$$

It is a consequence of the continuity of f that $f(x, u_{t_0}) - f(x, u_{t_0} + \theta) + t - t_0 \geq 0$ for $\theta > 0$ small enough. Thus $-\Delta_p w \geq f(x, u_{t_0} + \theta) + t$, and the claim is proved.

The rest of the proof follows as in the previous case.

□

ACKNOWLEDGEMENT. The authors have been supported by CNPq and FAPESP. Part of this work was done while the first author (F.O.P.) was working at IMECC-UNICAMP. We would like to thank the referee for helpful comments and suggestions.

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