ASYMPTOTIC PROPERTIES IN PARABOLIC PROBLEMS DOMINATED BY p-LAPLACIAN OPERATOR WITH LOCALIZED LARGE DIFFUSION

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ABSTRACT. This paper is concerned with upper semicontinuity of the family of attractors associated to nonlinear reaction diffusion equations with principal part governed by degenerate p-Laplacian in which the diffusion d_λ blows up in localized regions inside the domain.

Keywords: p-laplacian; localized large diffusion; attractor; upper semicontinuity

1. INTRODUCTION

Let $\Omega \subset \mathbb{R}^n$, $n \geqslant 1$, be a smooth bounded domain with smooth boundary $\Gamma = \partial\Omega$, and $\lambda \in (0, 1]$ a parameter. In this work we study the asymptotic behavior of the solutions of the family of parabolic equations

$$\begin{cases} u_t^\lambda - \operatorname{div}(d_\lambda(x)|\nabla u^\lambda|^{p-2}\nabla u^\lambda) + |u^\lambda|^{p-2}u^\lambda = B(u^\lambda) & \text{in } \Omega \\ u^\lambda = 0 & \text{on } \Gamma, \\ u^\lambda(0) = u_0^\lambda, \end{cases} \tag{1.1}$$

as $\lambda \to 0$. The parameter λ represents the fact that, as $\lambda \to 0$, the diffusion d_λ is going to infinity in a localized region Ω_0 inside the physical domain Ω. We assume $p > 2$ and that B is globally Lipschitz and uniformly integrable.

Next we introduce some notations following [1]. Let Ω_0 be a smooth subdomain of Ω, with $\bar{\Omega}_0 \subset \Omega$, $\Omega_0 = \bigcup_{i=1}^{m} \Omega_{0,i}$, where m is a positive integer and $\Omega_{0,i}$ are connected smooth subdomains of Ω with $\bar{\Omega}_{0,i} \cap \bar{\Omega}_{0,j} = \varnothing$, for $i \neq j$. Denote $\Omega_1 = \Omega \setminus \bar{\Omega}_0$, and $\Gamma_{0,i} = \partial\Omega_{0,i}$, $\Gamma_0 = \bigcup_{i=1}^{m} \Gamma_{0,i}$ the boundaries of $\Omega_{0,i}$ and Ω_0, respectively. Notice that $\partial\Omega_1 = \Gamma \cup \Gamma_0$.

The diffusion coefficients $d_\lambda : \Omega \subset \mathbb{R}^n \to \mathbb{R}$ are bounded and smooth functions in Ω, satisfying

$$0 < m_0 \leqslant d_\lambda(x) \leqslant M_\lambda, \tag{1.2}$$

for all $x \in \Omega$ and $0 < \lambda \leqslant 1$. We also assume that the diffusion is large in Ω_0 as $\lambda \to 0$, more precisely,

$$d_\lambda(x) \to \begin{cases} d_0(x), & \text{uniformly on } \Omega_1, (d_0 \in \mathcal{C}^1(\bar{\Omega}_1, (0,\infty))); \\ \infty, & \text{uniformly on compact subsets of } \Omega_0. \end{cases}$$

It is important to notice here that the assumption $\Gamma \cap \Gamma_0 = \varnothing$, that is, the diffusion is large in the interior of Ω, is crucial in the development of our analysis.

If in a reaction-diffusion process the diffusion coefficient behaves as expressed above, intuitively we expect that the solutions will tend to become homogeneous in the regions where the diffusion becomes large, that is, for small values of λ, we expect that the solution of the problem (1.1) will become approximately constant on Ω_0 as it occurs in semilinear problems (see [2] and [3]). For this reason, suppose that u^λ converges to u as $\lambda \to 0$, in some sense, and that u takes on Ω_0 a time dependent spatially constant value, which we will denote by $u_{\Omega_0}(t)$.

Next we intend to obtain the equation that describes the limiting problem. Notice that, since the limit function u is in $W^{1,p}(\Omega)$, its constant value in Ω_0, $u_{\Omega_0}(t)$, cannot be arbitrary. Also, in the boundary $\Gamma_0 = \partial\Omega_0$, we must have $u_{|_{\Gamma_0}} = u_{\Omega_0}(t)$. In Ω_1, we have

$$u_t^\lambda - \operatorname{div}(d_\lambda(x)|\nabla u^\lambda|^{p-2}\nabla u^\lambda) + |u^\lambda|^{p-2}u^\lambda = B(u^\lambda).$$

From properties of convergence of the function $d_\lambda(x)$ in Ω_1 it seems reasonable to have in the limit

$$u_t - \operatorname{div}(d_0(x)|\nabla u|^{p-2}\nabla u) + |u|^{p-2}u = B(u), \text{ for } u \in W^{1,p}(\Omega).$$

In Ω_0, we have

$$\int_{\Omega_0} u_t^\lambda(x,t)\,\mathrm{d}x - \int_{\Omega_0} \operatorname{div}(d_\lambda(x)|\nabla u^\lambda|^{p-2}\nabla u^\lambda)\,\mathrm{d}x + \int_{\Omega_0} |u^\lambda|^{p-2}u^\lambda\,\mathrm{d}x = \int_{\Omega_0} B(u^\lambda(x,t))\,\mathrm{d}x.$$

From Gauss' Divergence Theorem it follows that

$$\int_{\Omega_0} \operatorname{div}(d_\lambda(x)|\nabla u^\lambda|^{p-2}\nabla u^\lambda)\,\mathrm{d}x = -\int_{\Gamma_0} d_\lambda(x)|\nabla u^\lambda|^{p-2}\frac{\partial u^\lambda}{\partial \vec{n}}\,\mathrm{d}x,$$

where $\vec{n}$ denotes the unit inward normal to Ω_0 in the surface integral and then

$$\int_{\Omega_0} u_t^\lambda(x,t)\,\mathrm{d}x + \int_{\Gamma_0} d_\lambda(x)|\nabla u^\lambda|^{p-2}\frac{\partial u^\lambda}{\partial \vec{n}}\,\mathrm{d}x + \int_{\Omega_0} |u^\lambda|^{p-2}u^\lambda\,\mathrm{d}x = \int_{\Omega_0} B(u^\lambda(x,t))\,\mathrm{d}x.$$

Taking the limit as $\lambda \to 0$, we obtain

$$\dot{u}_{\Omega_0}(t)|\Omega_0| + \int_{\Gamma_0} d_0(x)|\nabla u|^{p-2}\frac{\partial u}{\partial \vec{n}}\,\mathrm{d}x + \int_{\Omega_0} |u_{\Omega_0}(t)|^{p-2}u_{\Omega_0}(t)\,\mathrm{d}x = |\Omega_0|B(u_{\Omega_0}(t)).$$

Dividing both sides by $|\Omega_0|$, we get the following ordinary differential equation:

$$\dot{u}_{\Omega_0}(t) + \frac{1}{|\Omega_0|}\left(\int_{\Gamma_0} d_0(x)|\nabla u|^{p-2}\frac{\partial u}{\partial \vec{n}}\,\mathrm{d}x + \int_{\Omega_0} |u_{\Omega_0}(t)|^{p-2}u_{\Omega_0}(t)\,\mathrm{d}x\right) = B(u_{\Omega_0}(t)).$$

With these heuristic considerations and assuming that in the limit we will work with a space of constant functions on Ω_0, we can write the limiting problem in the following way:

$$\begin{cases} u_t - \operatorname{div}(d_0(x)|\nabla u|^{p-2}\nabla u) + |u|^{p-2}u = B(u) & \text{in } \Omega_1 \\ u_{|\Omega_{0,i}} := u_{\Omega_{0,i}} & \text{in } \Omega_{0,i}, \\ \dot{u}_{\Omega_{0,i}} + \dfrac{1}{|\Omega_{0,i}|}\left[\displaystyle\int_{\Gamma_{0,i}} d_0(x)|\nabla u|^{p-2}\frac{\partial u}{\partial \vec{n}}\,\mathrm{d}x + \int_{\Omega_{0,i}} |u_{\Omega_{0,i}}|^{p-2}u_{\Omega_{0,i}}\,\mathrm{d}x\right] = B(u_{\Omega_{0,i}}) & \\ u = 0 & \text{on } \Gamma \\ u(0) = u_0, & \end{cases} \tag{1.3}$$

which is a singular limit problem due to the variation of the parameter λ.

The study of upper semicontinuity of attractors for localized large diffusion semilinear problems has been considered in [4] and lower semicontinuity has been proved in [5]. Attractors for parabolic problems governed by p-Laplacian operators, when $p > 2$, have been appearing in the literature since the nineties, and these systems usually present behavior similar to the Laplacian problems, despite the results have to be, in general, proved by different tools, avoiding linear arguments [6, 7]. The existence of global solutions and a great number of useful properties enjoyed by this kind of problem can be obtained through the theory for monotone operators [8], and one of the first obstacle to study this class of quasilinear equation was the lack of uniqueness when considering non globally Lipschitz perturbations of p-Laplacian. At that time, when firstly parabolic p-Laplacian started to be considered, it was not clear how to deal with multivalued dynamical systems but, from then on, a great understanding of these problems was achieved and, by the union of several isolated efforts, a well organized theory has been arising, [9, 10].

The first work considering large diffusion for p-Laplacian problems is [11], where it is proved that there exists a positive time from which the spatial gradients of solutions go to zero as the diffusion goes to infinity and, as a simple consequence of the Poincaré-Wirtinger Inequality, all the relevant elements to describe the asymptotic behavior are around their own spatial average if the diffusion is large enough. It is also proved that the attractors continuously approach the attractor of an ordinary equation as the diffusion increases to infinity in the whole domain, but the authors only deal with globally Lipschitz perturbations of p-Laplacian in order to avoid multivalued systems. Problems involving both arbitrarily large diffusion and multivalued systems are studied in [12], where authors consider a coupled system admiting non globally Lipschitz perturbations of p-Laplacian, and obtain the lower and upper semicontinuity of attractors.

The singular case of p-Laplacian have been left aside once sensitive points arise when considering $1 < p < 2$. Depending on the conditions imposed, we can not have local estimates for $|\nabla u|$, what suggests that solutions of this kind of problem can exhibit phenomena which can not be incorporated by the classical weak formulation [13]. However, it seems that the same homogenization process must occur in this case and even fast, since the term $|\nabla u|^{p-2}$ can be understood as a nonlinear diffusion coefficient, which is large within the region where $|\nabla u|$ is near zero.

Degenerate p-Laplacian diffusion problems have been applied in several interesting problems. In [14] for example, authors exhibits a connection between p-Laplacian diffusion and the Monge-Kantorovich mass transfer problem by regarding limits as $p \to \infty$ of the flow governed by the p-Laplacian as providing a model for the collapse of an initially unstable sandpile. Another example of application can be found in [15] where the p-Laplacian operator is used in a climatological model and the authors deal with the sensitivity in time of the problem with respect to small changes in solar constant.

In this work we are considering singlevalued systems and the degenerated case of p-Laplacian. Since we guess that the limiting problem is giving by (1.3), in the Section 2 we lay emphasis on the properties of the operators A_λ, $\lambda \in [0,1]$, to obtain the existence of the family of attractors associated. We reach the main result, the upper semicontinuity of attractors, guaranteeing uniform estimates of the solutions, which are easily obtained in Section 3, and continuity properties of the flow. The steps we must follow to show the continuity are entirely described in Section 4 and here we outline them. We first show that, for $u_0 \in L^2(\Omega)$ and $T > 0$, the family of flows $\{T_\lambda(\cdot)u_0\}$ associated to the equation (1.1) is a pre-compact set in $C([0,T], L^2(\Omega))$. This compactness property is proved by using a slightly different version of the Bara's Theorem [16] and it is completely proved here. Then we obtain the continuity of the flow by proving that the limit of any subsequence of $\{T_\lambda(\cdot)u_0\}$ must be $T(\cdot)u_0$, which is associated to (1.3). After this we conclude the upper semicontinuity of attractors.

It is worth to note that in [11, 12] the lower-semicontinuity is obtained in a trivial way by an inclusion argument. However, if the limiting asymptotic set of states is not included inside each attractor belonging to the approaching sequence, this stronger continuity property remains opened for quasilinear evolution problems.

2. Abstract formulation and background results

For simplicity of notation, throughout this paper, V will stand for the space $W_0^{1,p}(\Omega)$ and V_0 for the space $W_{\Omega_0,0}^{1,p}(\Omega) := \{u \in W_0^{1,p}(\Omega) : u \text{ is constant in } \Omega_0\}$. We will also write V^*

and V_0^* to denote the dual spaces of V and V_0, respectively, and H to denote the space $L^2(\Omega)$. We consider

$$\mathcal{D}(A_\lambda) = \left\{u \in V : -\mathrm{div}(d_\lambda(x)|\nabla u|^{p-2}\nabla u) \in \mathrm{L}^2(\Omega)\right\},$$

and for $u \in \mathcal{D}(A_\lambda)$,

$$A_\lambda u = -\mathrm{div}(d_\lambda(x)|\nabla u|^{p-2}\nabla u) + |u|^{p-2}u.$$

If $\lambda = 0$,

$$\mathcal{D}(A_0) = \{u \in V_0 : -\mathrm{div}(d_0(x)|\nabla u|^{p-2}\nabla u) \in \mathrm{L}^2(\Omega_1)\},$$

and for $u \in \mathcal{D}(A_0)$,

$$\begin{aligned} A_0 u &= (-\mathrm{div}(d_0(x)|\nabla u|^{p-2}\nabla u) + |u|^{p-2}u)\chi_{\Omega_1} \\ &\quad + \sum_{i=1}^{m} \frac{1}{|\Omega_{0,i}|}\left(\int\limits_{\Gamma_{0,i}} d_0(x)|\nabla u|^{p-2}\frac{\partial u}{\partial \vec{n}}\,\mathrm{d}x + \int\limits_{\Omega_{0,i}} |u_{\Omega_{0,i}}|^{p-2}u_{\Omega_{0,i}}\,\mathrm{d}x\right)\chi_{\Omega_{0,i}}, \end{aligned}$$

where χ_E is the characteristic function of the set E.

We suppose that $B : H \to H$ is globally Lipschitz and uniformly integrable in $L^1((0,T),H)$, by meaning, given $\epsilon > 0$, there exists $\delta > 0$ such that

$$\int_E \|Bu(\cdot)\|_H\,\mathrm{d}t < \epsilon \quad \text{whenever} \quad \mu(E) < \delta. \tag{2.1}$$

The operators A_λ also can be seen as subdifferential type, meaning $A_\lambda = \partial\varphi^\lambda$, where $\varphi^\lambda : H \to (-\infty,\infty]$ are lower semicontinuous convex functions defined by

$$\varphi^\lambda(u) = \begin{cases} \dfrac{1}{p}\displaystyle\int_\Omega d_\lambda(x)|\nabla u|^p\,\mathrm{d}x + \dfrac{1}{p}\displaystyle\int_\Omega |u|^p\,\mathrm{d}x, & \text{if } u \in V \\ \infty, & \text{otherwise} \end{cases} \tag{2.2}$$

for $\lambda \in (0,1]$ and, for $\lambda = 0$,

$$\varphi^0(u) = \begin{cases} \dfrac{1}{p}\displaystyle\int_{\Omega_1} d_0(x)|\nabla u|^p\,\mathrm{d}x + \dfrac{1}{p}\displaystyle\int_\Omega |u|^p\,\mathrm{d}x, & \text{if } u \in V_0 \\ \infty, & \text{otherwise.} \end{cases} \tag{2.3}$$

The problems (1.1) and (1.3) can be written abstractly as

$$\begin{cases} u_t^\lambda + A_\lambda u^\lambda = B(u^\lambda) \\ u^\lambda(0) = u_0^\lambda \end{cases} \tag{2.4}$$

for $\lambda \in [0,1]$.

Tartar's inequality will be useful in the proof of some properties of the operators A_λ. We present this result here for the reader's convenience.

Lemma 2.1 (Tartar's Inequality)**.** *For any $a, b \in \mathbb{R}^m$, $m \in \mathbb{N}$, there exists a constant $\gamma > 0$, depending on p and m such that*

$$\langle \|a\|^{p-2}a - \|b\|^{p-2}b, a-b\rangle \geqslant \gamma\|a-b\|^p, \text{ if } p \geqslant 2$$

and

$$\langle \|a\|^{p-2}a - \|b\|^{p-2}b, a-b\rangle \leqslant \gamma\|a-b\|^p, \text{ if } 1 < p < 2.$$

Theorem 2.2. *The operators $A_\lambda : \mathcal{D}(A_\lambda) \subset V \to V^*$, $\lambda \in (0,1]$ and $A_0 : \mathcal{D}(A_0) \subset V_0 \to V_0^*$ are monotone, hemicontinuous and coercive.*

Proof. We will prove just the case $\lambda = 0$. For $\lambda \in (0,1]$ the result can be proved analogously. Since

$$\langle A_0 u, v\rangle_{V_0^*,V_0} := \int\limits_{\Omega_1} -\mathrm{div}(d_0(x)|\nabla u|^{p-2}\nabla u)v \ \mathrm{d}x + \int\limits_{\Omega} |u|^{p-2}uv \ \mathrm{d}x + \int\limits_{\Gamma_0} d_0(x)|\nabla u|^{p-2}\frac{\partial u}{\partial \vec{n}} v \ \mathrm{d}x,$$

for all $v \in V_0$, then

$$\begin{aligned}\langle A_0 u - A_0 v, u-v\rangle_{V_0^*,V_0} = &\int\limits_{\Omega_1} - \mathrm{div}(d_0(x)|\nabla u|^{p-2}\nabla u)(u-v) \ \mathrm{d}x + \int\limits_{\Omega} |u|^{p-2}u(u-v) \ \mathrm{d}x \\ &+ \int\limits_{\Gamma_0} d_0(x)|\nabla u|^{p-2}\frac{\partial u}{\partial \vec{n}}(u-v) \ \mathrm{d}x + \int\limits_{\Omega_1} \mathrm{div}(d_0(x)|\nabla v|^{p-2}\nabla v)(u-v) \ \mathrm{d}x \\ &- \int\limits_{\Omega} |v|^{p-2}v(u-v) \ \mathrm{d}x - \int\limits_{\Gamma_0} d_0(x)|\nabla v|^{p-2}\frac{\partial v}{\partial \vec{n}}(u-v) \ \mathrm{d}x\end{aligned}$$

$$= +\int_{\Omega_1} d_0(x)|\nabla u|^{p-2}\nabla u(\nabla u - \nabla v)\ \mathrm{d}x - \int_{\Omega_1} d_0(x)|\nabla v|^{p-2}\nabla v(\nabla u - \nabla v)\ \mathrm{d}x$$
$$+ \int_{\Omega} (|u|^{p-2}u - |v|^{p-2}v)(u-v)\ \mathrm{d}x$$

$$= \int_{\Omega_1} d_0(x)(|\nabla u|^{p-2}\nabla u - |\nabla v|^{p-2}\nabla v)\cdot(\nabla u - \nabla v)\ \mathrm{d}x$$
$$+ \int_{\Omega} (|u|^{p-2}u - |v|^{p-2}v)(u-v)\ \mathrm{d}x$$
$$\geqslant m_0\gamma_0 \int_{\Omega_1} |\nabla u - \nabla v|^p\ \mathrm{d}x + \tilde{\gamma}_0 \int_{\Omega} |u-v|^p\ \mathrm{d}x$$
$$\geqslant \min\{m_0\gamma_0, \tilde{\gamma}_0\} \left[\int_{\Omega_1} |\nabla(u-v)|^p\ \mathrm{d}x + \int_{\Omega} |u-v|^p\ \mathrm{d}x\right]$$
$$= \min\{m_0\gamma_0, \tilde{\gamma}_0\}\|u-v\|_{V_0}^p \geqslant 0,$$

where the constants γ_0 and $\tilde{\gamma}_0$ are derived from Tartar's inequality. Therefore A_0 is monotone.

To obtain that A_0 is hemicontinuous, notice that

$$A_0u = (-\mathrm{div}(d_0(x)|\nabla u|^{p-2}\nabla u)\chi_{\Omega_1} + |u|^{p-2}u + \sum_{i=1}^{m}\frac{1}{|\Omega_{0,i}|}\left(\int_{\Gamma_{0,i}} d_0(x)|\nabla u|^{p-2}\frac{\partial u}{\partial \vec{n}}\,\mathrm{d}x\right)\chi_{\Omega_{0,i}}$$
$$= -\sum_{j=1}^{n}\partial_j a_j(\nabla u) + |u|^{p-2}u + \sum_{i=1}^{m}\frac{1}{|\Omega_{0,i}|}\left(\int_{\Gamma_{0,i}} d_0(x)|\nabla u|^{p-2}\frac{\partial u}{\partial \vec{n}}\,\mathrm{d}x\right)\chi_{\Omega_{0,i}},$$

where $a_j(\nabla u) = (d_0(x)|\nabla u|^{p-2}\nabla u)_j$. From convexity of the gradient operator and continuity of the functions $\partial_j a_j(\cdot)$, we have that

$$\lim_{t\to 0} A_0((1-t)\xi + t\varphi) = -\sum_{j=1}^{n}\partial_j a_j(\nabla\xi) + |\xi|^{p-2}\xi$$

$$+\sum_{i=1}^{m}\frac{1}{|\Omega_{0,i}|}\left(\int_{\Gamma_{0,i}} d_0(x)|\nabla\xi|^{p-2}\frac{\partial\xi}{\partial\vec{n}}\,\mathrm{d}x\right)\chi_{\Omega_{0,i}} = A_0\xi,$$

for $\xi,\varphi\in V_0$.

Furthermore,

$$\begin{aligned}\langle A_0u,u\rangle_{V_0^*,V_0} &= \int_{\Omega_1} -\operatorname{div}(d_0(x)|\nabla u|^{p-2}\nabla u)u\ \mathrm{d}x + \int_{\Omega}|u|^{p-2}uu\ \mathrm{d}x + \int_{\Gamma_0} d_0(x)|\nabla u|^{p-2}\frac{\partial u}{\partial\vec{n}}u\ \mathrm{d}x\\ &= \int_{\Omega_1} d_0(x)|\nabla u|^p\ \mathrm{d}x + \int_{\Omega}|u|^p\ \mathrm{d}x\\ &\geqslant \min\{m_0,1\}\left[\int_{\Omega_1}|\nabla u|^p\ \mathrm{d}x + \int_{\Omega}|u|^p\ \mathrm{d}x\right]\\ &= C_2\|u\|_{V_0}^p,\end{aligned}$$

thus the coercivity of A_0 follows from

$$\frac{\langle A_0u,u\rangle}{\|u\|_{V_0}} \geqslant C_2\|u\|_{V_0}^{p-1}\to+\infty \text{ as } \|u\|_{V_0}\to+\infty, \tag{2.5}$$

since $p>2$. □

The last result allows us to conclude that the hypothesis (H1) in [6] is satisfied. To obtain the existence of weak solutions of (2.4), we define the sets

$$\mathcal{D}(A_\lambda^H) := \{v\in V : A_\lambda v\in H\},\quad \text{for}\quad \lambda\in(0,1],$$

$$\mathcal{D}(A_0^H) := \{v\in V_0 : A_0 v\in H\},$$

and consider the operators $A_\lambda^H:\mathcal{D}(A_\lambda^H)\subset H\to H$ given by

$$A_\lambda^H(u) = A_\lambda u \quad \text{for all} \quad u\in\mathcal{D}(A_\lambda^H),\quad \text{for}\quad \lambda\in[0,1].$$

Then it follows from Proposition 1 in [6] that the equation (2.4) has a global weak solution $u^\lambda(\cdot,u_0^\lambda)$ starting in $u_0^\lambda\in\overline{D(A_\lambda^H)}$. So we can define in $\overline{\mathcal{D}(A_\lambda^H)}$ a semigroup $\{\mathrm{T}_\lambda(t):t\geqslant 0\}$ of nonlinear operators, associated to (2.4) by $\mathrm{T}_\lambda(t)u_0^\lambda = u^\lambda(t,u_0^\lambda)$, $t\geqslant 0$. Moreover, if

$u_0^\lambda \in \mathcal{D}(A_\lambda^H)$, then $u^\lambda(\cdot, u_0^\lambda) = \mathrm{T}_\lambda(\cdot)u_0^\lambda$ is a Lipschitz continuous strong solution of (2.4). To simplify we will denote the solution $u^0(t, u_0^0)$ just by $u(t, u_0)$.

Next we want to ensure the existence of the family of global attractors for the equation (2.4). To achieve this purpose, we are going to estimate $\|A_0 v\|_{V_0^*}$ and later use Theorem 1 from [6].

$$
\begin{aligned}
|\langle A_0 u, v\rangle_{V_0^*,V_0}| &\leqslant \int_{\Omega_1} d_0(x)|\nabla u|^{p-2}|\nabla u||\nabla v|\,\mathrm{d}x + \int_{\Omega} |u|^{p-2}|u||v|\,\mathrm{d}x \\
&\leqslant (\sup_{x\in\bar{\Omega}_1} d_0(x)) \int_{\Omega_1} |\nabla u|^{p-1}|\nabla v|\,\mathrm{d}x + \int_{\Omega} |u|^{p-1}|v|\,\mathrm{d}x \\
&\leqslant (\sup_{x\in\bar{\Omega}_1} d_0(x)) \| |\nabla u|^{p-1}\|_{L^q(\Omega_1)} \|\nabla v\|_{L^p(\Omega_1)} + \| |u|^{p-1}\|_{L^q(\Omega)}\|v\|_{L^p(\Omega)} \\
&= (\sup_{x\in\bar{\Omega}_1} d_0(x)) \left(\int_{\Omega_1} (|\nabla u|^{p-1})^q\,\mathrm{d}x\right)^{1/q} \|\nabla v\|_{L^p(\Omega_1)} + \left(\int_{\Omega} (|u|^{p-1})^q\,\mathrm{d}x\right)^{1/q} \|v\|_{L^p(\Omega)} \\
&= (\sup_{x\in\bar{\Omega}_1} d_0(x)) \left[\left(\int_{\Omega_1} |\nabla u|^p \mathrm{d}x\right)^{1/p}\right]^{p-1} \|\nabla v\|_{L^p(\Omega_1)} + \left[\left(\int_{\Omega} |u|^p \mathrm{d}x\right)^{1/p}\right]^{p-1} \|v\|_{L^p(\Omega)} \\
&= (\sup_{x\in\bar{\Omega}_1} d_0(x)) \|\nabla u\|_{L^p(\Omega_1)}^{p-1} \|\nabla v\|_{L^p(\Omega_1)} + \|u\|_{L^p}^{p-1}\|v\|_{L^p(\Omega)} \\
&\leqslant \max\{(\sup_{x\in\bar{\Omega}_1} d_0(x)), 1\} \left[\left(\|\nabla u\|_{L^p(\Omega_1)} + \|u\|_{L^p(\Omega)}\right)^{p-1} \|\nabla v\|_{L^p(\Omega_1)}\right. \\
&\quad \left. + \left(\|\nabla u\|_{L^p(\Omega_1)} + \|u\|_{L^p(\Omega)}\right)^{p-1} \|v\|_{L^p(\Omega)}\right] \\
&= K\,\|u\|_{V_0}^{p-1}(\|\nabla v\|_{L^p(\Omega_1)} + \|v\|_{L^p(\Omega)})] \\
&= K\|u\|_{V_0}^{p-1}\|v\|_{V_0},
\end{aligned}
$$

where $K := \max\{(\sup_{x\in\bar{\Omega}_1} d_0(x)), 1\}$.

Then

$$
\|A_0 u\|_{V_0^*} = \sup_{\|v\|_{V_0}\leqslant 1} |\langle A u, v\rangle_{V_0^*,V_0}| \leqslant w_2\|u\|_{V_0}^{p-1} \leqslant w_2(1 + \|u\|_{V_0}^{p-1}). \tag{2.6}
$$

When $\lambda \in (0,1]$, it is possible to obtain analogous conditions to (2.5) and (2.6) for the operators A_λ. Thus we can apply Theorem 1 from [6] to guarantee the existence of a global attractor $\mathcal{A}_\lambda$ for the semigroup $T_\lambda(t)$, for $\lambda \in [0,1]$.

Theorem 2.3. *The semigroup* $\mathrm{T}_\lambda(t)$ *associated to problem* (2.4) *has a global attractor* $\mathcal{A}_\lambda$ *in* $\overline{(D(A_\lambda^H))}$*, for* $\lambda \in [0,1]$.

3. Uniform estimates of solutions

In this section we guarantee, by usual methods, the essential uniform estimates in H and V in order to prove the upper semicontinuity of the family of attractors $\{\mathcal{A}_\lambda : \lambda \in [0,1]\}$.

Lemma 3.1. *Let* u^λ *be a solution of* (2.4)*. Then*

(1) *There exist positive constants* r_0 *and* t_0 *such that* $\|u^\lambda\|_H \leqslant r_0$*, for any* $t \geqslant t_0$*, uniformly in* $(0,1]$.

(2) *There exist positive constants* r_1 *and* $t_1 > t_0$ *such that* $\|u^\lambda\|_V \leqslant r_1$*, for any* $t \geqslant t_1$*, uniformly in* $(0,1]$.

Proof. (i) Taking the scalar product by u^λ in (2.4), we get

$$\langle u_t^\lambda, u^\lambda\rangle_H + \langle A_\lambda u^\lambda, u^\lambda\rangle_H = \langle B(u^\lambda), u^\lambda\rangle_H$$

and

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u^\lambda\|_H^2 + \int_\Omega d_\lambda(x)|\nabla u^\lambda|^p\,\mathrm{d}x + \int_\Omega |u^\lambda|^p\,\mathrm{d}x &\leqslant \langle B(u^\lambda) - B(0), u^\lambda\rangle_H + \langle B(0), u^\lambda\rangle_H \\ &\leqslant L_B\|u^\lambda\|_H^2 + C_0\|u^\lambda\|_H,\end{aligned}$$

where L_B is the Lipschitz constant of B and $C_1 = \|B(0)\|_H$. Then

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u^\lambda\|_H^2 &\leqslant L_B\|u^\lambda\|_H^2 + C_0\|u^\lambda\|_H - \int_\Omega d_\lambda(x)|\nabla u^\lambda|^p\,\mathrm{d}x - \int_\Omega |u^\lambda|^p\,\mathrm{d}x \\ &\leqslant L_B\|u^\lambda\|_H^2 + C_0\|u^\lambda\|_H - \min\{m_0, 1\}\|u^\lambda\|_V^p \\ &\leqslant L_B\|u^\lambda\|_H^2 + C_0\|u^\lambda\|_H - C_1\|u^\lambda\|_H^p,\end{aligned}$$

where $C_1 = \min\{m_0, 1\}$. Using Young's Inequality[1] for $L_B\|u^\lambda\|_H^2$ and $C_0\|u^\lambda\|_H$ with $\varepsilon = \frac{C_1}{4}$, we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u^\lambda\|_H^2 \leqslant -\frac{C_1}{2}\|u^\lambda\|_H^p + C_2 L_B^{\frac{p}{p-2}} + C_3 C_0^{\frac{p}{p-2}},$$

where the constants C_2 and C_3 depend on C_1.

So we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u^\lambda\|_H^2 + C_1\|u^\lambda\|_H^p \leqslant C_4,$$

where $C_4 = 2(C_2 L_B^{\frac{p}{p-2}} + C_3 C_0^{\frac{p}{p-2}})$. By Lemma 5.1 in [17], we have

$$\|u^\lambda(t)\|_H \leqslant \left[\left(\frac{C_4}{C_1}\right)^{\frac{2}{p}} + \frac{1}{(C_1(\frac{p}{2}-1)t)^{\frac{2}{p-2}}}\right]^{\frac{1}{2}} := r(t).$$

Since $r(t)$ is a non-increasing function, we obtain

$$\|u^\lambda(t)\|_H \leqslant r(t_0) := r_0, \quad \text{for all } t \geqslant t_0,$$

for some $t_0 > 0$ fixed.

(ii) Notice that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\varphi^\lambda(u^\lambda) &= \langle \partial\varphi^\lambda(u^\lambda), u_t^\lambda\rangle_H = \langle B(u^\lambda) - u_t^\lambda, u_t^\lambda\rangle_H \\ &= \langle B(u^\lambda) - u_t^\lambda, u_t^\lambda - B(u^\lambda) + B(u^\lambda)\rangle_H \\ &= -\|B(u^\lambda) - u_t^\lambda\|_H^2 + \langle B(u^\lambda) - u_t^\lambda, B(u^\lambda)\rangle_H.\end{aligned}$$

Thus

$$\begin{aligned}\|B(u^\lambda) - u_t^\lambda\|_H^2 + \frac{\mathrm{d}}{\mathrm{d}t}\varphi^\lambda(u^\lambda) &\leqslant \|B(u^\lambda) - u_t^\lambda\|_H\|B(u^\lambda)\|_H \\ &\leqslant \frac{1}{2}\|B(u^\lambda) - u_t^\lambda\|_H^2 + \frac{1}{2}\|B(u^\lambda)\|_H^2\end{aligned}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t}\varphi^\lambda(u^\lambda) \leqslant \|B(u^\lambda)\|_H^2 \leqslant \frac{1}{2}k_1^2,$$

uniformly in $(0, 1]$, since $\|u^\lambda\|_H \leqslant r_0$ and B is globally Lipschitz.

[1] $ab \leqslant \varepsilon a^p + \varepsilon^{\frac{-1}{p-1}} b^{p'}$ for $a, b \geqslant 0$ and $\frac{1}{p} + \frac{1}{p'} = 1$.

Furthermore, we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u^\lambda\|_H^2 + \varphi^\lambda(u^\lambda) \leqslant \langle u_t^\lambda, u^\lambda\rangle_H + \langle \partial\varphi^\lambda(u^\lambda), u^\lambda\rangle_H = \langle B(u^\lambda), u^\lambda\rangle_H$$
$$\leqslant \|B(u^\lambda)\|_H\|u^\lambda\|_H \leqslant k_1 r_0$$

and it follows that

$$\begin{aligned}\int_t^{t+r} \varphi^\lambda(u^\lambda)\,\mathrm{d}s &\leqslant \frac{1}{2}\|u^\lambda(t+r)\|_H^2 + \int_t^{t+r} \varphi^\lambda(u^\lambda)\,\mathrm{d}s \\ &\leqslant \frac{1}{2}\|u^\lambda(t+r)\|_H^2 + \int_t^{t+r}\left(k_1 r_0 - \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}s}\|u^\lambda\|_H^2\right)\mathrm{d}s \\ &\leqslant \frac{1}{2}\|u^\lambda(t)\|_H^2 + k_1 r_0 r := a_3.\end{aligned}$$

Using the Uniform Gronwall Lemma, Lemma 1.1, [17], we obtain

$$\varphi^\lambda(u^\lambda(t+r)) \leqslant \left(\frac{a_3}{r} + \frac{1}{2}k_1^2 r\right) := \tilde{r}_1, \qquad \text{for all } t \geqslant t_0. \tag{3.1}$$

Finally, denoting $\tau = t + r$, it follows that

$$\begin{aligned}\min\{m_0, 1\}\frac{1}{p}\|u^\lambda(\tau)\|_V^p &\leqslant \frac{1}{p}\int_\Omega m_0|\nabla u^\lambda|^p\,\mathrm{d}x + \frac{1}{p}\int_\Omega |u^\lambda|^p\,\mathrm{d}x \\ &\leqslant \frac{1}{p}\int_\Omega d_\lambda(x)|\nabla u^\lambda|^p\,\mathrm{d}x + \frac{1}{p}\int_\Omega |u^\lambda|^p\,\mathrm{d}x = \varphi^\lambda(u^\lambda) \leqslant \tilde{r}_1, \end{aligned}\tag{3.2}$$

for any $\tau \leqslant t_0 + r := t_1$, uniformly in $(0, 1]$. □

4. CONTINUITY PROPERTIES

The upper semicontinuity of the attractors is easily obtained from the uniform estimates on last section and the continuity of the flow. We start this section by proving a compactness result from which we can obtain that $\mathrm{T}_\lambda(t)u^\lambda$ behaves continuously in $C([0,T],H)$ as λ goes to zero.

Lemma 4.1. *Let* $\{\mathrm{T}_\lambda(t) : H \to H, t \geqslant 0\}$ *be the semigroup generated by* $\partial\varphi^\lambda$, $\lambda \in [0,1]$ *and* u^λ *a solution of the problem* (2.4)*. Then*

$$\|\mathrm{T}_\lambda(h)u^\lambda(t) - u^\lambda(t)\|_H \to 0$$

as $h \to 0$, uniformly in $[0,1]$, for each $t > 0$. Moreover, if $T > 0$, we have

$$\|u^\lambda(T-h) - \mathrm{T}_\lambda(h)u^\lambda(T-h)\|_H \to 0,$$

as $h \to 0$, uniformly in $[0,1]$.

Proof. For any $\lambda \in [0,1]$, it follows from Theorem 1 in [18] that

$$\begin{aligned}\|\mathrm{T}_\lambda(h)u^\lambda - u^\lambda\| &= \|\mathrm{T}_\lambda(h)u^\lambda - J_h^{\varphi^\lambda}u^\lambda + J_h^{\varphi^\lambda}u^\lambda - u^\lambda\| \leqslant \|\mathrm{T}_\lambda(h)u^\lambda - J_h^{\varphi^\lambda}u^\lambda\| + \|J_h^{\varphi^\lambda}u^\lambda - u^\lambda\| \\ &\leqslant 2\|J_h^{\varphi^\lambda}u^\lambda - u^\lambda\| + \|J_h^{\varphi^\lambda}u^\lambda - u^\lambda\| = 3\|J_h^{\varphi^\lambda}u^\lambda - u^\lambda\|\end{aligned}$$

where $J_h^{\varphi^\lambda} = (I + h\partial\varphi^\lambda)^{-1}$.

From Proposition 2.11 in [8], for any $\mu > 0$ and $u \in H$, we have

$$\frac{1}{2\mu}\|J_\mu^{\varphi^\lambda}u - u\|_H + \varphi^\lambda(J_\mu^{\varphi^\lambda}u) = \min_{v\in H}\left[\frac{1}{2\mu}\|v-u\|_H^2 + \varphi^\lambda(v)\right],$$

and we obtain from (3.1),

$$\frac{1}{2h}\|u^\lambda(t) - J_h^{\varphi^\lambda}u^\lambda(t)\| + \varphi^\lambda(J_h^{\varphi^\lambda}u^\lambda(t)) \leqslant \varphi^\lambda(u^\lambda(t)) \leqslant K_2$$

with K_2 uniform in $\lambda \in [0,1]$. Therefore $\|T_\lambda(h)u^\lambda - u^\lambda\| \to 0$ as $h \to 0$.

Using similar arguments, we obtain

$$\|u^\lambda(T-h) - \mathrm{T}_\lambda(h)u^\lambda(T-h)\|_H \leqslant \sqrt{2hK_2},$$

with K_2 uniform with respect to $\lambda \in [0,1]$.

□

Theorem 4.2. *The set $S_\lambda := \{u^\lambda : \lambda \in [0,1], u^\lambda$ is a solution of (1.1)$\}$ is relatively compact in $\mathcal{C}([0,T],H)$.*

Proof. It follows from Lemma 2.3.1 in [16] that if $t \in [0,T)$ and $h > 0$ are such that $T-h, t+h \in [0,T]$, we obtain

$$\begin{aligned}\|u^\lambda(t+h) - u^\lambda(t)\|_H &\leqslant \|u^\lambda(t+h) - \mathrm{T}_\lambda(h)u^\lambda(t)\|_H + \|\mathrm{T}_\lambda(h)u^\lambda(t) - u^\lambda(t)\|_H \\ &\leqslant \int_t^{t+h}\|Bu^\lambda(s)\|_H\,\mathrm{d}s + \|\mathrm{T}_\lambda(h)u^\lambda(t) - u^\lambda(t)\|_H \qquad (4.1)\end{aligned}$$

and

$$\begin{aligned}\|u^\lambda(T)-u^\lambda(T-h)\|_H &\leqslant \|u^\lambda(T)-T_\lambda(h)u^\lambda(T-h)\|_H+\|T_\lambda(h)u^\lambda(T-h)-u^\lambda(T-h)\|_H\\ &\leqslant \int_{T-h}^{T}\|Bu^\lambda(s)\|_H\,\mathrm{d}s+\|\mathrm{T}_\lambda(h)u^\lambda(T-h)-u^\lambda(T-h)\|_H.\end{aligned} \tag{4.2}$$

Lemma 4.1 guarantees that for any $\lambda\in(0,1]$, given $\eta>0$, there exists $\bar{\delta}>0$ such that

$$\|\mathrm{T}_\lambda(h)u^\lambda(t)-u^\lambda(t)\|_H<\frac{\eta}{2}$$

for all $0<|h|<\bar{\delta}$. Since $\{Bu^\lambda : \lambda\in[0,1],\, u^\lambda \text{ is a solution of (1.1)}\}$ is a uniformly integrable subset of $\mathrm{L}^1([0,T],H)$, then there exists $\gamma(\bar{\delta})>0$ such that

$$\int_t^{t+h}\|Bu^\lambda(s)\|_H\,\mathrm{d}s<\frac{\eta}{2},$$

for all $|h|<\gamma(\bar{\delta})$. By taking $\delta=\min\{\gamma(\bar{\delta}),\bar{\delta}\}$ it follows from (4.1) that

$$\|u^\lambda(t+h)-u^\lambda(t)\|_H<\eta,$$

for $0<|h|<\delta$. Arguing as above and using (4.2) we have

$$\|u^\lambda(T-h)-u^\lambda(T)\|_H<\eta,$$

for $0<|h|<\delta$. Therefore S_λ is equicontinuous in $\mathcal{C}([0,T];H)$.

Next we show that for each $t\in(0,T]$,

$$S_\lambda(t):=\{u^\lambda(t):\lambda\in[0,1],u^\lambda \text{ is a solution of (1.1)}\}$$

is relatively compact in H.

For $t\in(0,T]$ fixed, consider $h>0$ such that $t-h\in[0,T]$. Again, using Lemma 2.3.1 in [16]

$$\|\mathrm{T}_\lambda(h)u^\lambda(t-h)-u^\lambda(t)\|_H\leqslant\int_{t-h}^{t}\|Bu^\lambda(s)\|_H\,\mathrm{d}s,$$

for $\lambda\in[0,1]$. Let $F_h : S_\lambda(t)\subset H\to H$ be the operator defined by $F_hu^\lambda(t)=\mathrm{T}_\lambda(h)u^\lambda(t-h)$. Since $\mathrm{T}_\lambda(h)$ is a compact semigroup, it follows from Lemma 3.1 that $S_\lambda(t-h)$ is bounded in H and thus F_h maps bounded subsets in relatively compact subsets of H. Furthermore,

$\lim F_h = I$ as $h \to 0$, uniformly in $S_\lambda(t)$. Therefore $I : S_\lambda(t) \to H$ is a compact operator and $S_\lambda(t)$ is relatively compact in H, for all $t \in (0, T]$.

Finally, noticing that $S_\lambda(0) = \{u_0\}$ is relatively compact, it follows from Arzelá-Ascoli Theorem that S_λ is relatively compact in $\mathcal{C}([0,T], H)$. □

Theorem 4.3. *Let $\{u^\lambda\} \subset V$ such that $u^\lambda \to u \in V_0$ in V as $\lambda \to 0$. Then*

$$\sup_{t\in[0,T]} \|\mathrm{T}_\lambda(t)u^\lambda - \mathrm{T}_0(t)u\| \to 0 \quad \textit{as} \quad \lambda \to 0.$$

Proof. Let $u^\lambda(t, u_0^\lambda) = \mathrm{T}_\lambda(t)u_0^\lambda$ be a solution of (1.1). It follows from Proposition 3.6 in [8] that the following estimate is satisfied

$$\frac{1}{2}\|\mathrm{T}_\lambda(t)u_0^\lambda - v\|^2 \leqslant \frac{1}{2}\|\mathrm{T}_\lambda(s)u_0^\lambda - v\|^2 + \int_s^t \langle B(\mathrm{T}_\lambda(\tau)u_0^\lambda) - A_\lambda v, \mathrm{T}_\lambda(\tau)u_0^\lambda - v\rangle \mathrm{d}\tau, \quad (4.3)$$

for $v \in \mathcal{D}(A_\lambda)$.

Since $\mathrm{T}_\lambda(t)u_0^\lambda \in S_\lambda \subset \bar{S}_\lambda$ then $\mathrm{T}_\lambda(t)u_0^\lambda$ has a convergent subsequence, that we denote by the same, $\mathrm{T}_\lambda(t)u_0^\lambda \to \xi(t, u_0)$ as $\lambda \to 0$. It follows from (2.1) that $B(\mathrm{T}_\lambda(t)u_0^\lambda) \to B(\xi(t, u_0))$ as $\lambda \to 0$.

Next we show that $A_\lambda v \rightharpoonup A_0 v$ as $\lambda \to 0$ for all $v \in \mathcal{D}(A_0)$. Let be $v, w \in \mathcal{D}(A_0)$ then

$$\begin{aligned}
\langle A_\lambda v, w\rangle &= \int_\Omega -\mathrm{div}(d_\lambda(x)|\nabla v|^{p-2}\nabla v) w \,\mathrm{d}x + \int_\Omega |v|^{p-2} v w \,\mathrm{d}x \\
&= -\int_{\partial\Omega} d_\lambda(x)|\nabla v|^{p-2}\frac{\partial v}{\partial \vec{n}} w \,\mathrm{d}x + \int_\Omega d_\lambda(x)|\nabla v|^{p-2}\nabla v \nabla w \,\mathrm{d}x + \int_\Omega |v|^{p-2} v w \,\mathrm{d}x \\
&= \int_{\Omega_1} d_\lambda(x)|\nabla v|^{p-2}\nabla v \nabla w \,\mathrm{d}x + \int_\Omega |v|^{p-2} v w \,\mathrm{d}x.
\end{aligned}$$

Since that

$$\begin{aligned}
\int_{\Omega_1} d_0(x)|\nabla v|^{p-2}\nabla v \nabla w \,\mathrm{d}x + \int_\Omega |v|^{p-2} v w \,\mathrm{d}x &= \int_{\partial\Omega_1} d_0(x)|\nabla v|^{p-2}\frac{\partial v}{\partial \vec{n}} w \mathrm{d}x \\
&\quad - \int_{\Omega_1} \mathrm{div}(d_0(x)|\nabla v|^{p-2}\nabla v) w \,\mathrm{d}x + \int_\Omega |v|^{p-2} v w \,\mathrm{d}x \\
&= \int_{\partial\Omega_0} d_0(x)|\nabla v|^{p-2}\frac{\partial v}{\partial \vec{n}} w \,\mathrm{d}x
\end{aligned}$$

$$
\begin{aligned}
&- \int_{\Omega_1} \operatorname{div}(d_0(x)|\nabla v|^{p-2}\nabla v) w \,\mathrm{d}x + \int_\Omega |v|^{p-2} v w \,\mathrm{d}x \\
&= \int_\Omega \left(\frac{1}{|\Omega_0|} \int_{\partial\Omega_0} d_0(y)|\nabla v|^{p-2} \frac{\partial v}{\partial \vec{n}} dy \right) \chi_{\Omega_0}(x) w \,\mathrm{d}x \\
&- \int_\Omega \operatorname{div}(d_0(x)|\nabla v|^{p-2}\nabla v) w \chi_{\Omega_1}(x) \,\mathrm{d}x + \int_\Omega |v|^{p-2} v w \,\mathrm{d}x \\
&= \langle A_0 v, w \rangle, \forall w \in \mathcal{D}(A_0),
\end{aligned}
$$

then taking limit as $\lambda \to 0$ we get

$$
\begin{aligned}
\langle A_\lambda v, w \rangle &= \int_{\Omega_1} d_\lambda(x)|\nabla v|^{p-2}\nabla v \nabla w \,\mathrm{d}x + \int_\Omega |v|^{p-2} v w \,\mathrm{d}x \\
&\to \int_{\Omega_1} d_0(x)|\nabla v|^{p-2}\nabla v \nabla w \,\mathrm{d}x + \int_\Omega |v|^{p-2} v w \,\mathrm{d}x = \langle A_0 v, w \rangle,
\end{aligned}
$$

for all $w \in \mathcal{D}(A_0)$ and then $A_\lambda v \rightharpoonup A_0 v$ as $\lambda \to 0$. Hence

$$
\langle B(\mathrm{T}_\lambda(\tau)u_0^\lambda) - A_\lambda v, \mathrm{T}_\lambda(\tau)u_0^\lambda - v \rangle \rightharpoonup \langle B(\xi) - A_0 v, \xi - v \rangle. \tag{4.4}
$$

Taking the limit in (4.3) and using (4.4) we get

$$
\frac{1}{2}\|\xi(t,u_0) - v\|^2 \leqslant \frac{1}{2}\|\xi(s,u_0) - v\|^2 + \int_s^t \langle B(\xi(t,u_0)) - A_0 v, \xi(\tau,u_0) - v \rangle \,\mathrm{d}\tau. \tag{4.5}
$$

Now, using (4.5) and Proposition 3.6 in [8] again, we obtain that $v(t,\cdot)$ is solution of

$$
\begin{cases}
u_t + A_0 u = B(u) \\
u(0) = u_0
\end{cases}
$$

From uniqueness we obtain $v(t,x) = T_0(t)u_0$. Thus

$$
\mathrm{T}_\lambda(t)u_0^\lambda \to \mathrm{T}_0(t)u_0, \quad \text{as} \quad \lambda \to 0.
$$

□

Theorem 4.4. *The family of attractors $\{\mathcal{A}_\lambda : \lambda \in [0,1]\}$ is upper semicontinuous in $\lambda = 0$.*

Proof. Let us consider the sequence $\{u^{\lambda_n}\}$, $u^{\lambda_n} \in \mathcal{A}_{\lambda_n}$, $n \in \mathbb{N}$ and $\lambda_n \to 0$. Since, from Lemma 3.1 (ii), $\sup\limits_{\lambda\in(0,1]} \|u^{\lambda}\|_V \leqslant \tilde{r}_1$ and V is reflexive, then there exists a subsequence, which we also denote by $\{u^{\lambda_n}\}$, and $u \in V$ such that $u^{\lambda_n} \to u$ weakly in V and strongly in H. It is easy to see that $u^{\lambda_n} \stackrel{n\to\infty}{\longrightarrow} u$ weakly in $W^{1,p}(K)$ for any open set K with $K \subset\subset \Omega$.

On the other hand, it follows from (3.2) that

$$\inf_{x\in K}\{d_{\lambda_n}(x)\}\int\limits_K |\nabla u|^p \mathrm{d}x \leqslant \int\limits_K d_{\lambda_n}(x)|\nabla u^{\lambda_n}|^p \mathrm{d}x \leqslant C_1,$$

for some positive constant $C_1 > 0$.

Since $d_{\lambda_n}(x) \to \infty$ uniformly on compact subsets of Ω_0, we have that

$$\int\limits_K |\nabla u|^p \mathrm{d}x \leqslant \liminf_{\lambda_n\to 0}\int\limits_K |\nabla u^{\lambda_n}|^p \mathrm{d}x = 0.$$

Consequently, u is constant in K for all $K \subset\subset \Omega_0$. From this and from the fact that $\Omega_0 = \bigcup\limits_{\substack{i=1 \\ K_i\subset\subset\Omega_0}}^{\infty} K_i$, we obtain that $u \in V_0$.

Next we set up a bounded complete orbit through u, which implies $u \in \mathcal{A}_0$. For each $n \in \mathbb{N}$, let $\gamma_n(\cdot, u^{\lambda_n}) : \mathbb{R} \to V$ be the bounded complete orbit through u^{λ_n}. Theorem 4.3 guarantees that $\gamma_n(t, u^{\lambda_n}) = \mathrm{T}_{\lambda_n}(t, u^{\lambda_n}) \to \mathrm{T}_0(t, u)$ for $t \geqslant 0$. For $0 > t \in (-j, -j+1]$, $j \in \mathbb{Z}^+$, we consider the sequence $\{\gamma_n(-j, u^{\lambda_n})\}_{n\in\mathbb{N}}$ in $\bigcup\limits_{\lambda\in(0,1]} \mathcal{A}_\lambda$, which we assume convergent to $\gamma_0(-j, u)$. Therefore

$$\gamma_n(t, u^{\lambda_n}) = \gamma_n(t + j, \gamma_n(-j, u^{\lambda_n})) \to \mathrm{T}_0(t + j, \gamma_0(-j, u)) := \tilde{\gamma}_0(t, u).$$

It follows that $\gamma_0(t, u) = \begin{cases} \mathrm{T}_0(t, u), & \text{for } t \geqslant 0; \\ \tilde{\gamma}_0(t, u), & \text{for } t < 0 \end{cases}$ is a bounded complete orbit through u. □

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