Global and local minimizers of the Cahn-Hilliard functional over a parallelepiped: with and without constraint.

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Abstract. This note is a review of results concerning the number, geometric profile and one-dimensional character of local and global minimizers of the Cahn-Hilliard functional over rectangles and parallelepipeds with and without constraint.

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1. Introduction

In this note we prove and announce some new results concerning the exact number and the geometric characterization of global and local minimizers of the so-called Cahn-Hilliard energy functional, with and without constraints, over a n-dimensional parallelepiped (see [10] and [11], for a physical background).

For the unconstrained problem we consider in $H^1(\Omega)$ the functional

$$E(u) = \int_\Omega \{\frac{1}{2}|\nabla u|^2 + F(u)\}dx \tag{1.1}$$

over domains $\Omega \subset \mathbb{R}^n$ $(2 \leq n < \infty)$ of the type

$$\Omega = (a_1, b_1) \times (a_2, b_2) \ldots \times (a_n, b_n).$$

Here F is a any real function in C^2. We prove that any local minimizer of E in $H^1(\Omega)$ must be a constant function.

As for the constrained case our goal is to find, for $\varepsilon > 0$ small, the exact number of solutions to

$$\inf_{u \in \mathcal{M}} \quad E_\varepsilon(u) \tag{1.2}$$

where

$$E_\varepsilon(u) \stackrel{\text{def}}{=} \int_\Omega [\frac{\varepsilon}{2}|\nabla u|^2 + \varepsilon^{-1}(1-u^2)^2]\, dx$$

and

$$\mathcal{M} \stackrel{\text{def}}{=} \{u \in H^1(\Omega) : \int_\Omega u = 0\}$$

(global minimizer of E_ε in $\mathcal{M}$) as well as local (but not-global) minimizers of E_ε in $\mathcal{M}$. For ε small enough and dimensions $n = 2, 3$ we present a complete picture of its symmetry properties, geometric profiles and locations of the limiting interfaces. In particular global as well as local minimizers are one-dimensional and odd functions.

Our approach relies on the theory of $\Gamma-$ convergence and the conclusions are established after a series of technical results of isoperimetric inequalities type in Geometric Measure Theory. The proofs of these results will appear elsewhere.

2. The unconstrained case

The following result seems to be known to the specialist but we could not find any reference. We render a proof for the sake of completeness since the constrained case is going to be considered as well.

Theorem 2.1. *Suppose that u is a local minimizer of E in $H^1(\Omega)$ where $\Omega = (a_1, b_1) \times (a_2, b_2) \ldots (a_n, b_n)$. Then u is a constant function.*

Proof. Suppose that E possesses a local minimizer $u \in H^1(\Omega)$ which is not a constant function. Hence

$$E'(u)\varphi = 0, \quad E''(u)(\varphi, \varphi) \geq 0, \quad \forall\, \varphi \in H^1(\Omega)$$

and (cf. [7]) $u \in H^3(\Omega) \cap C^{3+\delta}(\Omega)$. Moreover since u is not constant $\exists\, j \in \{1, \ldots, n\}$ such that

$$v \stackrel{\text{def}}{=} \partial_{x_j} u \not\equiv 0 \tag{2.1}$$

satisfies

$$\triangle v - f'(u)v = 0 \text{ in } \Omega \tag{2.2}$$

where $F' = f$. From now on, for simplicity in notation, by $\partial^*\Omega$ we mean the topological boundary of Ω leaving out the edges and vertices of Ω. Moreover

$$v = 0 \text{ on } \mathcal{D} \stackrel{\text{def}}{=} \partial^*\Omega \cap \{x : x_j = a_j\} \cap \{x : x_j = b_j\} \tag{2.3}$$

and

$$\partial_\nu v = 0 \text{ on } \partial^*\Omega \setminus \mathcal{D}. \tag{2.4}$$

Note that $v \in C^{2+\delta}(\partial^*\Omega)$ for some $0 < \delta < 1$ (cf. [7]). The regularity of v allows us to compute

$$\begin{aligned} E''(u)(v, v) &= \int_\Omega [|\nabla v|^2 + f'(u)v^2] dx \\ &= \int_{\partial\Omega} v\partial_\nu v \, d\mathcal{H}^{n-1} - \int_\Omega v\triangle v \, dx + \int_\Omega f'(u)v^2 \, dx \\ &= \int_\Omega [-\triangle v + f'(u)v]v. \end{aligned} \tag{2.5}$$

where, due to (2.3) and (2.4), the integral on the boundary vanishes.

By its turn (2.2) implies $E''(u)(v,v) = 0$. But given that $E''(u)(\varphi,\varphi) \geq 0,\ \forall\,\varphi \in H^1(\Omega)$, we conclude

$$\begin{cases} \triangle v - f'(u)v = 0 & \text{in } \Omega \\ \partial_\nu v = 0 & \text{on } \partial^*\Omega. \end{cases} \tag{2.6}$$

In particular $v \equiv \partial_\nu v \equiv 0$ on $\mathcal{D}$ and this implies, using the Unique Continuation Principle (see [3], for instance), that $v \equiv \partial_{x_j} u \equiv 0$ in Ω, contradicting (2.1). This completes the proof. □

A similar result has been obtained in [9] for periodic solutions in the variable corresponding to a infinite cylinder axis. This proof differs from ours only in that whereas in [9] a variant of Krein-Rutman's Theorem is used we instead resort to the Unique Continuation Principle.

3. The constrained case

We first review some of the closest related works in order to set our own into perspective. The authors in [2] showed existence of global and local minimizers of E_ε, under a mass constraint $\int_\Omega u$ fixed, over the flat torus $\mathbb{T}^n$ ($n = 2, 3$), in which case it is called the periodic Cahn-Hilliard problem.

In [8], for smooth domains $\Omega \subset \mathbb{R}^2$ and $\int_\Omega u = m \approx 0$ or $m \approx \mathcal{L}^2(\Omega)$, the authors showed existence of local minimizers of E_ε (for $\varepsilon > 0$ small) whose transition layers are close to circular arcs and intersect the boundary orthogonally. In particular, under these hypotheses, they conclude that there are at least two such local minimizers and each of these arcs encloses a point on the boundary where the curvature of $\partial\Omega$ attains its local maximum.

The problem of minimizing $\Lambda_\varepsilon = \varepsilon E_\varepsilon$ over the unit square $\Omega = (0\,,1) \times (0\,,1)$ has been considered in [1] via a bifurcation approach where either ε or the total mass $\int_\Omega u = m \in \mathbb{R}$ is the bifurcation parameter. Bifurcation of critical points from a particular class of eigenfunctions is considered and in particular for m fixed the author shows that the pointwise limit, as $\varepsilon \to 0$, of some conditionally critical points are global minimizers of the limiting problem Λ_0.

Monotonicity, regularity and other symmetry properties of local minimizers of E_ε under the constraint $\int_\Omega u = m$ over cylindrical domains has been studied in [7] but existence is not considered.

In [9] the author gives some accounts on symmetry and monotonicity of local minimizers of some variational problems with a integral constraint in cylinders and annuli with periodic phase separation. Again existence is not the issue.

First we state our main result when the domain is a rectangle.

Theorem 3.1. *Let $\Omega = R = (-l, l) \times (-r, r)$ with $l \geq r > 0$. Then, for ε small, (1.2) has a family of solutions $\{u_\epsilon\}_{\varepsilon>0}$ in $C^2(R) \cap C^0(\overline{R})$ (global minimizers of E_ε in $\mathcal{M}$) satisfying:*

- $u_\epsilon(x_1,x_2) = u_\epsilon(x_1), x \in R$,
- $u_\epsilon(x_1) = -u_\epsilon(-x_1)$ *and*
- u_ϵ *is increasing in* $(-l,l)$.

(3.1.i) *If $l > r$, problem (1.2) has, for ε small, only two solutions: u_ε (given above) and $-u_\varepsilon$. Moreover E_ε has only two other local (but not global) minimizers in $\mathcal{M}$: v_ε and $-v_\varepsilon$, where v_ε depends only on the second variable and is increasing and odd in $(-r,r)$.*

In particular $E_\varepsilon(u_\varepsilon) = E_\varepsilon(-u_\varepsilon) < E_\varepsilon(v_\varepsilon) = E_\varepsilon(-v_\varepsilon)$.

(3.1.ii) *If R is a square, then problem (1.2) has, for ε small, only four solutions $u_\varepsilon(x_1)$, $-u_\varepsilon(x_1)$, $v_\varepsilon(x_2)$, $-v_\varepsilon(x_2)$ where v_ε is obtained from u_ε by a rotation of $\pi/2$ with respect to x_1-axis. There is no other local minimizer.*

Let us label the points $x = (x_1, x_2, x_3) \in \mathbb{R}^3$ as $x = (x_1, x')$ where $x' = (x_2, x_3)$.

Theorem 3.2. *Let $\Omega = (-l,l) \times (-r,r) \times (-q,q)$. Then, for ε small, (1.2) has a family of solutions $\{u_\epsilon\}_{\varepsilon>0}$ in $C^2(\Omega) \cap C^0(\overline{\Omega})$ satisfying*

- $u_\epsilon(x_1, x') = u_\epsilon(x_1),\ x_1 \in (-l,l)$,
- $u_\epsilon(x_1) = -u_\epsilon(-x_1)$ *in* $(-l,l)$ *and*
- u_ϵ *is increasing in* $(-l,l)$

(3.2.i) *If $l > r > q$ then, for ε small, (1.2) has only two solutions: u_ϵ (given above) and $-u_\epsilon$. Moreover E_ε has only four other local (but not global) minimizers in $\mathcal{M}$: v_ε, $-v_\varepsilon$, w_ε and $-w_\varepsilon$. Here v_ε depends only on the second variable and is increasing and odd in $(-r,r)$ whereas w_ε depends only on the third variable and is increasing and odd in $(-q,q)$.*

(3.2.ii) *In case $l = r > q$ the functions u_ϵ, $-u_\epsilon$, v_ε and $-v_\varepsilon$ are the only solutions to (1.2) and w_ε, $-w_\varepsilon$ the only other local minimizers.*

(3.2.iii) *If $l = r = q$ then problem (1.2) has only six solutions $u_\varepsilon(x_1)$, $-u_\varepsilon(x_1)$, $v_\varepsilon(x_2)$, $-v_\varepsilon(x_2)$, $w_\varepsilon(x_3)$ and $-w_\varepsilon(x_3)$ and there is no other local minimizer.*

The results above regarding existence are sharp and show that, for ε small, the geometry of global and local minimizers of the energy functional restrict to $\mathcal{M}$ is quite simple as they present no oscillation, a type of Sturm-Liouville property. The selection of global minimizers by the energy functional is made via the smallest interface area criterion.

The underlying approach is based on the concept of $\Gamma-$convergence and following result which is due to De Giorgi [5]. It relates isolated local minimizers of the Γ-limit to local minimizers of the original problem.

Theorem 3.3. *Suppose that a family of extended-real functionals $\{\Lambda_\varepsilon\}_{\varepsilon>0}$ defined in $L^1(\Omega)$ Γ-converges, as $\varepsilon \to 0$, to a extended functional Λ_0 and that the following hypotheses are satisfied:*

(3.3.i) $\forall\ \{v_\varepsilon\}_{\varepsilon>0} : \Lambda_\varepsilon(v_\varepsilon) \le$ *constant* $< \infty$ *is compact in* $L^1(\Omega)$.

(3.3.ii) *There exists an isolated L^1-local minimizer v_0 of Λ_0.*

Then there exists $\varepsilon_0 > 0$ and a family $\{v_\varepsilon\}_{0<\varepsilon\le\varepsilon_0}$ such that

- v_ε *is an L_1-local minimizer of Λ_ε,*

- $\|v_\varepsilon - v_0\|_{L^1(\Omega)} \overset{\varepsilon\to 0}{\longrightarrow} 0$.

In order to work in $L^1(\Omega)$, a space whose topology has the compactness property which we need with respect to $BV(\Omega)$, we define $E_\varepsilon : L^1(\Omega) \longmapsto \mathbb{R} \cup \{\infty\}$ by

$$E_\varepsilon(u) \overset{\text{def}}{=} \begin{cases} \int_\Omega [\varepsilon|\nabla u|^2 + \varepsilon^{-1}(1-u^2)^2]\,dx, & \text{if } u \in \mathcal{M} \\ \infty, & \text{otherwise.} \end{cases} \tag{3.1}$$

It is well-known that the family of functionals $\{E_\varepsilon\}_{0<\varepsilon<\varepsilon_0}$ $\Gamma-$converges in $L^1(\Omega)$, as $\varepsilon \to 0$, to

$$E_0(v) = \begin{cases} \frac{8}{3}\mathrm{Per}_\Omega\{v=1\}, & v \in BV(\Omega,\{\pm 1\}) \text{ and } \int_\Omega v = 0 \\ \infty, & \text{otherwise.} \end{cases} \tag{3.2}$$

A proof can be found in [6], for instance.

Existence of local minimizers is obtained by proving that E_0 has an isolated $L^1(\Omega)$-local minimizer, say u_0, and then De Giorgi's theorem assures that close (in $L^1(\Omega)$ norm) to u_0 there is a local minimizer of the original problem. In particular we prove that when Ω is a rectangle or a parallelepiped the admissible function

$$u_0(x) \overset{\text{def}}{=} \chi_{\Omega_r}(x) - \chi_{\Omega_l}(x), \quad x \in \Omega \tag{3.3}$$

where $\Omega_l \overset{\text{def}}{=} \{x \in \Omega : x_1 < 0\}$ and $\Omega_r \overset{\text{def}}{=} \{x \in \Omega : x_1 > 0\}$ is indeed a local minimizer of E_0. This gives a positive answer to a question raised in [4], p. 80, for a rectangle.

Lemma 3.4. *Let $\Omega = (-l,l)\times(-r,r)\times(-q,q)$. Then the function u_0, given by (3.3), is a local isolated minimizer of E_0 given by (3.2), i.e., there is $\rho > 0$ such that for any $v \in BV(\Omega,\{\pm1\})$, satisfying $0 < \|v-u_0\|_{L^1(\Omega)} < \rho$ and $\int_\Omega v = 0$, we have $E_0(u_0) < E_0(v)$.*

Its proof for $n = 3$ constitutes the most difficult part of the work and it is built on a series of technical results based on some type of isoperimetric inequalities. We state the results with the purpose of giving an idea of how the radius ρ of the ball in L^1, appearing in the above lemma, is found. We set for simplicity in notation

$$D \overset{\text{def}}{=} (-r,r)\times(-q,q).$$

Lemma 3.5. *Let $\Omega = (-l,l)\times D$. There exists $\theta > 0$ such that for any $A \subset D$ of finite perimeter satisfying $\mathcal{H}^{n-1}(A) < \frac{1}{2}\mathcal{H}^{n-1}(D)$ it holds that*

$$0 < \theta \le \frac{\mathcal{H}^1(\partial^* A \backslash \partial^* D)}{\mathcal{H}^1(\partial^* A)}, \tag{3.4}$$

where ∂^* stands for the reduced boundary of a set.

A well-known isoperimetric inequality states: if $E \subset \mathbb{R}^n$ $(n \ge 2)$ is a set of finite perimeter then

$$\mathcal{H}^n(E)^{\frac{n-1}{n}} \le c_1 \mathcal{H}^{n-1}(\partial^* E) \tag{3.5}$$

where the constant c_1 depends only on n.

Lemma 3.6. *Let Ω be a parallelepiped as above, c_1 as in (3.5) and θ as in Lemma 3.5. Then there exists a real number $\xi \in (\frac{l}{2}, l)$ and a continuous function $\eta : [\frac{l}{2}, \xi] \to [0, \infty)$ satisfying*

$$\begin{cases} \eta(t) = \frac{\theta}{2c_1} \int_t^\xi \eta(s)^{\frac{n-2}{n-1}} ds \\ \eta(\xi) = 0 \\ \eta(\frac{l}{2}) \le \frac{1}{2}\mathcal{H}^{n-1}(D) \\ \eta > 0 \ \ on \ [\frac{l}{2}, \xi). \end{cases} \tag{3.6}$$

With ξ and η as above we define

$$\rho = \min\{\int_{l/2}^{\xi} \eta(t)dt, \frac{(l-\xi)}{2}\mathcal{H}^{n-1}(D)\}, \ n = 2, 3. \tag{3.7}$$

Lemma 3.7. *Let Ω, c_1 and θ be as in Lemma* 3.6 *and $\rho > 0$ as in (3.7). Then for any $\mathcal{L}$-measurable function $\psi : [\frac{l}{2}, l] \to (0, \mathcal{H}^{n-1}(D)]$ $(n = 2, 3)$ satisfying*

$$\int_{\frac{l}{2}}^{l} \psi(s) d\mathcal{H}^1 < \rho$$

there exists $t_0 \in (\frac{l}{2}, l)\setminus I$, where $I = \{t \in [\frac{l}{2}, l] : \psi(t) \ge \frac{1}{2}\mathcal{H}^{n-1}(D)\}$, such that

$$\psi(t_0) < \frac{\theta}{2c_1} \int_{[t_0,l]\setminus I} \psi(s)^{\frac{n-2}{n-1}} ds.$$

The function ψ, by its turn, is used to prove the following fundamental result.

Lemma 3.8. *Let $\Omega = (-l, l) \times D$ and ρ as in (3.7). Then*

(4.6.i) *Given $A^+ \subset \overline{D \times \left[\frac{l}{2}, l\right]}$, with $\mathcal{H}^n(A^+) < \rho$, there exists $t_0 \in \left(\frac{l}{2}, l\right)$ such that*

$$\mathcal{H}^{(n-1)}(A^+ \cap (D \times \{t_0\})) \le$$
$$\frac{1}{2}\mathcal{H}^{(n-1)}(\partial^*(A^+ \cap (D \times [t_0, l]))\setminus\partial^*(D \times [t_0, l]));$$

(4.6.ii) *Given $A^- \subset \overline{D \times \left[-l, \frac{-l}{2}\right]}$ with $\mathcal{H}^n(A^-) < \rho$ there exists $t_0' \in \left(-l, -\frac{l}{2}\right)$ such that*

$$\mathcal{H}^{(n-1)}(A^- \cap (D \times \{t_0'\})) \le$$
$$\frac{1}{2}\mathcal{H}^{(n-1)}(\partial^*(A^- \cap (D \times [-l, t_0']))\setminus\partial^*(D \times [-l, t_0'])).$$

At last, with this result in hands, we can prove Lemma 3.4.

Going back to Lemma 3.4 we remark that if $l > r > q$ then u_0 is the only global minimizer of E_0 in $\mathcal{M}$. Henceforth u_0 gives rise, via Theorem 3.3, to a family $\{u_\varepsilon\}$ of global minimizers of E_ε. Existence of the other local minimizers are obtained in a similar manner by considering instead the admissible functions

$$v_0(x) \stackrel{\text{def}}{=} \chi_{\Omega_r}(x) - \chi_{\Omega_l}(x), \quad x \in \Omega \tag{3.8}$$

where $\Omega_l \stackrel{\text{def}}{=} \{x \in \Omega : x_2 < 0\}$ and $\Omega_r \stackrel{\text{def}}{=} \{x \in \Omega : x_2 > 0\}$ and

$$w_0(x) \stackrel{\text{def}}{=} \chi_{\Omega_r}(x) - \chi_{\Omega_l}(x), \quad x \in \Omega \ \text{ and} \tag{3.9}$$

where $\Omega_l \stackrel{\text{def}}{=} \{x \in \Omega : x_3 < 0\}$ and $\Omega_r \stackrel{\text{def}}{=} \{x \in \Omega : x_3 > 0\}$.

The one-dimension character of any local minimizer u_ε (thus any global one) as well as the proof that the minimizer is an odd function of this variable is obtained via a refined application of the Unique Continuation Principle to the function $w_\varepsilon = 1 - u_\varepsilon^2$ and using the fact that $u_\varepsilon \longrightarrow u_0$ in $L^1(\Omega)$. When it is the case u_0 is replaced with v_0 or w_0.

The distinction between global and local minimizers are made using the symmetry properties of the domain and the well-known property of $\Gamma-$convergence: if a family of functionals $\{E_\varepsilon\}_{0<\varepsilon<\varepsilon_0}$ $\Gamma-$converges in $L^1(\Omega)$, as $\varepsilon \to 0$, to E_0 and a family $\{u_\varepsilon\}$ of global minimizers of E_ε satisfies $u_\varepsilon \longrightarrow u_0$ in $L^1(\Omega)$ then u_0 is also a global minimizer of E_0.

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