Real-valued non-analytic solutions for the generalized Korteweg-de Vries equation

A. Alexandrou Himonas* & Gerson Petronilho**

Abstract

In both the periodic and non-periodic cases non-analytic in time solutions to the Cauchy problem of the gKdV equation are constructed with real-valued analytic initial data when k is not a multiple of four. In the case that $k = 4\ell$, that is the nonlinearity is of the form $u^{4\ell}\partial_x u$, where ℓ is a positive integer, then non-analytic in time solutions are available only for complex-valued initial data.

1 Introduction

For $k \in \{1, 2, 3, \cdots\}$ we consider the Cauchy problem for the generalized Korteweg-de Vries (gKdV) equation

$$\begin{cases} \partial_t u = \partial_x^3 u + u^k \partial_x u, \quad x \in \mathbb{T} \text{ or } \mathbb{R},\, t \in \mathbb{R} \\ u(x,0) = \varphi(x), \end{cases} \tag{1.1}$$

and construct solutions with real-valued analytic initial data $\varphi(x)$ which are not analytic in the time variable. For $k = 1$ and $k = 2$ we obtain the KdV and mKdV respectively. These are the most important members of the gKdV family of equations since they are both integrable and can be solved by using inverse scattering. Furthermore, KdV has a celebrated history beginning with its derivation by Korteweg and de Vries as a model for long water waves in a channel [KdV].

In the periodic case existence and uniqueness of solution to this Cauchy problem with analytic initial data follows from the well-posedness of gKdV in Sobolev spaces, since such data belong to every Sobolev space. In [HHP] it is shown that this solution is analytic in the space variable x and Gevrey G^3 in the time variable t. Here we show that the regularity of the solution in time is not better than G^3 in both the periodic and the non-periodic cases by choosing appropriate real-valued analytic initial data. Well-posedness for the non-periodic gKdV equation in spaces of analytic functions has been proved by Grujić and Kalisch [GK].

*Corresponding author: himonas.1@nd.edu, Phone: 574-631-7583, FAX: 574-631-6579

**The second author was partially supported by CNPq and Fapesp.

2000 Mathematics Subject Classification. Primary 35Q53; Secondary 35B65

Key words and phrases. Generalized Korteweg de Vries equation, gKdV, initial value problem, periodic, non-periodic, analytic solutions

Furthermore, Bona, Grujić and Kalisch [BGK] have shown that solutions to the non-periodic gKdV equation with initial data that are analytic in a strip in the complex plane continue to be analytic in a strip as time progresses. Also, they proved algebraic lower bounds on the possible rate of decrease in time of the uniform radius of spatial analyticity.

The well-posedness of gKdV in Sobolev spaces has been studied by many authors. Kenig, Ponce and Vega [KPV1] proved that on the real line gKdV is locally well-posed in $H^s(\mathbb{R})$ for all $s \geq \frac{1}{2} - \frac{2}{k}$ if $k \geq 4$. When $k = 1$ (KdV), $k = 2$ (mKdV), $k = 3$ and $k = 4$ well-posedness is proved for $s > 3/4$, $s \geq 1/4$, $s \geq 1/12$ and $s \geq 0$ respectively. The periodic case was studied by Bourgain in [B1], where he proves local and global well-posedness in $H^s(\mathbb{T})$ for all $s \geq 0$ when $k = 1$. When $k = 2$ he proves local well-posedness in $H^s(\mathbb{T})$ for all $s \geq 1/2$. When $k > 2$ he proves local well-posedness for small $H^1(\mathbb{T})$ data. Also, he proves global well-posedness in $H^s(\mathbb{T})$, $s > 3/2$, with sufficiently small H^1 data. For the KdV and for complex-valued initial data these results were improved in [KPV2], where local-wellposedness is proved in $H^s(\mathbb{T})$ for all $s \geq -1/2$ and in $H^s(\mathbb{R})$ for all $s > -3/4$. Colliander, Keel, Staffilani, Takaoka and Tao [CKSTT1] proved the Global well-posedness for real-valued functions and for the same range of Sobolev indices. Also, it is shown that the real-valued Cauchy problem for mKdV is globally well-posed in $H^s(\mathbb{R})$ for all $s > 1/4$ and $H^s(\mathbb{T})$ for all $s \geq 1/2$. Kappeler and Topalov [KT1] proved well-posedness for KdV in $H^s(\mathbb{T})$, $s \geq -1$, in a weaker sense, using inverse scattering techniques. Also, in [KT2] they proved global well-posedness of mKdV in $L^2(\mathbb{T})$. When the nonlinearity $\partial_x[u^{k+1}]$ of gKdV is replaced by the more general form $\partial_x[F(u)]$, where F is a polynomial of degree $k+1$, then local well-posedness for the corresponding equation in $H^s(\mathbb{T})$ for all $s > 1/2$ and sufficiently small data has been established in [CKSTT2].

For more results about the well-posedness and ill-posedness of gKdV for various values of k we refer the reader to Birnir, Kenig, Ponce, Svanstedt and Vega [BKPSV], Kenig, Ponce and Vega [KPV3], Bona and Smith [BS], Bourgain [B2] and [B3], Ginibre and Tsutsumi [GT], Kato [K], Saut and Temam [ST], Sjöberg [S], Tao [T], Christ, Colliander and Tao [CCT], and the references therein. For analytic and Gevrey regularity results we refer the reader to Trubowitz [Tr], [GH2], Kato and Masuda [KM], Hayashi [H], De Bouard, Hayashi and Kato [DHK], Kato and Ogawa [KO], Tarama [Ta] and the references therein.

In this paper we show that the solution of the gKdV Cauchy problem (1.1) with real-valued analytic initial data may not be analytic in the time variable, in both the periodic and non-periodic cases. More precisely we prove the following results.

Theorem 1.1 *Suppose that for* $k \in \{1, 3, 5, \cdots\}$

$$u(x,0) = -\mathrm{Re}\left(\sum_{n=1}^{\infty} \widehat{\psi}(n) e^{inx}\right) \tag{1.2}$$

and for $k = 4r + 2$, $r = 0, 1, 2, \cdots$

$$u(x,0) = Re\left(i \sum_{n=1}^{\infty} \widehat{\psi}(n) e^{inx}\right), \tag{1.3}$$

where $\widehat{\psi}(n) = e^{-n}$, *then the solution* $u(x,t)$ *to the periodic initial value problem* (1.1) *is not analytic in* t, *for* t *near zero.*

The case $k = 4r$, $r = 1, 2, 3\cdots$ is missing. Real-valued non-analytic solutions in time with analytic initial data are not available in this case. However, complex valued non-analytic solutions on the circle have been constructed in [GH1] for all k. More precisely, if

$$\varphi(x) = \frac{i^{\frac{2}{k}} e^{-ix}}{M - e^{-ix}}, \tag{1.4}$$

for some $M > 1$, it is shown that the corresponding solution $u(x,t)$ is not analytic in t near $t = 0$.

In the non-periodic case we have the following result.

Theorem 1.2 *Suppose that for* $k \in \{1, 3, 5, \cdots\}$

$$u(x,0) = Re\left[(i-x)^{-2}\right] \tag{1.5}$$

and for $k = 4r+2$, $r = 0, 1, 2, \cdots$

$$u(x,0) = Re\left[(i-x)^{-1}\right], \tag{1.6}$$

then the solution $u(x,t)$ *of the non-periodic Cauchy problem* (1.1) *is not analytic in* t*, for* t *near zero.*

In this case and when k is a multiple of four real-valued non-analytic solutions are not available either. Complex-valued non-analytic solutions on the line have been constructed for all k in [GH1] by choosing the following analytic initial data

$$\varphi(x) = (a+x)^{-2/k}, \tag{1.7}$$

where $a \in \mathbb{C} - \mathbb{R}$.

Note that all of our initial data in Theorems 1.1 and 1.2 are real-valued, analytic and belong to H^s for any s. Therefore, the existence and uniqueness of solution follows from the gKdV well-posedness results mentioned above.

2 Proof of Theorems 1.1 and 1.2

We begin the proofs with the following result which expresses a time derivative of a gKdV solution as a linear combination of spacial derivatives with non-negative coefficients. The proof is based on Leibniz rule (see [GH1] for a similar result).

Proposition 2.1 *If* u *is a solution to the equation* (1.1) *then the following formula holds true*

$$\partial_t^j u = \partial_x^{3j} u + \sum_{q=1}^{j} \sum_{|\alpha|+2q=3j} C_\alpha^q \partial_x^{\alpha_1} u \cdots \partial_x^{\alpha_{qk+1}} u, \tag{2.1}$$

for $j \in \{1, 2, 3, \cdots\}$*, where* $C_\alpha^q \geq 0$*.*

Next, we give the proof of Theorem 1.1.

Periodic case: k odd. Differentiating (1.2) with respect to x gives

$$\partial_x^q u(x,0) = -\mathrm{Re}\left(\sum_{n=1}^{\infty} \widehat{\psi}(n)(in)^q e^{inx}\right).$$

Therefore,

$$\partial_x^q u(0,0) = -\mathrm{Re}(i^q)A_q,$$

where

$$A_q \doteq \sum_{n=1}^{\infty} \widehat{\psi}(n) n^q > 0. \tag{2.2}$$

For $j \in \mathbb{N}$, using the formula (2.1), we obtain

$$\partial_t^j u(0,0) = -Re(i^{3j})A_{3j} + \sum_{q=1}^{j} \sum_{|\alpha|+2q=3j} C_\alpha^q (-Re(i^{\alpha_1}))A_{\alpha_1} \cdots (-Re(i^{\alpha_{qk+1}}))A_{\alpha_{qk+1}}.$$

Since $Re(i^{3j}) \neq 0$ only if j is even, and the terms in the sum that are non-zero only happen when all α_ν are even it follows from the last equality that for j even we have

$$\begin{aligned}
\partial_t^j u(0,0) &= (-1)^{\frac{3j+2}{2}} A_{3j} + \sum_{q=1}^{j} \sum_{|\alpha|+2q=3j} C_\alpha^q (-1)^{\frac{\alpha_1+2}{2}} A_{\alpha_1} \cdots (-1)^{\frac{\alpha_{qk+1}+2}{2}} A_{\alpha_{qk+1}} \\
&= (-1)^{\frac{3j+2}{2}} A_{3j} + \sum_{q=1}^{j} \sum_{|\alpha|+2q=3j} C_\alpha^q (-1)^{\frac{|\alpha|+2(qk+1)}{2}} A_{\alpha_1} \cdots A_{\alpha_{qk+1}} \\
&= (-1)^{\frac{3j+2}{2}} A_{3j} + \sum_{q=1}^{j} \sum_{|\alpha|+2q=3j} C_\alpha^q (-1)^{\frac{3j-2q+2(qk+1)}{2}} A_{\alpha_1} \cdots A_{\alpha_{qk+1}} \\
&= (-1)^{\frac{3j+2}{2}} \left(A_{3j} + \sum_{q=1}^{j} \sum_{|\alpha|+2q=3j} C_\alpha^q A_{\alpha_1} \cdots A_{\alpha_{qk+1}} \right)
\end{aligned}$$

since in the last equality we have used the fact that k is odd. Since for j even we have $\left|(-1)^{\frac{3j+2}{2}}\right| = 1$ and $C_\alpha^q \geq 0$ it follows from the last equality and (2.2) that

$$\left|\partial_t^j u(0,0)\right| \geq A_{3j} = \sum_{n=1}^{\infty} \widehat{\psi}(n) n^{3j} = \sum_{n=1}^{\infty} e^{-n} n^{3j} \geq e^{-3j}(3j)^{3j}. \tag{2.3}$$

By using the fact that $k^k \geq k!$ and $(k+\ell)! \geq k!\ell!$ it follows from the last inequality that

$$\left|\partial_t^j u(0,0)\right| \geq e^{-3j}(j!)^3,$$

which shows that $u(0,\cdot)$ cannot be analytic near zero. □

Periodic case: $k = 4r+2$. As before, we have

$$\partial_t^j u(0,0) = Re(i^{3j+1})A_{3j} + \sum_{q=1}^{j} \sum_{|\alpha|+2q=3j} C_\alpha^q Re(i^{\alpha_1+1})A_{\alpha_1} \cdots Re(i^{\alpha_{qk+1}+1})A_{\alpha_{qk+1}}.$$

Since $Re(i^{3j+1}) \neq 0$ only if j is odd, and the terms in the sum that are non-zero only happen when all α_μ are odd it follows from the last equality that for j odd we have

$$\partial_t^j u(0,0) = (-1)^{\frac{3j+1}{2}} A_{3j} + \sum_{q=1}^{j} \sum_{|\alpha|+2q=3j} C_\alpha^q (-1)^{\frac{\alpha_1+1}{2}} A_{\alpha_1} \cdots (-1)^{\frac{\alpha_{qk+1}+1}{2}} A_{\alpha_{qk+1}}.$$

Thus, for j odd and by using our hypothesis we have

$$\begin{aligned}
\partial_t^j u(0,0) &= (-1)^{\frac{3j+1}{2}} A_{3j} + \sum_{q=1}^{j} \sum_{|\alpha|+2q=3j} C_\alpha^q (-1)^{\frac{|\alpha|+qk+1}{2}} A_{\alpha_1} \cdots A_{\alpha_{qk+1}} \\
&= (-1)^{\frac{3j+1}{2}} A_{3j} + \sum_{q=1}^{j} \sum_{|\alpha|+2q=3j} C_\alpha^q (-1)^{\frac{3j-2q+qk+1}{2}} A_{\alpha_1} \cdots A_{\alpha_{qk+1}} \\
&= (-1)^{\frac{3j+1}{2}} A_{3j} + \sum_{q=1}^{j} \sum_{|\alpha|+2q=3j} C_\alpha^q (-1)^{\frac{3j+1}{2}} (-1)^{\frac{-2q+q(4r+2)}{2}} A_{\alpha_1} \cdots A_{\alpha_{qk+1}} \\
&= (-1)^{\frac{3j+1}{2}} \left(A_{3j} + \sum_{q=1}^{j} \sum_{|\alpha|+2q=3j} C_\alpha^q A_{\alpha_1} \cdots A_{\alpha_{qk+1}} \right)
\end{aligned}$$

since in the last equality we have used the fact that $(-1)^{2rq} = 1$. It follows from this that for j odd we have

$$|\partial_t^j u(0,0)| \geq A_{3j}$$

which shows, as in first case, that $u(0,\cdot)$ cannot be analytic near zero. □

Now, we prove Theorem 1.2.

Non-periodic case: k odd. In this case we have

$$\begin{aligned}
\partial_t^j u(0,0) &= (3j+1)! Re(i^{-(3j+2)}) \\
&+ \sum_{q=1}^{j} \sum_{|\alpha|+2q=3j} C_\alpha^q (\alpha_1+1)! Re(i^{-(\alpha_1+2)}) \cdots (\alpha_{qk+1}+1)! Re(i^{-(\alpha_{qk+1}+2)}).
\end{aligned}$$

Since $Re(i^{-(3j+2)}) \neq 0$ only for j even, and the terms in the sum that are non-zero only happen when all α_μ are even it follows from the last equality that for j even we have

$$\begin{aligned}
\partial_t^j u(0,0) &= (3j+1)!(-1)^{-\frac{3j+2}{2}} \\
&+ \sum_{q=1}^{j} \sum_{|\alpha|+2q=3j} C_\alpha^q (\alpha_1+1)!(-1)^{-\frac{\alpha_1+2}{2}} \cdots (\alpha_{qk+1}+1)!(-1)^{-\frac{\alpha_{qk+1}+2}{2}}.
\end{aligned}$$

Thus, for j even, using the hypothesis we obtain

$$\begin{aligned}
\partial_t^j u(0,0) &= (3j+1)!(-1)^{-\frac{3j+2}{2}} \\
&+ \sum_{q=1}^{j} \sum_{|\alpha|+2q=3j} C_\alpha^q (-1)^{-\frac{|\alpha|+2(qk+1)}{2}} (\alpha_1+1)! \cdots (\alpha_{qk+1}+1)! \\
&= (3j+1)!(-1)^{-\frac{3j+2}{2}} \\
&+ \sum_{q=1}^{j} \sum_{|\alpha|+2q=3j} C_\alpha^q (-1)^{-\frac{3j-2q+2qk+2}{2}} (\alpha_1+1)! \cdots (\alpha_{qk+1}+1)! \\
&= (3j+1)!(-1)^{-\frac{3j+2}{2}} \\
&+ \sum_{q=1}^{j} \sum_{|\alpha|+2q=3j} C_\alpha^q (-1)^{-\frac{3j+2}{2}} (-1)^{-\frac{2q(k-1)}{2}} (\alpha_1+1)! \cdots (\alpha_{qk+1}+1)! \\
&= (-1)^{-\frac{3j+2}{2}} \left((3j+1)! + \sum_{q=1}^{j} \sum_{|\alpha|+2q=3j} C_\alpha^q (\alpha_1+1)! \cdots (\alpha_{qk+1}+1)! \right)
\end{aligned}$$

since in the last equality we have used the fact that $q(k-1)$ is even. It follows from this that for j even we have

$$|\partial_t^j u(0,0)| \geq (3j+1)!$$

which shows that $u(0,\cdot)$ cannot be analytic near zero. □

Non-periodic case: $k = 4r+2$**.** We have

$$\begin{aligned}
\partial_t^j u(0,0) &= (3j)! Re(i^{-(3j+1)}) \\
&+ \sum_{q=1}^{j} \sum_{|\alpha|+2q=3j} C_\alpha^q (\alpha_1)! Re(i^{-(\alpha_1+1)}) \cdots (\alpha_{qk+1})! Re(i^{-(\alpha_{qk+1}+1)}).
\end{aligned}$$

Since $Re(i^{-(3j+1)}) \neq 0$ only if j is odd, and the terms in the sum that are non-zero only happen when all α_μ are odd it follows from the last equality that for j odd we have

$$\partial_t^j u(0,0) = (3j)!(-1)^{-\frac{3j+1}{2}} + \sum_{q=1}^{j} \sum_{|\alpha|+2q=3j} C_\alpha^q (\alpha_1)!(-1)^{-\frac{\alpha_1+1}{2}} \cdots (\alpha_{qk+1})!(-1)^{-\frac{\alpha_{qk+1}+1}{2}}.$$

Thus, for j odd, using the hypothesis we have

$$\begin{aligned}
\partial_t^j u(0,0) &= (3j)!(-1)^{-\frac{3j+1}{2}} + \sum_{q=1}^{j} \sum_{|\alpha|+2q=3j} C_\alpha^q \alpha!(-1)^{-\frac{|\alpha|+qk+1}{2}} \\
&= (3j)!(-1)^{-\frac{3j+1}{2}} + \sum_{q=1}^{j} \sum_{|\alpha|+2q=3j} C_\alpha^q \alpha!(-1)^{-\frac{3j-2q+qk+1}{2}} \\
&= (3j)!(-1)^{-\frac{3j+1}{2}} + \sum_{q=1}^{j} \sum_{|\alpha|+2q=3j} C_\alpha^q \alpha!(-1)^{-\frac{3j+1}{2}}(-1)^{-\frac{-2q+qk}{2}}.
\end{aligned}$$

By using the fact that $k = 4r + 2$ it follows from the last equality that

$$\partial_t^j u(0,0) = (-1)^{-\frac{3j+1}{2}} \left((3j)! + \sum_{q=1}^{j} \sum_{|\alpha|+2q=3j} C_\alpha^q \alpha! \right).$$

It follows from this that for j odd we have

$$|\partial_t^j u(0,0)| \geq (3j)!$$

which shows that $u(0,\cdot)$ cannot be analytic near $t = 0$. □

References

[BGK] J. Bona, Z. Grujić and H. Kalisch, *Algebraic lower bounds for the uniform radius of spatial analyticity for the generalized KdV equation.* Ann. Inst. H. Poincaré Anal. Non Linéaire **22**, No. 6, (2005), 783–797.

[BS] J. Bona and R. Smith, *The initial-value problem for the Korteweg-de Vries equation.* Phil. Trans. Roy. Soc. London A **278**, No. 1287, (1975), 555–601.

[BKPSV] B. Birnir, C. Kenig, G. Ponce, N. Svanstedt, and L. Vega, *On the ill-posedness of the IVP for the generalized Korteweg-de Vries and nonlinear Schrödinger equations.* J. London Math. Soc. (2) **53**, No. 3, (1996), 551–559.

[B1] J. Bourgain, *Fourier transform restriction phenomena for certain lattice subsets and applications to nonlinear evolution equations, II, The KdV-equation.* Geom. Funct. Anal. **3**, No. 3, (1993), 209–262.

[B2] J. Bourgain, *On the Cauchy problem for periodic KdV-type equations.* Proceedings of the Conference in Honor of Jean-Pierre Kahane (Orsay, 1993), J. Fourier Anal. Appl. 1995, Special Issue, 17–86.

[B3] J. Bourgain, *Periodic Korteweg–de Vries equation with measures as initial data.* Selecta Math. (N.S.) **3**, No.2, (1997), 115–159.

[CCT] M. Christ, J. Colliander and T. Tao, *Asymptotics, frequency modulation, and low regularity ill-posedness for canonical defocusing equations.* Amer. J. Math. **125**, No. 6, (2003), 1235–1293.

[CKSTT1] J. Colliander, M. Keel, G. Staffilani, H. Takaoka, and T. Tao, *Sharp global well-posedness for KdV and modified KdV on* $\mathbb{R}$ *and* $\mathbb{T}$. J. Amer. Math. Soc. **16**, No. 3, (2003), 705–749.

[CKSTT2] J. Colliander, M. Keel, G. Staffilani, H. Takaoka, and T. Tao, *Multilinear estimates for periodic KdV equations, and applications.* J. Funct. Anal. **211**, (2004),173–218.

[DHK] A. De Bouard, N. Hayashi, and K. Kato, *Gevrey regularizing efect for the (generalized) Korteweg-de Vries equation and nonlinear Schrödinger equations.* Ann. Inst. H. Poincaré Anal. Non Linéaire **12**, No. 6, (1995), 673-725.

[GT] J. Ginibre and Y. Tsutsumi, *Uniqueness of solutions for the generalized Korteweg-de Vries equation*. SIAM J. Math. Anal. **20**, No. 6, (1989), 1388–1425.

[GH1] J. Gorsky and A. Himonas, *Construction of non-analytic solutions for the generalized KdV equation*. J. Math. Anal. Appl. **303**, No. 2, (2005), 522–529.

[GH2] J. Gorsky and A. Himonas, *On analyticity in space variable of solutions to the KdV equation*. Geometric analysis of PDE and several complex variables, Contemp. Math. **368**, Amer. Math. Soc., (2005), 233–247.

[GK] Z. Grujić and H. Kalisch, *Local well-posedness of the generalized Korteweg-de Vries equation in spaces of analytic functions*. Differential Integral Equations **15**, No. 11, (2002), 1325–1334.

[HHP] H. Hannah, A. Himonas and G. Petronilho, *Gevrey regularity for the periodic gKdV equation*. Preprint, (2010).

[H] N. Hayashi, *Analyticity of solutions of the Korteweg-de Vries equation*. SIAM J. Math. Anal. **22**, No. 6, (1991), 1738–1743.

[KT1] T. Kappeler and P. Topalov, *Global well-posedness of KdV in* $H^{-1}(\mathbb{T},\mathbb{R})$. Duke Math. J. **135**, No. 2, (2006), 327–360.

[KT2] T. Kappeler and P. Topalov, *Global well-posedness of mKdV in* $L^2(\mathbb{T},\mathbb{R})$. Comm. Partial Differential Equations **30**, No. 1-3, (2005), 435–449.

[K] T. Kato, *On the Cauchy problem for the (generalized) Korteweg-de Vries equation*. Adv. Math. Suppl. Studies **8**, (1983), 93–128.

[KM] T. Kato and K. Masuda, *Nonlinear evolution equations and analyticity I*. Ann. Inst. H. Poincaré Anal. Non Linéaire **3**, No. 6, (1986), 455-467.

[KO] K. Kato and T. Ogawa, *Analyticity and smoothing effect for the Korteweg–de Vries equation with a single point singularity*. Math. Ann. **316**, No. 3, (2000), 577–608.

[KPV1] C. Kenig, G. Ponce and L. Vega, *Well-posedness and scattering results for the generalized Korteweg-de Vries equation via the contraction principle*. Comm. Pure Appl. Math. **46**, (1993), 527–620.

[KPV2] C. Kenig, G. Ponce and L. Vega, *A bilinear estimate with applications to the KdV equation*. J. Amer.Math. Soc. **9**, (1996), 573–603.

[KPV3] C. Kenig, G. Ponce and L. Vega, *On the ill-posedness of some canonical dispersive equations*. Duke Math. J. **106**, No. 3, (2001), 617–633.

[KdV] D. J. Korteweg and G. de Vries, *On the change of form of long waves advancing in a rectangular canal, and on a new type of long stationary waves*. Philos. Mag. **5** 39, (1895), 422–443.

[ST] J. Saut and R. Temam, *Remarks on the Korteweg-de Vries equation*. Israel J. Math. **24**, No. 1, (1976), 78–87.

[S] A. Sjöberg, *On the Korteweg-de Vries equation: existence and uniqueness*. J. Math. Anal. Appl. **29**, (1970), 569–579.

[Ta] S. Tarama, *Analyticity of solutions of the Korteweg-de Vries equation*. J. Math. Kyoto Univ. **44**, No. 1, (2004), 1–32.

[T] T. Tao, *Nonlinear dispersive equations: local and global solutions*. American Mathematical Society, Providence, RI (2006).

[Tr] E. Trubowitz, *The inverse problem for periodic potentials*. Comm. Pure Appl. Math. **30**, (1977), 321–337.

A. Alexandrou Himonas (Corresponding author)
Department of Mathematics
University of Notre Dame
Notre Dame, IN 46556
E-mail: *himonas.1@nd.edu*

Gerson Petronilho
Departamento de Matemática
Universidade Federal de São Carlos
São Carlos, SP 13565-905, Brazil
E-mail: *gersonpetro@gmail.com*