

A COINCIDENCE THEOREM FOR COMMUTING INVOLUTIONS

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ABSTRACT. Let M^m be an m -dimensional, closed and smooth manifold, and $S, T : M^m \rightarrow M^m$ two smooth and commuting diffeomorphisms of period 2. Suppose that $S \neq T$ on each component of M^m . Denote by F_S and F_T the respective sets of fixed points. In this paper we prove the following coincidence theorem: if F_T is empty and the number of points of F_S is of the form $2p$, with p odd, then $\text{Coinc}(S, T) = \{x \in M^m \mid S(x) = T(x)\}$ has at least some component of dimension $m - 1$. This generalizes the classic example given by $M^m = S^m$, the m -dimensional sphere, $S(x_0, x_1, \dots, x_m) = (-x_0, -x_1, \dots, -x_{m-1}, x_m)$ and T the antipodal map.

1. Introduction.

Coincidence theory concerns itself with the following basic question: given two maps $f, g : X \rightarrow Y$ of topological spaces, when is there a coincidence point, that is, a point $x \in X$ such that $f(x) = g(x)$? If so, another question concerns with the size of the set of coincidence points, $\text{Coinc}(f, g)$. For example, one direction is looking for maps $f', g' : X \rightarrow Y$, with f' homotopic to f and g' homotopic to g , so that, in some sense, the size of $\text{Coinc}(f', g')$ is minimal. Coincidence theory includes the Fixed Point theory, with $X = Y$ and g the identity map. Also, it includes questions of Borsuk-Ulam type: if X is equipped with a free involution $T : X \rightarrow X$, one asks for $\text{Coinc}(f, fT)$. As is well known, there is a lot of articles in the literature dealing with such questions, and it is unnecessary to list references.

In the generality under which the coincidence question is above stated, one can say almost nothing of interest; good questions arise when we focus

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attention on specific situations. In this setting, consider $X = Y = M^m$ an m -dimensional, closed and smooth manifold, and S and T two smooth commuting diffeomorphisms of finite period on M^m . If $T : M^m \rightarrow M^m$ is a smooth diffeomorphism of finite period, the fixed point set of T , F_T , is either empty or a finite and disjoint union of closed submanifolds of M^m , $F_T = \bigcup_{j=0}^n F^j$ ($n < m$), where F^j denotes the union of those components of F_T having dimension j (see [1, page 206]). We denote by $\dim(F_T)$ the maximum dimension of a component of F_T ; $\dim(F_T)$ is a reasonable parameter to measure the size of F_T . Here, we are excluding components of M^m on which T acts as the identity map. Note that $\text{Coinc}(S, T)$ is the fixed point set of $S^{-1}T$, which is a diffeomorphism with period the least common multiple of the periods of S and T . We raise the following specific coincidence question: how far can $\dim(S)$ and $\dim(T)$ influence $\dim(S^{-1}T)$? In this paper we deal with a particular case of this question, taking S and T with period 2, that is, with S and T being involutions. Our result is motivated by the following classic example: consider $S, T : S^m \rightarrow S^m$, where S^m is the m -dimensional sphere, $S(x_0, x_1, \dots, x_m) = (-x_0, -x_1, \dots, -x_{m-1}, x_m)$ and $T(x_0, x_1, \dots, x_m) = (-x_0, -x_1, \dots, -x_m)$ (the antipodal map). Then F_S consists of two points, F_T is empty and $\dim(F_{ST}) = m - 1$. We will see that, in a much more general situation, the behavior is very similar to the classic example. Specifically, we will prove the following:

Theorem. *Let M^m be an m -dimensional, closed and smooth manifold, and S and T two smooth and commuting involutions on M^m with $S \neq T$ on each component of M^m . Suppose that F_T is empty and the number of points of F_S is of the form $2p$, with p odd. Then $\text{Coinc}(S, T)$ has at least some component of dimension $m - 1$; that is, $\dim(F_{ST}) = m - 1$.*

Remark 1. As an example of $S : M^m \rightarrow M^m$ with $2p$ fixed points, take the disjoint union of p copies of the above described $S : S^m \rightarrow S^m$.

Remark 2. The result is not valid if F_S has $2p$ points with p not odd. In fact, consider again the involutions of the classic example, $S, T : S^m \rightarrow S^m$. Let $(S^m)^r$ be the cartesian product of r copies of S^m , where $r \geq 2$. On $(S^m)^r$ we consider the product involutions $\mathcal{S} = S \times S \times \dots \times S$ and $\mathcal{T} = T \times T \times \dots \times T$ (r times). Then $F_{\mathcal{T}}$ is empty and $F_{\mathcal{S}}$ has 2^r points; however, $\dim(F_{\mathcal{S}\mathcal{T}}) = mr - r \neq mr - 1$.

Remark 3. The theorem stated above may suggest a kind of compensation law determining the possibilities for $\dim(F_S)$, $\dim(F_T)$ and $\dim(F_{ST})$ (for example, $\dim(F_S) + \dim(F_T) + \dim(F_{ST}) \geq m - 1$). However, in general there need be no relation among the three dimensions. In fact, one may consider any involution $U : N^n \rightarrow N^n$ on a closed smooth n -dimensional manifold N^n and let $M = N^n \times S^j$ with $S = U \times Id$, $T = Id \times -Id$; here, $j \geq 1$ and Id is the identity map. Then F_T and F_{ST} are empty and $F_S = F_U \times S^j$. Special choices of j and (N^n, U) give an essentially arbitrary choice of dimensions.

2. Proof of the result.

The main tool to be used lies in the context started with the famous work [3] of R. Thom about cobordism theory, which gives to him the Field Medal at 1958, and continued ten years later with the monumental work [2] of P. Conner and E. Floyd, which extended the previous Thom results to the setting of singular cobordism of spaces.

Write $F_S = \{P_1, P_2, \dots, P_{2p}\}$, $F_{ST} = \bigcup_{j=0}^n F^j$; at this stage, F_{ST} could be empty. Because T is free, $P_i \notin F^j$, for every $i = 1, 2, \dots, 2p$ and $j = 0, 1, \dots, n$. Because of the commutativity of the involutions, ST acts freely on F_S and S acts freely on F_{ST} . Without loss, we can suppose $ST(P_{2i-1}) = P_{2i}$, $i = 1, 2, \dots, p$. Choose disjoint closed m -balls B_1, B_2, \dots, B_{2p} with center P_1, P_2, \dots, P_{2p} , respectively. The radius $\epsilon > 0$ of these balls can be selected so that: i) each B_i is invariant under S , and up to an equivariant diffeomorphism S acts on B_i as the antipodal map; ii) ST acts on $B_1 \cup B_2 \cup \dots \cup B_{2p}$ as an orthogonal bundle map covering the action of ST on F_S . That is, each B_{2i-1} is orthogonally mapped by ST into B_{2i} . Denote by $\eta_j \rightarrow F^j$ the normal bundle of F^j in M^m , $j = 0, 1, \dots, n$, and by $D(\eta_j)$ the associated disk bundle. By identifying disk bundles associated to normal bundles to suitable tubular neighborhoods of the base spaces, each $D(\eta_j)$ is identified to a tubular neighborhood D_j of F^j so that $\bigcup_{j=0}^n D_j$ is the union of a pairwise disjoint collection of tubular neighborhoods of the various components of F_{ST} . Again, the radius $\epsilon > 0$ of these tubular neighborhoods can be selected so that: i) $D_j \cap B_i = \emptyset$ for every $j = 0, 1, \dots, n$ and $i = 1, 2, \dots, 2p$; ii) each D_j is invariant under ST , and up to an equivariant diffeomorphism ST acts on each fiber of D_j as the antipodal map; iii) each D_j is invariant under S , and up to an equivariant diffeomorphism S acts on D_j as an orthogonal bundle map covering the action of S on F^j . These choices are made possible by the fact that S and

ST define a smooth $Z_2 \oplus Z_2$ -action on M^m , and thus there is a Riemannian metric on M^m with respect to which S and ST act as isometries; see [2, Chapter 3, Section 22, pages 57 and 58].

Consider the manifold W obtained by removing the interior of all these tubular neighborhoods from M^m . That is,

$$W = M^m - ((\cup_{i=1}^{2p} \text{int}(B_i)) \cup (\cup_{j=0}^n \text{int}(D_j)));$$

W is an m -dimensional and compact manifold with boundary

$$\partial(W) = (\cup_{i=1}^{2p} \partial(B_i)) \cup (\cup_{j=0}^n \partial(D_j)).$$

Call $W' = \frac{W}{ST}$, the orbit space of W by ST . Since ST acts freely on W , W' is an m -dimensional and compact manifold with boundary

$$\partial(W') = \frac{\partial(W)}{ST}.$$

Call $W'' = \frac{W'}{S}$. Since S and T act freely on W , S acts freely on W' , so W'' is an m -dimensional and compact manifold with boundary

$$\partial(W'') = \frac{\partial(W')}{S}.$$

Set $\mu \rightarrow W''$ for the line bundle of the double cover $W' \rightarrow W''$, and take $F : W'' \rightarrow BO(1)$ a classifying map of this bundle. Set

$$f = F|_{\partial(W'')} : \partial(W'') \rightarrow BO(1),$$

the restriction map. Write $\xi \rightarrow \partial(W'')$ for the restriction of the bundle μ to $\partial(W'')$; obviously, f is a classifying map of ξ . Following Conner and Floyd, the pair $(\partial(W''), f)$ is a *singular manifold* in $BO(1)$, and represents an unoriented cobordism class $[(\partial(W''), f)]$ in the $(m-1)$ -dimensional unoriented cobordism group of $BO(1)$, $\mathcal{N}_{m-1}(BO(1))$; see [2, Chapter 1, Section 4]. Denoting the generator of the first cohomology group $H^1(BO(1), Z_2)$ by c , the characteristic class of $\xi \rightarrow \partial(W'')$ is then

$$w_1(\xi) = f^*(c) \in H^1(\partial(W''), Z_2).$$

Because $\partial(W'')$ is a boundary and the restriction of F is f , $[(\partial(W''), f)]$ bounds as a singular manifold in $BO(1)$; see [2, Chapter 1, Section 4]. Therefore the Whitney number obtained by evaluating the cohomology class

$$f^*(c^{m-1}) = (w_1(\xi))^{m-1} \in H^{m-1}(\partial(W''), Z_2)$$

on the fundamental homology class $[\partial(W'')]$,

$$(w_1(\xi))^{m-1}[\partial(W'')],$$

gives a (modulo 2) zero characteristic number; see [2, Theorem 17.2, page 47]. The next step is to analyze more closely this characteristic number, and the first analysis is over F_S . Note that $\frac{\cup_{i=1}^{2p} \partial(B_i)}{ST}$ is the disjoint union of p spheres of dimension $m - 1$,

$$\frac{\cup_{i=1}^{2p} \partial(B_i)}{ST} = S_1^{m-1} \cup S_2^{m-1} \cup \dots \cup S_p^{m-1},$$

where

$$S_i^{m-1} = \frac{\partial(B_{2i-1}) \cup \partial(B_{2i})}{ST}.$$

Since S acts on each $\partial(B_i)$ as the antipodal map, it acts on each S_i^{m-1} in the same way; in this way, the part of $\partial(W'')$ corresponding to F_S is the union of p real projective spaces of dimension $m - 1$,

$$\frac{S_1^{m-1}}{S} \cup \frac{S_2^{m-1}}{S} \cup \dots \cup \frac{S_p^{m-1}}{S}.$$

If $\alpha_i \in H^1(\frac{S_i^{m-1}}{S}, Z_2) \cong Z_2$ is the generator, it is known that α_i is the characteristic class of the double cover $S_i^{m-1} \rightarrow \frac{S_i^{m-1}}{S}$ and $\alpha_i^{m-1} \in H^{m-1}(\frac{S_i^{m-1}}{S}, Z_2) \cong Z_2$ is the generator. Since p is odd, it follows that, on F_S , the characteristic number $(w_1(\xi))^{m-1}[\partial(W'')]$ becomes

$$\sum_{i=1}^p \alpha_i^{m-1} [\frac{S_i^{m-1}}{S}] = 1.$$

Hence some component of F_{ST} must provide a nonzero characteristic number, which in particular implies that F_{ST} is nonempty. Take $j < m - 1$; we will analyze $(w_1(\xi))^{m-1}[\partial(W'')]$ on F^j . Note that $\frac{\partial(D_j)}{ST}$ is the total space of the real projective space bundle $p_j : RP(\eta_j) \rightarrow F_j$ associated to η_j , p_j the projection map. Further, S acts as a bundle map and freely on $RP(\eta_j)$, and this action covers the free action of S on F_j . Hence p_j induces the projection map

$$p'_j : \frac{RP(\eta_j)}{S} \rightarrow \frac{F_j}{S}.$$

Write $\xi_j \rightarrow \frac{RP(\eta_j)}{S}$ for the restriction of the line bundle $\xi \rightarrow \partial(W'')$ to $\frac{RP(\eta_j)}{S}$. Denoting by $\nu \rightarrow \frac{F_j}{S}$ the line bundle of the double cover $F_j \rightarrow \frac{F_j}{S}$, one has that $\xi_j \rightarrow \frac{RP(\eta_j)}{S}$ is the pullback

$$p_j'^*(\nu) \rightarrow \frac{RP(\eta_j)}{S}.$$

So

$$(w_1(\xi_j))^{m-1} = (p_j'^*(w_1(\nu)))^{m-1} = p_j'^*(w_1(\nu)^{m-1}) \in H^{m-1}\left(\frac{RP(\eta_j)}{S}, Z_2\right).$$

Because $w_1(\nu)^{m-1} \in H^{m-1}(F_j, Z_2)$ and $m-1 > j$, one has that $w_1(\nu)^{m-1} = 0$. It follows that, on $\bigcup_{j=0}^{m-2} F^j$, the characteristic number $(w_1(\xi))^{m-1}[\partial(W'')]$ is zero. Therefore some component of F^{m-1} must provide a nonzero characteristic number, which ends the proof.

Remark. Taking into account our result and Remarks 2 and 3, we raise the question of finding conditions on the pairs (M^m, T) , (M^m, S) , under which some relation among the three dimensions could be valid (e.g., the one stated in Remark 3). We also ask for similar results when S and T are diffeomorphisms with other periods.

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