NONNEGATIVE SOLUTIONS FOR INDEFINITE SUBLINEAR ELLIPTIC PROBLEMS

FRANCISCO ODAIR DE PAIVA

Abstract. This paper is devoted to the study of existence, nonexistence and multiplicity of positive solutions for the semilinear elliptic problem

$$-\Delta u = \lambda u + g(x, u) \quad \text{in} \quad \Omega,$$

$$u = 0 \quad \text{on} \quad \partial \Omega,$$

where $\Omega$ is a bounded domain of $\mathbb{R}^N$, $\lambda \in \mathbb{R}$ and $g(x, u)$ is a Carathéodory function. The obtained results apply to the following classes of nonlinearities:

$$a(x)u^q + b(x)u^p \quad \text{and} \quad c(x)(1 + u)^p \quad (0 \leq q < 1 < p).$$

The proofs rely on the sub-super solution method and the mountain pass theorem.

1. Introduction

In this paper we deal with nonnegative solutions of the semilinear elliptic problems of the type

$$\begin{cases}
-\Delta u = \lambda u + g(x, u) & \text{in} \quad \Omega \\
u \geq 0 & \text{in} \quad \Omega \\
u = 0 & \text{on} \quad \partial \Omega.
\end{cases}$$

(\(Q_\lambda\))

Here $\Omega$ is a bounded domain of $\mathbb{R}^N$, $\lambda$ is a real parameter and $g : \Omega \times \mathbb{R} \to \mathbb{R}$ is Carathéodory function such that $g(x, t) = g(x, 0) \geq 0$ for all $t \leq 0$. Our goal is to show that, under some assumptions on $g$, there is a $\lambda^*$ such that for all $\lambda < \lambda^*$ the problem (\(Q_\lambda\)) has at least two nontrivial solutions. By solutions we mean weak solutions in $H^1_0(\Omega)$.

Our assumptions on the function $g$ are motivated by similar ones in the papers [11, 12], where the case $\lambda = 0$ was studied. The proofs are based on an idea that appeared in [9]. First, we prove the existence of a solution by the sub-super solution method. Then, we prove that this solution is a local minimum of the associated functional. Finally, we get the second solution invoking the mountain pass theorem.

We are also interested and motivated by the model problem:

$$\begin{cases}
-\Delta u = \lambda u + a(x)u^q + b(x)u^p & \text{in} \quad \Omega \\
u \geq 0 & \text{in} \quad \Omega \\
u = 0 & \text{on} \quad \partial \Omega.
\end{cases}$$

(\(S_\lambda\))

where $0 \leq q < 1 < p < 2^* - 1$ and $a, b \in \mathcal{L}^\infty$ (called concave-convex problem). This kind of problem comes from the celebrated work [2], where problem (\(S_\lambda\)) with $\lambda = 0$, $b \equiv 1$ and $a \equiv \gamma > 0$ was considered. In [12], (\(S_\lambda\)) was considered with $\lambda = 0$ and $a(x) = \gamma c(x)$ ($\gamma > 0$ and $c(x) \geq 0$). Many others authors have studied problems

2000 Mathematics Subject Classification. 35J65 (35J20).

Key words and phrases. positive solution, indefinite sublinear nonlinearity, concave-convex nonlinearity.

This work was supported by CNPq, FAPESP and Université Libre de Bruxelles (ULB).
of this form, see for instance [6, 11, 12, 14, 15, 20, 21]. The common feature, in all these works, is the presence of a parameter in the nonlinearity. Problem \((S_\lambda)\), under Neumann boundary conditions, was considered in [1] with \(b \equiv \gamma > 0\) and \(a(x)\) changes sign.

We remark that our multiplicity result for \((Q_\lambda)\) only applies to \((S_\lambda)\) when \(a(x) \geq 0\). In the last section, following [1], we prove a multiplicity result for \((S_\lambda)\) when \(a(x)\) changes its sign.

Let us also note that our results cover the case \(g(x, u) = c(x)(1 + u)^p\), where \(p > 1\) and \(c\) is a bounded function. This kind of equation is also interesting and has been studied by several authors, see for instance [9, 12, 16, 18, 19].

The paper is organized in the following way. In the next section, we state our results relative to \((Q)\) and we comment the consequences in the special cases. The proofs of main theorems are presented in Sections 3, 4 and 5. Section 6 is devoted to problem \((S_\lambda)\).

2. Statement of main results

In this section, we present our results relative to \((Q_\lambda)\). For the existence result, the requirements on \(g\) are that:

\(H_0\) For any \(u \in H^1_0(\Omega)\), there exists \(K(x) \in L^{2^*'}\) such that

\[ |g(x, t)| \leq K(x), \quad \text{for a.e. } x \in \Omega \text{ and for all } 0 \leq t \leq u(x); \]

\(H_1\) There exist \(p > 1\) and \(c_0 > 0\) such that

\[ g(x, t) \leq c_0(1 + t^p) \quad \text{for a.e. } x \in \Omega \text{ and for all } t \geq 0; \text{ and} \]

\(H_1'\) There exist \(0 < q < 1\) and \(t_0, c_0' > 0\) such that

\[ g(x, t) \leq c_0't^q \quad \text{for a.e. } x \in \Omega \text{ and for all } 0 \leq t \leq t_0; \text{ or} \]

\(H_2\) There exist a nonempty sub domain \(\Omega_1 \subset \Omega, 0 \leq \alpha < 1, c_1 \in L^{\infty}\) with \(c_1(x) > 0\ a.e. \ x \in \Omega_1, \) and \(b_0, t_1 > 0\) such that

\[ g(x, t) \geq c_1(x)t^\alpha - b_0t \quad \text{for a.e. } x \in \Omega_1 \text{ and for all } 0 \leq t \leq t_1. \]

The energy functional is

\[ F_\lambda(u) = \int_\Omega |\nabla u|^2 dx - \frac{\lambda}{2} \int_\Omega (u^+)^2 dx - \int_\Omega G(x, u) dx, \]

where \(G(x, t) = \int_0^t g(x, s) ds\), and under the assumption \((H_0)\), \(F_\lambda\) is well defined in \(H^1_0(\Omega)\).

The assumption \((H_2)\) is a local sublinearity condition at the origin. The subdomain \(\Omega_1\) plays an important role in the solvability of \((Q_\lambda)\). Actually, we only are able to find solutions of \((Q_\lambda)\) which are nontrivial in \(\Omega_1\). Moreover, if \(u \in H^1_0(\Omega)\) is a solution of \((Q_\lambda)\) and is nontrivial in \(\Omega_1\) then, by the Vazquez’s maximum principle and \((H_2)\), we have that \(u(x) > 0\ a.e. \ x \in \Omega_1\). what motivates us to consider the following problem:

\[ -\Delta u + \lambda u + g(x, u) \quad \text{in } \Omega \]

\[ u > 0 \quad \text{in } \Omega_1, \]

\[ u = 0 \quad \text{on } \partial \Omega. \]

We will look for solutions of the problem \((P_\lambda)\). In what follows, we state the main results.
2.1. Existence of one solution.

**Theorem 1.** Assume that either (H₁) or (H₃') and (H₂a) holds. Moreover, assume that \( g(x,0) \geq 0 \). Then there is \( \lambda^* \in [−\infty, \infty] \) such that for all \( \lambda < \lambda^* \) problem \((P_\lambda)\) has at least one solution, and no solution for \( \lambda > \lambda^* \).

2.2. Nonexistence for large \( \lambda \).

**Theorem 2.** Assume the hypotheses of Theorem 1, with \( c_1(x) \equiv c_1 \) in (H₂) (i.e. the function \( c_1(x) \) is a constant). Moreover, suppose that there exist a nonempty subdomain \( \Omega'_1 \subset \Omega_1 \), \( \Omega_1 \) as in (H₂), and \( c_1' \in \mathbb{R} \) such that

\[
(H_{ne}) \quad g(x,t) \geq c_1't \quad \text{for a.e.} \quad x \in \Omega'_1 \quad \text{and all} \quad t \geq 0.
\]

Then \( \lambda^* < \infty \).

2.3. Existence for \( \lambda = \lambda^* \).

In order to get some a-priori estimates, we assume the following:

\( (H_3) \) There exist \( \sigma \leq 2^* \), \( d_1 > 0 \) such that

\[
g(x,t) \leq \sigma(t^{p-1}) \quad \text{for a.e.} \quad x \in \Omega \quad \text{and all} \quad t \geq 0.
\]

\( (H_4) \) There exist \( \theta > 2, 1 < r < 2 \) and \( d, t_2 > 0 \) such that

\[
\theta g(x,t) \leq t^r \text{ for a.e.} \quad x \in \Omega \quad \text{and all} \quad t \geq t_2.
\]

\( (H_5) \) There exist a nonempty sub domain \( \Omega_2 \subset \Omega_1 \), \( t_2 > 0 \) and \( c_2 > 0 \) such that

\[
G(x,t) \geq c_2t^2 \quad \text{for a.e.} \quad x \in \Omega_2 \quad \text{and all} \quad t \geq t_2.
\]

**Theorem 3.** Assume the hypotheses of Theorem 1. Moreover, assume that \((H_3 - H_4)\) hold and \( \Omega \) is \( C^2 \). If either \( \lambda^* < \lambda_1(\Omega) \) or \( \Omega_2 = \Omega \) in \((H_3)\), then problem \((P_\lambda)\) has at least one solution for \( \lambda = \lambda^* \).

2.4. Multiplicity for \( \lambda < \lambda^* \).

In addition to the previous hypotheses, we assume that:

\( (H_6) \) For any \( s > 0 \) there is \( B > 0 \) such that

\[
t \to g(x,t) + Bt \quad \text{is nondecreasing on} \quad [0,s], \quad \text{for a.e.} \quad x \in \Omega.
\]

\( (H_7) \) Either \( \Omega \) is \( C^2 \) or for a.e. \( x \in \Omega \) the function \( t \in (0, \infty) \mapsto g(x,t) \) is \( C^1 \) and

\[
|g_t(x,t)| \leq C(1 + t^{\beta-2}),
\]

for some \( 2 \leq \beta < 2^* \).

**Theorem 4.** Assume the hypotheses of Theorem 1. Moreover, assume that \((H_3 - H_4 - H_5 - H_6 - H_7)\) hold. If either \( \lambda < \min\{\lambda_1(\Omega), \lambda^*\} \) or \( \lambda < \lambda^* \) and \( \Omega_2 = \Omega \) in \((H_5)\), then problem \((P_\lambda)\) has at least two nontrivial solutions.

2.5. Applications.

Let \( 0 < q < 1 < p \leq 2^* - 1 \) and assume that \( a, b, c \in L^\infty(\Omega) \) and \( c(x) \geq 0 \). For \( (x,t) \in \Omega \times \mathbb{R}^+ \), define \( g_1(x,t) = a(x)t^q + b(x)t^p \), \( g_2(x,t) = c(x) + b(x)t^p \) and \( g_3(x,t) = c(x)(1 + t)^p \). It follows that \( g_1, g_2 \) and \( g_3 \) satisfy \((H_0 - H_1)\).

Assuming that \( a(x) > 0 \) a.e. in some ball \( B_a \), and \( c(x) > 0 \) a.e. in some ball \( B_c \), then it is easy to see that \( g_1, g_2 \) and \( g_3 \) satisfy \((H_2)\). Moreover, if \( b(x) \geq 0 \) in \( B_a \) (respectively, \( b(x) \geq 0 \) in \( B_c \)), it is clear that \( g_1 \) (respectively, \( g_2 \)) satisfies \((H_{ne})\) in Theorem 2. This condition is, obviously, satisfied by \( g_3 \).
The hypotheses \((H_3 - H_4)\) are satisfied by \(g_1, g_2\) and \(g_3\) if \(p < 2^* - 1\) \(((H_3)\) is obvious, the verification of \((H_4)\) for \(g_1, g_2\) and \(g_3\) follows as in [11, p. 457] and [12, p. 285], respectively). Assumption \((H_5)\) is satisfied by \(g_1, g_2\) and \(g_3\) if \(b(x) \geq \epsilon_6 > 0\) a.e. in some ball \(B_0\), and \(c(x) \geq \epsilon_c > 0\) a.e. in some ball \(B_c\), respectively.

Assuming that \(a(x) \geq 0\) a.e. in \(\Omega\), it follows that \(g_1\) satisfies \((H_6)\). It is clear that \((H_6)\) is satisfied by \(g_2\) and \(g_3\). Moreover, if \(\Omega\) is \(C^2\) (and so \((H_7)\) holds), then we are ready to apply Theorem 4. On the other hand, \(g_2\) and \(g_3\) satisfy the second alternative in \((H_7)\), provided \(p < 2^* - 1\). When \(\Omega\) is not regular, then \((H_7)\) in not satisfied by \(g_1\). That problem is studied in Section 6.

The exact statement of the results are left to the reader.

3. PROOF OF THEOREM 1

First, we shall prove that under \((H_1)\) or \((H'_1)\) the problem \((P_\lambda)\) has a supersolution for \(\lambda < \lambda_0 - 1\).

Claim 1. There is \(\tilde{\lambda}\) such that for \(\lambda \leq \tilde{\lambda}\) the problem \((P_\lambda)\) has a supersolution.

In fact, let \(e\) be a solution of
\[-\Delta u = 1 \quad \text{in} \quad \Omega,\]
\[u = 0 \quad \text{on} \quad \partial\Omega.\]

First, suppose that \((H_1)\) holds. Let \(m\) be greater than \(c_0\). There exists \(\delta > 0\) such that \(c_0 + c_0 m^p e(x)^p \leq m\) for \(x \in \Omega_\delta = \{x \in \Omega : e(x) < \delta\}\). Thus
\[\lambda m e(x) + c_0 + c_0(m e(x))^p \leq m, \quad \text{for all } x \in \Omega_\delta,
\]
provided \(\lambda < 0\). We can choose \(\tilde{\lambda} < 0\) such that
\[m \geq \lambda m \delta + c_0 + c_0 \|m e\|_\infty^p \quad \text{for all } \lambda \leq \tilde{\lambda}.
\]
It follows that, if \(\lambda \leq \tilde{\lambda}\),
\[-\Delta (m e) = m \geq \lambda m \delta + c_0 + c_0 \|m e\|_\infty^p \geq \lambda m e(x) + c_0 + c_0(m e)^p\]
for \(x \in \Omega \setminus \Omega_\delta\). Thus, by \((H_1)\),
\[-\Delta (m e) \geq \lambda m e + g(x, m e) \quad \text{in} \quad \Omega.
\]
So \(m e\) is a supersolution to \((P_\lambda)\), since \(\lambda \leq \tilde{\lambda}\), q.e.d.

Now, suppose that \((H'_1)\) holds, take \(m' > 0\) such that \(m'(e(x)) \leq t_0\) for all \(x \in \Omega\) \((t_0\) as in \((H'_1)\)). Then there exists \(\delta > 0\) such that \(c'_0(m'(\delta)^q \leq m'\), and so \(c'_0(m'(e(x))^q \leq m'\) for \(x \in \Omega_\delta = \{x \in \Omega : e(x) < \delta\}\). Thus
\[\lambda m' e(x) + c'_0(m'(e(x))^q \leq m', \quad \text{for all } x \in \Omega_\delta,
\]
provided \(\lambda < 0\). Hence
\[-\Delta (m' e) = m' \geq \lambda m' \delta + c_0(m' e)^q \quad \text{in} \quad \Omega_\delta.
\]
Choosing \(\tilde{\lambda} < 0\) such that
\[m' \geq \lambda m' \delta + c_0 \|m' e\|_\infty^q \quad \text{for all } \lambda < \tilde{\lambda},
\]
it follows that
\[-\Delta (m' e) = m' \geq \lambda m' \delta + c_0 \|m' e\|_\infty^q \geq \lambda m' e + c'_0(m' e)^q\]
for \(x \in \Omega \setminus \Omega_\delta\), if \(\lambda \leq \tilde{\lambda}\). Thus, by \((H'_1)\),
\[-\Delta (m' e) \geq \lambda m' e + g(x, m' e) \quad \text{in} \quad \Omega.
\]
and so \( m'e \) is a supersolution to \((P_\lambda)\) if \( \lambda \leq \bar{\lambda} \), q.e.d.

**Conclusion of the proof of Theorem 1.**

**Step 1.** Existence of a nontrivial solution for \( \lambda \leq \bar{\lambda} \).

Let \( \lambda \) be such that \( \lambda \leq \bar{\lambda} \), where \( \bar{\lambda} \) as in Claim 1. Defining \( \varpi := me \), where \( m \) and \( e \) are as in Claim 1, we have that \( \varpi \) is a supersolution for \((Q_\lambda)\). Moreover, \( \varpi = 0 \) is a subsolution, since \( g(x, 0) \geq 0 \). Consider the following minimization problem

\[
\inf_M F_\lambda, \quad \text{where} \quad M = \{ u \in H^1_0 \mid \varpi(x) \leq u(x) \leq \varpi(x) \text{ a.e. } x \in \Omega \}.
\]

By Theorem I.2.4 from [23], the above infimum is achieved at \( u_1 \in M \) and, in addition, \( u_1 \) is a solution of \((Q_\lambda)\). It remains to show that \( u_1 \) solves \((P_\lambda)\). Suppose, by contradiction, that \( u \equiv 0 \) a.e. \( x \in \Omega \). Let \( \varphi \in C^1_0(\Omega_1) \) be nonnegative and nontrivial. Therefore \( u_1 + s\varphi \in M \) and \( s\varphi \leq t_1 \) (\( t_1 \) as in \((H_2)\)), for sufficiently small \( s > 0 \). Using \((H_2)\), we get

\[
F_\lambda(u_1 + s\varphi) = F_\lambda(u_1) + F_\lambda(s\varphi) \\
= F_\lambda(u_1) + \frac{s^2}{2}||\varphi|| - \lambda \frac{s^2}{2}||\varphi||_2 - \int_\Omega G(x, s\varphi)dx \\
\leq F_\lambda(u_1) + \frac{s^2}{2}(\text{Const.}) - s^{1+\alpha} \int_\Omega c_1(x)\varphi^{\alpha+1}dx, \\
= F_\lambda(u_1) + s^{\alpha+1} \left( \frac{s^{1-\alpha}}{2}(\text{Const.}) - \int_\Omega c_1(x)\varphi^{\alpha+1}dx \right).
\]

It follows that \( F_\lambda(u_1 + s\varphi) < F_\lambda(u_1) \) if \( s > 0 \) is small enough, since \( 0 < \alpha < 1 \) and \( \int_\Omega c_1\varphi^{\alpha+1}dx > 0 \). This contradicts the definition of \( u_1 \), and so \( u_1 \) is a solution of \((P_\lambda)\).

Now, we define

\[
\Lambda := \{ \lambda \in \mathbb{R} : (P_\lambda) \text{ has a solution} \}, \quad \text{and} \quad \lambda^* := \sup \Lambda.
\]

By Step 1, we have that \( \Lambda \neq \emptyset \).

**Step 2.** Existence of a nontrivial solution for \( \lambda < \lambda^* \).

Let \( \lambda \) be such that \( \bar{\lambda} < \lambda < \lambda^* \), where \( \bar{\lambda} \in \Lambda \). Set \( \varpi \) the solution of \((P_{\lambda^*})\), then

\[
-\Delta \varpi = \lambda \varpi + g(x, \varpi) \geq \lambda \varpi + g(x, \varpi),
\]

and so \( \varpi \) is a supersolution for \((P_\lambda)\). Consider \( M = \{ u \in H^1_0 \mid 0 \leq u \leq \varpi \} \).

Let \( u_1 \in M \) such that \( F_\lambda(u_1) = \inf_M F_\lambda \), as in Step 1, \( u_1 \) is a solution of \((Q_\lambda)\). Suppose, by contradiction, that \( u_1 \) does not solve \((P_\lambda)\), i.e. \( u_1 \equiv 0 \) a.e. \( x \in \Omega_1 \). Let \( \varphi \in C^1_0(\Omega_1) \) be nonnegative and nontrivial such that \( \varphi \varpi \gtrsim 0 \) a.e. \( x \in \Omega_1 \). So we get \( u_1 + s\varphi \varpi \in M \) for \( s > 0 \) that is sufficiently small. Then, by a similar argument as in Step 1, we have that \( F_\lambda(u_1 + s\varphi \varpi) < F_\lambda(u_1) \) if \( s > 0 \) is small enough, which contradicts the definition of \( u_1 \). Thus \( u_1 \) is a solution of \((P_\lambda)\).
4. Proof of Theorem 2 and 3

4.1. Proof of Theorem 2.

We will prove by contradiction that \((P_\lambda)\) has no solution for large \(\lambda\). To do this, suppose that \(u\) is a nontrivial solution of \((P_\lambda)\). Let \(\mu_1'\) be the first eigenvalue of \((-\Delta, H^1_0(\Omega))\) and \(\phi_1\) the associated eigenfunction. We have

\[
\int_{\Omega_1} \nabla u \nabla \phi_1' dx = \mu_1' \int_{\Omega_1} u \phi_1' dx.
\]

On the other hand

\[
\int_{\Omega_1} \nabla u \nabla \phi_1' dx = \lambda \int_{\Omega_1} u \phi_1' dx + \int_{\Omega_1} g(x, u) \phi_1' dx.
\]

Then

\[
c_1' \int_{\Omega_1} u \phi_1' \leq \int_{\Omega_1} g(x, u) \phi_1' \leq (\mu_1' - \lambda) \int_{\Omega_1} u \phi_1' dx,
\]

which is a contradiction if \(\lambda > \mu_1 - c_1'\).

\(\Box\)

4.2. Proof of Theorem 3.

We begin by recalling that, under the assumption \((H_3)\) and the regularity of \(\Omega\), the solutions of \((P_\lambda)\) are in \(C^1_0(\Omega)\).

By the definition of \(\lambda^*\), there is a sequence \(\lambda_k \in \Lambda\) such that \(\lambda_k \nearrow \lambda^*\) and \((P_{\lambda_k})\) has a solution. Let \(u_k\) be a solution of \((P_{\lambda_k})\) with \(F(u_k) < 0\) (it is a consequence of Step 2 in the proof of Theorem 1). Let us suppose for a moment that \(u_k \equiv 0\) in \(\Omega_1\). Then we can assume that \(u_k \rightharpoonup u\) in \(H^1_0(\Omega)\). Thus \(u\) solves \((Q_{\lambda^*})\) and \(F(u) \leq 0\). Moreover, by standard bootstrap, we can assert that \(u_k \to u\) in \(C^1_0(\Omega)\).

We have to prove that \(u\) is a solution of \((P_\lambda)\). For this purpose, assume, by contradiction, that \(u = 0\) in \(\Omega_1\), \(\Omega_1\) as in \((H_2)\). Let \(\varphi_1 > 0\) be the associated eigenfunction to the eigenvalue \(\lambda_1(\Omega_1)\). We can assume that \(u_k(x) \leq t_1\) for all \(x \in \Omega_1\) if \(k\) is large enough, \(t_1\) is as in \((H_2)\). We have

\[
\lambda_1(\Omega_1) \int_{\Omega_1} u_k \varphi_1 dx = \int_{\Omega_1} \nabla u_k \nabla \varphi_1 dx
\]

\[
= \lambda_k \int_{\Omega_1} u_k \varphi_1 dx + \int_{\Omega_1} g(x, u_k) \varphi_1 dx
\]

\[
\geq (\lambda_k - b_0) \int_{\Omega_1} u_k \varphi_1 dx + \int_{\Omega_1} c_1 u_k^p \varphi_1 dx.
\]

Then

\[
\int_{\Omega_1} c_1 u_k^{p} \varphi_1 dx \leq (\lambda_1(\Omega_1) + b_0 - \lambda_k) \int_{\Omega_1} u_k \varphi_1 dx.
\]

It is a contradiction, if \(k\) is large enough, since \(c_1 u_k(x)^{p} \geq (\lambda_1(\Omega_1) + b_0 - \lambda_k) u_k\) for a.e. \(x \in \Omega_1\) (here we use that \(c_1(x)\), in \((H_2)\), is a constant).

It remains to prove that \(||u_k||\) is bounded. Since \(F_{\lambda_k}(u_k) = 0\) and \(F(u_k) < 0\), we have

\[
\theta F(u_k) - F'(u_k) \cdot u_k \leq 0,
\]

where \(\theta\) is from \((H_4)\). At this point we divided the proof in two cases.
Case 1. Assume that $\lambda^* < \lambda_1(\Omega)$. Using $(H_4)$, we can rewrite (1) as

$$\left(\frac{\theta}{2} - 1\right) \left(||u_k||^2 - \lambda_k||u_k||^2_2\right) \leq C||u_k||^r + C.$$

It follows that $||u_k||$ is bounded, provided $\lambda_k \leq \lambda^* < \lambda_1(\Omega)$.

Case 2. Assume that $\lambda^* \geq \lambda_1(\Omega)$. Assuming $\Omega = \Omega_2$ in $(H_5)$, there exist $t_1, \theta_1 > 0$ such that

$$G(x, t) \geq \theta_1 t^2$$

for a.e. $x \in \Omega$ and all $t \geq t_1$.

As show in [11], the above condition and $(H_4)$ imply that for some $s, c > 0$,

$$G(x, t) \geq ct$$

for a.e. $x \in \Omega$ and all $t \geq s$.

where $\theta$ is as in $(H_4)$. We define $h_k(x, t) = g(x, t) + \lambda_k t$. We claim that for $\epsilon > 0$, such that $2 < \theta - \epsilon$, we have

$$\left(\theta - \epsilon\right)H_k(x, t) \leq \theta_1 t^2$$

for a.e. $x \in \Omega$ and all $t \geq s_2$.

But, it follows easily from (2).

Using (3), we can rewrite (1) as

$$\left(\frac{\theta - \epsilon}{2} - 1\right) ||u_k||^2 \leq C||u_k||^r + C.$$

It follows that $||u_k||$ is bounded. This concludes the proof.

\[\square\]

5. Proof of Theorem 4

We want to prove that there exists a second solution of $(P_\lambda)$ for each $\lambda < \min\{\lambda_1(\Omega), \lambda^*\}$ or $\lambda < \lambda^*$ if $\Omega_2 = \Omega$ in $(H_5)$. To this end, fixe $\lambda$, with this restrictions, and set $\lambda < \lambda < \lambda^*$ and $u_1$ a solution of $(P_{\lambda^*})$. Let $u_1$ be the solution of $(P_\lambda)$ constructed in the proof of Theorem 1. First, we will show that $u_1$ is a local minimum in $H_0^1$. After this, we get the second solution applying the mountain pass theorem.

Assume that $\Omega$ is $C^2$. We know that $\pi \geq u_1$ and $\pi \not\equiv u_1$. Moreover, there is $B > 0$ such that

$$-\Delta(\pi - u_1) \geq \pi - u_1 + g(x, \pi) - g(x, u_1) \geq (\pi - B)(\pi - u_1) \text{ in } \Omega.$$

Applying the Vazquez’s maximum principle, we can conclude that

$$\pi > u_1 \text{ in } \Omega \text{ and } \partial_\nu u_1 < \partial_\nu \pi \text{ on } \partial\Omega.$$

Analogously, we can prove that

$$u_1 > 0 \text{ in } \Omega \text{ and } \partial_\nu u_1 < 0 \text{ on } \partial\Omega.$$
It follows that \( M = \{ u \in H_0^1 : \underline{u} \leq u \leq \overline{u} \} \) contains a \( C^1 \) neighborhood of \( u_1 \), and so \( u_1 \) is a local minimizer for \( F_\lambda \) in \( C^1_0 \). Thus \( u_1 \) is a local minimizer for \( F_\lambda \) in \( H_0^1 \) (see, for instance, [7, Theorem 1]).

In the other hand, if we do not assume regularity of \( \Omega \), the maximum principle implies only that
\[
\underline{u} < u_1 < \overline{u} \quad \text{in} \quad \Omega.
\]

But, since \( u_1 \) is obtained as the minimum of \( F \) in \( M \), Proposition 1, proved below, asserts that \( u_1 \) is a local minimizer for \( F_\lambda \) in \( H_0^1 \).

The second solution will be obtained by applying the mountain pass theorem in convex sets. More specifically, we will look for solution in the set
\[
V = \{ u \in H_0^1 : u \geq u_1 \}.
\]

We recall:

**Definition 1.** We say that \( u \in V \) is a critical point of \( F_\lambda \) in \( V \) if
\[
f_\lambda (u) = \sup \{ F_\lambda (u) \cdot (u - v) : v \in V, ||v - u|| \leq 1 \} = 0.
\]

Since \( u_1 \) is a critical point of \( F_\lambda \), then a critical point of \( F_\lambda \) in \( V \) will be also a critical point of \( F_\lambda \) in \( H \) (see [23, p. 168]).

The next theorem can be proved as the classical versions of the mountain pass theorem, see for instance [20]. We just replace the deformation lemma by a version of deformation lemma in convex subsets proved in [23, Chapter 2, Theorem 12.7].

**Theorem 5.** Let \( W \) be closed and convex subset of a Hilbert space \( H \) and \( F \in C^1 (H, \mathbb{R}) \). Suppose that:

1. There exist \( v_1, v_2 \in W \) and \( r, \rho > 0 \) such that \( ||v_1 - v_2|| > \rho \) and \( F(w) \geq r > \max \{ F(v_1), F(v_2) \} \) for all \( w \in W \) with \( ||w - v|| = \rho \).
2. For any sequence \( \{ w_n \} \subset W \) are such that \( \{ F(w_n) \} \) is bounded and
\[
f(w_n) = \sup \{ F(w_n) \cdot (w_n - w) : w \in W, ||w - w_n|| \leq 1 \} \rightarrow 0,
\]
then \( \{ w_n \} \) is relatively compact.

Then \( F \) has a critical point in \( W \) with critical value \( c \geq r \).

In this point, we will verify the conditions of the Theorem 5 (this concludes the proof of Theorem 4). We start showing the item (1). Since \( u_1 \) is a local minimizer for \( F_\lambda \) in \( H_0^1 \), we have that there is \( r, \rho > 0 \) such that
\[
F_\lambda (v) \geq r > F_\lambda (u_1), \quad \text{for all} \quad v \in V \quad \text{with} \quad ||v - u_1|| = \rho.
\]

Now we look for some \( v_2 \in V \) such that \( F_\lambda (v_2) \leq F_\lambda (v_1) \) and \( ||v_2 - u_1|| > \rho \). Note that, as shown in [11], we have \( t_3 \) and \( c > 0 \) such that
\[
G(x, t) \geq c t^a, \quad \text{for a.e.} \quad x \in \Omega_2 \quad \text{and all} \quad t \geq t_3.
\]
Let $v_0$ be nontrivial and smooth function, which is supported $\Omega_2$. Then
\[
F_\lambda(u_1 + sv_0) \leq C + Cs + Cs^2 - \int_\Omega G(x, u_1 + sv_0)dx
\]
\[
= O(s^2) - \int_{\Omega \setminus \Omega_2} G(x, u_1)dx - \int_{\Omega_2} G(x, u_1 + sv_0)dx
\]
\[
\leq O(s^2) - \int_{\Omega_2} G(x, u_1 + sv_0)dx
\]
\[
\leq O(s^2) - \int_{\{x \in \Omega_2: u_1 + sv_0 \geq s_3\}} G(x, u_1 + sv_0)dx
\]
\[
\leq O(s^2) - c \int_{\{x \in \Omega_2: u_1 + sv_0 \geq s_3\}} (u_1 + sv_0)^\theta dx
\]
\[
\leq O(s^2) - cs^\theta \int_{\{x \in \Omega_2: u_1 + sv_0 \geq s_3\}} v_0 dx \to -\infty \quad \text{as} \quad s \to \infty,
\]
since $\theta > 2$. So we can then choose $s_0 > 0$ such that $F_\lambda(u_1 + s_0v_0) \leq F_\lambda(u_0)$ and $\|sv_0\| > \rho$. The function $v_2 = u_0 + s_0v_0$ satisfies the required properties.

For proving the item (2), let $u_n \in V$ be a sequence such that $\{F_\lambda(u_n)\}$ is bounded and
\[
f(u_n) = \sup \{F_\lambda'(u_n) \cdot (u_n - v) : v \in V, \|v - u_n\| \leq 1\} \to 0.
\]
For $\theta$ as in $(H_3)$, $\epsilon_n \to 0$ and $v \in V$ we have
\[
\theta F_\lambda(u_n) + F_\lambda'(u_n) \cdot (u_n - v) \leq C + \epsilon_n \|u_n - v\|.
\]
Choosing $v = 2u_n$, we get
\[
\theta F_\lambda(u_n) - F_\lambda'(u_n) \cdot (u_n) \leq C + C\|u_n\|,
\]
it means that
\[
\left(\frac{\theta}{2} - 1\right) (\|u_n\|^2 - \lambda\|u_n\|_2^2) - \int_\Omega (\theta G(x, u_n) - u_ng(x, u_n)) \leq C + C\|u_n\|.
\]
Consequently, by $(H_4)$,
\[
\|u_n\|^2 - \lambda\|u_n\|_2^2 \leq C + C\|u_n\|^\gamma + C\|u_n\|.
\]
It follows that $\|u_n\|$ is bounded, provided $\lambda < \lambda_1(\Omega)$. For showing in the case $\lambda > \lambda_1$, remember that, we have assumed that $\Omega_2 = \Omega$ in $(H_5)$, So we can use the same argument used in the end of the proof of Theorem 3, and we also conclude that $\|u_n\|$ is bounded.

Finely, $(H_3)$ implies that $\{u_n\}$ has a convergent subsequence, by standard arguments.

\[\square\]

5.1. Appendix.

Let $u$ and $v$ be a subsolution and a supersolution of $(P_\lambda)$, respectively.

**Proposition 1.** Assume that for a.e. $x \in \Omega$ the function $t \in (0, \infty) \to g(x, t)$ is $C^1$ and
\[
|g_t(x, t)| \leq C(1 + t^{\beta-2}),
\]

NONNEGATIVE SOLUTIONS 9
for some $2 \leq \beta \leq 2^*$. Moreover, suppose that $u_0$ satisfies $\underline{u} < u_0 < \overline{u}$ and

$$F_\lambda(u_0) = \inf_M F_\lambda,$$

where $M = \{ u \in H^1_0 : \underline{u}(x) \leq u(x) \leq \overline{u}(x) \text{ a.e. } x \in \Omega \}$. Then $u_0$ is a local minimum of $F_\lambda$ in $H^1_0$.

**Proof.** Suppose that there is $u_n \in H^1_0$ with $\|u_n - u_0\| \to 0$ and $F_\lambda(u_n) < F(u_0)$. Define

$$v_n = \max\{ \underline{u}, \min\{ u_n, \overline{u} \} \}, \quad w_n = (u_n - \overline{u})^+, \quad z_n = (\underline{u} - u_n)^+,$$

it follows that $u_n = v_n - z_n + w_n$, $v_n \in M$, and $w_n$ and $z_n$ have disjoint support. Define the sets $R_n = \{ x \in \Omega : \underline{u} \leq u_n(x) \leq \overline{u} \}$, $S_n = \text{supp}(w_n)$ and $T_n = \text{supp}(z_n)$, and the functions

$$h(x, t) = \lambda t^+ + g(x, t) \quad \text{and} \quad H(x, t) = \frac{\lambda}{2} (t^+)^2 + G(x, t).$$

Then, we can rewrite $F(u_n)$ as

$$F_\lambda(u_n) = \int_{S_n \cup T_n} \left( \frac{\| \nabla u_n \|^2}{2} - H(x, u_n) \right) + \int_{R_n} \left( \frac{\| \nabla v_n \|^2}{2} - H(x, v_n) \right).$$

Observe that

$$\int_{S_n} \left( \frac{\| \nabla u_n \|^2}{2} - H(x, u_n) \right) = \int_{S_n} \left( \frac{\| \nabla (\overline{u} + w_n) \|^2}{2} - H(x, \overline{u} + w_n) \right),$$

$$\int_{T_n} \left( \frac{\| \nabla u_n \|^2}{2} - H(x, u_n) \right) = \int_{T_n} \left( \frac{\| \nabla (\underline{u} - z_n) \|^2}{2} - H(x, \underline{u} - z_n) \right),$$

and

$$\int_{R_n} \left( \frac{\| \nabla v_n \|^2}{2} - H(x, v_n) \right) = F(u_n) - \int_{S_n} \left( \frac{\| \nabla \overline{u} \|^2}{2} - H(x, \overline{u}) \right) - \int_{T_n} \left( \frac{\| \nabla \underline{u} \|^2}{2} - H(x, \underline{u}) \right).$$

Therefore

$$F(u_n) = F(v_n) + \int_{S_n} \left( \frac{\| \nabla (\overline{u} + w_n) \|^2}{2} - \| \nabla \overline{u} \|^2 \right) - (H(x, \overline{u} + w_n) - H(x, \overline{u}))$$

$$+ \int_{T_n} \left( \frac{\| \nabla (\underline{u} - z_n) \|^2}{2} - \| \nabla \underline{u} \|^2 \right) - (H(x, \underline{u} - z_n) - H(x, \underline{u})).$$

Moreover

$$\int_{\Omega} \nabla \overline{u} \cdot \nabla w_n dx \geq \int_{\Omega} h(x, \overline{u}) w_n dx, \quad \text{and}$$

$$\int_{\Omega} \nabla \underline{u} \cdot \nabla z_n dx \leq \int_{\Omega} h(x, \underline{u}) z_n dx.$$
Thus
\[
F(u_n) \geq F(v_n) + \frac{1}{2} \int_{\Omega} |\nabla w_n|^2 dx + \frac{1}{2} \int_{\Omega} |\nabla z_n|^2 dx
- \int_{S_n} (H(x, \overline{\nu} + w_n) - H(x, \overline{\nu}) - h(x, \overline{\nu})w_n) dx
- \int_{T_n} (H(x, \overline{\mu} - z_n) - H(x, \overline{\mu}) - h(x, \overline{\mu})(-z_n)) dx.
\]

We can then conclude that
\[
\frac{1}{2}||w_n||^2 + \frac{1}{2}||z_n||^2 < \int_{S_n} (H(x, \overline{\nu} + w_n) - H(x, \overline{\nu}) - h(x, \overline{\nu})w_n) dx
+ \int_{T_n} (H(x, \overline{\mu} - z_n) - H(x, \overline{\mu}) - h(x, \overline{\mu})(-z_n)) dx.
\]

The conclusion of the proof of the proposition follows from the next claim.

Claim. We have
\[
\int_{S_n} (H(x, \overline{\nu} + w_n) - H(x, \overline{\nu}) - h(x, \overline{\nu})w_n) dx \leq o(1)||w_n||^2, \quad \text{and}
\int_{T_n} (H(x, \overline{\mu} - z_n) - H(x, \overline{\mu}) - h(x, \overline{\mu})(-z_n)) dx \leq o(1)||z_n||^2.
\]

Assuming for a moment that the above claim is true, we have
\[
\frac{1}{2}(||z_n||^2 - o(1)||z_n||^2) + \frac{1}{2}(||w_n||^2 - o(1)||w_n||^2) < 0,
\]
which implies that \(w_n = z_n = 0\) for \(n\) large enough. It follows that \(w_n = \nu_n \in M\), for \(n\) large, and so we conclude that \(F(\nu_n) \geq F(\nu_0)\), what is a contradiction.

Proof of the Claim. We will prove only the first statement, the second one is similar. First, define \(H_n(x) = H(x, \overline{\nu} + w_n) - H(x, \overline{\nu}) - H_n(x, \overline{\nu})w_n\) and consider the following splitting for \(H_n = H_{0n} + H_{1n}\), where
\[
H_{0n}(x) = \frac{\lambda}{2}[(\overline{\nu} + w_n)^2 - \overline{\nu}^2] - \lambda \overline{\nu} w_n, \quad \text{and}
H_{1n}(x) = G(x, \overline{\nu} + w_n) - G(x, \overline{\nu}) - g(x, \overline{\nu})w_n.
\]

Note that \(H_{0n}(x) = \frac{\lambda}{2} w_n^2\), so we have
\[
\frac{\lambda}{2} \int_{S_n} w_n^2 dx \leq \frac{\lambda}{2}|S_n|^\frac{1}{1} \left( \frac{1}{\Omega} \int_{\Omega} \frac{w_n^2}{x} dx \right)^{\frac{N-2}{2}} \leq \frac{\lambda}{2}|S_n|^\frac{1}{1} ||w_n||^2.
\]

Now, we claim that \(|S_n| = o(1)\). Actually, given \(\epsilon > 0\) we can choose \(\delta > 0\) such that \(|\{\overline{\nu} \leq u_0 + \delta\}| < \epsilon\), since \(u_0 < \overline{\nu}\) in \(\Omega\). However, we have
\[
S_n \subset \{\overline{\nu} \leq u_0 + \delta\} \cup \{u_n > \overline{\nu} > u_0 + \delta\}.
\]

And since \(u_n \to u_0\) in \(L^2\) there is \(n_0\) such that for all \(n \geq n_0\),
\[
\epsilon \delta^2 \geq \int_{\Omega} (u_n - u_0)^2 \geq \int_{\{u_n > u_0 + \delta\}} (u_n - u_0)^2
\geq \int_{\{u_n > u_0 + \delta\}} \delta^2 |\{u_n > u_0 + \delta\}|.
\]
Thus $|S_n| \leq |\{u \leq u_0 + \delta\}| + |\{u_n > u_0 + \delta\}| \leq 2\epsilon$. It follows that
\[
\int_{S_n} H_{u_n}(x) dx \leq o(1)||w_n||^2.
\]
Now, there are $s(x), t(x) \in (0,1)$ such that
\[
H_{u_n}(x) = g(x, \pi + s(x)w_n) - g(x, \pi)w_n \leq Cw_n^2 + Cw_n^\beta.
\]
Thus
\[
\int_{S_n} H_{u_n}(x) dx \leq o(1)||w_n||^2 + C||w_n||^\beta \leq o(1)||w_n||^2,
\]
since $\beta \geq 2$. It finishes the proof of the Claim.

6. Multiplicity with indefinite sublinear nonlinearities

Consider the following problem
\[
(S_\lambda) \quad \begin{cases}
-\Delta u = \lambda u + a(x)u^q + b(x)u^p & \text{in } \Omega \\
u \geq 0 & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega.
\end{cases}
\]
where $0 < q < 1 < p < 2^*$ and $a, b \in L^\infty$. Our goal here is to consider the case when $a$ changes its sign. We assume that:

(a) The sets $\Omega^+_a = \{x \in \Omega : a(x) > 0\}$ and $\Omega^-_a = \{x \in \Omega : a(x) < 0\}$ are nonempty, $\Omega^+_a$ is open, moreover,
\[
\overline{\Omega^+_a} \cap \overline{\Omega^-_a} = \emptyset \quad \text{and} \quad \overline{\Omega^-_a} \subset \Omega.
\]

(ab) The sets $\Omega_a = \{x \in \Omega : a(x) = 0\}$ and $\Omega_b = \{x \in \Omega : b(x) = 0\}$ are nonempty and $\Omega_a \cap \Omega_b$ has nonempty interior.

(b) The set $\Omega^+_b = \{x \in \Omega : b(x) > 0\}$ has nonempty interior.

Moreover, we define $\Omega^+_b = \Omega \setminus \{b \neq 0\}.$

Remark 1. i) Note that, by the Maximum Principle, if $u$ is a solution of $(S_\lambda)$ such that $u$ is nontrivial in the components of $\Omega_a$, then $u > 0$ in $\text{int}(\Omega_a) \supset \Omega^+_a$.

ii) Assumption $\Omega^+_a \cap \overline{\Omega^-_a} = \emptyset$ has been considered by many authors in the study of elliptic problems with indefinite nonlinearities, see for instance [3, 4]. Assumption $\Omega^+_a \subset \Omega$ was considered in [5].

Theorem 6. Assume (a), (ab) and (b). Then there is $\lambda^* \in \mathbb{R}$ such that:

(1) For all $\lambda < \lambda^*$ problem $(S_\lambda)$ has at least one nontrivial solution, and has no solution for $\lambda > \lambda^*$;

(2) If we assume that $b(x) > 0$ a.e. $x \in \Omega$ and $a(x) \geq a_0 > 0$ in some ball $B_a$, then for $\lambda = \lambda^*$ problem $(S_\lambda)$ has at least one nontrivial solution;

(3) If we assume that
\[
\text{int}(\Omega_a) = \bigcup_i U_i, \quad U_i \text{ connected, } U_i \cap \Omega^+_a \neq \emptyset,
\]
then for all $\lambda < \min\{\lambda^*, \lambda_1(\Omega^+_b)\}$ problem $(S_\lambda)$ has at least two nontrivial solutions.
Remark 2. The statements (1) and (2) are consequence of Theorem 1 and Theorems 2 and 3, respectively. The statement (3) is not a consequence of Theorem 4, because the assumption \((H_0)\) is not satisfied when \(|\Omega_0^*| > 0\). Moreover, the requirement on \(\lambda\) is different.

Proof of Theorem 6.

Proof of (1) and (2). The verification of \((H_1)\) is a easy consequence of the assumptions \(a, b \in L^\infty\) and \(0 < p < 1 < p\). To verify \((H_2)\) note that \(g(x, t) = a(x)(t^+)^q + b(x)(t^+)p \ge a(x)(t^+)^q - \|b\|_\infty t^+\) for all \(x \in \Omega_0^*\) and \(0 \le t \le 1\). Then we can apply Theorem 1. We also can apply Theorem 2 with \(\Omega_1 = \text{int}(\Omega_0 \cap \Omega_b)\) and \(c_1 = 0\), since by \((ab)\) we have \(g(x, u) \ge 0\) in \(\text{int}(\Omega_0 \cap \Omega_b)\). Thus we get a \(\lambda^* \in \mathbb{R}\) which fulfills the requirements of (1). Moreover, denoting by \(u_1\), the solution obtained by Theorem 1, we have that \(u_1 > 0\) in \(\Omega_0^*\) (see the proof of Theorem 1).

The item (2) follows as the proof of Theorem 3, we need only to show that: if \(\lambda_n \not> \lambda^*\) and \(u_n\) is the solution of \((F_{\lambda_n})\) obtained in (1), then \(|u_n|\) is bounded. We have that

\[(p + 1)F_{\lambda_n}(u_n) - F_{\lambda_n}'(u_n) \cdot u_n \le 0,\]

it follows that

\[|u_n|^2 \le C|u_n|^q + C|u_n|^{q+1}.\]

From the above inequality, it enough to show that \(|u_n|_2\) is bounded. To do this, suppose, by contradiction, that \(|u_n|_2 \to \infty\). Then \(v_n = u_n/|u_n|_2\) is bounded, and so we can assume that \(v_n \to v_0\) in \(H_0^1\), \(v_n \to v_0\) in \(L^r\), for \(1 \le r < 2N/(N-2)\). In particular, we have \(|v_0|_2 = 1\). For a function \(\phi \in C_0^1\) we consider the identity \((1/|u_n|_2)F'(u_n)\phi = 0\), that means

\[|u_n|^2 = \int_\Omega b(x)v_n^p \phi dx = \int_\Omega \left(\nabla v_n \nabla \phi - \lambda_n v_n \phi - \frac{a(x)}{|u_n|^2} u_n^q \phi \right) dx = \int_\Omega \left(\nabla v_n \nabla \phi - \lambda_n v_n \phi \right) dx + o(1).

It follows that

\[\int_\Omega b(x)v_n^p \phi dx = \liminf_{n \to \infty} \int_\Omega b(x)v_n^p \phi dx = 0.

Which implies that \(v_0 = 0\) a.e. \(x \in \Omega\), that is a contradiction with \(|v_0|_2 = 1\). Now, we can proceed as the proof of Theorem 3.

Proof of (3). Fixe \(\lambda < \lambda^*\) and let \(\lambda < \bar \lambda < \lambda^*\) and \(\pi\) a nontrivial solution of \((S_\bar \lambda)\). Let \(u_1\) the solution of \((S_\lambda)\) constructed at the proof of Theorem 1. Note that \(u_1 > 0\) in the connected components of \(\Omega_0^*\), see the proof of Theorem 1. By the Proposition 2 below, \(u_1\) is a local minimum in \(H_0^1\). Now, in order to get the second solution, consider the functional

\[I_\lambda(u) = \frac{1}{2} \int_\Omega |\nabla (u_1 + u)|^2 dx - \int_\Omega H(x, u_1 + u) dx,

where

\[H(x, t) = \frac{\lambda}{2}(t^+)^2 + \frac{1}{q + 1}a(x)(t^+)^{q+1} + \frac{1}{p + 1}b(x)(t^+)^{p+1}.

Note that, \(I_\lambda(u) = F_\lambda(u + u_1)\), where \(F_\lambda\) is the associated functional of \((S_\lambda)\), that here reads

\[F_\lambda(u) = \frac{1}{2} \int_\Omega |\nabla u|^2 dx - \int_\Omega H(x, u) dx.

NONNEGATIVE SOLUTIONS 13
Moreover, if $u \in H_0^1$ is a critical point of $I_\lambda$, then $u_1 + u$ is a nonnegative critical point of $F_\lambda$, and so a solution of $(S_\lambda)$. Thus, we have to prove that $I_\lambda$ has a nontrivial critical point. To this end, we will show that $I_\lambda$ satisfies the assumptions of the relaxed mountain pass theorem, see [17, Corollary 5.11].

First, observe that $I_\lambda(0) = F_\lambda(u_1)$. Thus we have to show that:

(i) there is $r > 0$ such that $I_\lambda(u) \geq F_\lambda(u_1)$ for all $u \in H_0^1$ with $\|u\| = r$;
(ii) there is $w_1 \in H_0^1$ such that $I(w_1) \leq F_\lambda(u_1)$ and $\|w_1\| > r$; and
(iii) $I_\lambda$ satisfies the $(PS)$ condition.

The item (i) is a consequence of $u_1$ being a local minimum of $F_\lambda$. In order to prove item (ii), let $v_1 \in C^1_0(\Omega_0)$ be nonnegative, nontrivial and such that $\int_\Omega b(x)v_1^{p+1} > 0$. We have, for large $s$,

$$F_\lambda(u_1 + sv_1) = O(s^2) - \int_\Omega \left( \frac{a(x)(u_1 + sv_1)^q}{q + 1} + \frac{b(x)(u_1 + sv_1)^{p+1}}{p + 1} \right)$$

$$= O(s^2) - s^{q+1} \int_\Omega a(x)(u_1/s + v_1)^q dx$$

$$- s^{p+1} \int_\Omega b(x)(u_1/s + v_1)^{p+1} dx$$

$$\to -\infty \text{ as } s \to \infty,$$

and so (ii) follows.

For (iii), first, we remark that $F_\lambda$ satisfies $(PS)$ condition if $\lambda < \lambda_1(\Omega_0^0)$. It follows as the proof of Lema 1.5 in [4]. Now, let $u_n$ be a $(PS)$ sequence of $I_\lambda$ at level $c$, it follows that $u_1 + u_n$ is a $(PS)$ sequence for $F_\lambda$, and so has a convergent subsequence. Thus, $I_\lambda$ satisfies the $(PS)$ condition.

6.1. Appendix.

In this appendix, we are assuming that $\overline{\lambda} > \lambda^*$ and $\overline{\pi}$ is a nontrivial solution of $(S_\pi)$, given by Theorem 1.

Lemma 1. Let $\lambda < \overline{\lambda}$ and $u$ solution of $(S_\lambda)$. If $u \leq \overline{\pi}$, then $u < \overline{\pi}$ in $A = \{\pi > 0\}$.

Proof. Let $v = \pi - u \geq 0$ a.e. in $\Omega$, then

$$-\Delta v + m(x)v \geq 0, \text{ where } m(x) := \left( a - \frac{\overline{\pi}^q - u^q}{\overline{\pi} - u} + b - \frac{\overline{\pi}^p - u^p}{\overline{\pi} - u} \right).$$

Suppose by contradiction that $v(x_0) = 0$ for some $x_0 \in A$. We can choose $r > 0$ such that the ball $B_r[x_0] \subset A$. We have that $m$ is uniformly bounded in $B_r(x_0)$, so by the Strong Maximum Principle we get $v = 0$ in $B_r(x_0)$. It means that $u = \overline{\pi}$ in $B_r(x_0)$, what contradicts the equations satisfied by these functions, since $\lambda < \overline{\lambda}$.

Proposition 2. Suppose that $u_0$ attains

$$F_\lambda(u_0) = \inf_M F_\lambda, \text{ where } M := \{u \in H_0^1 : 0 \leq u \leq \overline{\pi} \text{ a.e. } x \in \Omega\}.$$

Then $u_0$ is a local minimum of $F_\lambda$ in $H_0^1$. 

\[\square\]
Proof. Suppose that there is $u_n \in H^1_0$ with $|u_n - u_0| \to 0$ and $F_\lambda(u_n) < F_\lambda(u_0)$. Let
\[ v_n = \max\{0, \min\{u_n, \overline{u}\}\}, \quad w_n = (u_n - \overline{u})^+, \]
so that $u_n = v_n - u_n^+ + u_n^-, \ v_n \in M$, and $u_n^-$ and $w_n$ have disjoint support. Define the sets $R_n = \{ x \in \Omega : 0 \leq u_n(x) \leq \overline{u}\}$, $S_n = \text{supp}(w_n)$ and $T_n = \text{supp}(u_n^-)$, and the functions
\[ h(x, t) = \lambda t^+ + a(x)(t^+)^q + b(x)(t^+)^p \quad \text{and} \quad H(x, t) = \int_0^t h(x, s)ds. \]
Arguing as in the proof of Proposition 1, we can show that
\[ \frac{1}{2}|w_n|^2 + \frac{1}{2}|u_n^-|^2 < \int_{S_n} (H(x, \overline{u} + w_n) - H(x, \overline{u}) - H_n(x, \overline{u})w_n) dx. \]

Claim. $\int_{S_n} (H(x, \overline{u} + w_n) - H(x, \overline{u}) - H_n(x, \overline{u})w_n) \leq o(1)|w_n|^2$.

Assuming the above claim, we have
\[ \frac{1}{2}|u_n^-|^2 + \frac{1}{2}|u_n|^2 < o(1)|w_n|^2 < 0, \]
which implies that $w_n, u_n^- = 0$ for $n$ large enough. It follows that $u_n = v_n \in M$, for $n$ large, and so $F_\lambda(u_n) \geq F_\lambda(u_0)$, what is a contradiction.

Proof of the Claim.

First, consider the following splitting for the function $H_n = H_{0n} + H_{1n} + H_{2n}$, where
\[ H_{0n}(x) = \frac{\lambda}{2}|(\overline{u} + w_n)^2 - \overline{u}^2| - \lambda \overline{u}w_n, \]
\[ H_{1n}(x) = \frac{b(x)}{p + 1}[(\overline{u} + w_n)^p - \overline{u}^p] - b(x)\overline{u}^pw_n, \quad \text{and} \]
\[ H_{2n}(x) = \frac{a(x)}{q + 1}[(\overline{u} + w_n)^q - \overline{u}^q] - a(x)\overline{u}^qw_n. \]

Superlinear term: Note that, there are $s(x), \theta(x) \in (0, 1)$ such that
\[ H_{1n}(x) = b(x)([\overline{u} + \theta w_n]^p - \overline{u}^p)w_n \leq C(\pi + s\theta w_n)^{p-1}\theta w_n^2. \]
Moreover, $(\pi + s\theta w_n)^{p-1}w_n^2 \leq w_{n+1}^2$ in $B = \Omega \setminus A$, where $A = \overline{\{ \pi > 0\}}$, then
\[ \int_{S_n \setminus A} H_{1n}(x)dx \leq C||w_n||^{p+1} \leq o(1)||w_n||^2. \]
On the other hand, $(\pi + s\theta w_n)^{p-1}w_n^2 \leq Cw_n^2 + Cw_{n+1}^2$ in $A$, then
\[ \int_{S_n \cap A} H_{1n}(x)dx \leq C \int_{S_n \cap A} w_n^2dx + C||w_n||^{p+1} \]
\[ \leq |S_n \cap A|^{\frac{2}{p+1}} \left( \int_{\Omega} w_n^{2q}dx \right)^{\frac{p+1}{2}} + C||w_n||^{p+1} \]
\[ \leq C|S_n \cap A|^{\frac{2}{p+1}}||w_n||^2 + o(1)||w_n||^2. \]
Now, we claim that \(|S_n \cap A| \to 0\) as \(n \to \infty\). Actually, given \(\epsilon > 0\), by Lemma 1, we can choose \(\delta > 0\) such that \(|A \cap \{\pi \leq u_0 + \delta\}| < \epsilon\). However, we have

\[
S_n \subset \{\pi \leq u_0 + \delta\} \cup \{u_n < \pi > u_0 + \delta\},
\]

and since \(u_n \to u_0\) in \(L^2\) there is \(n_0\) such that for all \(n \geq n_0\),

\[
\epsilon \delta^2 \geq \int_{\Omega} (u_n - v_0)^2 \geq \int_{\{u_n > u_0 + \delta\}} (u_n - v_0)^2 \geq \int_{\{u_n > u_0 + \delta\}} \delta^2 |\{u_n > u_0 + \delta\}|.
\]

Thus \(|S_n \cap A| \leq |A \cap \{\pi \leq u_0\}| + |\{u_n > u_0 + \delta\}| \leq 2\epsilon\). It follows that

\[
\int_{S_n \cap A} H_{1,n}(x) dx \leq o(1)||w_n||^2.
\]

**Sublinear term:** First, observe that \(\Omega_n^+ \subset A\) and \(B \subset \Omega \setminus \Omega_n^+\). We have, for \(x \in A \setminus \Omega_n^+\),

\[
H_{2n}(x) = a(x)(|\pi + \theta w_n|^q - \pi^q) w_n \leq 0.
\]

Thus

\[
\int_{S_n \cap (A \setminus \Omega_n^+)} H_{2n}(x) dx \leq o(1)||w_n||^2.
\]

On the other hand, note that \(\Omega_n^+ \subset int(\Omega_n)\) and \(\pi > 0\) in \(int(\Omega_n)\), and so there is \(\delta > 0\) such that \(\pi(x) \geq \delta \in \Omega_n^+\). Then, for \(x \in \Omega_n^+\),

\[
H_{2n}(x) = a(x)(|\pi + \theta w_n|^q - \pi^q) w_n = a(x)(\pi + s \theta w_n)^{q-1} \theta w_n^2 \leq C \delta^{q-1} w_n^2 \leq C w_n^2,
\]

with \(\theta(x), s(x) \in (0, 1)\). Thus

\[
\int_{S_n \cap \Omega_n^+} H_{2n}(x) dx \leq C \int_{S_n \cap \Omega_n^+} w_n^2 \leq |S_n \cap \Omega_n^+| \pi \|w_n\|^2 \leq o(1)\|w_n\|^2,
\]

where the last inequality is a consequence of \(S_n \cap \Omega_n^+ \subset S_n \cap A\) and \(|S_n \cap A| = o(1)\).

Now, for \(x \in B\) we have \(H_{2n}(x) = -\frac{a^{-1}(x)}{q+1} w_n^{q+1}\) since \(B \subset \Omega_n^-\). Thus

\[
\int_B H_{2n}(x) dx = -\int_B \frac{a^{-1}(x)}{q+1} w_n^{q+1} dx \leq 0.
\]

**Linear term:** Note that \(H_{0n}(x) = \frac{\lambda}{2} w_n^2\), we have

\[
\frac{\lambda}{2} \int_{S_n \cap A} w_n^2 dx \leq \frac{\lambda}{2} |S_n \cap A| \frac{1}{q+2} \left(\int_{\Omega} \frac{\partial w_n}{\partial x} dx \right)^{\frac{q+2}{q+1}} \leq o(1)\|w_n\|^2.
\]

The integral in \(B\) will be estimate by the integral in (6). Define the set

\[
U_n = \left\{ x \in \Omega_n^- : w_n(x) \geq \left(\frac{2a^{-1}(x)}{\lambda(q+1)}\right)^{\frac{1}{q+1}} \right\}.
\]
It follows that $|U_n| \to 0$ as $n \to \infty$, since $w_n \to 0$ in $H_0^1$. Therefore
\[
\frac{\lambda}{2} \int_B w_n(x)^2 \, dx \leq \int_{B \setminus U_n} \frac{a^-(x)}{q + 1} w_n^{q+1} \, dx + \frac{\lambda}{2} \int_{U_n} w_n^2 \, dx \\
\leq \int_{B} \frac{a^-(x)}{q + 1} w_n^{q+1} \, dx + \frac{\lambda}{2} |U_n|^\frac{N}{q-2} \left( \int_{\Omega} w_n^{\frac{2N}{N-2}} \, dx \right)^{\frac{N-2}{N}} \\
\leq \int_{B} \frac{a^-(x)}{q + 1} w_n^{q+1} \, dx + o(1)||w_n||^2.
\]
The claim follows from (6) and the above estimates.

7. Acknowledgements

This work was done while the author was visiting the Mathematics Department of Université Libre de Bruxelles (ULB). The author thanks Professor Jean Pierre-Gossez and Humberto Ramos Quoirin for helpful discussion as well as the Mathematics Department of ULB for hospitality and a stimulating scientific atmosphere. Finally, we would like to thank the referee for helpful comments and suggestions.

References


Universidade Federal de São Carlos, Departamento de Matemática, 13565-905, São Carlos - SP, Brazil
E-mail address: odair@dm.ufscar.br