NONNEGATIVE SOLUTIONS FOR INDEFINITE SUBLINEAR ELLIPTIC PROBLEMS

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Abstract. This paper is devoted to the study of existence, nonexistence and multiplicity of positive solutions for the semilinear elliptic problem

$$\begin{array}{lll} -\Delta u = \lambda u + g(x,u) & \text{in} & \Omega \\ u = 0 & \text{on} & \partial\Omega, \end{array}$$

where Ω is a bounded domain of $\mathbb{R}^N$, $\lambda \in \mathbb{R}$ and $g(x,u)$ is a Carathéodory function. The obtained results apply to the following classes of nonlinearities: $a(x)u^q + b(x)u^p$ and $c(x)(1+u)^p$ $(0 \le q < 1 < p)$. The proofs rely on the sub-super solution method and the mountain pass theorem.

1. Introduction

In this paper we deal with nonnegative solutions of the semilinear elliptic problems of the type

$$(Q_\lambda) \qquad \begin{cases} -\Delta u = \lambda u + g(x,u) & \text{in} \quad \Omega \\ u \ge 0 & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega. \end{cases}$$

Here Ω is a bounded domain of $\mathbb{R}^N$, λ is a real parameter and $g : \Omega \times \mathbb{R} \to \mathbb{R}$ is Carathéodory function such that $g(x,t) = g(x,0) \ge 0$ for all $t \le 0$. Our goal is to show that, under some assumptions on g, there is a λ^* such that for all $\lambda < \lambda^*$ the problem (Q_λ) has at least two nontrivial solutions. By solutions we mean weak solutions in $H_0^1(\Omega)$.

Our assumptions on the function g are motivated by similar ones in the papers [11, 12], where the case $\lambda = 0$ was studied. The proofs are based on an idea that appeared in [9]. First, we prove the existence of a solution by the sub-super solution method. Then, we prove that this solution is a local minimum of the associated functional. Finally, we get the second solution invoking the mountain pass theorem.

We are also interested and motivated by the model problem:

$$(S_\lambda) \qquad \begin{cases} -\Delta u = \lambda u + a(x)u^q + b(x)u^p & \text{in} \quad \Omega \\ u \ge 0 & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases}$$

where $0 \le q < 1 < p < 2^* - 1$ and $a, b \in L^\infty$ (called concave-convex problem). This kind of problem comes from the celebrated work [2], where problem (S_λ) with $\lambda = 0$, $b \equiv 1$ and $a \equiv \gamma > 0$ was considered. In [12], (S_λ) was considered with $\lambda = 0$ and $a(x) = \gamma c(x)$ ($\gamma > 0$ and $c(x) \ge 0$). Many others authors have studied problems

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of this form, see for instance [6, 11, 12, 14, 15, 20, 21]. The common feature, in all these works, is the presence of a parameter in the nonlinearity. Problem (S_λ), under Neumann boundary conditions, was considered in [1] with $b \equiv \gamma > 0$ and $a(x)$ changes sign.

We remark that our multiplicity result for (Q_λ) only applies to (S_λ) when $a(x) \geq 0$. In the last section, following [1], we prove a multiplicity result for (S_λ) when $a(x)$ changes its sign.

Let us also note that our results cover the case $g(x,u) = c(x)(1+u)^p$, where $p > 1$ and c is a bounded function. This kind of equation is also interesting and has been studied by several authors, see for instance [9, 12, 16, 18, 19].

The paper is organized in the following way. In the next section, we state our results relative to (Q_λ) and we comment the consequences in the special cases. The proofs of main theorems are presented in Sections 3, 4 and 5. Section 6 is devoted to problem (S_λ).

2. Statement of main results

In this section, we present our results relative to (Q_λ). For the existence result, the requirements on g are that:

(H_0) For any $u \in H_0^1(\Omega)$, there exists $K(x) \in L^{2^{*\prime}}$ such that

$$|g(x,t)| \leq K(x), \quad \text{for a.e. } x \in \Omega \text{ and for all } 0 \leq t \leq u(x);$$

(H_1) There exist $p > 1$ and $c_0 > 0$ such that

$$g(x,t) \leq c_0(1+t^p) \quad \text{for a.e. } x \in \Omega \text{ and for all } t \geq 0; \quad \text{and}$$

(H_1') There exist $0 < q < 1$ and $t_0, c_0' > 0$ such that

$$g(x,t) \leq c_0' t^q \quad \text{for a.e. } x \in \Omega \text{ and for all } 0 \leq t \leq t_0; \quad \text{or}$$

(H_2) There exist a nonempty sub domain $\Omega_1 \subset \Omega$, $0 \leq \alpha < 1$, $c_1 \in L^\infty$ with $c_1(x) > 0$ a.e. $x \in \Omega_1$, and $b_0, t_1 > 0$ such that

$$g(x,t) \geq c_1(x)t^\alpha - b_0 t \quad \text{for a.e. } x \in \Omega_1 \text{ and for all } 0 \leq t \leq t_1.$$

The energy functional is

$$F_\lambda(u) = \int_\Omega |\nabla u|^2 dx - \frac{\lambda}{2}\int_\Omega (u^+)^2 dx - \int_\Omega G(x,u)dx,$$

where $G(x,t) = \int_0^t g(x,s)ds$, and under the assumption (H_0), F_λ is well defined in $H_0^1(\Omega)$.

The assumption (H_2) is a local sublinearity condition at the origin. The subdomain Ω_1 plays an important role in the solvability of (Q_λ). Actually, we only are able to find solutions of (Q_λ) which are nontrivial in Ω_1. Moreover, if $u \in H_0^1(\Omega)$ is a solution of (Q_λ) and is nontrivial in Ω_1 then, by the Vazquez's maximum principle and (H_2), we have that $u(x) > 0$ a.e. $x \in \Omega_1$. what motivates us to consider the following problem:

$$(P_\lambda) \qquad \begin{cases} -\Delta u = \lambda u + g(x,u) & \text{in} \quad \Omega \\ u > 0 & \text{in} \quad \Omega_1, \\ u = 0 & \text{on} \quad \partial\Omega. \end{cases}$$

We will look for solutions of the problem (P_λ). In what follows, we state the main results.

2.1. Existence of one solution.

Theorem 1. *Assume that either* (H_1) *or* (H_1') *and* $(H_0), (H_2)$ *holds. Moreover, assume that* $g(x, 0) \geq 0$*. Then there is* $\lambda^* \in (-\infty, \infty]$ *such that for all* $\lambda < \lambda^*$ *problem* (P_λ) *has at least one solution, and no solution for* $\lambda > \lambda^*$*.*

2.2. Nonexistence for large λ.

Theorem 2. *Assume the hypotheses of Theorem 1, with* $c_1(x) \equiv c_1$ *in* (H_2) *(i.e. the function* $c_1(x)$ *is a constant). Moreover, suppose that there exist a nonempty subdomain* $\Omega_1' \subset \Omega_1$*,* Ω_1 *as in* (H_2)*, and* $c_1' \in \mathbb{R}$ *such that*

$$(H_{ne}) \qquad g(x, t) \geq c_1' t \quad \text{for a.e. } x \in \Omega_1' \text{ and all } t \geq 0.$$

Then $\lambda^* < \infty$*.*

2.3. Existence for $\lambda = \lambda^*$.

In order to get some *a-priori* estimates, we assume the following:

(H_3) There exist $1 < \sigma \leq 2^*$, $d_1 > 0$ such that

$$|g(x, t)| \leq d_1(1 + |t|^{\sigma-1}) \quad \text{for a.e. } x \in \Omega \text{ and all } t \geq 0.$$

(H_4) There exist $\theta > 2$, $1 \leq r < 2$ and $d, t_2 > 0$ such that

$$\theta G(x, t) \leq t g(x, t) + d t^r \quad \text{for a.e. } x \in \Omega \text{ and all } t \geq t_2.$$

(H_5) There exist a nonempty sub domain $\Omega_2 \subset \Omega$, $t_2 > 0$ and $c_2 > 0$ such that

$$G(x, t) \geq c_2 t^2 \quad \text{for a.e. } x \in \Omega_2 \text{ and all } t \geq t_2.$$

Theorem 3. *Assume the hypotheses of Theorem 1. Moreover, assume that* $(H_3 - H_4)$ *hold and* Ω *is* C^2*. If either* $\lambda^* < \lambda_1(\Omega)$ *or* $\Omega_2 = \Omega$ *in* (H_5)*, then problem* (P_λ) *has at least one solution for* $\lambda = \lambda^*$*.*

2.4. Multiplicity for $\lambda < \lambda^*$.

In addition to the previous hypotheses, we assume that:

(H_6) For any $s > 0$ there is $B > 0$ such that

$$t \to g(x, t) + Bt \text{ is nondecreasing on } [0, s], \quad \text{for a.e. } x \in \Omega.$$

(H_7) Either Ω is C^2 or for a.e. $x \in \Omega$ the function $t \in (0, \infty) \mapsto g(x, t)$ is C^1 and

$$|g_t(x, t)| \leq C(1 + t^{\beta-2}),$$

for some $2 \leq \beta < 2^*$.

Theorem 4. *Assume the hypotheses of Theorem 1. Moreover, assume that* $(H_3 - H_4 - H_5 - H_6 - H_7)$ *hold. If either* $\lambda < \min\{\lambda_1(\Omega), \lambda^*\}$ *or* $\lambda < \lambda^*$ *and* $\Omega_2 = \Omega$ *in* (H_5)*, then problem* (P_λ) *has at least two nontrivial solutions.*

2.5. Applications.

Let $0 < q < 1 < p \leq 2^* - 1$ and assume that $a, b, c \in L^\infty(\Omega)$ and $c(x) \geq 0$. For $(x, t) \in \Omega \times \mathbb{R}^+$, define $g_1(x, t) = a(x)t^q + b(x)t^p$, $g_2(x, t) = c(x) + b(x)t^p$ and $g_3(x, t) = c(x)(1 + t)^p$. It follows that g_1, g_2 and g_3 satisfy $(H_0 - H_1)$.

Assuming that $a(x) > 0$ a.e. in some ball B_a, and $c(x) > 0$ a.e. in some ball B_c, then it is easy to see that g_1, g_2 and g_3 satisfy (H_2). Moreover, if $b(x) \geq 0$ in B_a (respectively, $b(x) \geq 0$ in B_c), it is clear that g_1 (respectively, g_2) satisfies (H_{ne}) in Theorem 2. This condition is, obviously, satisfied by g_3.

The hypotheses $(H_3 - H_4)$ are satisfied by g_1, g_2 and g_3 if $p < 2^* - 1$ ((H_3) is obvious, the verification of (H_4) for g_1, g_2 and g_3 follows as in [11, p. 457] and [12, p. 285], respectively). Assumption (H_5) is satisfied by g_1, g_2 and g_3 if $b(x) \geq \epsilon_b > 0$ a.e. in some ball B_b, and $c(x) \geq \epsilon_c > 0$ a.e. in some ball B_c, respectively.

Assuming that $a(x) \geq 0$ a.e. in Ω, it follows that g_1 satisfies (H_6). It is clear that (H_6) is satisfied by g_2 and g_3. Moreover, if Ω is C^2 (and so (H_7) holds), then we are ready to apply Theorem 4. On the other hand, g_2 and g_3 satisfy the second alternative in (H_7), provided $p < 2^* - 1$. When Ω is not regular, then (H_7) in not satisfied by g_1. That problem is studied in Section 6.

The exact statement of the results are left to the reader.

3. Proof of Theorem 1

First, we shall prove that under (H_1) or (H_1') the problem (P_λ) has a supersolution for $\lambda << -1$.

Claim 1. There is $\widetilde{\lambda}$ such that for $\lambda \leq \widetilde{\lambda}$ the problem (P_λ) has a supersolution.

In fact, let e be a solution of

$$\begin{aligned} -\Delta u = 1 \quad &\text{in} \quad \Omega \\ u = 0 \quad &\text{on} \quad \partial\Omega. \end{aligned}$$

First, suppose that (H_1) holds. Let m be greater than c_0. There exists $\delta > 0$ such that $c_0 + c_0 m^p e(x)^p \leq m$ for $x \in \Omega_\delta = \{x \in \Omega : e(x) < \delta\}$. Thus

$$\lambda m e(x) + c_0 + c_0 (m e(x))^p \leq m, \text{ for all } x \in \Omega_\delta,$$

provided $\lambda < 0$. We can choose $\widetilde{\lambda} < 0$ such that

$$m \geq \lambda m \delta + c_0 + c_0 ||me||_\infty^p \quad \text{for all} \quad \lambda \leq \widetilde{\lambda}.$$

It follows that, if $\lambda \leq \widetilde{\lambda}$,

$$-\Delta(me) = m \geq \lambda m \delta + c_0 + c_0 ||me||_\infty^p \geq \lambda m e(x) + c_0 + c_0 (me)^p$$

for $x \in \Omega \setminus \Omega_\delta$. Thus, by (H_1),

$$-\Delta(me) \geq \lambda m e + g(x, me) \quad \text{in} \quad \Omega.$$

So me is a supersolution to (P_λ), since $\lambda \leq \widetilde{\lambda}$, q.e.d.

Now, suppose that (H_1') holds, take $m' > 0$ such that $m' e(x) \leq t_0$ for all $x \in \Omega$ (t_0 as in (H_1')). Then there exists $\delta > 0$ such that $c_0'(m'\delta)^q \leq m'$, and so $c_0'(m' e(x))^q \leq m'$ for $x \in \Omega_\delta = \{x \in \Omega : e(x) < \delta\}$. Thus

$$\lambda m' e(x) + c_0' (m' e(x))^q \leq m', \text{ for all } x \in \Omega_\delta,$$

provided $\lambda < 0$. Hence

$$-\Delta(m'e) = m' \geq \lambda m' e + c_0 (m'e)^q \quad \text{in} \quad \Omega_\delta.$$

Choosing $\widetilde{\lambda} < 0$ such that

$$m' \geq \lambda m' \delta + c_0 ||m'e||_\infty^q \quad \text{for all} \quad \lambda < \widetilde{\lambda},$$

it follows that

$$-\Delta(m'e) = m' \geq \lambda m' \delta + c_0' ||m'e||_\infty^q \geq \lambda m' e + c_0' (m'e)^q$$

for $x \in \Omega \setminus \Omega_\delta$, if $\lambda \leq \widetilde{\lambda}$. Thus, by (H_1'),

$$-\Delta(m'e) \geq \lambda m' e + g(x, m'e) \quad \text{in} \quad \Omega,$$

and so $m'e$ is a supersolution to (P_λ) if $\lambda \leq \widetilde{\lambda}$, q.e.d.

Conclusion of the proof of Theorem 1.

Step 1. Existence of a nontrivial solution for $\lambda \leq \widetilde{\lambda}$.

Let λ be such that $\lambda \leq \widetilde{\lambda}$, where $\widetilde{\lambda}$ as in Claim 1. Defining $\overline{u} := me$, where m and e are as in Claim 1, we have that $\overline{u}$ is a supersolution for (Q_λ). Moreover, $\underline{u} = 0$ is a subsolution, since $g(x,0) \geq 0$. Consider the following minimization problem

$$\inf_M F_\lambda, \quad \text{where} \quad M = \{u \in H_0^1 : \underline{u}(x) \leq u(x) \leq \overline{u}(x) \text{ a.e. } x \in \Omega\}.$$

By Theorem I.2.4 from [23], the above infimum is achieved at $u_1 \in M$ and, in addition, u_1 is a solution of (Q_λ). It remains to show that u_1 solves (P_λ). Suppose, by contradiction, that $u \equiv 0$ a.e. $x \in \Omega_1$. Let $\varphi \in C_0^1(\Omega_1)$ be nonegative and nontrivial. Therefore $u_1 + s\varphi \in M$ and $s\varphi \leq t_1$ (t_1 as in (H_2)), for sufficiently small $s > 0$. Using (H_2), we get

$$\begin{aligned} F_\lambda(u_1 + s\varphi) &= F_\lambda(u_1) + F_\lambda(s\varphi) \\ &= F_\lambda(u_1) + \frac{s^2}{2}||\varphi|| - \lambda\frac{s^2}{2}||\varphi||_2 - \int_\Omega G(x, s\varphi)dx \\ &\leq F_\lambda(u_1) + \frac{s^2}{2}(Const.) - s^{\alpha+1}\int_\Omega c_1(x)\varphi^{\alpha+1}dx, \\ &= F_\lambda(u_1) + s^{\alpha+1}\left(\frac{s^{1-\alpha}}{2}(Const.) - \int_\Omega c_1(x)\varphi^{\alpha+1}dx\right). \end{aligned}$$

It follows that $F_\lambda(u_1 + s\varphi) < F_\lambda(u_1)$ if $s > 0$ is small enough, since $0 < \alpha < 1$ and $\int_\Omega c_1\varphi^{\alpha+1}dx > 0$. This contradicts the definition of u_1, and so u_1 is a solution of (P_λ).

Now, we define

$$\Lambda := \{\lambda \in \mathbb{R} : (P_\lambda) \text{ has a solution}\}, \quad \text{and} \quad \lambda^* := \sup \Lambda.$$

By Step 1, we have that $\Lambda \neq \emptyset$.

Step 2. Existence of a nontrivial solution for $\lambda < \lambda^*$.

Let λ be such that $\overline{\lambda} < \lambda < \lambda^*$, where $\overline{\lambda} \in \Lambda$. Set $\overline{u}$ the solution of $(P_{\overline{\lambda}})$, then

$$-\Delta\overline{u} = \overline{\lambda}\overline{u} + g(x,\overline{u}) \geq \lambda\overline{u} + g(x,\overline{u}),$$

and so $\overline{u}$ is a supersolution for (P_λ). Consider $M = \{u \in H_0^1 : 0 \leq u \leq \overline{u}\}$. Let $u_1 \in M$ such that $F_\lambda(u_1) = \inf_M F_\lambda$, as in Step 1, u_1 is a solution of (Q_λ). Suppose, by contradiction, that u_1 does not solve (P_λ), i.e. $u_1 \equiv 0$ a.e. $x \in \Omega_1$. Let $\varphi \in C_0^1(\Omega_1)$ be nonnegative and nontrivial such that $\varphi\overline{u} \gneqq 0$ a.e. $x \in \Omega_1$. So we get $u_1 + s\varphi\overline{u} \in M$ for $s > 0$ that is sufficiently small. Then, by a similar argument as in Step 1, we have that $F_\lambda(u_1 + s\varphi\overline{u}) < F_\lambda(u_1)$ if $s > 0$ is small enough, which contradicts the definition of u_1. Thus u_1 is a solution of (P_λ).

□

4. Proof of Theorem 2 and 3

4.1. **Proof of Theorem 2.**

We will prove by contradiction that (P_λ) has no solution for large λ. To do this, suppose that u is a nontrivial solution of (P_λ). Let μ_1' be the first eigenvalue of $(-\Delta, H_0^1(\Omega_1'))$ and ϕ_1' the associated eigenfunction. We have

$$\int_{\Omega_1} \nabla u \nabla \phi_1' dx = \mu_1' \int_{\Omega_1} u\phi_1' dx.$$

On the other hand

$$\int_{\Omega_1} \nabla u \nabla \phi_1' dx = \lambda \int_{\Omega_1} u\phi_1' dx + \int_{\Omega_1} g(x,u)\phi_1' dx.$$

Then

$$c_1' \int_{\Omega_1} u\phi_1' \leq \int_{\Omega_1} g(x,u)\phi_1' dx \leq (\mu_1' - \lambda) \int_{\Omega_1} u\phi_1' dx,$$

which is a contradiction if $\lambda > \mu_1 - c_1'$.

□

4.2. **Proof of Theorem 3.**

We begin by recalling that, under the assumption (H_3) and the regularity of Ω, the solutions of (P_λ) are in $C_0^1(\overline{\Omega})$.

By the definition of λ^*, there is a sequence $\lambda_k \in \Lambda$ such that $\lambda_k \nearrow \lambda^*$ and (P_{λ_k}) has a solution. Let u_k be a solution of (P_{λ_k}) with $F(u_k) < 0$ (it is a consequence of Step 2 in the proof of Theorem 1). Let us suppose for a moment that $||u_n||$ is bounded. Then we can assume that $u_k \rightharpoonup u$ in H_0^1. Thus u solves (Q_{λ^*}) and $F(u) \leq 0$. Moreover, by standard bootstrap, we can assert that $u_k \to u$ in $C_0^1(\Omega)$.

We have to prove that u is a solution of (P_λ). For this purpose, assume, by contradiction, that $u = 0$ in Ω_1, Ω_1 as in (H_2). Let $\varphi_1 > 0$ be the associated eigenfunction to the eigenvalue $\lambda_1(\Omega_1)$. We can assume that $u_k(x) \leq t_1$ for all $x \in \Omega_1$ if k is large enough, t_1 is as in (H_2). We have

$$\begin{aligned} \lambda_1(\Omega_1) \int_{\Omega_1} u_k \varphi_1 dx &= \int_{\Omega_1} \nabla u_k \nabla \varphi_1 dx \\ &= \lambda_k \int_{\Omega_1} u_k \varphi_1 dx + \int_{\Omega_1} g(x,u_k)\varphi_1 dx \\ &\geq (\lambda_k - b_0) \int_{\Omega_1} u_k \varphi_1 dx + \int_{\Omega_1} c_1 u_k^\alpha \varphi_1 dx. \end{aligned}$$

Then

$$\int_{\Omega_1} c_1 u_k^\alpha \varphi_1 dx \leq (\lambda_1(\Omega_1) + b_0 - \lambda_k) \int_{\Omega_1} u_k \varphi_1 dx.$$

It is a contradiction, if k is large enough, since $c_1 u_k(x)^\alpha \geq (\lambda_1(\Omega_1) + b_0 - \lambda_k)u_k$ for a.e. $x \in \Omega_1$ (here we use that $c_1(x)$, in (H_2), is a constant).

It remains to prove that $||u_k||$ is bounded. Since $F'_{\lambda_n}(u_k) = 0$ and $F(u_k) < 0$, we have

$$\theta F(u_k) - F'(u_k) \cdot u_k \leq 0, \tag{1}$$

where θ is from (H_4). At this point we divided the proof in two cases.

Case 1. Assume that $\lambda^* < \lambda_1(\Omega)$. Using (H_4), we can rewrite (1) as

$$\left(\frac{\theta}{2} - 1\right)(||u_k||^2 - \lambda_k||u_k||_2^2) \leq C||u_k||^r + C.$$

It follows that $||u_k||$ is bounded, provided $\lambda_k \leq \lambda^* < \lambda_1(\Omega)$.

Case 2. Assume that $\lambda^* \geq \lambda_1(\Omega)$. Assuming $\Omega = \Omega_2$ in (H_5), there exist $t_1, \theta_1 > 0$ such that

$$G(x,t) \geq \theta_1 t^2 \quad \text{for a.e.} \quad x \in \Omega \quad \text{and all} \quad t \geq t_1.$$

As show in [11], the above condition and (H_4) imply that for some $s, c > 0$,

$$G(x,t) \geq ct^\theta, \quad \text{for a.e.} \quad x \in \Omega \quad \text{and all} \quad t \geq s. \tag{2}$$

where θ is as in (H_4). We define $h_k(x,t) = g(x,t) + \lambda_k t$. We claim that for $\epsilon > 0$, such that $2 < \theta - \epsilon$, we have

$$(\theta - \epsilon)H_k(x,t) \leq th_k(x,t) + Ct^r, \quad \text{for a.e.} \quad x \in \Omega \quad \text{and all} \quad t \geq s_2, \tag{3}$$

for some constants $C, s_2 > 0$ and r is as in (H_4). In other words, the function h_k satisfies (H_4) with θ replaced by $\theta - \epsilon$. Actually, it is enough to show that

$$\left(\frac{\theta - \epsilon}{2} - 1\right)\lambda_k t^2 \leq \epsilon G(x,t), \quad \text{for a.e.} \quad x \in \Omega \quad \text{and all} \quad t \geq s_2.$$

But, it follows easily from (2).

Using (3), we can rewrite (1) as

$$\left(\frac{\theta - \epsilon}{2} - 1\right)||u_k||^2 \leq C||u_k||^r + C.$$

It follows that $||u_k||$ is bounded. This concludes the proof.

□

5. Proof of Theorem 4

We want to prove that there exists a second solution of (P_λ) for each $\lambda < \min\{\lambda_1(\Omega), \lambda^*\}$ or $\lambda < \lambda^*$ if $\Omega_2 = \Omega$ in (H_5). To this end, fixe λ, with this restrictions, and set $\lambda < \overline{\lambda} < \lambda^*$ and $\overline{u}$ a solution of $(P_{\overline{\lambda}})$. Let u_1 be the solution of (P_λ) constructed in the proof of Theorem 1. First, we will show that u_1 is a local minimum in H_0^1. After this, we get the second solution applying the mountain pass theorem.

Assume that Ω is C^2. We know that $\overline{u} \geq u_1$ and $\overline{u} \not\equiv u_1$. Moreover, there is $B > 0$ such that

$$-\Delta(\overline{u} - u_1) \geq \overline{\lambda}(\overline{u} - u_1) + g(x, \overline{u}) - g(x, u_1) \geq (\overline{\lambda} - B)(\overline{u} - u_1) \quad \text{in} \quad \Omega.$$

Applying the Vazquez's maximum principle, we can conclude that

$$\overline{u} > u_1 \quad \text{in} \quad \Omega \quad \text{and} \quad \partial_\nu u_1 < \partial_\nu \overline{u} \quad \text{on} \quad \partial\Omega.$$

Analogously, we can prove that

$$u_1 > 0 \quad \text{in} \quad \Omega \quad \text{and} \quad \partial_\nu u_1 < 0 \quad \text{on} \quad \partial\Omega.$$

It follows that $M = \{u \in H_0^1 : \underline{u} \leq u \leq \overline{u}\}$ contains a C_0^1 neighborhood of u_1, and so u_1 is a local minimizer for F_λ in C_0^1. Thus u_1 is a local minimizer for F_λ in H_0^1 (see, for instance, [7, Theorem 1]).

In the other hand, if we do not assume regularity of Ω, the maximum principle implies only that

$$\underline{u} < u_1 < \overline{u} \text{ in } \Omega.$$

But, since u_1 is obtained as the minimum of F in M, Proposition 1, proved below, asserts that u_1 is a local minimizer for F_λ in H_0^1.

The second solution will be obtained by applying the mountain pass theorem in convex sets. More specifically, we will look for solution in the set

$$V = \{u \in H_0^1 : u \geq u_1\}.$$

We recall:

Definition 1. *We say that $u \in V$ is a critical point of F_λ in V if*

$$f_\lambda(u) = \sup\{F'_\lambda(u) \cdot (u - v) : v \in V, ||v - u|| \leq 1\} = 0.$$

Since u_1 is a critical point of F_λ, then a critical point of F_λ in V will be also a critical point of F_λ in H (see [23, p. 168]).

The next theorem can be proved as the classical versions of the mountain pass theorem, see for instance [20]. We just replace the deformation lemma by a version of deformation lemma in convex subsets proved in [23, Chapter 2, Theorem 12.7].

Theorem 5. *Let W be closed and convex subset of a Hilbert space H and $F \in C^1(H, \mathbb{R})$. Suppose that:*

1. *There exist $v_1, v_2 \in W$ and $r, \rho > 0$ such that $||v_1 - v_2|| > \rho$ and $F(w) \geq r > \max\{F(v_1), F(v_2)\}$ for all $w \in W$ with $||w - v_1|| = \rho$.*
2. *For any sequence $\{w_n\} \subset W$ are such that $\{F(w_n)\}$ is bounded and*

$$f(w_n) = \sup\{F(w_n) \cdot (w_n - w) : w \in W, ||w - w_n|| \leq 1\} \to 0,$$

then $\{w_n\}$ is relatively compact.

Then F has a critical point in W with critical value $c \geq r$.

In this point, we will verify the conditions of the Theorem 5 (this concludes the proof of Theorem 4). We start showing the item (1). Since u_1 is a local minimizer for F_λ in H_0^1, we have that there is $r, \rho > 0$ such that

$$F_\lambda(v) \geq r > F_\lambda(u_1), \text{ for all } v \in V \text{ with } ||v - u_1|| = \rho.$$

Now we look for some $v_2 \in V$ such that $F_\lambda(v_2) \leq F_\lambda(v_1)$ and $||v_2 - u_1|| > \rho$. Note that, as shown in [11], we have t_3 and $c > 0$ such that

$$G(x,t) \geq ct^\theta, \quad \text{for a.e. } x \in \Omega_2 \text{ and all } t \geq t_3.$$

Let v_0 be nontrivial and smooth function, which is supported Ω_2. Then

$$\begin{aligned} F_\lambda(u_1+sv_0) &\leq C + Cs + Cs^2 - \int_\Omega G(x,u_1+sv_0)dx \\ &= O(s^2) - \int_{\Omega\setminus\Omega_2} G(x,u_1)dx - \int_{\Omega_2} G(x,u_1+sv_0)dx \\ &\leq O(s^2) - \int_{\Omega_2} G(x,u_1+sv_0)dx \\ &\leq O(s^2) - \int_{\{x\in\Omega_2:u_1+sv_0\geq s_3\}} G(x,u_1+sv_0)dx \\ &\leq O(s^2) - c\int_{\{x\in\Omega_2:u_1+sv_0\geq s_3\}} (u_1+sv_0)^\theta dx \\ &\leq O(s^2) - cs^\theta \int_{\{x\in\Omega_2:u_1+sv_0\geq s_3\}} v_0 dx \to -\infty \quad \text{as} \quad s\to\infty, \end{aligned}$$

since $\theta > 2$. So we can then choose $s_0 > 0$ such that $F_\lambda(u_1+s_0v_0) \leq F_\lambda(u_0)$ and $||s_0v_0|| > \rho$. The function $v_2 = u_0 + s_0v_0$ satisfies the required properties.

For proving the item (2), let $u_n \in V$ be a sequence such that $\{F_\lambda(u_n)\}$ is bounded and

$$f(u_n) = \sup\{F'_\lambda(u_n)\cdot(u_n - v) : v\in V,\ ||v-u_n|| \leq 1\} \to 0.$$

For θ as in (H_3), $\epsilon_n \to 0$ and $v \in V$ we have

$$\theta F_\lambda(u_n) + F'_\lambda(u_n)\cdot(u_n - v) \leq C + \epsilon_n||u_n - v||.$$

Choosing $v = 2u_n$, we get

$$\theta F_\lambda(u_n) - F'_\lambda(u_n)\cdot(u_n) \leq C + C||u_n||,$$

it means that

$$\left(\frac{\theta}{2}-1\right)(||u_n||^2 - \lambda||u_n||_2^2) - \int_\Omega (\theta G(x,u_n) - u_n g(x,u_n)) \leq C + C||u_n||.$$

Consequently, by (H_4),

$$||u_n||^2 - \lambda||u_n||_2^2 \leq C + C||u_n||^r + C||u_n||. \tag{4}$$

It follows that $||u_n||$ is bounded, provided $\lambda < \lambda_1(\Omega)$. For showing in the case $\lambda > \lambda_1$, remember that, we have assumed that $\Omega_2 = \Omega$ in (H_5). So we can use the same argument used in the end of the proof of Theorem 3, and we also conclude that $||u_n||$ is bounded.

Finely, (H_3) implies that $\{u_n\}$ has a convergent subsequence, by standard arguments.

□

5.1. **Appendix.**

Let $\underline{u}$ and $\overline{u}$ be a subsolution and a supersolutions of (P_λ), respectively.

Proposition 1. *Assume that for a.e.* $x\in\Omega$ *the function* $t\in(0,\infty)\mapsto g(x,t)$ *is* C^1 *and*

$$|g_t(x,t)| \leq C(1+t^{\beta-2}),$$

for some $2 \leq \beta \leq 2^*$. *Moreover, suppose that* u_0 *satisfies* $\underline{u} < u_0 < \overline{u}$ *and*

$$F_\lambda(u_0) = \inf_M F_\lambda, \text{ where } M = \{u \in H_0^1 : \underline{u}(x) \leq u(x) \leq \overline{u}(x) \text{ a.e. } x \in \Omega\}.$$

Then u_0 *is a local minimum of* F_λ *in* H_0^1.

Proof. Suppose that there is $u_n \in H_0^1$ with $||u_n - u_0|| \to 0$ and $F_\lambda(u_n) < F(u_0)$. Defining

$$v_n = \max\{\underline{u}, \min\{u_n, \overline{u}\}\}, \quad w_n = (u_n - \overline{u})^+, \quad z_n = (\underline{u} - u_n)^+,$$

it follows that $u_n = v_n - z_n + w_n$, $v_n \in M$, and w_n and z_n have disjoint support. Define the sets $R_n = \{x \in \Omega : \underline{u} \leq u_n(x) \leq \overline{u}\}$, $S_n = \text{supp}(w_n)$ and $T_n = \text{supp}(z_n)$, and the functions

$$h(x,t) = \lambda t^+ + g(x,t) \quad \text{and} \quad H(x,t) = \frac{\lambda}{2}(t^+)^2 + G(x,t).$$

Then, we can rewrite $F(u_n)$ as

$$F_\lambda(u_n) = \int_{S_n \cup T_n} \left(\frac{|\nabla u_n|^2}{2} - H(x,u_n) \right) + \int_{R_n} \left(\frac{|\nabla v_n|^2}{2} - H(x,v_n) \right).$$

Observe that

$$\int_{S_n} \left(\frac{|\nabla u_n|^2}{2} - H(x,u_n) \right) = \int_{S_n} \left(\frac{|\nabla(\overline{u} + w_n)|^2}{2} - H(x,\overline{u} + w_n) \right),$$

$$\int_{T_n} \left(\frac{|\nabla u_n|^2}{2} - H(x,u_n) \right) = \int_{T_n} \left(\frac{|\nabla(\underline{u} - z_n)|^2}{2} - H(x,\underline{u} - z_n) \right),$$

and

$$\begin{aligned} \int_{R_n} \left(\frac{|\nabla v_n|^2}{2} - H(x,v_n) \right) &= F(v_n) - \int_{S_n} \left(\frac{|\nabla \overline{u}|^2}{2} - H(x,\overline{u}) \right) \\ &- \int_{T_n} \left(\frac{|\nabla \underline{u}|^2}{2} - H(x,\underline{u}) \right). \end{aligned}$$

Therefore

$$\begin{aligned} F(u_n) &= F(v_n) \\ &+ \int_{S_n} \left(\frac{(|\nabla(\overline{u} + w_n)|^2 - |\nabla \overline{u}|^2)}{2} - (H(x,\overline{u} + w_n) - H(x,\overline{u})) \right) \\ &+ \int_{T_n} \left(\frac{(|\nabla(\underline{u} - z_n)|^2 - |\nabla \underline{u}|^2)}{2} - (H(x,\underline{u} - z_n) - H(x,\underline{u})) \right). \end{aligned}$$

Moreover

$$\begin{aligned} &\int_\Omega \nabla \overline{u} \nabla w_n dx \geq \int_\Omega h(x,\overline{u}) w_n dx, \quad \text{and} \\ &\int_\Omega \nabla \underline{u} \nabla z_n dx \leq \int_\Omega h(x,\underline{u}) z_n dx. \end{aligned}$$

Thus

$$\begin{aligned} F(u_n) \quad &\geq \quad F(v_n) + \frac{1}{2}\int_\Omega |\nabla w_n|^2 dx + \frac{1}{2}\int_\Omega |\nabla z_n|^2 dx \\ &- \quad \int_{S_n} (H(x, \overline{u} + w_n) - H(x, \overline{u}) - h(x, \overline{u}) w_n) dx \\ &- \quad \int_{T_n} (H(x, \underline{u} - z_n) - H(x, \underline{u}) - h(x, \underline{u})(-z_n)) dx. \end{aligned}$$

We can then conclude that

$$\begin{aligned} \frac{1}{2}||w_n||^2 + \frac{1}{2}||z_n||^2 \quad &< \quad \int_{S_n} (H(x, \overline{u} + w_n) - H(x, \overline{u}) - h(x, \overline{u}) w_n) dx \\ &+ \quad \int_{T_n} (H(x, \underline{u} - z_n) - H(x, \underline{u}) - h(x, \underline{u})(-z_n)) dx. \end{aligned}$$

The conclusion of the proof of the proposition follows from the next claim.

Claim. We have

$$\int_{S_n} (H(x, \overline{u} + w_n) - H(x, \overline{u}) - h(x, \overline{u}) w_n) dx \leq o(1)||w_n||^2, \quad \text{and}$$
$$\int_{T_n} (H(x, \underline{u} - z_n) - H(x, \underline{u}) - h(x, \underline{u})(-z_n)) dx \leq o(1)||z_n||^2.$$

Assuming for a moment that the above claim is true, we have

$$\frac{1}{2}(||z_n||^2 - o(1)||z_n||^2) + \frac{1}{2}(||w_n||^2 - o(1)||w_n||^2) < 0,$$

which implies that $w_n = z_n = 0$ for n large enough. It follows that $u_n = v_n \in M$, for n large, and so we conclude that $F_\lambda(u_n) \geq F_\lambda(u_0)$, what is a contradiction.

Proof of the Claim. We will prove only the first statement, the second one is similar. First, define $H_n(x) = H(x, \overline{u} + w_n) - H(x, \overline{u}) - H_u(x, \overline{u}) w_n$ and consider the following splitting for $H_n = H_{0n} + H_{1n}$, where

$$H_{0n}(x) = \frac{\lambda}{2}[(\overline{u} + w_n)^2 - \overline{u}^2] - \lambda \overline{u} w_n, \quad \text{and}$$
$$H_{1n}(x) = G(x, \overline{u} + w_n) - G(x, \overline{u}) - g(x, \overline{u}) w_n.$$

Note that $H_{0n}(x) = \frac{\lambda}{2} w_n^2$, so we have

$$\frac{\lambda}{2}\int_{S_n} w_n^2 dx \leq \frac{\lambda}{2}|S_n|^{\frac{2}{N}} \left(\int_\Omega w_n^{\frac{2N}{N-2}} dx\right)^{\frac{N-2}{N}} \leq \frac{\lambda}{2}|S_n|^{\frac{2}{N}} ||w_n||^2.$$

Now, we claim that $|S_n| = o(1)$. Actually, given $\epsilon > 0$ we can choose $\delta > 0$ such that $|\{\overline{u} \leq u_0 + \delta\}| < \epsilon$, since $u_0 < \overline{u}$ in Ω. However, we have

$$S_n \subset \{\overline{u} \leq u_0 + \delta\} \cup \{u_n > \overline{u} > u_0 + \delta\}.$$

And since $u_n \to u_0$ in L^2 there is n_0 such that for all $n \geq n_0$,

$$\begin{aligned} \epsilon\delta^2 \quad &\geq \quad \int_\Omega (u_n - u_0)^2 \geq \int_{\{u_n > u_0 + \delta\}} (u_n - u_0)^2 \\ &\geq \quad \int_{\{u_n > u_0 + \delta\}} \delta^2 = \delta^2 |\{u_n > u_0 + \delta\}|. \end{aligned}$$

Thus $|S_n| \le |\{\overline{u} \le u_0 + \delta\}| + |\{u_n > u_0 + \delta\}| \le 2\epsilon$. It follows that

$$\int_{S_n} H_{0n}(x)dx \le o(1)||w_n||^2.$$

Now, there are $s(x), t(x) \in (0,1)$ such that

$$\begin{aligned} H_{1n}(x) &= g(x, \overline{u} + s(x)w_n)w_n - g(x, \overline{u})w_n \\ &= g'(x, \overline{u} + t(x)s(x)w_n)w_n^2 \\ &\le Cw_n^2 + Cw_n^\beta. \end{aligned}$$

Thus

$$\int_{S_n} H_{1n}(x)dx \le o(1)||w_n||^2 + C||w_n||^\beta \le o(1)||w_n||^2,$$

since $\beta \ge 2$. It finishes the proof of the Claim.

□

6. Multiplicity with indefinite sublinear nonlinearities

Consider the following problem

$$(S_\lambda) \qquad \begin{cases} -\Delta u = \lambda u + a(x)u^q + b(x)u^p & \text{in} \quad \Omega \\ u \ge 0 & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega. \end{cases}$$

where $0 < q < 1 < p < 2^*$ and $a, b \in L^\infty$. Our goal here is to consider the case when a changes its sign. We assume that:

(a) The sets $\Omega_a^+ = \{x \in \Omega : a(x) > 0\}$ and $\Omega_a^- = \{x \in \Omega : a(x) < 0\}$ are nonempty, Ω_a^+ is open, moreover,

$$\overline{\Omega_a^+} \cap \overline{\Omega_a^-} = \emptyset \text{ and } \overline{\Omega_a^+} \subset \Omega.$$

(ab) The sets $\Omega_a = \{x \in \Omega : a(x) \ge 0\}$ and $\Omega_b = \{x \in \Omega : b(x) \ge 0\}$ are nonempty and $\Omega_a \cap \Omega_b$ has nonempty interior.

(b) The set $\Omega_b^+ = \{x \in \Omega : b(x) > 0\}$ has nonempty interior.

Moreover, we define $\Omega_b^0 = \Omega \setminus \overline{\{b \neq 0\}}$.

Remark 1. i) Note that, by the Maximum Principle, if u is a solution of (S_λ) such that u is nontrivial in the components of Ω_a, then $u > 0$ in $int(\Omega_a) \supset \Omega_a^+$. Notice that the case $|\Omega_a^-| = 0$ is included in the above assumptions.

ii) Assumption $\overline{\Omega_a^+} \cap \overline{\Omega_a^-} = \emptyset$ has been considered by many authors in the study of elliptic problems with indefinite nonlinearities, see for instance [3, 4]. Assumption $\overline{\Omega_a^+} \subset \Omega$ was considered in [5].

Theorem 6. *Assume (a), (ab) and (b). Then there is $\lambda^* \in \mathbb{R}$ such that:*

(1) *For all $\lambda < \lambda^*$ problem (S_λ) has at least one nontrivial solution, and has no solution for $\lambda > \lambda^*$;*

(2) *If we assume that $b(x) > 0$ a.e. $x \in \Omega$ and $a(x) \ge a_0 > 0$ in some ball B_a, then for $\lambda = \lambda^*$ problem (S_λ) has at least one nontrivial solution;*

(3) *If we assume that*

$$int(\Omega_a) = \cup_1^k U_i, \quad U_i \ \textit{conected}, \quad U_i \cap \Omega_a^+ \neq \emptyset,$$

then for all $\lambda < \min\{\lambda^*, \lambda_1(\Omega_b^0)\}$ problem (S_λ) has at least two nontrivial solutions.

Remark 2. The statements (1) and (2) are consequence of Theorem 1 and Theorems 2 and 3, respectively. The statement (3) is not a consequence of Theorem 4, because the assumption (H_6) is not satisfied when $|\Omega_a^-| > 0$. Moreover, the requirement on λ is different.

Proof of Theorem 6.

Proof of (1) *and* (2). The verification of (H_1) is a easy consequence of the assumptions $a, b \in L^\infty$ and $0 \le p < 1 < p$. To verify (H_2) note that $g(x,t) = a(x)(t^+)^q + b(x)(t^+)^p \ge a(x)(t^+)^q - ||b||_\infty t^+$ for all $x \in \Omega_a^+$ and $0 \le t \le 1$. Then we can apply Theorem 1. We also can apply Theorem 2 with $\Omega_1' = int(\Omega_a \cap \Omega_b)$ and $c_1 = 0$, since by (ab) we have $g(x,u) \ge 0$ in $int(\Omega_a \cap \Omega_b)$. Thus we get a $\lambda^* \in \mathbb{R}$ which fulfills the requirements of (1). Moreover, denoting by u_1, the solution obtained by Theorem 1, we have that $u_1 > 0$ in Ω_a^+ (see the proof of Theorem 1).

The item (2) follows as the proof of Theorem 3, we need only to show that: if $\lambda_n \nearrow \lambda^*$ and u_n is the solution of (P_{λ_n}) obtained in (1), then $||u_n||$ is bounded. We have that

$$(p+1)F_{\lambda_n}(u_n) - F'_{\lambda_n}(u_n) \cdot u_n \le 0,$$

it follows that

$$||u_n||^2 \le C||u_n||_2^2 + C||u_n||^{q+1}.$$

From the above inequality, it enough to show that $||u_n||_2$ is bounded. To do this, suppose, by contradiction, that $||u_n||_2 \to \infty$. Then $v_n = u_n/||u_n||_2$ is bounded, and so we can assume that $v_n \rightharpoonup v_0$ in H_0^1, $v_n \to v_0$ in L^r, for $1 \le r < 2N/(N-2)$. In particular, we have $||v_0||_2 = 1$. For a function $\phi \in C_0^1$ with $\phi > 0$ in Ω, we consider the identity $(1/||u_n||_2)F'(u_n)\phi = 0$, that means

$$\begin{aligned} ||u_n||_2^{p-1} \int_\Omega b(x) v_n^p \phi dx &= \int_\Omega \left(\nabla v_n \nabla \phi - \lambda_n v_n \phi - \frac{a(x)}{||u_n||_2} u_n^q \phi \right) dx \\ &= \int_\Omega (\nabla v_n \nabla \phi - \lambda_n v_n \phi)\, dx + o(1). \end{aligned}$$

It follows that

$$\int_\Omega b(x) v_0^p \phi dx = \liminf_{n\to\infty} \int_\Omega b(x) v_n^p \phi dx = 0.$$

Which implies that $v_0 = 0$ a.e. $x \in \Omega$, that is a contradiction with $||v_0||_2 = 1$. Now, we can proceed as the proof of Theorem 3.

Proof of (3). Fixe $\lambda < \lambda^*$ and let $\lambda < \overline{\lambda} < \lambda^*$ and $\overline{u}$ a nontrivial solution of $(S_{\overline{\lambda}})$. Let u_1 the solution of (S_λ) constructed at the proof of Theorem 1. Note that $u_1 > 0$ in the connected components of Ω_a^+, see the proof of Theorem 1. By the Proposition 2 below, u_1 is a local minimum in H_0^1. Now, in order to get the second solution, consider the functional

$$I_\lambda(u) = \frac{1}{2}\int_\Omega |\nabla(u_1+u)|^2 dx - \int_\Omega H(x, u_1+u) dx,$$

where

$$H(x,t) = \frac{\lambda}{2}(t^+)^2 + \frac{1}{q+1}a(x)(t^+)^{q+1} + \frac{1}{p+1}b(x)(t^+)^{p+1}.$$

Note that, $I_\lambda(u) = F_\lambda(u+u_1)$, where F_λ is the associated functional of (S_λ), that here reads

$$F_\lambda(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 dx - \int_\Omega H(x,u) dx.$$

Moreover, if $u \in H_0^1$ is a critical point of I_λ, then $u_1 + u$ is a nonnegative critical point of F_λ, and so a solution of (S_λ). Thus, we have to prove that I_λ has a nontrivial critical point. To this end, we will show that I_λ satisfies the assumptions of the relaxed mountain pass theorem, see [17, Corolarry 5.11].

First, observe that $I_\lambda(0) = F_\lambda(u_1)$. Thus we have to show that:

(i) there is $r > 0$ such that $I_\lambda(u) \geq F_\lambda(u_1)$ for all $u \in H_0^1$ with $||u|| = r$;
(ii) there is $w_1 \in H_0^1$ such that $I(w_1) \leq F_\lambda(u_1)$ and $||w_1|| > r$; and
(iii) I_λ satisfies the (PS) condition.

The item (i) is a consequence of u_1 being a local minimum of F_λ. In order to prove item (ii), let $v_1 \in C_0^1(\Omega_b)$ be nonnegative, nontrivial and such that $\int_\Omega b(x)v_1^{p+1} > 0$. We have, for large s,

$$\begin{aligned} F_\lambda(u_1 + sv_1) &= O(s^2) - \int_\Omega \left(\frac{a(x)(u_1 + sv_1)^{q+1}}{q+1} + \frac{b(x)(u_1 + sv_1)^{p+1}}{p+1} \right) \\ &= O(s^2) - s^{q+1} \int_\Omega \frac{a(x)(u_1/s + v_1)^{q+1}}{q+1} dx \\ &- s^{p+1} \int_\Omega \frac{b(x)(u_1/s + v_1)^{p+1}}{p+1} dx \\ &\to -\infty \text{ as } s \to \infty, \end{aligned}$$

and so (ii) follows.

For (iii), first, we remark that F_λ satisfies (PS) condition if $\lambda < \lambda_1(\Omega_b^0)$. It follows as the proof of Lema 1.5 in [4]. Now, let u_n be a (PS) sequence of I_λ at level c, it follows that $u_1 + u_n$ is a (PS) sequence for F_λ, and so has a convergent subsequence. Thus, I_λ satisfies the (PS) condition.

□

6.1. **Appendix.**

In this appendix, we are assuming that $\overline{\lambda} < \lambda^*$ and $\overline{u}$ is a nontrivial solution of $(S_{\overline{\lambda}})$, given by Theorem 1.

Lemma 1. *Let $\lambda < \overline{\lambda}$ and u solution of (S_λ). If $u \leq \overline{u}$, then $u < \overline{u}$ in $A = \{\overline{u} > 0\}$.*

Proof. Let $v = \overline{u} - u \geq 0$ a.e. in Ω, then

$$-\Delta v + m(x)v \geq 0, \quad \text{where} \quad m(x) := \left(a^- \frac{\overline{u}^q - u^q}{\overline{u} - u} + b^- \frac{\overline{u}^p - u^p}{\overline{u} - u} \right).$$

Suppose by contradiction that $v(x_0) = 0$ for some $x_0 \in A$. We can choose $r > 0$ such that the ball $B_r[x_0] \subset A$. We have that m is uniformly bounded in $B_r(x_0)$, so by the Strong Maximum Principle we get $v = 0$ in $B_r(x_0)$. It means that $u = \overline{u}$ in $B_r(x_0)$, what contradicts the equations satisfied by these functions, since $\lambda < \overline{\lambda}$.

□

Proposition 2. *Suppose that u_0 attains*

$$F_\lambda(u_0) = \inf_M F_\lambda, \quad \text{where} \quad M := \{u \in H_0^1 : 0 \leq u \leq \overline{u} \quad \text{a.e.}\, x \in \Omega\}.$$

Then u_0 is a local minimum of F_λ in H_0^1.

Proof. Suppose that there is $u_n \in H_0^1$ with $||u_n - u_0|| \to 0$ and $F_\lambda(u_n) < F_\lambda(u_0)$. Let

$$v_n = \max\{0, \min\{u_n, \overline{u}\}\}, \quad w_n = (u_n - \overline{u})^+,$$

so that $u_n = v_n - u_n^- + w_n$, $v_n \in M$, and u_n^- and w_n have disjoint support. Define the sets $R_n = \{x \in \Omega : 0 \le u_n(x) \le \overline{u}\}$, $S_n = \mathrm{supp}(w_n)$ and $T_n = \mathrm{supp}(u_n^-)$, and the functions

$$h(x,t) = \lambda t^+ + a(x)(t^+)^q + b(x)(t^+)^p \quad \text{and} \quad H(x,t) = \int_0^t h(x,s)ds.$$

Arguing as in the proof of Proposition 1, we can show that

$$\frac{1}{2}||w_n||^2 + \frac{1}{2}||u_n^-||^2 < \int_{S_n} (H(x, \overline{u} + w_n) - H(x,\overline{u}) - H_u(x,\overline{u})w_n)\, dx.$$

Claim. $\int_{S_n} (H(x, \overline{u} + w_n) - H(x,\overline{u}) - H_u(x,\overline{u})w_n) \le o(1)||w_n||^2$.

Assuming the above claim, we have

$$\frac{1}{2}||u_n^-||^2 + \frac{1}{2}(||w_n||^2 - o(1)||w_n||^2) < 0,$$

which implies that $w_n, u_n^- = 0$ for n large enough. It follows that $u_n = v_n \in M$, for n large, and so $F_\lambda(u_n) \ge F_\lambda(u_0)$, what is a contradiction.

Proof of the Claim.

First, consider the following splitting for the function $H_n = H_{0n} + H_{1n} + H_{2n}$, where

$$\begin{aligned} H_{0n}(x) &= \frac{\lambda}{2}[(\overline{u} + w_n)^2 - \overline{u}^2] - \lambda \overline{u} w_n, \\ H_{1n}(x) &= \frac{b(x)}{p+1}[(\overline{u} + w_n)^{p+1} - \overline{u}^{p+1}] - b(x)\overline{u}^p w_n, \quad \text{and} \\ H_{2n}(x) &= \frac{a(x)}{q+1}[(\overline{u} + w_n)^{q+1} - \overline{u}^{q+1}] - a(x)\overline{u}^q w_n. \end{aligned}$$

Superlinear term: Note that, there are $s(x), \theta(x) \in (0,1)$ such that

$$\begin{aligned} H_{1n}(x) &= b(x)[(\overline{u} + \theta w_n)^p - \overline{u}^p] w_n \\ &\le C(\overline{u} + s\theta w_n)^{p-1} \theta w_n^2. \end{aligned}$$

Moreover, $(\overline{u} + s\theta w_n)^{p-1} w_n^2 \le w_n^{p+1}$ in $B = \Omega \setminus A$, where $A = \{\overline{u} > 0\}$, then

$$\int_{S_n \setminus A} H_{1n}(x)dx \le C||w_n||^{p+1} \le o(1)||w_n||^2. \tag{5}$$

On the other hand, $(\overline{u} + s\theta w_n)^{p-1} w_n^2 \le Cw_n^2 + Cw_n^{p+1}$ in A, then

$$\begin{aligned} \int_{S_n \cap A} H_{1n}(x)dx &\le C \int_{S_n \cap A} w_n^2 dx + C||w_n||^{p+1} \\ &\le |S_n \cap A|^{\frac{2}{N}} \left(\int_\Omega w_n^{\frac{2N}{N-2}} dx \right)^{\frac{N-2}{N}} + C||w_n||^{p+1} \\ &\le C|S_n \cap A|^{\frac{2}{N}} ||w_n||^2 + o(1)||w_n||^2. \end{aligned}$$

Now, we claim that $|S_n \cap A| \to 0$ as $n \to \infty$. Actually, given $\epsilon > 0$, by Lemma 1, we can choose $\delta > 0$ such that $|A \cap \{\overline{u} \le u_0 + \delta\}| < \epsilon$. However, we have

$$S_n \subset \{\overline{u} \le u_0 + \delta\} \cup \{u_n > \overline{u} > u_0 + \delta\},$$

and since $u_n \to u_0$ in L^2 there is n_0 such that for all $n \ge n_0$,

$$\begin{aligned} \epsilon\delta^2 &\ge \int_\Omega (u_n - v_0)^2 \ge \int_{\{u_n > u_0+\delta\}} (u_n - v_0)^2 \\ &\ge \int_{\{u_n > u_0+\delta\}} \delta^2 = \delta^2 |\{u_n > u_0 + \delta\}|. \end{aligned}$$

Thus $|S_n \cap A| \le |A \cap \{\overline{u} \le u_0 + \delta\}| + |\{u_n > u_0 + \delta\}| \le 2\epsilon$. It follows that

$$\int_{S_n \cap A} H_{1,n}(x)dx \le o(1)||w_n||^2.$$

Sublinear term: First, observe that $\Omega_a^+ \subset A$ and $B \subset \Omega \setminus \Omega_a^+$. We have, for $x \in A \setminus \Omega_a^+$,

$$H_{2n}(x) \le a(x)[(\overline{u} + \theta w_n)^q - \overline{u}^q]w_n \le 0.$$

Thus

$$\int_{S_n \cap (A \setminus \Omega_a^+)} H_{2n}(x)dx \le 0 \le o(1)||w_n||^2.$$

On the other hand, note that $\overline{\Omega_a^+} \subset int(\Omega_a)$ and $\overline{u} > 0$ in $int(\Omega_a)$, and so there is $\delta > 0$ such that $\overline{u}(x) \ge \delta$ in Ω_a^+. Then, for $x \in \Omega_a^+$,

$$\begin{aligned} H_{2n}(x) &= a(x)[(\overline{u} + \theta w_n)^q - \overline{u}^q]w_n \\ &= a(x)(\overline{u} + s\theta w_n)^{q-1}\theta w_n^2 \\ &\le C\delta^{q-1} w_n^2 \le C w_n^2, \end{aligned}$$

with $\theta(x), s(x) \in (0,1)$. Thus

$$\int_{S_n \cap \Omega_a^+} H_{2n}(x)dx \le C \int_{S_n \cap \Omega_a^+} w_n^2 \le |S_n \cap \Omega_a^+|^{\frac{2}{N}} ||w_n||^2 \le o(1)||w_n||^2,$$

where the last inequality is a consequence of $S_n \cap \Omega_a^+ \subset S_n \cap A$ and $|S_n \cap A| = o(1)$.

Now, for $x \in B$ we have $H_{2n}(x) = -\frac{a^-(x)}{q+1} w_n^{q+1}$ since $B \subset \Omega_a^-$. Thus

$$\int_B H_{2n}(x)dx = -\int_B \frac{a^-(x)}{q+1} w_n^{q+1} dx \le 0. \tag{6}$$

Linear term: Note that $H_{0n}(x) = \frac{\lambda}{2} w_n^2$, we have

$$\frac{\lambda}{2} \int_{S_n \cap A} w_n^2 dx \le \frac{\lambda}{2} |S_n \cap A|^{\frac{2}{N}} \left(\int_\Omega w_n^{\frac{2N}{N-2}} dx \right)^{\frac{N-2}{N}} \le o(1)||w_n||^2.$$

The integral in B will be estimate by the integral in (6). Define the set

$$U_n = \left\{ x \in \Omega_a^- : w_n(x) \ge \left(\frac{2a^-(x)}{\lambda(q+1)} \right)^{\frac{1}{1-q}} \right\}.$$

It follows that $|U_n| \to 0$ as $n \to \infty$, since $w_n \to 0$ in H_0^1. Therefore

$$\begin{aligned}\frac{\lambda}{2}\int_B w_n(x)^2dx &\leq \int_{B\setminus U_n}\frac{a^-(x)}{q+1}w_n^{q+1}dx+\frac{\lambda}{2}\int_{U_n}w_n^2dx\\ &\leq \int_B\frac{a^-(x)}{q+1}w_n^{q+1}dx+\frac{\lambda}{2}|U_n|^{\frac{2}{N}}\left(\int_\Omega w_n^{\frac{2N}{N-2}}dx\right)^{\frac{N-2}{N}}\\ &\leq \int_B\frac{a^-(x)}{q+1}w_n^{q+1}dx+o(1)||w_n||^2.\end{aligned}$$

The claim follows from (6) and the above estimates.

□

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