NONNEGATIVE SOLUTIONS OF ELLIPTIC PROBLEMS WITH SUBLINEAR INDEFINITE NONLINEARITY

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Abstract. We are concerned with existence, nonexistence and multiplicity of nonnegative solutions for the elliptic problem

$$\begin{array}{lll} -\Delta u = a(x)u^q + \lambda b(x)u^p & \text{in} & \Omega \\ u = 0 & \text{on} & \partial\Omega, \end{array}$$

where Ω is a bounded domain in $\mathbb{R}^N$, $\lambda \in \mathbb{R}$, $0 < q < 1 < p \leq 2^* - 1$ and a, b are bounded functions, with $b(x) \geq 0$ and $a(x)$ changes its sign.

1. Introduction

There has recently been increasing interest in questions about positive solutions of semilinear elliptic problem of the type:

$$-\Delta u = f(x, u, \lambda),$$

$x \in \Omega \subset \mathbb{R}^N$ and $\lambda \in \mathbb{R}$. One interesting situation appears when f is sublinear at the origin in some open subset $\Omega' \subset \Omega$, i.e., if the following condition,

$$\lim_{u \to 0^+} \frac{f(x, u, \lambda)}{u} = \infty,$$

holds uniformly for $x \in \Omega'$ and $\lambda \in \mathbb{R}$. One direction of research is looking for an interval $\Lambda \subset \mathbb{R}$, such that $-\Delta u = f(x, u, \lambda)$ has two solution for $\lambda \in \Lambda$.

In this paper we deal with the following class of parameterized elliptic problems

$$(Q_\lambda) \qquad \left\{ \begin{array}{lll} -\Delta u = a(x)u^q + \lambda b(x)u^p & \text{in} & \Omega \\ u \geq 0 & \text{in} & \Omega, \\ u = 0 & \text{on} & \partial\Omega, \end{array} \right.$$

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where Ω is a bounded domain in $\mathbb{R}^N$, $\lambda \in \mathbb{R}$, $0 < q < 1 < p \leq 2^* - 1$ and a, b are bounded functions. Here we will assume that $a(x)$ changes its sign in Ω, so the Maximum Principle is not applicable, thus the solutions can vanish on parts of Ω (see for instance [11]). By solutions we mean weak solutions in $H_0^1(\Omega)$, i.e. the critical points of the associated C^1 functional $F_\lambda : H_0^1(\Omega) \to \mathbb{R}$, given by

$$(1)\quad F_\lambda(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 - \frac{1}{q+1}\int_\Omega a(x)(u^+)^{q+1} - \frac{\lambda}{p+1}\int_\Omega b(x)(u^+)^{p+1}.$$

Our general assumptions concerning the functions $a(x)$ and $b(x)$ are that $a, b \in L^\infty(\Omega)$ and the sets

$$\Omega_a = \{x \in \Omega : a(x) \geq 0\}, \quad \Omega_a^+ = \{x \in \Omega : a(x) > 0\},$$
$$\Omega_a^- = \{x \in \Omega : a(x) < 0\} \text{ and } \Omega_b^+ = \{x \in \Omega : b(x) > 0\},$$

are nonempty. Moreover, we will make the following assumptions:

(a) Ω_a^+ is open, $|\Omega_a^-| > 0$ and $\overline{\Omega_a^+} \cap \overline{\Omega_a^-} = \emptyset$;
(b) $int(\Omega_b^+) \neq \emptyset$ and $b \geq 0$;
(c) $\Omega_a^+ \subset \Omega_b^+$ and $\overline{\Omega_a^+} \subset \Omega$;
(d) $int(\Omega_a) = \cup_1^k U_i$, U_i connected, and $U_i \cap \Omega_a^+ \neq \emptyset$;

As a consequence of assumption (d), by the Maximum Principle, if u is a solution of (Q_λ) such that u is nontrivial in the components of Ω_a, then $u > 0$ in $int(\Omega_a) \supset \Omega_a^+$. This motivates the following definition:

Definition 1. *We say that $u \in H_0^1(\Omega)$ is a solution to (P_λ) if u solves (Q_λ) in the weak sense and $u(x) > 0$ a.e. $x \in \Omega_a^+$.*

The aim of this paper is to obtain, assuming the above hypotheses, a global result in the following sense: if

$$\lambda^* = \sup\{\lambda > 0;\ (P_\lambda)\text{ has a solution}\},$$

then for all $0 < \lambda < \lambda^*$, (P_λ) has at least two nontrivial solutions (in the critical case we have an additional hypothesis). This kind of result was proved in [2] for the case $a \equiv b \equiv 1$. When $a \geq 0$ and b is indefinite, the same result was obtained in [9].

Before presenting our results, we notice that elliptic problems with indefinite nonlinearities have been widely studied recently. For instance, problems where the nonlinearity are composed by a linear part and an indefinite superlinear were considered in [3, 4, 5, 20]. For concave-convex nonlinearities, in addition to [9], we can cite [10], for p-Laplacian problems, and [1] for a semilinear equation with Neumann boundary condition (see also [12, 13, 19]). Moreover, we refer to [8, 16, 17, 18] for local results, i.e., when λ is in a small neighborhood of the origin. Notice that in [8] both coefficients, a and b, could change sign, but the authors have proved only local results.

The main results of this paper are stated in the theorems below. First, we consider existence results:

Theorem 1. *Assume (a)-(d) and $0 < q < 1 < p \leq 2^* - 1$. Then there is $\lambda^* \in (0, \infty)$ such that problem (P_λ) has at least one solution for $0 < \lambda < \lambda^*$, moreover, this solution is a local minimum of F_λ. Furthermore, (P_λ) has no solution for $\lambda > \lambda^*$; and, if Ω is smooth then (P_{λ^*}) has at least one solution.*

Remark 1. Assumption $\overline{\Omega_a^+} \cap \overline{\Omega_a^-} = \emptyset$ has been considered by many authors in the study of elliptic problems with indefinite nonlinearities, see for instance [3, 4]. Assumption (d), that appears in [1, 6] in the study of a problem with Neumann boundary condition, will be essential to prove that the solution obtained in Theorem 1 is a local minimum in $H_0^1(\Omega)$.

Now we consider multiplicity in the subcritical case. Let λ^* be given in previous theorem.

Theorem 2. *Assume (a)-(d) and $0 < q < 1 < p < 2^* - 1$. Then problem (P_λ) has at least two solutions for $0 < \lambda < \lambda^*$.*

Aiming now to multiplicity in the critical case, we will assume that $N \geq 3$ and, without loss of generality, that $0 \in \Omega$. Moreover, in addition to the above hypotheses, we assume that b satisfies:

(e) for some $\delta > 0$, $M > 0$ and some $\gamma > 2$, one has

$$0 \leq ||b||_\infty - b(x) \leq M|x|^\gamma \quad a.e. \;\; x \in B_\delta(0).$$

Let λ^* be given in Theorem 1, we have:

Theorem 3. *Assume $(a) - (e)$ and $0 < q < 1 < p = 2^* - 1$. Then for all $0 < \lambda < \lambda^*$ problem (P_λ) has at least two solutions.*

Remark 2. Assumption (e) appears in [10], but there γ depends on the dimension in the following way: $\gamma > 2^*$ when $N \geq 5$, $\gamma \geq 2^*$ when $N = 4$ and $\gamma > 3/5$ when $N = 3$.

Throughout this paper, we will use the following notations: $|| \cdot ||$ to the norm of $H_0^1(\Omega)$, $|| \cdot ||_p$ to the norm of $L^p(\Omega)$ and C to several different positive constants.

2. Proof of Theorem 1

Existence for $0 < \lambda < \lambda^*$:

First, we shall prove that problem (P_λ) has a supersolution for $\lambda > 0$ small enough.

Claim 1. There is $\epsilon_0 > 0$ such that for $0 < \lambda \leq \epsilon_0$, problem (P_λ) has a supersolution.

In fact, let e be a solution of

$$\begin{array}{lcc} -\Delta u = 1 & \text{in} & \Omega \\ u = 0 & \text{on} & \partial\Omega. \end{array}$$

Since $0 < q < 1 < p$, we can find $m > 0$ and $\epsilon_0 > 0$ such that

$$m \geq ||a||_\infty ||me||_\infty^q + \epsilon_0 ||b||_\infty ||me||_\infty^p.$$

It follows that me is a supersolution to (P_λ), since $0 < \lambda \leq \epsilon_0$, q.e.d.

Let λ be such that $0 < \lambda \leq \epsilon_0$, where ϵ_0 is as in Claim 1. Defining $\overline{u} := me$, where m and e are as in Claim 1, we have that $\overline{u}$ is a supersolution for (Q_λ). Moreover, $\underline{u} = 0$ is a solution, and so a subsolution. Consider the following minimization problem

$$\inf_M F_\lambda, \quad \text{where} \quad M = \{u \in H_0^1 : \underline{u}(x) \leq u(x) \leq \overline{u}(x) \text{ a.e. } x \in \Omega\}.$$

By Theorem I.2.4 from [21], the above infimum is achieved at $u_\lambda \in M$ and, in addition, u_λ is a solution of (Q_λ). It remains to show that u_λ solves (P_λ) (see Definition 1). Suppose, by contradiction, that $u_\lambda \equiv 0$ a.e. $x \in \Omega_a^+$. Let $\varphi \in C_0^1(\Omega_a^+)$ be nonnegative and nontrivial. Therefore $u_\lambda + s\varphi \in M$, for sufficiently small $s > 0$, and

$$\begin{array}{rcl} F_\lambda(u_\lambda + s\varphi) & = & F_\lambda(u_\lambda) + F_\lambda(s\varphi) \\ & = & F_\lambda(u_\lambda) + \dfrac{s^2}{2}||\varphi||^2 \\ & & -\dfrac{s^{q+1}}{q+1}\displaystyle\int_\Omega a(x)\varphi^{q+1}dx - \dfrac{s^{p+1}\lambda}{p+1}\displaystyle\int_\Omega b(x)\varphi^{p+1}dx. \end{array}$$

It follows that $F_\lambda(u_\lambda + s\varphi) < F_\lambda(u_\lambda)$ if $s > 0$ is small enough. This contradicts the definition of u_λ, and so u_λ is a solution of (P_λ).

Now, we can define

$$\Lambda := \{\lambda > 0 : (P_\lambda) \text{ has a solution}\}, \quad \text{and} \quad \lambda^* := \sup \Lambda. \tag{2}$$

By the previous paragraph, we have $\Lambda \neq \emptyset$, and so λ^* is well defined.

Now, we will prove existence of a solution for $0 < \lambda < \lambda^*$. Let λ be such that $\lambda < \overline{\lambda} < \lambda^*$, with $\overline{\lambda} \in \Lambda$. Let $\overline{u}$ be a solution of $(P_{\overline{\lambda}})$, then

$$-\Delta\overline{u} = a(x)\overline{u}^q + \overline{\lambda}b(x)\overline{u}^p \geq a(x)\overline{u}^q + \lambda b(x)\overline{u}^p,$$

and so $\overline{u}$ is a supersolution for (P_λ). Consider $M = \{u \in H_0^1 : 0 \leq u \leq \overline{u}\}$. Let $u_\lambda \in M$ be such that $F_\lambda(u_\lambda) = \inf_M F_\lambda$. As before, u_λ is a solution of (Q_λ). Suppose, by contradiction, that u_λ does not solve (P_λ), i.e. $u_\lambda \equiv 0$ a.e. $x \in \Omega_a^+$. Let $\varphi \in C_0^1(\Omega_a^+)$ be nonnegative and nontrivial such that $\varphi\overline{u} \gneqq 0$ a.e. $x \in \Omega_a^+$. So we get $u_\lambda + s\varphi\overline{u} \in M$ for $s > 0$ and sufficiently small. Arguing as in Claim 1, we can conclude that $F_\lambda(u_\lambda + s\varphi\overline{u}) < F_\lambda(u_\lambda)$ if $s > 0$ is small enough, which contradicts the definition of u_λ. Thus u_λ is a solution of (P_λ). The proof that u_λ is a local minimum of F_λ is deferred to an appendix. q.e.d.

Proof of $\lambda^* < \infty$:

First, note that

$$a(x)t^q + b(x)t^p \geq \lambda^{\frac{1-q}{p-q}} m(x)t, \quad \text{a.e. } x \in \Omega_a \quad \text{and for all} \quad t \geq 0,$$

where $m(x) = C(p,q)a(x)^{\frac{p-1}{p-q}}b(x)^{\frac{1-q}{p-q}}$ (see [8, Lemma 3.6]).

Let u be a solution of (P_λ). Let B be a ball in Ω_a^+. Let μ_1 be the first eigenvalue of $(-\Delta, H_0^1(B))$, with the weight $m(x)$, and ϕ_1 the associated eigenfunction, i.e. $-\Delta\phi_1 = \mu_1 m(x)\phi_1$ in B. We have

$$\int_B \nabla u \nabla \phi_1 dx = \mu_1 \int_B m(x)u\phi_1 dx.$$

On the other hand

$$\int_B \nabla u \nabla \phi_1 dx = \int_B (a(x)u^q + b(x)u^p)\phi_1 dx.$$

It follows that

$$\lambda^{\frac{1-q}{p-q}} \int_B m(x)u\phi_1 dx \leq \mu_1 \int_B m(x)u\phi_1 dx,$$

which implies that $\lambda^{\frac{1-q}{p-q}} \leq \mu_1$. Thus λ^* is finite and, by the definition of λ^*, it follows that (P_λ) has no solution for $\lambda > \lambda^*$. q.e.d.

Existence for $\lambda = \lambda^*$:

We begin by recalling that, under the assumption $0 < q < p \leq 2^* - 1$, the solutions of (P_λ) are in $C_0^1(\overline{\Omega})$ (see [21, p. 245]).

By the definition of λ^*, there is a sequence $\lambda_k \in \Lambda$ such that $\lambda_k \nearrow \lambda^*$ and (P_{λ_k}) has a solution. Let u_k be a solution of (P_{λ_k}). First, let us show that $||u_k||$ is bounded. Actually, since $F'(u_k) = 0$ and $F(u_k) \leq 0$ with $F(u_k) \leq 0$ (see the first part of the proof), we have

$$pF(u_k) - F'(u_k) \cdot u_k \leq C||u_k||,$$

that means

$$\left(\frac{p}{2} - 1\right) ||u_k||^2 \leq C||u_k||^{q+1} + C.$$

It follows that $||u_k||$ is bounded, since $q < 1$.

Thus we can assume that $u_k \rightharpoonup u_*$ in H_0^1. Hence u solves (Q_{λ^*}) and $F(u) \leq 0$. Moreover, by standard bootstrap, we can assert that $u_k \to u_*$ in $C_0^1(\Omega)$.

We have to prove that u_* is a solution of (P_{λ^*}). For this purpose, assume, by contradiction, that $u_* = 0$ in Ω_a^+. Let $\varphi_1 > 0$ be the associated eigenfunction to the eigenvalue $\lambda_1(B_1)$, where B_1 is an open ball in Ω_a^+. We have

$$\begin{aligned}\lambda_1(B_1)\int_{B_1} u_k\varphi_1 dx &= \int_{B_1} \nabla u_k \nabla\varphi_1 dx \\ &= \int_{B_1} (a(x)u_k^q + \lambda^* b(x)u_k^p)\varphi_1 dx.\end{aligned}$$

Then

$$\int_\Omega c_1 u_k^q \varphi_1 dx \leq \int_{B_1} (\lambda_1(B_1)u_k - \lambda^* b(x)u_k^p)\varphi_1 dx.$$

It is a contradiction, if k is large enough, provided

$$c_1 u_k(x)^q \geq \lambda_1(B_1)u_k - \lambda^* b(x)u_k^p$$

for a.e. $x \in B_1$, since $u_k \to u_*$ in $C_0^1(\Omega)$. This concludes the proof of Theorem 1. □

3. Proof of Theorem 2

It follows, by Theorem 1, that for each $\lambda \in (0, \lambda^*)$ the functional (F_λ) has a local minimum u_λ, that satisfies $F_\lambda(u_\lambda) \leq 0$. We will look for a second solution of the form

$$v = u_\lambda + u, \quad \text{with} \quad u \geq 0.$$

It is equivalent to find a critical point of the following functional, defined in $H_0^1(\Omega)$,

$$\begin{aligned} I_\lambda(u) &= \frac{1}{2}\int |\nabla u|^2 \\ &- \frac{1}{q+1}\int a(x)[(u^+ + u_\lambda)^{q+1} - (u_\lambda)^{q+1} - (q+1)u_\lambda^q u^+] \\ &- \frac{\lambda}{p+1}\int b(x)[(u^+ + u_\lambda)^{p+1} - (u_\lambda)^{p+1} - (p+1)u_\lambda^p u^+]. \end{aligned}$$

Indeed, if u is a critical point of I_λ, then $u+u_\lambda$ is a critical point of F_λ. Moreover, using $-u^-$ as a test function, we can conclude that $u \geq 0$. Thus, we have to prove that I_λ has a nontrivial critical point. To this end, we will show that I_λ satisfies the assumptions of the relaxed mountain pass theorem, see [14, Corollary 5.11].

First, observe that $I_\lambda(0) = F_\lambda(u_\lambda)$. Thus we have to show that:

(i) there is $r > 0$ such that $I_\lambda(u) \geq F_\lambda(u_\lambda)$ for all $u \in H_0^1$ with $||u|| = r$;
(ii) there is $w_1 \in H_0^1$ such that $I(w_1) \leq F_\lambda(u_\lambda)$ and $||w_1|| > r$; and
(iii) I_λ satisfies the (PS) condition.

The item (i) is a consequence of u_λ being a local minimum of F_λ. In order to prove item (ii), let $v_1 \in C_0^1(\Omega_b^+)$ be nonnegative, nontrivial and such that $\int_\Omega b(x)v_1^{p+1} > 0$. We have, for large s,

$$\begin{aligned} I_\lambda(sv_1) &= \frac{s^2}{2}\int_\Omega |\nabla v_1|^2 \\ &- \frac{s^{q+1}}{q+1}\int_\Omega a(x)[(v_1 + \frac{u_\lambda}{s})^{q+1} - (\frac{u_\lambda}{s})^{q+1} - \frac{(q+1)u_\lambda^q v_1}{s^q}] \\ &- \frac{\lambda s^{q+1}}{p+1}\int_\Omega b(x)[(v_1 + \frac{u_\lambda}{s})^{p+1} - (\frac{u_\lambda}{s})^{p+1} + \frac{(p+1)u_\lambda^p v_1}{s^p}] \\ &= O(s^{q+1}) - \frac{\lambda s^{q+1}}{p+1}\int_\Omega b(x)(v_1 + \frac{u_\lambda}{s})^{p+1} \\ &\to -\infty \quad \text{as} \quad s \to \infty, \end{aligned}$$

and so (ii) follows.

For (iii), notice that F_λ satisfies (PS), see for instance [8]. Now, let u_n be a (PS) sequence of I_λ at level c, it follows that $u_1 + u_n$ is a (PS) sequence for F_λ, and so has a convergent subsequence. Thus, I_λ satisfies the (PS) condition. This concludes the proof of Theorem 2.

□

4. Proof of Theorem 3

As in the previous section, we have to prove that the functional I_λ has a nontrivial critical point. Again, I_λ has a local minimum at the origin and we can find $e \in H_0^1$, with $||e||$ large enough, such that $I_\lambda(e) \leq 0$. We are assuming that $p = 2^* - 1$, then I_λ fails to satisfies (PS) condition. In order to avoid this difficulty we follow the ideas in [7].

We argue by contradiction, i.e., suppose that 0 is the unique critical point of I_λ. Consider the mountain pass level

$$c_\lambda := \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} I_\lambda(\gamma(t)),$$

where $\Gamma = \{\gamma \in C([0,1], H_0^1) : \gamma(0) = 0, \gamma(1) \leq 0\}$. The next two lemmas will be proved below.

Lemma 1. *The following is true:*

$$c_\lambda < \frac{S^{\frac{N}{2}}}{N||b||_\infty^{\frac{N-2}{2}}}.$$

Lemma 2. *I_λ satisfies the $(PS)_c$ condition for all $c < \frac{S^{\frac{N}{2}}}{N||b||_\infty^{\frac{N-2}{2}}}$.*

Now, by the mountain pass theorem, there is $w_n \in H_0^1(\Omega)$ a sequence such that

$$I'_\lambda(w_n) \to 0 \;\text{ and }\; I_\lambda(w_n) \to c_\lambda, \tag{3}$$

where $0 \leq c_\lambda < \frac{S^{\frac{N}{2}}}{N||b||_\infty^{\frac{N-2}{2}}}$, by Lemma 1. Lemma 2 implies that $w_n \to w_0$ in $H_0^1(\Omega)$. Thus w_0 is a critical point of I_λ, and so, by our assumption, $w_0 = 0$. We have that, by (3),

$$\frac{1}{p+1} I'_\lambda(w_n) \cdot (u_\lambda + w_n) - I_\lambda(w_n) = \frac{1}{N}||w_n||^2 + o(1) \to c_\lambda. \tag{4}$$

If $c_\lambda = 0$, using the version of mountain pass theorem by Ghoussoub & Preiss [15, Theorem 1], we can take w_n satisfying $||w_n|| \to r$, $r > 0$, what is a contradiction with (4). Thus we should have $c_\lambda > 0$. Since $I'(w_n) \cdot w_n \to 0$, we conclude that

$$\lim_{n\to\infty} ||w_n||^2 = \lim_{n\to\infty} \int b(x)(w_n^+)^{2^*} = N c_\lambda.$$

By definition of S, we have

$$||w_n||^2 \geq S \left(\int |w_n|^{2^*} \right)^{\frac{2}{2^*}} \geq \frac{S}{||b||_\infty^{\frac{2}{2^*}}} \left(\int b(x)|w_n^+|^{2^*} \right)^{\frac{2}{2^*}}.$$

Passing to the limit in the above inequality, we get

$$c_\lambda N \geq \frac{S}{||b||_\infty^{\frac{2}{2^*}}} (c_\lambda N)^{\frac{2}{2^*}}.$$

It follows that $c_\lambda \geq \frac{S^{\frac{N}{2}}}{N||b||_\infty^{\frac{N-2}{2}}}$, provided $c > 0$. This contradicts Lemma 2. Thus Theorem 3 is proved.

□

Proof of Lemma 1: As usual, we will follow the approach from [7]. Define

$$v_\epsilon(x) = \frac{C_N \epsilon^{\frac{N-2}{2}}}{(\epsilon^2 + |x|^2)^{\frac{N-2}{2}}},$$

where $C_N = (N(N-2))^{\frac{N-2}{4}}$, so that v_ϵ satisfies

$$-\Delta v_\epsilon = v_\epsilon^{2^*-1} \ \text{ in } \mathbb{R}^N.$$

Pick a function $\eta \in C_0^\infty(B_\rho(0))$ such that $0 \leq \eta(x) \leq 1$ and $\eta(x) = 1$ for all $x \in B_{\rho/2}(0)$ (ρ as in (c)). Then set

$$u_\epsilon(x) = \eta(x) v_\epsilon(x).$$

It is easy to see that for ϵ_0, sufficiently small, there is $R > 0$ such that

$$I_\lambda(Ru_\epsilon) < 0, \quad \text{for all} \quad \epsilon \in (0, \epsilon_0).$$

It means that if we put $\gamma(t) = tRu_\epsilon$, $t \in [0,1]$, then $\gamma \in \Gamma$, and hence

$$c_\lambda \leq \max_{t \in [0,1]} I_\lambda(tu_\epsilon).$$

Thus we need to show that

$$\max_{t \in [0,R]} I_\lambda(tu_\epsilon) < \frac{S^{\frac{N}{2}}}{N||b||_\infty^{\frac{N-2}{2}}}.$$

First, we remark some standard estimates,

$$||v_\epsilon||^2 = S^{\frac{N}{2}} + O(\epsilon^{N-2}), \quad ||v_\epsilon||_{2^*}^{2^*} = S^{\frac{N}{2}} + O(\epsilon^N), \tag{5}$$

and, for some constants K_1, K_2 and K_3,

$$||v_\epsilon||_2^2 = \begin{cases} K_1\epsilon^2 + O(\epsilon^{N-2}) & \text{if} \quad N \geq 5, \\ K_2\epsilon^2|\ln \epsilon^2| + O(\epsilon^2) & \text{if} \quad N = 4, \\ K_3\epsilon + O(\epsilon^2) & \text{if} \quad N = 3. \end{cases} \tag{6}$$

Moreover,

$$\begin{aligned}\int_\Omega |u_\epsilon|^{q+1}dx &\leq \int_{B_\epsilon} \frac{(C_N\epsilon)^{\frac{N-2}{2}(q+1)}}{\epsilon^{(N-2)(q+1)}} + \int_{B_\rho\setminus B_\epsilon} \frac{(C_N\epsilon)^{\frac{N-2}{2}(q+1)}}{|x|^{(N-2)(q+1)}}\\ &\leq C\epsilon^{\frac{(N-2)(1-q)+4}{2}} + C\epsilon^{\frac{N-2}{2}(q+1)}\int_\epsilon^\rho r^{q(2-N)+1}dr\end{aligned}$$

and so

$$\int_\Omega |u_\epsilon|^{q+1}dx \leq$$

$$\begin{cases} C\epsilon^{\frac{(N-2)(1-q)+4}{2}} + C\epsilon^{\frac{N-2}{2}(q+1)}, & \text{if } \ q \neq \frac{2}{(N-2)}\\ C\epsilon^{\frac{(N-2)(1-q)+4}{2}} + C\epsilon^{\frac{N-2}{2}(q+1)} + C\epsilon^{\frac{N-2}{2}(q+1)}|\ln\epsilon|, & \text{if } \ q = \frac{2}{(N-2)}.\end{cases}$$

Thus

$$\int_\Omega |u_\epsilon|^{q+1}dx \leq \begin{cases} o(\epsilon^2), & \text{if } \ N\geq 6\\ o(\epsilon^{\frac{N-2}{2}}), & \text{if } \ 3\leq N\leq 5.\end{cases} \tag{7}$$

Now,

$$\int b(x)u_\epsilon^{2^*} = ||b||_\infty\int_{B_\rho(0)} u_\epsilon^{2^*} - \int_{B_\rho(0)}(||b||_\infty - b(x))u_\epsilon^{2^*}.$$

Using (c) and doing a change of variables, one has

$$\begin{aligned}\int_{B_\rho(0)}(||b||_\infty - b(x))u_\epsilon^{2^*} &\leq \int_{B_\rho(0)}|x|^\eta u_\epsilon^{2^*}\\ &= \int_{B_{\rho/\epsilon}(0)}|\epsilon x|^\eta v_\epsilon^{2^*}(\epsilon x) + O(\epsilon^N)\\ &= \epsilon^\eta\int_{B_{\rho/\epsilon}(0)}\frac{|x|^\gamma C_N^{2^*}}{(1+|x|)^N} + O(\epsilon^N)\\ &= O(\epsilon^\eta) + O(\epsilon^N).\end{aligned}$$

Thus

$$\int b(x)u_\epsilon^{2^*} = ||b||_\infty||u_\epsilon||_{2^*}^{2^*} + O(\epsilon^\eta) + O(\epsilon^N). \tag{8}$$

Now, we divide the proof in two cases:

Case $N\geq 6$: Note that, see Appendix B, we have

$$a(x)\left[\frac{(tu_\epsilon+u_\lambda)^{q+1} - u_\lambda^{q+1}}{q+1} - u_\lambda^q(tu_\epsilon)\right] \geq -C(tu_\epsilon)^{q+1}$$

and

$$b(x)\left[\frac{(tu_\epsilon+u_\lambda)^{p+1} - u_\lambda^{p+1}}{p+1} - u_\lambda^p(tu_\epsilon)\right] \geq b(x)\left[\frac{(tu_\epsilon)^{p+1}}{p+1} + u_\lambda^{p-1}\frac{(tu_\epsilon)^2}{2}\right].$$

Thus

$$\begin{aligned} I_\lambda(tu_\epsilon) &\leq \frac{t^2}{2}(||u_\epsilon||^2 - C||u_\epsilon||_2^2) - \frac{t^{2^*}}{2^*}\int b(x)u_\epsilon^{2^*} + Ct^{q+1}\int u_\epsilon^{q+1} \\ &\leq \frac{1}{N}\left[\frac{||u_\epsilon||^2 - C||u_\epsilon||_2^2}{\left(\int b(x)u_\epsilon^{2^*}\right)^{\frac{2}{2^*}}}\right]^{\frac{N}{2}} + Ct^{q+1}\int u_\epsilon^{q+1} \end{aligned}$$

Using (5)-(8), one has

$$\begin{aligned} I_\lambda(tu_\epsilon) &\leq \frac{1}{N}\left[\frac{S^{\frac{N}{2}} - C\epsilon^2 + O(\epsilon^{N-2})}{\left(||b||_\infty S^{\frac{N}{2}} + O(\epsilon^\eta) + O(\epsilon^N)\right)^{\frac{2}{2^*}}}\right]^{\frac{N}{2}} + o(\epsilon^2) \\ &= \frac{1}{N}\left[\frac{S^{\frac{N}{2}} - C\epsilon^2 + o(\epsilon^2)}{\left(||b||_\infty S^{\frac{N}{2}} + o(\epsilon^2)\right)^{\frac{2}{2^*}}}\right]^{\frac{N}{2}} + o(\epsilon^2) \\ &= \frac{S^{\frac{N}{2}}}{N||b||_\infty^{\frac{N-2}{2}}}[1 - C\epsilon^2 + o(\epsilon^2)] + o(\epsilon^2) \\ &< \frac{S^{\frac{N}{2}}}{N||b||_\infty^{\frac{N-2}{2}}}, \end{aligned}$$

for $\epsilon > 0$ sufficiently small (above, we used that $N \geq 6$ and $\eta > 2$).

Case $3 \leq N \leq 5$: Using (7) and (12), in Appendix B, we get

$$I_\lambda(tu_\epsilon) \leq \frac{t^2}{2}||u_\epsilon||^2 - \frac{t^{2^*}}{2^*}\int b(x)u_\epsilon^{2^*} - C_0\frac{t^p}{p}||u_\epsilon||_p^p + o(\epsilon^{\frac{N-2}{2}}), \quad C_0 > 0.$$

Now, noting that $||u_\epsilon||_{2^*-1}^{2^*-1} = C_1\epsilon^{\frac{N-2}{2}} + O(\epsilon^{\frac{N+2}{2}})$, $C_1 > 0$, it follows that

$$\begin{aligned} I_\lambda(tu_\epsilon) &\leq \frac{t^2}{2}S^{\frac{N}{2}} - \frac{t^{2^*}}{2^*}||b_\infty||S^{\frac{N}{2}} - C\frac{t^p}{p}\epsilon^{\frac{N-2}{2}} \\ &\quad + o(\epsilon^{\frac{N-2}{2}}) + O(\epsilon^{\frac{N+2}{2}}) + O(\epsilon^N) + O(\epsilon^\eta) \\ &= \frac{t^2}{2}S^{\frac{N}{2}} - \frac{t^{2^*}}{2^*}||b_\infty||S^{\frac{N}{2}} - C\frac{t^p}{p}\epsilon^{\frac{N-2}{2}} + o(\epsilon^{\frac{N-2}{2}}) \end{aligned}$$

where we used (5), (8) and that $\eta > (N-2)/2$. Calling t_ϵ the maximum of the right-hand side for $t \in [0, 1]$, then t_ϵ satisfy

$$S^{\frac{N}{2}} = t_\epsilon^{2^*-2}||b||_\infty S^{\frac{N}{2}} + t_\epsilon^{2^*-3}C\epsilon^{\frac{N-2}{2}} + o(\epsilon^{\frac{N-2}{2}}).$$

It follows that

$$t_\epsilon = \frac{1}{||b||_\infty^{\frac{N-2}{4}}} - C\epsilon^{\frac{N-2}{2}} t_\epsilon^{2^*-3} + o(\epsilon^{\frac{N-2}{2}}).$$

Thus

$$\begin{aligned}
\max_{t\in[0,R]} I_\lambda(tu_\epsilon) &\leq \frac{t_\epsilon^2}{2} S^{\frac{N}{2}} - \frac{t_\epsilon^{2^*}}{2^*}||b_\infty||S^{\frac{N}{2}} - Ct_\epsilon^{2^*-3}\epsilon^{\frac{N-2}{2}} + o(\epsilon^{\frac{N-2}{2}}) \\
&= \frac{1}{2}\frac{S^{\frac{N}{2}}}{||b||_\infty^{\frac{N-2}{2}}} - \frac{1}{2^*}\frac{S^{\frac{N}{2}}}{||b||_\infty^{\frac{N-2}{2}}} - Ct_\epsilon^{2^*-3}\epsilon^{\frac{N-2}{2}} + o(\epsilon^{\frac{N-2}{2}}) \\
&= \frac{1}{N}\frac{S^{\frac{N}{2}}}{||b||_\infty^{\frac{N-2}{2}}} - Ct_\epsilon^{2^*-3}\epsilon^{\frac{N-2}{2}} + o(\epsilon^{\frac{N-2}{2}}) \\
&< \frac{1}{N}\frac{S^{\frac{N}{2}}}{||b||_\infty^{\frac{N-2}{2}}},
\end{aligned}$$

for ϵ sufficiently small. This completes the proof of Lemma 1.

□

Proof of Lemma 2: Let $w_n \in H_0^1(\Omega)$ be a sequence such that

$$I'_\lambda(w_n) \to 0 \quad \text{and} \quad I_\lambda(w_n) \to c < \frac{S^{\frac{N}{2}}}{N||b||_\infty^{\frac{N-2}{2}}}.$$

Notice that w_n is bounded, actually, we have

$$\frac{1}{p+1} I'_\lambda(w_n)\cdot(u_\lambda + w_n) - I_\lambda(w_n) \leq \epsilon_n ||u_\lambda + w_n||, \quad \epsilon_n \to 0.$$

In the above expression the terms of power $p+1$ are cancelled, then it can be rewriten as

$$||w_n||^2 \leq C(||w_n||^{q+1} + ||w_n|| + 1),$$

which yields the boundedness of $||w_n||$. Passing to a subsequence, $w_n \rightharpoonup w_0$ in $H_0^1(\Omega)$, $w_n \to w_0$ in $L^r(\Omega)$, $1 < r < 2^*$. Moreover, $u_\lambda + w_0$ is a solution of (Q_λ) and so a critical point of F_λ. Thus w_0 is a critical point of I_λ. By assumption, we have $w_0 = 0$. Now,

$$\frac{1}{p+1} I'_\lambda(w_n)\cdot(u_\lambda + w_n) - I_\lambda(w_n) = \frac{1}{N}||w_n||^2 + o(1) \to c.$$

If $c = 0$ then $w_n \to 0$ in $H_0^1(\Omega)$ and the proof is finished. We claim that $c = 0$ is the unique possibility. Assume, by contradiction, that

$c \neq 0$. We can assume that $||w_n||$ converges and since $I'(w_n) \cdot w_n \to 0$, we conclude that

$$\lim_{n\to\infty} ||w_n||^2 = \lim_{n\to\infty} \int b(x)(w_n^+)^{2^*} = Nc.$$

By definition of S, we have

$$||w_n||^2 \geq S \left(\int |w_n|^{2^*} \right)^{\frac{2}{2^*}} \geq \frac{S}{||b||_\infty^{\frac{2}{2^*}}} \left(\int b(x)|w_n^+|^{2^*} \right)^{\frac{2}{2^*}}.$$

Passing to the limit in the above inequality, we get

$$cN \geq \frac{S}{||b||_\infty^{\frac{2}{2^*}}} (cN)^{\frac{2}{2^*}}.$$

It follows that $c \geq \frac{S^{\frac{N}{2}}}{N||b||_\infty^{\frac{N-2}{2}}}$, provided $c > 0$. This contradiction completes the proof of Lemma 2.

□

5. Appendix

5.1. **Appendix A.**

This appendix has the purpose of proving that the solution u_λ, obtained in the first part of the proof of Theorem 1, is a local minimum of F_λ for $0 < \lambda < \lambda^*$, where λ^* is defined by (2).

Let us remember that

$$F_\lambda(u_\lambda) = \inf_M F_\lambda \quad \text{where} \quad M = \{u \in H_0^1(\Omega); 0 \leq u \leq \overline{u}\}.$$

Here $\overline{u}$ is a solution of $(P_{\overline{\lambda}})$ for some $\lambda < \overline{\lambda} < \lambda^*$.

Lemma 3. *We have that* $u_\lambda < \overline{u}$ *in* $U = \{\overline{u} > 0\} \cap \Omega_b^+$.

Proof. Let $v = \overline{u} - u_\lambda \geq 0$ a.e. in Ω, then

$$-\Delta v + m(x)v \geq 0, \quad \text{where} \quad m(x) := a^- \frac{\overline{u}^q - u^q}{\overline{u} - u}.$$

Suppose, by contradiction, that $v(x_0) = 0$ for some $x_0 \in U$. We can choose $r > 0$ such that the ball $B_r[x_0] \subset U$. We have that m is uniformly bounded in $B_r(x_0)$, so by the Strong Maximum Principle we get $v = 0$ in $B_r(x_0)$. It means that $u_\lambda = \overline{u}$ in $B_r(x_0)$, what contradicts the equations satisfied by these functions, since $\lambda < \overline{\lambda}$ and $b > 0$ in U. □

Suppose, by contradiction, that u_λ is not a local minimum of F_λ. Then we can choose $u_n \in H_0^1$ with $||u_n - u_\lambda|| \to 0$ and $F_\lambda(u_n) < F_\lambda(u_\lambda)$. Let

$$v_n = \max\{0, \min\{u_n, \overline{u}\}\}, \quad w_n = (u_n - \overline{u})^+,$$

so that $u_n^+ = v_n + w_n$ and $v_n \in M$. Define the sets $R_n = \{x \in \Omega : 0 \le u_n(x) \le \overline{u}\}$, $S_n = \mathrm{supp}(w_n)$ and $T_n = \mathrm{supp}(u_n^-)$, and the functions

$$h(x,t) = a(x)(t^+)^q + \lambda b(x)(t^+)^p \quad \text{and} \quad H(x,t) = \int_0^t h(x,s)ds.$$

Then, we can rewrite $F_\lambda(u_n^+)$ as

$$F_\lambda(u_n^+) = \int_{S_n} \left(\frac{|\nabla u_n|^2}{2} - H(x,u_n) \right) + \int_{R_n} \left(\frac{|\nabla v_n|^2}{2} - H(x,v_n) \right).$$

Notice that

$$\int_{S_n} \left(\frac{|\nabla u_n|^2}{2} - H(x,u_n) \right) = \int_{S_n} \left(\frac{|\nabla (\overline{u} + w_n)|^2}{2} - H(x,\overline{u} + w_n) \right),$$

and moreover

$$\int_{R_n} \left(\frac{|\nabla v_n|^2}{2} - H(x,v_n) \right) \;=\; F_\lambda(v_n) - \int_{S_n} \left(\frac{|\nabla \overline{u}|^2}{2} - H(x,\overline{u}) \right).$$

Therefore

$$\begin{aligned} F_\lambda(u_n) &= F_\lambda(u_n^+) + F_\lambda(u_n^-) \\ &= \int_{S_n} \left(\frac{(|\nabla(\overline{u} + w_n)|^2 - |\nabla \overline{u}|^2)}{2} - (H(x,\overline{u} + w_n) - H(x,\overline{u})) \right) \\ &\quad + F_\lambda(v_n) + \int_{T_n} \left(\frac{|\nabla u_n|^2}{2} - H(x,u_n) \right). \end{aligned}$$

By using that

$$\int_\Omega \nabla \overline{u} \nabla w_n dx \ge \int_\Omega h(x,\overline{u}) w_n dx,$$

we get

$$\begin{aligned} F_\lambda(u_n) \;&\ge\; F_\lambda(v_n) + \frac{1}{2}\int_\Omega |\nabla w_n|^2 dx + \frac{1}{2}\int_\Omega |\nabla u_n^-|^2 dx \\ &\quad - \int_{S_n} (H(x,\overline{u} + w_n) - H(x,\overline{u}) - h(x,\overline{u}) w_n) dx. \end{aligned}$$

We can then conclude that

$$\frac{1}{2}||w_n||^2 + \frac{1}{2}||u_n^-||^2 < \int_{S_n} (H(x,\overline{u} + w_n) - H(x,\overline{u}) - h(x,\overline{u}) w_n) dx.$$

The proof follows from the next claim.

Claim. We have

$$\int_{S_n} (H(x, \overline{u} + w_n) - H(x, \overline{u}) - h(x, \overline{u})w_n)dx \leq o(1)||w_n||^2.$$

Assuming the above claim, we have

$$\frac{1}{2}||u_n^-|| + \frac{1}{2}(||w_n||^2 - o(1)||w_n||^2) < 0,$$

which implies that $w_n, u_n^- = 0$ for n large enough. It follows that $u_n = v_n \in M$, for n large, and so $F_\lambda(u_n) \geq F_\lambda(u_\lambda)$, what is a contradiction. □

Proof of the Claim.

First, consider the following splitting for the function $H_n = H_{1n} + H_{2n}$, where

$$H_{1n}(x) = \frac{\lambda b(x)}{p+1}[(\overline{u} + w_n)^{p+1} - \overline{u}^{p+1}] - \lambda b(x)\overline{u}^p w_n, \quad \text{and}$$
$$H_{2n}(x) = \frac{a(x)}{q+1}[(\overline{u} + w_n)^{q+1} - \overline{u}^{q+1}] - a(x)\overline{u}^q w_n.$$

<u>Superlinear term:</u> Note that, there are $s(x), \theta(x) \in (0, 1)$ such that

$$\begin{aligned} H_{1n}(x) &= \lambda b(x)[(\overline{u} + \theta w_n)^p - \overline{u}^p]w_n \\ &\leq C(\overline{u} + s\theta w_n)^{p-1}\theta w_n^2. \end{aligned}$$

Moreover, $(\overline{u} + s\theta w_n)^{p-1}w_n^2 \leq w_n^{p+1}$ in $B = \Omega \setminus A$, where $A = \{\overline{u} > 0\}$, then

$$\int_{S_n \setminus A} H_{1n}(x)dx \leq C||w_n||^{p+1} \leq o(1)||w_n||^2. \tag{9}$$

On the other hand, $(\overline{u} + s\theta w_n)^{p-1}w_n^2 \leq Cw_n^2 + Cw_n^{p+1}$ in A, then

$$\begin{aligned} \int_{S_n \cap A} H_{1n}(x)dx &\leq \int_{S_n \cap A \cap \Omega_b^+} H_{1n}(x)dx \\ &\leq C\int_{S_n \cap A \cap \Omega_b^+} w_n^2 dx + C||w_n||^{p+1} \\ &\leq |S_n \cap A \cap \Omega_b^+|^{\frac{2}{N}} \left(\int_\Omega w_n^{\frac{2N}{N-2}}dx\right)^{\frac{N-2}{N}} + C||w_n||^{p+1} \\ &\leq C|S_n \cap A \cap \Omega_b^+|^{\frac{2}{N}}||w_n||^2 + o(1)||w_n||^2. \end{aligned}$$

Now, we claim that $|S_n \cap A \cap \Omega_b^+| \to 0$ as $n \to \infty$. Actually, given $\epsilon > 0$, by Lemma 3, we can choose $\delta > 0$ such that $|A \cap \Omega_b^+ \cap \{\overline{u} \leq u_\lambda + \delta\}| < \epsilon$.

However, we have

$$S_n \subset \{\overline{u} \leq u_\lambda + \delta\} \cup \{u_n > \overline{u} > u_\lambda + \delta\},$$

and since $u_n \to u_\lambda$ in L^2 there is n_0 such that for all $n \geq n_0$,

$$\begin{aligned} \epsilon\delta^2 &\geq \int_\Omega (u_n - u_\lambda)^2 \geq \int_{\{u_n > u_\lambda+\delta\}} (u_n - u_\lambda)^2 \\ &\geq \int_{\{u_n > u_\lambda+\delta\}} \delta^2 = \delta^2 |\{u_n > u_\lambda + \delta\}|. \end{aligned}$$

Thus $|S_n \cap A \cap \Omega_b^+| \leq |A \cap \Omega_b^+ \cap \{\overline{u} \leq u_\lambda + \delta\}| + |\{u_n > u_\lambda + \delta\}| \leq 2\epsilon$. It follows that

$$\int_{S_n \cap A} H_{1,n}(x)dx \leq o(1)||w_n||^2.$$

<u>*Sublinear term:*</u> First, observe that $\Omega_a^+ \subset A$ and $B \subset \Omega \setminus \Omega_a^+$. We have, for $x \in A \setminus \Omega_a^+$,

$$H_{2n}(x) \leq a(x)[(\overline{u} + \theta w_n)^q - \overline{u}^q]w_n \leq 0.$$

Thus

$$\int_{S_n \cap (A \setminus \Omega_a^+)} H_{2n}(x)dx \leq 0 \leq o(1)||w_n||^2.$$

In the other hand, note that $\overline{\Omega_a^+} \subset int(\Omega_a)$ and $\overline{u} > 0$ in $int(\Omega_a)$, and so there is $\delta > 0$ such that $\overline{u}(x) \geq \delta$ in Ω_a^+. Then, for $x \in \Omega_a^+$,

$$\begin{aligned} H_{2n}(x) &= a(x)[(\overline{u} + \theta w_n)^q - \overline{u}^q]w_n \\ &= a(x)(\overline{u} + s\theta w_n)^{q-1}\theta w_n^2 \\ &\leq C\delta^{q-1} w_n^2 \leq C w_n^2, \end{aligned}$$

with $\theta(x), s(x) \in (0,1)$. Thus

$$\int_{S_n \cap \Omega_a^+} H_{2n}(x)dx \leq C \int_{S_n \cap \Omega_a^+} w_n^2 \leq |S_n \cap \Omega_a^+|^{\frac{2}{N}} ||w_n||^2 \leq o(1)||w_n||^2,$$

where the last inequality is a consequence of $S_n \cap \Omega_a^+ \subset S_n \cap A \cap \Omega_b^+$ and $|S_n \cap A \cap \Omega_b^+| = o(1)$.

Now, for $x \in B$ we have $H_{2n}(x) = -\frac{a^-(x)}{q+1} w_n^{q+1}$ since $B \subset \Omega_a^-$. Thus

$$\int_B H_{2n}(x)dx = -\int_B \frac{a^-(x)}{q+1} w_n^{q+1} dx \leq 0. \tag{10}$$

The claim follows from the above estimates.

□

5.2. **Appendix B.**

Lemma 4. (1) *There is a constant $C(p) > 0$ such that*

$$\frac{(r+s)^{p+1} - r^{p+1}}{p+1} - r^p s \geq \frac{s^{p+1}}{p+1} + Cr^{p-1}s^2, \quad r, s \geq 0, \tag{11}$$

(2) *for $r, s \geq 0$, we claim that there is a constant $C(q) > 0$ such that*

$$\frac{(r+s)^{q+1} - r^{q+1}}{q+1} - r^q s \leq C(q) s^{q+1}. \tag{12}$$

Proof. For (1), see [2, pag. 537]. Item (2) is left as an exercise to the reader.

□

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