THE ANALYTIC TORSION OF A DISC

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Abstract. We study the Reidemeister torsion and the analytic torsion of the m dimensional disc, with the Ray and Singer homology basis [20]. We prove that the Reidemeister torsion coincides with a power of the volume of the disc. We study the additional terms arising in the analytic torsion due to the boundary, using generalizations of the Cheeger-Müller theorem. We use a formula proved by Brüning and Ma [2], that predicts a new anomaly boundary term beside the known term proportional to the Euler characteristic of the boundary [15]. Some of our results extend to the case of the cone over a sphere, in particular we evaluate directly the analytic torsion for a cone over the circle and over the two sphere. We compare the results obtained in the low dimensional cases. We also consider a different formula for the boundary term given by Dai and Fang [8], and we compare the results. The results of these work were announced in [11].

1. Introduction

The Reidemeister torsion is an important topological invariant introduced originally by Reidemeister, Franz and de Rham to classify lens spaces. For non acyclic spaces, the R torsion depends on the homology. However, dealing with Riemannian manifolds, Ray and Singer [20] introduced a geometric torsion invariant by using the Riemannian structure to fixing the dependence on the homology of the R torsion. In the same work, in searching for an analytic description of the R torsion, Ray and Singer also introduced the analytic torsion, that soon became an important geometric invariant on its own, and has been deeply investigated by various authors (see for example [1] and the references therein). The equivalence between the R torsion and the analytic torsion, conjectured by Ray and Singer, was eventually proved by Cheeger [4] and Müller [18], for closed manifold. Cheeger also discussed the case of manifolds with boundary, showing that in this case an extra term could appear. Much later, this boundary term was explicitly given by Lück [15], for the case of manifolds with a product metric structure near the boundary. Only in 2000, Dai and Fang [8] gave a formula for the difference of the R torsion and the analytic torsion on a manifold with boundary without any assumption for the metric near the boundary. In this formula some new terms appear. However, in a recent work of Brüning and Ma [2] on Ray-Singer metrics on manifolds with boundary, a further formula is given, where a different boundary contribution appears. The results given in Theorem 2 below are obtained using the formula of Brüning and Ma. The results obtained using the formula of Dai and Fang are given at the end of Section 4. Beside the intensive investigation and the large literature available, comparably few results exist on the quantitative side, namely explicit evaluations

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of the analytic torsion [21] [30] [9]. Continuing along this line of investigation, we study in this work the simplest case of a manifold with boundary, namely the case of a disc. Let (W, g) be a compact connected Riemannian manifold with boundary ∂W, and metric g, and $\rho : \pi_1(W) \to O(k, \mathbb{R})$ an orthogonal representation of the fundamental group of W. We denote by $\tau_{\rm R}((W, g); \rho)$ the R torsion, by $\tau_{\rm R}((W, \partial W, g); \rho)$ the R torsion of the pair $(W, \partial W)$. We denote by $T_{\rm abs}((W, g); \rho)$ the analytic torsion of (W, g) with absolute boundary conditions on ∂W, and by $T_{\rm rel}((W, g); \rho)$, the analytic torsion of (W, g) with relative boundary condition, both with respect to the representation ρ (see Section 2 for the precise definitions). Let $D^m_l = \{x \in \mathbb{R}^m \mid |x| \leq l\}$, the disc of radius $l > 0$ in the euclidian space $\mathbb{R}^m$, and with the standard metric g_E induced by the immersion, and ρ an orthogonal representation of the fundamental group. With this notation, we now state our main results.

Theorem 1. *The R torsion of the disc D^m_l is:*

$$\tau_{\rm R}((D^m_l, g_E); \rho) = \left(\sqrt{{\rm Vol}_{g_E}(D^m_l)}\right)^{{\rm rk}(\rho)}.$$

In the same situation, the R torsion of the pair (D^m_l, S^{m-1}_l) is:

$$\tau_{\rm R}((D^m_l, S^{m-1}_l, g_E); \rho) = \left(\sqrt{{\rm Vol}_{g_E}(D^m_l)}\right)^{(-1)^{m-1}{\rm rk}(\rho)}.$$

Proof. The results follow from Propositions 1 and 2 of Section 3, taking $\alpha = \frac{\pi}{2}$. □

Theorem 2. *The analytic torsion of the disc D^m_l is ($p > 0$):*

$$\log T_{\rm abs}((D^{2p-1}_l, g_E); \rho) = \frac{1}{2}{\rm rk}(\rho) \log {\rm Vol}_{g_E}(D^{2p-1}_l) + \frac{1}{2}{\rm rk}(\rho) \log 2 + \frac{1}{4}{\rm rk}(\rho) \sum_{n=1}^{p-1} \frac{1}{n},$$

$$\log T_{\rm abs}((D^{2p}_l, g_E); \rho) = \frac{1}{2}{\rm rk}(\rho) \log {\rm Vol}_{g_E}(D^{2p}_l) + \frac{1}{2}{\rm rk}(\rho) \sum_{n=1}^{p} \frac{1}{2n-1}.$$

Proof. The results follow from Theorem 1, using Lemmas 1 and 2 of Section 4. □

Beside our main results concern the case of the discs, that are smooth manifolds, our technique easily extends, at least formally, to cover the case of the completed cone over a sphere $C_\alpha S^n_{l \sin\alpha}$, of angle α, length $l > 0$, and with the standard metric g_E induced by the immersion (see the beginning of Section 3 for the definition). And this generalization contains the case of the discs. For this reason, we develop our analysis in the more general case of the cone, whenever this is possible. The main problem, to deal with the cone, is the extension of the Hodge theory to the space of L^2-forms near the singularity at the tip of the cone. This theory has been developed in the work of J. Cheeger [6], and we will assume his results in the definition of the Laplacian on forms, necessary in order to define the analytic torsion appearing in the following theorems. More details on this aspect, are at the beginning of Section 4 and of Section 5.

Theorem 3. *The analytic torsion of the cone $C_\alpha S^1_{l \sin\alpha}$, with ${\rm rk}(\rho_0) = 1$, is:*

$$\log T_{\rm abs}((C_\alpha S^1_{l \sin\alpha}, g_E); \rho_0) = -\log T_{\rm rel} = \frac{1}{2}\log(\pi l^2 \sin\alpha) + \frac{1}{2}\sin\alpha.$$

In particular, for the disc D^2_l we have:

$$\log T_{\rm abs}((D^2_l, g_E);\rho_0) = -\log T_{\rm rel}((D^2_l, g_E);\rho_0) = \frac{1}{2}\log \pi l^2 + \frac{1}{2}.$$

Proof. The proof is in Section 5.2. □

Theorem 4. *The analytic torsion of the cone $C_\alpha S^2_{l\sin\alpha}$, with $\mathrm{rk}(\rho_0)=1$, is:*

$$\log T_{\rm abs}((C_\alpha S^2_{l\sin\alpha}, g_E);\rho_0) = \log T_{\rm rel} = \frac{1}{2}\log\frac{4}{3}l^3 - \frac{1}{2}F(0,\csc\alpha) + \frac{1}{4}\sin^2\alpha,$$

where the function $F(0,x)$ is given in Appendix 7. In particular, for the disc D^3_l:

$$\log T_{\rm abs}((D^3_l, g_E);\rho_0) = \log T_{\rm rel}((D^3_l, g_E);\rho_0) = \frac{1}{2}\log\frac{4\pi l^3}{3} + \frac{1}{2}\log 2 + \frac{1}{4}.$$

Proof. The proof is in Section 5.3. □

We note that all the results contained in the previous theorems, up to Theorem 4 when $\alpha \neq \pi/2$, are particular instances of the Cheeger-Müller theorem for a manifold with boundary, i.e. of the following formula:

$$\log T_{\rm abs}((W,g);\rho) = \log \tau_{\rm R}((W,g);\rho) + \frac{1}{4}\mathrm{rk}(\rho)\chi(\partial W)\log 2 + A_{\rm BM}(\partial W), \tag{1}$$

where $A_{\rm BM}(\partial W)$ is the anomaly boundary term of Brüning and Ma (see Section 4 for details). This is an expected results for the discs, that are manifolds, while is more surprising for the cone over a circle. On the other side, a simple calculation using the formula given in Theorem 4 for the cone over the sphere shows that an extra term appears in this case in the analytic torsion, beside the ones predicted by the formula in equation (1).

2. Preliminary and notation

We recall briefly the definition of the torsion of a finite chain complex of finite dimensional $\mathbb{F}$-vectors spaces (where $\mathbb{F}$ is a field of characteristic 0)

$$\mathcal{C}: \qquad C_m \xrightarrow{\partial_m} C_{m-1} \xrightarrow{\partial_{m-1}} \dots \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0.$$

Let $Z_q = \ker\partial_q$, $B_q = \mathrm{Im}\partial_{q+1}$, and $H_q = Z_q/B_q$. We assume that preferred bases $c_q = \{c_{q,j}\}$ and $h_q = \{h_{q,j}\}$ are given for C_q and H_q, respectively, for each q. Let $b_q = \{b_{q,j}\}$ be a set of independent vectors in C_q with $\partial_q(b_q) \neq 0$, and let $z_q = \{z_{q,j}\}$ be a set of independent vectors in Z_q with $p(z_{q,j}) = h_{q,j}$. Then, considering the sequence

$$0 \longrightarrow B_q \longrightarrow Z_q \xrightarrow{p} H_q \longrightarrow 0,$$

a basis for Z_q is given by the basis $\partial_{q+1}(b_{q+1})$ of B_q and the set z_q. We denote this basis by b_q, z_q (see [17] for details). By the same argument, the sequence

$$0 \longrightarrow Z_q \longrightarrow C_q \xrightarrow{\partial_q} B_{q-1} \longrightarrow 0,$$

determine the basis $\partial_{q+1}(b_{q+1}), z_q, b_q$ of C_q. Let $(\partial_{q+1}(b_{q+1}), z_q, b_q/c_q)$ denote the matrix of the change of basis. Then, the torsion of $\mathcal{C}$ is the class

$$\tau(\mathcal{C};\mathrm{v}) = \prod_{q=0}^{n}[\det(\partial_{q+1}(b_{q+1}), z_q, b_q/c_q)]^{(-1)^q}, \tag{2}$$

in $\mathbb{F}^\times/\{\pm 1\}$. It is easy to see that the torsion is independent of the graded bases $b = \{b_q\}$ and on the lifts $z = \{z_q\}$, but depends on the graded homology basis

$h = \{h_q\}$. More precisely, $\tau(\mathcal{C}; \mathrm{v})$ depends on the volume element $\mathrm{v} = \otimes_{q=0}^m h_q^{(-1)^q}$ in $\otimes_{q=0}^m \Lambda^{r_q} H_q$, where $r_q = \mathrm{rk} H_q$, and this explain the notation.

Next, let (K, L) be a pair of connected finite cell complexes of dimension m, and $(\tilde{K}, \tilde{L})$ its universal covering complex pair, and identify the fundamental group $\pi = \pi_1(K)$ with the group of the covering transformations of $\tilde{K}$. Note that covering transformations are cellular. Let $\mathcal{C}((\tilde{K}, \tilde{L}); \mathbb{Z})$ be the chain complex of $(\tilde{K}, \tilde{L})$ with integer coefficients. The action of the group of covering transformations makes each chain group $C_q((\tilde{K}, \tilde{L}); \mathbb{Z})$ into a module over the group ring $\mathbb{Z}\pi$, and each of these modules is $\mathbb{Z}\pi$-free and finitely generated with preferred basis given by the natural choice of the q-cells of $K - L$. Since K is finite it follows that $\mathcal{C}((\tilde{K}, \tilde{L}); \mathbb{Z})$ is free and finitely generated over $\mathbb{Z}\pi$. We obtain a complex of free finitely generated modules over $\mathbb{Z}\pi$ that we denote by $\mathcal{C}((K, L); \mathbb{Z}\pi)$. Let $\rho : \pi \to O(\mathbb{F}, k)$ be an orthogonal representation of the fundamental group on a $\mathbb{F}$-vector space V of dimension k, and consider the twisted complex $\mathcal{C}((K, L); V_\rho) = V \otimes_{\mathbb{Z}\pi} \mathcal{C}((K, L); \mathbb{Z}\pi)$. Then, the torsion of (K, L) with respect to the representation ρ is the class of $\mathbb{F}^\times / \{\pm 1\}$:

$$\tau((K, L); \rho, \mathrm{v}) = \tau(\mathcal{C}((K, L); V_\rho); \mathrm{v}).$$

Now, let W be an m dimensional orientable compact connected Riemannian manifold with metric g and possible boundary ∂W. The torsion of W can be defined taking any smooth triangulation or cellular decomposition of W. Moreover, the volume element v can also be fixed by using the metric structure. More precisely, given a graded orthonormal basis a_q for the space of harmonic forms $\mathcal{H}^q(W, V_\rho)$, either with absolute or relative BC, and applying the de Rham map (see [20])

$$\mathcal{A}_q^{\mathrm{abs}} = (-1)^q \mathcal{P}^{-1} \mathcal{A}_{\mathrm{rel}}^{m+1-q} * : \mathcal{H}^q_{\mathrm{abs}}(W, V_\rho) \to H_q(W; V_\rho), \tag{3}$$

we obtain a preferred homology graded basis $h = \mathcal{A}(a)$, that fix the volume element $\mathrm{w} = \mathcal{A}(\alpha)$, where α is the volume element determined by a. This gives the R torsion of W, and the relative R torsion of $(W, \partial W)$:

$$\tau_{\mathrm{R}}((W, g); \rho) = \tau(\mathcal{C}(W; V_\rho); \mathcal{A}(\alpha)), \quad \tau_{\mathrm{R}}((W, \partial W, g); \rho) = \tau(\mathcal{C}((W, \partial W); V_\rho); \mathcal{A}(\alpha)).$$

We conclude this section recalling the definition of the analytic torsion [20]. The zeta function of Laplace operator $\Delta^{(q)}$ on q forms in $\Omega^q(W, V_\rho)$ is defined by the meromorphic extension (analytic at $s = 0$) of the series

$$\zeta(s, \Delta^{(q)}) = \sum_{\lambda \in \mathrm{Sp}_+ \Delta^{(q)}} \lambda^{-s},$$

convergent for $\mathrm{Re}(s) > \frac{m}{2}$, and where Sp_+ denotes the positive part of the spectrum. If W has no boundary, the analytic torsion of (W, g) is

$$\log T((W, g); \rho) = \frac{1}{2} \sum_{q=1}^m (-1)^q q \zeta'(0, \Delta^{(q)}). \tag{4}$$

If W has a boundary, we denote by $T_{\mathrm{abs}}((W, g); \rho)$ the number defined by equation (4) with Δ satisfying absolute boundary conditions, and by $T_{\mathrm{rel}}((W, g); \rho)$ the number defined by the same equation with Δ satisfying relative boundary conditions.

3. The Reidemeister torsion of the cone over a sphere

In this section we compute the R torsion of the m dimensional disc, D_l^m. However, we will consider the slightly more general case of a cone. Namely we consider the cone of angle α, $C_\alpha S^n$, constructed in $\mathbb{R}^{n+2}$ over the sphere S^n, $m = n+1$, as defined below. In general, $C_\alpha S^n$ is not a smooth Riemannian manifold, but is a space with a singularity of conical type as defined by Cheeger in [4] (2.1). More precisely, $C_\alpha S^n$ coincides with the completed finite metric cone of Cheeger over the sphere of radius $\sin\alpha$. Note that we are adding a point at the tip of the cone, in order to have a simply connected space. The resulting space is compact, but obviously is not a smooth Riemannian manifold. The space obtained removing the tip, is an open smooth Riemannian manifold with the metric induced by the immersion, as in [4]. In order to define the R torsion some care is necessary, since we do not know how the de Rham theory extends. More precisely, we do not know if we have the de Rham maps $\mathcal{A}^q$ for spaces with conical singularities, in general. However, we show that we can define these maps in the particular case of $C_\alpha S^n$, and therefore we define the R torsion accordingly. In particular, the construction cover the smooth case of the disc.

Let S_b^n be the standard sphere of radius $b>0$ in $\mathbb{R}^{n+1}$, $S_b^n = \{x \in \mathbb{R}^{n+1} \mid |x| = b\}$ (we write S^n for S_1^n). Let $C_\alpha S^n_{l\sin\alpha}$ be the cone of angle α over $S^n_{l\sin\alpha}$ in $\mathbb{R}^{n+2}$. Note that the disc corresponds to $D_l^{n+1} = C_{\frac{\pi}{2}} S_l^n$. Parameterize $C_\alpha S^n_{l\sin\alpha}$ by

$$C_\alpha S^n_{l\sin\alpha} = \begin{cases} x_1 &= r\sin\alpha\sin\theta_n\sin\theta_{n-1}\cdots\sin\theta_3\sin\theta_2\cos\theta_1 \\ x_2 &= r\sin\alpha\sin\theta_n\sin\theta_{n-1}\cdots\sin\theta_3\sin\theta_2\sin\theta_1 \\ x_3 &= r\sin\alpha\sin\theta_n\sin\theta_{n-1}\cdots\sin\theta_3\cos\theta_2 \\ &\vdots \\ x_{n+1} &= r\sin\alpha\cos\theta_n \\ x_{n+2} &= r\cos\alpha \end{cases}$$

with $r\in[0,l]$, $\theta_1\in[0,2\pi]$, $\theta_2,\dots,\theta_n\in[0,\pi]$, α is a fixed positive real number, and $0 < a = \frac{1}{\nu} = \sin\alpha \le 1$. The induced metric is $(r>0)$

$$\begin{aligned} g_E &= dr\otimes dr + r^2a^2 g_{S_1^n} \\ &= dr\otimes dr + r^2a^2\left(\sum_{i=1}^{n-1}\left(\prod_{j=i+1}^{n}\sin^2\theta_j\right)d\theta_i\otimes d\theta_i + d\theta_n\otimes d\theta_n\right), \end{aligned}$$

and $\sqrt{|\det g_E|} = (r\sin\alpha)^n(\sin\theta_n)^{n-1}(\sin\theta_{n-1})^{n-2}\cdots(\sin\theta_3)^2(\sin\theta_2)$. Let K be the cellular decomposition of $C_\alpha S^n_{l\sin\alpha}$, with one top cell, one n-cell and one 0-cell, $K = c^1_{n+1}\cup c^1_n\cup c^1_0$, and let the subcomplex L of K be the cellular decomposition of $S^n_{l\sin\alpha}$, $L = c^1_n\cup c^1_0$. Let ρ be a real (trivial) representation.

Consider first the case of relative boundary conditions. Then the relevant complex of real vector spaces reads

$$\mathcal{C}_{\rm rel}:\ 0 \longrightarrow \mathbb{R}[c^1_{n+1}] \longrightarrow 0 \longrightarrow \cdots \longrightarrow 0 \longrightarrow 0 \longrightarrow 0\ ,$$

with preferred base $c_{n+1} = \{c^1_{n+1}\}$. To fix the base for the homology, we need a graded orthonormal base a for the harmonic forms. Since a base for $\Omega^{n+1}(C_\alpha S^n_{l\sin\alpha})$

is $\{\sqrt{|\det g_E|}dr\wedge d\theta_1\wedge\cdots\wedge d\theta_n\}$, we get $a_{n+1}=\left\{\frac{\sqrt{|\det g_E|}dr\wedge d\theta_1\wedge\cdots\wedge d\theta_n}{\sqrt{\mathrm{Vol}_{g_E}(C_\alpha S^n_{l\sin\alpha})}}\right\}$. Applying the formula in equation (3) for the de Rham map, we obtain $z_{n+1}=\{z^1_{n+1}\}$, with

$$\begin{aligned} z^1_{n+1}=\mathcal{A}^{\rm rel}_{n+1}(a^1_{n+1})&=\frac{1}{\sqrt{\mathrm{Vol}_{g_E}(C_\alpha S^n_{l\sin\alpha})}}\int_{\rm pt}*\sqrt{|\det g_E|}dr\wedge d\theta_1\wedge\cdots\wedge d\theta_n c^1_{n+1}\\ &=\frac{1}{\sqrt{\mathrm{Vol}_{g_E}(C_\alpha S^n_{l\sin\alpha})}}c^1_{n+1}. \end{aligned}$$

As $b_q=\emptyset$, for all q, we have that

$$|\det(z_{n+1}/c_{n+1})|=\frac{1}{\sqrt{\mathrm{Vol}_{g_E}(C_\alpha S^n_{l\sin\alpha})}},\qquad |\det(b_q/c_q)|=1,\ 0\leq q\leq n.$$

Applying the definition in equation (2), this proves the following result.

Proposition 1.

$$\tau_{\rm R}((C_\alpha S^n_{l\sin\alpha},S^n_{l\sin\alpha},g_E);\rho)=\left(\sqrt{\mathrm{Vol}_{g_E}(C_\alpha S^n_{l\sin\alpha})}\right)^{(-1)^n{\rm rk}(\rho)}.$$

Next, consider the case of absolute boundary conditions. The relevant complex is

$$\mathcal{C}_{\rm abs}:\ 0\longrightarrow\mathbb{R}[c^1_{n+1}]\longrightarrow\mathbb{R}[c^1_n]\longrightarrow 0\longrightarrow\cdots\longrightarrow 0\longrightarrow\mathbb{R}[c^1_0]\longrightarrow 0\ ,$$

with preferred bases $c_{n+1}=\{c^1_{n+1}\}$, $c_n=\{c^1_n\}$ and $c_0=\{c^1_0\}$. Hence, $H_p(K)=0$, for $p>1$, and $H_0(K)=\mathbb{R}[c^1_0]$. Since a base for $\Omega^0(C_\alpha S^n_{l\sin\alpha})$ is the constant form $\{1\}$, we have $a_0=\left\{\frac{1}{\sqrt{\mathrm{Vol}_{g_E}(C_\alpha S^n_{l\sin\alpha})}}\right\}$. Applying the formula in equation (3) for the de Rham map, we obtain $z_0=\{z^1_0\}$, with

$$z^1_0=\mathcal{A}^{\rm abs}_0(a^1_0)=\frac{1}{\sqrt{\mathrm{Vol}_{g_E}(C_\alpha S^n_{l\sin\alpha})}}\int_{C_\alpha S^n_{l\sin\alpha}}*1c^1_0=\sqrt{\mathrm{Vol}_{g_E}(C_\alpha S^n_{l\sin\alpha})}c^1_0.$$

As $b_q=\emptyset$ for $q=0,\ldots,n$, $b^1_{n+1}=c^1_{n+1}$ and $\partial(b^1_{n+1})=c_n$, we have that

$$|\det(z_0/c_0)|=\sqrt{\mathrm{Vol}_{g_E}(C_\alpha S^n_{l\sin\alpha})},\quad |\det(\partial(b^1_{n+1})/c_n)|=1,\quad |\det(b_{n+1}/c_{n+1})|=1.$$

Applying the definition in equation (2), this proves the following result.

Proposition 2.

$$\tau_{\rm R}((C_\alpha S^n_{l\sin\alpha},g_E);\rho)=\left(\sqrt{\mathrm{Vol}_{g_E}(C_\alpha S^n_{l\sin\alpha})}\right)^{{\rm rk}(\rho)}.$$

4. The anomaly boundary term and the analytic torsion of a disc

The aim of this section is to give the proof of Theorem 2. For we need a formula for the ratio between the analytic torsion and the Reidemeister torsion, that we call anomaly boundary contribution. Observe that, by result of Cheeger [4], this ratio depends only on some geometric terms coming from the geometry of the manifold near the boundary. Since the singularity at the tip of the cone does not affect the geometry near the boundary, we are allowed to perform our calculation for the general case of the cone $C_\alpha S^n_{l\sin\alpha}$. However, this would imply some technical difficulties that are beside the aim of the present work. As observed in the introduction, two different formulas for this anomaly are available at the moment. One is given in Theorem 1 of [8], and the second one comes from Theorem 1 of [2]. We first proceed to evaluate the anomaly boundary contribution using the formula of [2]. Then, at the end of the section, we will describe the contribution appearing using the formula of [8] (see also the Appendix of [16] for other examples). We proceed in two steps. First we give in Lemma 1 formulas for the anomaly in terms of some geometric invariants. This follows directly from Theorem 1 of [2], and gives, in the odd dimensional case, the anomaly in terms of the Euler characteristic of the boundary. The even case is harder, and needs the introduction of some machinery and notation from [2] and [1]. This is done in the course of the proof, and, as a result, the anomaly in the even case is written as some integral. The second step is accomplished in Lemma 2, where we give all the geometric invariants necessary to compute the integral appearing in the formula obtained in Lemma 1, and we conclude the calculation for the even case.

Before to start, we need some notation, that will be used without further comments in this section. The parameterization of the cone and the induced metric g_E are given in Section 3. Define the metrics

$$g_1 = g_E = dr\otimes dr + r^2 g_{S^n}, \qquad\qquad g_0 = dr\otimes dr + l^2 g_{S^n}.$$

Let ω_j, $j=0,1$ be the connection one forms associated to g_j, and $\Omega_j = d\omega_j + \omega_j\wedge\omega_j$ the curvature two forms. Let $e(W,g)$ denotes the Euler class of (W,g).

Lemma 1. *The ratio of the analytic torsion and the Reidemeister torsion of the disc D^m_l, of $m=2p-1$ odd dimension, and $m=2p$ even dimension ($p>0$), with absolute boundary conditions are, respectively:*

$$\log\frac{T_{\rm abs}((D_l^{2p-1},g_E);\rho)}{\tau_{\rm R}((D_l^{2p-1},g_E);\rho)} = \frac{1}{4}{\rm rk}(\rho)\chi(S_l^{2p-2},g_E)\left(\log 2+\frac{1}{2}\sum_{n=1}^{p-1}\frac{1}{n}\right),$$

$$\log\frac{T_{\rm abs}((D_l^{2p},g_E);\rho)}{\tau_{\rm R}((D_l^{2p},g_E);\rho)} = \frac{1}{2}{\rm rk}(\rho)\frac{2^{p-1}}{\sqrt{\pi}(2p-1)!!}\sum_{j=1}^{p}\frac{1}{2j-1}\int_{S_l^{2p-1}}\int^B \mathcal{S}_1^{2p-1}.$$

Proof. The proof is based on Theorem 0.1 of [2]. Note that we are in the particular case of the flat trivial bundle F, since the unique representations are the trivial ones. Therefore, we have from equation (0.6) and Section 4.1 of [2],

$$\log\frac{T_{\rm abs}((D_l^m,g_1);\rho)}{T_{\rm abs}((D_l^m,g_0);\rho)} = \frac{1}{2}{\rm rk}(\rho)\int_{S_l^{m-1}}\left(B(\nabla_1^{TD_l^m})-B(\nabla_0^{TD_l^m})\right), \tag{5}$$

where the forms $B(\nabla_j^{TX})$ are defined in equation (1.17) of [2] (see equation (8) below, and observe that we take the opposite sign with respect to the definition in [2], since we are considering left actions instead of right actions). Since g_0 is a product near the boundary, by the results of [15]

$$\log\frac{T_{\rm abs}((D_l^m,g_0);\rho)}{\tau_{\rm R}((D_l^m,g_E);\rho)} = \frac{1}{4}{\rm rk}(\rho)\chi(S^{m-1},g_E)\log 2,$$

and it just remains to evaluate the anomaly boundary term, on the right side of equation (5). For we first recall some notation from [1] Chapter III and [2] Section 1.1. For two $\mathbb{Z}/2$-graded algebras $\mathcal{A}$ and $\mathcal{B}$, let $\mathcal{A}\hat\otimes\mathcal{B}=\mathcal{A}\wedge\hat{\mathcal{B}}$ denotes the $\mathbb{Z}/2$-graded tensor product. For two real finite dimensional vector spaces V and E, of dimension m and n, with E Euclidean and oriented, the Berezin integral is the linear map

$$\int^B : \Lambda V^* \hat{\otimes} \Lambda E^* \to \Lambda V^*,$$

$$\int^B : \alpha \hat{\otimes} \beta \mapsto \frac{(-1)^{\frac{n(n+1)}{2}}}{\pi^{\frac{n}{2}}} \beta(e_1, \dots, e_n)\alpha,$$

where $\{e_j\}_{j=1}^n$ is an orthonormal base of E. Let A be an antisymmetric endomorphism of E. Consider the map

$$\hat{}: A \mapsto \hat{A} = \frac{1}{2} \sum_{j,l=1}^{n} (e_j, Ae_l) \hat{e}^j \wedge \hat{e}^l.$$

Note that

$$\int^B \mathrm{e}^{-\frac{\hat{A}}{2}} = Pf\left(\frac{A}{2\pi}\right), \tag{6}$$

and this vanishes if $\dim E = n$ is odd.

Let ω_j be the connection one form over D_l^m associated to the metric g_j, and Ω_j the curvature two form. Let Θ be the curvature two form of the boundary S^{m-1} (with radius 1) with the standard Euclidean metric. Let $(\omega_j)^a{}_b$ denotes the entries with line a and column b of the matrix of one forms ω_j. Then, introduce the following quantities (see [2] equations (1.8) and (1.15)), where $i: S_l^{m-1} \to D_l^m$ denotes the inclusion of the boundary,

$$\mathcal{S}_j = \frac{1}{2} \sum_{k=1}^{m-1} (i^*\omega_j - i^*\omega_0)^r{}_{\theta_k} \hat{e}^{\theta_k}, \quad \mathcal{R} = \hat{\Theta} = \frac{1}{2} \sum_{k,l=1}^{m-1} \Theta^{\theta_k}{}_{\theta_l} \hat{e}^{\theta_k} \wedge \hat{e}^{\theta_l}. \tag{7}$$

Then, we define

$$B(\nabla_j^{TD_l^m}) = \frac{1}{2} \int_0^1 \int^B \mathrm{e}^{-\frac{1}{2}\mathcal{R} - u^2 \mathcal{S}_j^2} \sum_{k=1}^{\infty} \frac{1}{\Gamma\left(\frac{k}{2}+1\right)} u^{k-1} \mathcal{S}_j^k du. \tag{8}$$

From this definition it follows that $B(\nabla_0^{TD_l^m})$ vanishes identically, since $\mathcal{S}_0$ does. It remains to evaluate $B(\nabla_1^{TD_l^m})$. For note that by equation (1.16) of [2] (or by direct calculation, since the curvature of the disc is null) $\mathcal{R} = -2\mathcal{S}_1^2$. Therefore, equation (8) gives

$$\begin{aligned} B(\nabla_1^{TD_l^m}) &= \frac{1}{2} \int_0^1 \int^B \mathrm{e}^{(1-u^2)\mathcal{S}_1^2} \sum_{k=1}^{\infty} \frac{1}{\Gamma\left(\frac{k}{2}+1\right)} u^{k-1} \mathcal{S}_1^k du \\ &= \frac{1}{2} \int^B \sum_{j=0,k=1}^{\infty} \frac{1}{j!\Gamma\left(\frac{k}{2}+1\right)} \int_0^1 (1-u^2)^j u^{k-1} du \mathcal{S}_1^{k+2j} \\ &= \frac{1}{2} \sum_{j=0,k=1}^{\infty} \frac{1}{k\Gamma\left(\frac{k}{2}+j+1\right)} \int^B \mathcal{S}_1^{k+2j}. \end{aligned}$$

Since the Berezin integral vanishes identically if $k + 2j \neq m - 1$, we obtain

$$B(\nabla_1^{TD_l^m}) = \frac{1}{2\Gamma\left(\frac{m+1}{2}\right)} \sum_{j=0}^{\left[\frac{m}{2}-1\right]} \frac{1}{m-2j-1} \int^B \mathcal{S}_1^{m-1}. \tag{9}$$

Now consider the two cases of even and odd m independently. First, assume $m = 2p+1$ ($p \geq 0$). Then, using equation (6), equation (9) gives

$$B(\nabla_1^{TD_l^{2p+1}}) = \frac{1}{2p!}\sum_{j=0}^{[p-\frac{1}{2}]}\frac{1}{2p-2j}\int^B \mathcal{S}_1^{2p} = \frac{1}{4}\sum_{n=1}^{p}\frac{1}{n}\int^B \mathrm{e}^{-\frac{\hat{\Theta}}{2}} = \frac{1}{4}\sum_{n=1}^{p}\frac{1}{n}Pf\left(\frac{\Theta}{2\pi}\right)$$
$$= \frac{1}{4}\sum_{n=1}^{p}\frac{1}{n}e(S^{2p}, g_E),$$

where $e(S^{2p}, g_E)$ is the Euler class of (S^{2p}, g_E), and we use the fact that

$$e(S_l^{2p}, g_l) = Pf\left(\frac{\Theta}{2\pi}\right) = \int^B \mathrm{e}^{-\frac{\hat{\Theta}}{2}}.$$

Therefore,

$$\log\frac{T_{\mathrm{abs}}((D_l^m, g_1);\rho)}{T_{\mathrm{abs}}((D_l^m, g_0);\rho)} = \frac{1}{2}\mathrm{rk}(\rho)\int_{S_l^{m-1}} B(\nabla_1^{TD_l^m}) = \frac{1}{8}\mathrm{rk}(\rho)\sum_{n=1}^{p}\frac{1}{n}\int_{S_l^{2p}} e(S^{2p}, g_E)$$
$$= \frac{1}{8}\mathrm{rk}(\rho)\sum_{n=1}^{p}\frac{1}{n}\chi(S^{2p}, g_E).$$

Second, assume $m = 2p$ ($p \geq 1$). Then, equation (9) gives

$$B(\nabla_1^{TD_l^{2p}}) = \frac{1}{2\Gamma(p+\frac{1}{2})}\sum_{j=0}^{p-1}\frac{1}{2p-2j-1}\int^B \mathcal{S}_1^{2p-1}$$
$$= \frac{2^{p-1}}{\sqrt{\pi}(2p-1)!!}\sum_{j=1}^{p}\frac{1}{2j-1}\int^B \mathcal{S}_1^{2p-1},$$

and substitution in equation (5) gives the formula stated in the Lemma. □

Lemma 2. *We have*

$$\frac{2^{p-1}}{\sqrt{\pi}(2p-1)!!}\int_{S_l^{2p-1}}\int^B \mathcal{S}_1^{2p-1} = 1.$$

Proof. First, we determine the connection one forms for the metric g_1 and g_0. We define the Christoffel symbols accordingly to $\nabla_{e_\alpha} e_\beta = \Gamma_{\alpha}{}^{\gamma}{}_{\beta} e_\gamma$, where $\{e_\alpha\}$ is an orthonormal base, and we use the formula

$$\Gamma_{\alpha}{}^{\gamma}{}_{\beta} = \frac{c_{\alpha\beta}{}^{\gamma} + c_{\gamma\alpha}{}^{\beta} + c_{\gamma\beta}{}^{\alpha}}{2}, \tag{10}$$

where the Cartan structure constant are defined by $[e_\alpha, e_\beta] = c_{\alpha\beta}{}^\gamma e_\gamma$. The orthonormal base and its dual with respect to g_1 are:

$$
\begin{aligned}
e_r &= \frac{\partial}{\partial r}, & e^r &= dr,\\
e_{\theta_1} &= (r\prod_{j=2}^n \sin\theta_j)^{-1}\frac{\partial}{\partial\theta_1}, & e^{\theta_1} &= r\prod_{j=2}^n \sin\theta_j d\theta_1,\\
&\vdots & &\vdots\\
e_{\theta_{n-1}} &= (r\sin\theta_n)^{-1}\frac{\partial}{\partial\theta_{n-1}}, & e^{\theta_{n-1}} &= r\sin\theta_n d\theta_{n-1},\\
e_{\theta_n} &= \frac{1}{r}\frac{\partial}{\partial\theta_n}, & e^{\theta_n} &= r d\theta_n.
\end{aligned}
$$

This gives $c_{\alpha\beta}{}^\gamma = -c_{\beta\alpha}{}^\gamma$, $c_{\alpha\alpha}{}^\gamma = 0, \forall\alpha,\gamma$, and the non zero are: $c_{\theta_i r}{}^{\theta_i} = r^{-1}$, and if $k > i$, $c_{\theta_i\theta_k}{}^{\theta_i} = \frac{\cos\theta_k}{r\prod_{j=k}^n \sin\theta_j}$. Using equation (10), the non zero Christoffel symbols are: $\Gamma_{\theta_i}{}^r{}_{\theta_i} = -\frac{1}{r}$, $\Gamma_{\theta_i}{}^{\theta_i}{}_r = \frac{1}{r}$, $\Gamma_{\theta_i}{}^{\theta_i}{}_{\theta_s} = \frac{\cos\theta_s}{r\prod_{j=s}^n \sin\theta_j}$, with $s > i$, and $\Gamma_{\theta_s}{}^{\theta_i}{}_{\theta_s} = \frac{-\cos\theta_i}{r\prod_{j=i}^n \sin\theta_j}$, with $i > s$. The connection one form is the matrix $\omega_1 = \Gamma_\gamma{}^\alpha{}_\beta e^\gamma$, with non zero entries

$$
(\omega_1)^{\theta_i}{}_{\theta_k} = \frac{\cos\theta_k}{r\prod_{j=k}^n \sin\theta_j} e^{\theta_i},\ i < k, \qquad (\omega_1)^r{}_{\theta_i} = -\frac{1}{r}e^{\theta_i}.
$$

The orthonormal base and its dual with respect to g_0 are:

$$
\begin{aligned}
e_r &= \frac{\partial}{\partial r} & e^r &= dr,\\
e_{\theta_1} &= (l\prod_{j=2}^n \sin\theta_j)^{-1}\frac{\partial}{\partial\theta_1}, & e^{\theta_1} &= (l\prod_{j=2}^n \sin\theta_j) d\theta_1,\\
&\vdots & &\vdots\\
e_{\theta_{n-1}} &= (l\sin\theta_n)^{-1}\frac{\partial}{\partial\theta_{n-1}}, & e^{\theta_{n-1}} &= (l\sin\theta_n) d\theta_{n-1},\\
e_{\theta_n} &= \frac{1}{l}\frac{\partial}{\partial\theta_n}, & e^{\theta_n} &= l d\theta_n.
\end{aligned}
$$

The non zero Cartan constants are: $c_{\theta_i\theta_k}{}^{\theta_i} = \frac{\cos\theta_k}{la\prod_{j=k}^n \sin\theta_j}$, with $k > i$. Using equation (10), the non zero Christoffel symbols are: $\Gamma_{\theta_i}{}^{\theta_i}{}_{\theta_s} = \frac{\cos\theta_s}{l\prod_{j=s}^n \sin\theta_j}$, when $s > i$, and $\Gamma_{\theta_s}{}^{\theta_i}{}_{\theta_s} = -\frac{\cos\theta_i}{l\prod_{j=i}^n \sin\theta_j}$, when $i > s$. The non zero entries of the connection one form matrix are

$$
(\omega_0)^{\theta_i}{}_{\theta_s} = \frac{\cos\theta_s}{l\prod_{j=s}^n \sin\theta_j} e^{\theta_i},\ i < s.
$$

It follows that the unique non zero entries of $\omega_1 - \omega_0$ are

$$
(\omega_1 - \omega_0)^r{}_{\theta_i} = -\frac{1}{r}e^{\theta_i} = -\prod_{j=i+1}^n \sin\theta_j d\theta_i.
$$

Second, we determine the curvature two form Θ. Since g_0 is a product metric, Θ is the restriction of Ω_0, and hence we compute $\Omega_0 = d\omega_0 + \omega_0 \wedge \omega_0$. We write ω_0 in the coordinate base

$$(\omega_0)^r{}_{\theta_i} = 0, \qquad (\omega_0)^{\theta_i}{}_{\theta_s} = \cos\theta_s \prod_{j=i+1}^{s-1} \sin\theta_j d\theta_i, \qquad i < s,$$

and hence $d\omega_0$ is

$$(d\omega_0)^r{}_{\theta_i} = 0, \qquad i \leq n,$$

$$(d\omega_0)^{\theta_i}{}_{\theta_k} = \prod_{j=i+1}^{k} \sin\theta_j d\theta_i \wedge d\theta_k - \sum_{s=i+1}^{k-1} \cos\theta_k \cos\theta_s \prod_{\substack{j=i+1,\\ j\neq s}}^{k-1} \sin\theta_j d\theta_i \wedge d\theta_s, \quad i < k,$$

and $\omega_0 \wedge \omega_0$ is

$$(\omega_0 \wedge \omega_0)^\alpha{}_\alpha = (\omega_0 \wedge \omega_0)^r{}_{\theta_i} = 0,$$

$$(\omega_0 \wedge \omega_0)^{\theta_i}{}_{\theta_k} = \sum_{s=i+1}^{k-1} \cos\theta_s \cos\theta_k \prod_{\substack{j=i+1,\\ j\neq s}}^{k-1} \sin\theta_j d\theta_i \wedge d\theta_s + \prod_{j=i+1}^{k} \sin\theta_j \left(\prod_{s=k+1}^{n} \sin^2\theta_s - 1 \right) d\theta_i \wedge d\theta_k, \qquad i < k.$$

Then, the curvature two form Ω_0 is

$$(\Omega_0)^r{}_{\theta_i} = 0, \qquad (\Omega_0)^{\theta_i}{}_{\theta_k} = \prod_{j=i+1}^{k} \sin\theta_j \prod_{s=k+1}^{n} \sin^2\theta_s d\theta_i \wedge d\theta_k, \qquad i < k,$$

and consequently $\Theta = i^*\Omega_0$ (where i denotes the inclusion of the boundary) is

$$\Theta^{\theta_i}{}_{\theta_k} = \prod_{j=i+1}^{k} \sin\theta_j \prod_{s=k+1}^{n} \sin^2\theta_s d\theta_i \wedge d\theta_k, \qquad i < k.$$

Third, recalling that $\mathcal{S}_1^2 = -\frac{1}{2}\mathcal{R}$,

$$\int^B \mathcal{S}_1^{2p-1} = \frac{(-1)^{p-1}}{2^{p-1}} \int^B \mathcal{S}_1 \mathcal{R}^{p-1},$$

and using the definitions in equation (7)

$$\int^B \mathcal{S}_1^{2p-1} = \frac{(-1)^{p-1}}{2^{2p-1}} \int^B \left(\sum_{k=1}^{2p-1} (\omega_1 - \omega_0)^r{}_{\theta_k} \hat{e}^{\theta_k} \right) \left(\sum_{i,j=1}^{2p-1} (\Omega_0)^{\theta_i}{}_{\theta_j} \hat{e}^{\theta_i} \wedge \hat{e}^{\theta_j} \right)^{p-1}$$

$$= \frac{(-1)^{p-1}}{2^{p-1}2^p} c_B \left(\sum_{\substack{\sigma\in S_{2p}\\ \sigma(1)=1}} \operatorname{sgn}(\sigma) (\omega_1 - \omega_0)^1{}_{\sigma(2)} (\Omega_0)^{\sigma(3)}{}_{\sigma(4)} \dots (\Omega_0)^{\sigma(2p-1)}{}_{\sigma(2p)} \right),$$

where $c_B = \frac{(-1)^{p(2p-1)}}{\pi^{\frac{2p-1}{2}}}$. Observe that $(\omega_1 - \omega_0)^1{}_{\sigma(2)}$ is a 1-form multiple of $d\theta_{\sigma(2)-1}$ and $(\Omega_0)^i{}_j$ is a 2-form multiple of $d_{\theta_{i-1}} \wedge d_{\theta_{j-1}}$. We can twist all the 2-forms $d_{\theta_{i-1}} \wedge d_{\theta_{j-1}}$, with $i > j$ in each term appearing in the last line of the equation above, as the matrix is skew-symmetric. Then, we can order the base element, in such a way that the top form appears in each term. This produces a sign coinciding with $\mathrm{sgn}(\sigma)$. Moreover, since the matrix of the curvature two form is skew-symmetric, the generic term in the sum in the last line of the above equation can be written in the following form

$$[\omega_1-\omega_0]^1{}_{\sigma(2)}[\Omega_0]^{\sigma(3)}{}_{\sigma(4)}\dots[\Omega_0]^{\sigma(2p-1)}{}_{\sigma(2p)}d\theta_1\wedge\dots\wedge d\theta_{2p-1},$$

where $[\xi]^i{}_j$ denotes the coefficient of the form $(\xi)^i{}_j$, and $\sigma\in S_{2p}$ is such that $\sigma(1)=1$ and $\sigma(2s-1)<\sigma(2s)$ for all s. We prove that

$$[\omega_1-\omega_0]^1{}_{\sigma(2)}[\Omega_0]^{\sigma(3)}{}_{\sigma(4)}\dots[\Omega_0]^{\sigma(2p-1)}{}_{\sigma(2p)}=-\prod_{i=2}^{2p}(\sin\theta_{\sigma(i)})^{\sigma(i)-1},$$

where $\sin\theta_{2p}=1$. The proof is by induction. If $p=1$ the equality holds trivially. Suppose it is true for $p-1$. By hypothesis, if $\tau\in S_{2p-2}$ with $\tau(1)=1$, then

$$[\omega_1-\omega_0]^1{}_{\tau(2)}[\Omega_0]^{\tau(3)}{}_{\tau(4)}\dots[\Omega_0]^{\tau(2p-3)}{}_{\tau(2p-2)}=-\prod_{i=2}^{2p-2}(\sin\theta_{\tau(i)})^{\tau(i)-1}.$$

Take $\sigma\in S_{2p}$ with $\sigma(1)=1$. It is clear that there are k_0,k_1,k_2 such that $\sigma(k_0)=2p-2$, $\sigma(k_1)=2p-1$ and $\sigma(k_2)=2p$. Factoring $\sin\theta_{\sigma(k_i)}$, $i=0,1,2$:

$$\begin{aligned}&[\omega_1-\omega_0]^1{}_{\sigma(2)}[\Omega_0]^{\sigma(3)}{}_{\sigma(4)}\dots[\Omega_0]^{\sigma(2p-1)}{}_{\sigma(2p)}\\&\qquad=(\sin\theta_{2p-2})^{2p-3}(\sin\theta_{2p-1})^{2p-2}\times\text{factor},\end{aligned}$$

where 'factor' is a product of $\sin\theta_j$, $j\neq\sigma(k_0),\sigma(k_1),\sigma(k_2)$. In this way we can rewrite 'factor' indexing it by a permutation $\tau\in S_{2p-2}$ such that the induction hypothesis holds. Then,

$$[\omega_1-\omega_0]^1{}_{\sigma(2)}[\Omega_0]^{\sigma(3)}{}_{\sigma(4)}\dots[\Omega_0]^{\sigma(2p-1)}{}_{\sigma(2p)}=-\prod_{j=2}^{2p}(\sin\theta_{\sigma(j)})^{\sigma(j)-1}.$$

We have proved that

$$\int^B\mathcal{S}_1^{2p-1}=c_B\frac{(-1)^p(2p-1)!}{2^{p-1}2^p}\prod_{j=2}^{2p-1}(\sin\theta_j)^{j-1}d\theta_1\wedge\dots\wedge d\theta_{2p-1}.$$

Then,

$$\begin{aligned}\frac{2^{p-1}}{\sqrt{\pi}(2p-1)!!}\int_{S_l^{2p-1}}\int^B\mathcal{S}_1^{2p-1}&=c_B\frac{2^{p-1}}{\sqrt{\pi}(2p-1)!!}\frac{(-1)^p(2p-1)!}{2^{p-1}2^pl^{2p-1}}\mathrm{Vol}(S_l^{2p-1})\\&=\frac{(2p-1)!\sqrt{\pi}}{2^{p-1}(p-1)!\sqrt{\pi}(2p-1)!!}.\end{aligned}$$

It is easy to see that

$$\frac{(2p-1)!}{(p-1)!(2p-1)!!}=2^{p-1},$$

and the thesis follows. □

Remark 1. *In the case $m=2$, namely the 2 dimensional disc D^2_l, the proof of Lemma 1 extends to the case of the cone $C_\alpha S^1_{l\sin\alpha}$. For the curvature of the cone vanishes identically in this case. Therefore we have that*

$$\log\frac{T_{\rm abs}((C_\alpha S^1_{l\sin\alpha},g_E);\rho)}{\tau_{\rm R}((C_\alpha S^1_{l\sin\alpha},g_E);\rho)}=\frac{1}{2}{\rm rk}(\rho)\frac{1}{\sqrt{\pi}}\int^B\mathcal{S}_1.$$

The integral can be evaluated proceeding as in the proof of Lemma 2, and we obtain

$$\log\frac{T_{\rm abs}((C_\alpha S^1_{l\sin\alpha},g_E);\rho)}{\tau_{\rm R}((C_\alpha S^1_{l\sin\alpha},g_E);\rho)}=\frac{1}{2}{\rm rk}(\rho)\sin\alpha.$$

We conclude this section by computing the anomaly boundary term using the formula given in Theorem 1 of [8]. In the even dimensional case we give the result for the more general case of the cone over the sphere. We need some more notation. Consider the homotopy $\omega_t=\omega_0+t(\omega_1-\omega_0)$, and let $\Omega_t=d\omega_t+\omega_t\wedge\omega_t$ be the corresponding curvature two form. The Chern-Simons class associated to the Euler class of Ω_t will be denoted by $\tilde e(g_0,g_1)$, and satisfies $d\tilde e(g_0,g_1)=e(g_1)-e(g_0)$, where $e(g_j)$ is the Euler class of Ω_j. Then, it is easy to see that Theorem 1 of [8] gives the following formulas:

$$\log\frac{T_{\rm abs}((D^{2p-1}_l,g_E);\rho)}{\tau_{\rm R}((D^{2p-1}_l,g_E);\rho)}=\frac{1}{4}{\rm rk}(\rho)\chi(S^{2p-2}_l,g_E)\log 2, \tag{11}$$

$$\log\frac{T_{\rm abs}((C_\alpha S^{2p-1}_{l\sin\alpha},g_E);\rho)}{\tau_{\rm R}((C_\alpha S^{2p-1}_{l\sin\alpha},g_E);\rho)}=\frac{1}{2}{\rm rk}(\rho)\int_{S^{2p-1}_{l\sin\alpha}}i^*\tilde e(g_0,g_E), \tag{12}$$

where i denotes the inclusion of the boundary. Proceeding in very similar way as in the proof of Lemma 2, we compute the integral appearing in equation (12). We obtain

$$\int_{S^{2p-1}_{l\sin\alpha}}i^*\tilde e(g_0,g_E)=\frac{(-1)^{p+1}(2p)!{\rm Vol}(S^{2p-1}_{l\sin\alpha})}{(4\pi)^p l^{2p-1}p!(\sin\alpha)^{2p-2}}\sum_{k=0}^{p-1}(-1)^k\frac{(\sin\alpha)^{2k}}{2k+1}\binom{p-1}{k},$$

and in particular for the disc ($\sin\alpha=1$)

$$\int_{S^{2p-1}_l}i^*\tilde e(g_0,g_E)=(-1)^{p+1}.$$

5. The analytic torsion of $C_\alpha S^1_{l\sin\alpha}$ and $C_\alpha S^2_{l\sin\alpha}$

In this section we compute the analytic torsion of the cones $C_\alpha S^1_{l\sin\alpha}$ and $C_\alpha S^2_{l\sin\alpha}$ by using the definition given in equation (4). For we need first the explicit knowledge of the spectrum of the Laplace operators on forms on these singular spaces, and second a suitable representation for the analytic extension of the associated zeta function, that allows to evaluate the derivative at zero. The first aspect of the problem was originally addressed by Cheeger in [6] (see also [24]). In the work of Cheeger, the Hodge-de Rham theory is developed for spaces with singularity of conical type. In particular, it is proved that the Laplacian on forms is a non negative self adjoint operator on the space of square integrable forms on the cone, if some set of appropriate boundary conditions at the tip of the cone are used. We recall this point briefly in the following Remark 2. We give the spectrum of $\Delta^{(q)}$ on $C_\alpha S^1_{l\sin\alpha}$ and on $C_\alpha S^2_{l\sin\alpha}$ in Lemma 3 and Lemma 4 below, respectively. Next, to deal with the second aspect, namely an analytic extension of the zeta functions and a method to evaluating the derivative at zero, we use a method introduced by Spreafico to deal with the zeta invariants of an abstract class of double zeta functions. In fact, the eigenvalues of $\Delta^{(q)}_{C_\alpha S^n_{l\sin\alpha}}$ can be identified with the zero $z_{\nu,k}$ of some combination of Bessel functions and their derivatives, and be enumerated with two positive indices as $\lambda^{(q)}_{n,k} = z^2_{u_n,k}$, where the u_n depends on the eigenvalues of the the Laplacian on some space of q-forms on the section of the cone. Using classical estimates for the zeros of the Bessel functions it is possible to prove that the relevant sequences U and S are contained in the class of abstract sequences introduced in [25] [27]. This means that we can use the method of [24] [26] [27], to evaluate the derivative at zero of the associated zeta functions. In particular, we will use the notation and the formula as given in Section 5 of [12], and all the reference of the following Sections 5.2 and 5.3 are to that paper.

5.1. **Spectrum of the Laplacian on forms.** In this section we compute the spectrum of the Laplacian on forms. The general form of the solutions of the eigenvalues equation are given in [6] and [7]. However, we present here the explicit form of the solutions in the case under study and some details on the calculation, that we were not able to find elsewhere. Furthermore we give, in the course of the proofs, the complete set of the eigenforms of the Laplace operator. We give a more detailed proof for the case of the circle. We denote by $\{k : \lambda\}$ the set of eigenvalues λ with multiplicity k.

Remark 2. *Decomposing with respect to the projections on the eigenspaces of the restriction of the Laplacian on the section of the cone (i.e. with respect to the angular momenta), the definition of an appropriate self adjoint extension of the Laplace operator (on functions) on a cone reduces to the analysis of the boundary values of a singular Sturm-Liouville ordinary second order differential equation. The problem was addressed already by Rellich in* [22]*, who parameterized the self adjoint extensions. In particular, it turns out that there are not boundary values for the non zero mode of the angular momentum, while a boundary condition is necessary for the zero modes, and the unique self adjoint extension defined by this boundary condition is the maximal extension, corresponding to the Friedrich extension (see* [3] *or* [6] *for the boundary condition). The same argument works for the Laplacian on forms. However, in the present situation we do not actually need boundary values for forms of positive degree, since the middle homology of the section of the cone is trivial (compare with* [5]*).*

Lemma 3. *The spectrum of the (Friedrich extension of the) Laplacian operator $\Delta^{(q)}_{C_\alpha S^1_{l\sin\alpha}}$ on q-forms with absolute boundary conditions is (where $\nu = \mathrm{cosec}\alpha$):*

$$
\begin{aligned}
\mathrm{Sp}\Delta^{(0)}_{C_\alpha S^1_{l\sin\alpha}} &= \left\{j^2_{1,k}/l^2\right\}_{k=1}^{\infty} \cup \left\{2 : (j'_{\nu n,k})^2/l^2\right\}_{n,k=1}^{\infty}, \\
\mathrm{Sp}\Delta^{(1)}_{C_\alpha S^1_{l\sin\alpha}} &= \left\{j^2_{0,k}/l^2\right\}_{k=1}^{\infty} \cup \left\{j^2_{1,k}/l^2\right\}_{k=1}^{\infty} \cup \left\{2 : j^2_{\nu n,k}/l^2\right\}_{n,k=1}^{\infty} \\
&\quad \cup \left\{2 : (j'_{\nu n,k})^2/l^2\right\}_{n,k=1}^{\infty}, \\
\mathrm{Sp}\Delta^{(2)}_{C_\alpha S^1_{l\sin\alpha}} &= \left\{j^2_{0,k}/l^2\right\}_{k=1}^{\infty} \cup \left\{2 : j^2_{\nu n,k}/l^2\right\}_{n,k=1}^{\infty}.
\end{aligned}
$$

The spectrum with relative boundary conditions is:

$$
\begin{aligned}
\mathrm{Sp}\Delta^{(0)}_{C_\alpha S^1_{l\sin\alpha}} &= \left\{j^2_{0,k}/l^2\right\}_{k=1}^{\infty} \cup \left\{2 : j^2_{\nu n,k}/l^2\right\}_{n,k=1}^{\infty},\\
\mathrm{Sp}\Delta^{(1)}_{C_\alpha S^1_{l\sin\alpha}} &= \left\{j^2_{0,k}/l^2\right\}_{k=1}^{\infty} \cup \left\{j^2_{1,k}/\, l^2\right\}_{k=1}^{\infty} \cup \left\{2 : j^2_{\nu n,k}/l^2\right\}_{n,k=1}^{\infty}\\
&\quad \cup \left\{2 : (j'_{\nu n,k})^2/l^2\right\}_{n,k=1}^{\infty},\\
\mathrm{Sp}\Delta^{(2)}_{C_\alpha S^1_{l\sin\alpha}} &= \left\{j^2_{1,k}/l^2\right\}_{k=1}^{\infty} \cup \left\{2 : (j'_{\nu n,k})^2/l^2\right\}_{n,k=1}^{\infty}.
\end{aligned}
$$

Proof. Recall we parameterize $C_\alpha S^1_{l\sin\alpha}$ by

$$
C_\alpha S^1_{l\sin\alpha} = \begin{cases} x_1 = x\sin\alpha\cos\theta\\ x_2 = x\sin\alpha\sin\theta\\ x_3 = x\cos\alpha \end{cases}
$$

where $(x,\theta)\in[0,l]\times[0,2\pi]$, l and α are fixed positive real numbers and $0 < a = \frac{1}{\nu} = \sin\alpha \leq 1$. The induced metric is

$$
g = dx\otimes dx + a^2x^2 d\theta\otimes d\theta,
$$

and the Hodge operator is

$$
*: 1 \mapsto ax dx\wedge d\theta; \quad *: dx \mapsto ax d\theta, \quad *: d\theta \mapsto -\frac{1}{ax}dx; \quad *: dx\wedge d\theta \mapsto \frac{1}{ax}.
$$

The Laplacian on forms is

$$
\begin{aligned}
\Delta^{(0)}(f) &= -\partial_x^2 f - \frac{1}{x}\partial_x f - \frac{1}{a^2x^2}\partial_\theta^2 f,\\
\Delta^{(1)}(f_x dx + f_\theta d_\theta) &= \left(-\partial_x^2 f_x - \frac{1}{a^2x^2}\partial_\theta^2 f_x + \frac{1}{x^2}f_x - \frac{1}{x}\partial_x f_x + \frac{2}{a^2x^3}\partial_\theta f_\theta\right)dx\\
&\quad + \left(-\partial_x^2 f_\theta - \frac{1}{a^2x^2}\partial_\theta^2 f_\theta + \frac{1}{x}\partial_x f_\theta - \frac{2}{x}\partial_\theta f_x\right)d\theta,\\
\Delta^{(2)}(f dx\wedge d\theta) &= -\partial_x^2 f + \frac{1}{x}\partial_x f - \frac{2}{x^2}f - \frac{1}{a^2x^2}\partial_\theta^2 f.
\end{aligned}
$$

Decomposing forms near the boundary into tangent and normal components, the BC gives the following set of equations. For the 0-forms:

$$
\text{rel.}: \quad \omega(l,\theta) = 0, \qquad \text{abs.}: \quad (\partial_x\omega)(l,\theta) = 0, \tag{13}
$$

and relative BC coincide with Dirichlet BC. For 2-forms

$$
\text{abs.}: \quad \omega(l,\theta) = 0, \qquad \text{abs.}: \quad \left(\partial_x\frac{\omega}{x}\right)(l,\theta) = 0, \tag{14}
$$

and absolute BC coincide with Dirichlet BC. For 1-forms:

$$
\text{abs.}: \begin{cases} \omega_x(l,\theta) = 0,\\ (\partial_x\omega_\theta)(l,\theta) = 0, \end{cases} \qquad \text{rel.}: \begin{cases} \omega_\theta(l,\theta) = 0,\\ \left(\partial_x(ax\omega_x) + \frac{1}{ax}\partial_\theta\omega_\theta\right)(l,\theta) = 0. \end{cases} \tag{15}
$$

Next, we solve the eigenvalues equations. Note that the Laplacian on 2-forms coincides with the one on 0-forms up to a Liouville transform $f = xh$. Consider the eigenvalues equation for the Laplacian on 0-forms

$$
\Delta^{(0)} f = \left(-\partial_x^2 - \frac{1}{x}\partial_x - \frac{1}{a^2x^2}\partial_\theta^2\right) f = \lambda^2 f. \tag{16}
$$

We can decompose the problem in the eigenspaces of $-\partial_\theta$. In fact, $\phi_n(\theta) = \mathrm{e}^{in\theta}$ is a complete system of eigenfunctions for $-d_\theta$ on the circle S^1, and the eigenvalue of ϕ_n is $\epsilon_n = n^2$, $n \in \mathbb{Z}$. Thus,

$$\Delta^{(0)} = \sum_{n\in\mathbb{Z}} L_n \Pi_n,$$

where Π_n is the projection onto the subspace generated by the eigenvector ϕ_n of the eigenspace relative to the eigenvalue ϵ_n and

$$L_n = -d_x^2 - \frac{1}{x} d_x + \frac{\nu^2 n^2}{x^2},$$

where $\nu = \frac{1}{a}$. Since $\epsilon_n = \epsilon_{-n}$, $\Pi_n = \Pi_{-n}$. Thus $-d_\theta$ has the complete system

$$\left\{\epsilon_n = n^2; \phi_{n,+}(\theta) = \mathrm{e}^{in\theta}, \phi_{n,-}(\theta) = \mathrm{e}^{-in\theta}\right\}_{n\in\mathbb{N}},$$

where all the eingenvalues are double up to $\epsilon_0 = 0$ that is simple; since $L_n = L_{-n}$,

$$\Delta^{(0)} = L_0 \Pi_0 \oplus \sum_{n=1}^{\infty} L_n (\Pi_{n,+} \oplus \Pi_{n,-}),$$

where $\Pi_{n,\pm}$ is the projection on the eigenspace generated by $\phi_{n,\pm}$ in the eigenspace of ϵ_n (in fact the eigenspace of ϵ_n is generated by the two eigenvectors $\phi_{n,\pm}$ for all $n \neq 0$). Now, we solve the eigenvalues equation for L_n on $L^2(0,l)$, namely

$$L_n u = \left(-d_x^2 - \frac{1}{x} d_x + \frac{\nu^2 n^2}{x^2}\right) u = \lambda_n^2 u. \tag{17}$$

This can be solved in terms of Bessel function. By classical result, equation (17) has the two linearly independent solutions (assume $\mu = \nu n$ is not an integer) $y_{\pm\mu}(x) = J_{\pm|\mu|}(\lambda_n x)$ (where we assume $\lambda_n > 0$). But $J_{-|\mu|}(x)$ diverges as $x^{-|\mu|}$ at $x = 0$, and therefore does not satisfy the BC at $x = 0$, or it is not in $L^2(0,l)$ (depending on the value of μ). Thus in any case we only have the solution y_+. This means that the eigenvalues equation (17) for L_n has the solution $\psi_n(x) = J_{|\nu n|}(\lambda_n x)$, for each $n \in \mathbb{Z}$; in particular it has solution $\psi_n(x) = J_{\nu n}(\lambda_n x)$, if $n \geq 0$, since $\nu \geq 0$. Therefore a system of linearly independent solutions of the eigenvalues equation (16) for $\Delta^{(0)}$ is

$$\begin{aligned}&\{\phi_0(\theta)\psi_0(x) = J_0(\lambda_0 x)\}\\ &\cup \left\{\phi_{n,+}(\theta)\psi_n(x) = \mathrm{e}^{in\theta} J_{\nu n}(\lambda_n x), \phi_{n,-}(\theta)\psi_n(x) = \mathrm{e}^{-in\theta} J_{\nu n}(\lambda_n x)\right\}_{n\in\mathbb{N}_0}.\end{aligned} \tag{18}$$

The solution for $\Delta^{(2)}$ are given by the inverse of the above Liouville transform,

$$\begin{aligned}&\{\phi_0(\theta)\psi_0(x) = x J_0(\lambda_0 x)\}\\ &\cup \left\{\phi_{n,+}(\theta)\psi_n(x) = x\mathrm{e}^{in\theta} J_{\nu n}(\lambda_n x), \phi_{n,-}(\theta)\psi_n(x) = x\mathrm{e}^{-in\theta} J_{\nu n}(\lambda_n x)\right\}_{n\in\mathbb{N}_0}.\end{aligned} \tag{19}$$

The eigenvalues equation for the Laplacian on 1-forms:

$$\Delta^{(1)}\omega = \lambda^2 \omega, \tag{20}$$

with $\omega = f_x dx + f_\theta d\theta$ corresponds to the system of partial differential equations

$$\begin{cases} -\partial_x^2 f_x - \dfrac{1}{x}\partial_x f_x + \dfrac{-\nu^2\partial_\theta^2 + 1}{x^2} f_x + \dfrac{2\nu^2}{x^3}\partial_\theta f_\theta = \lambda^2 f_x, \\ -\partial_x^2 f_\theta + \dfrac{1}{x}\partial_x f_\theta + \dfrac{-\nu^2\partial_\theta^2}{x^2} f_\theta - \dfrac{2}{x}\partial_\theta f_x = \lambda^2 f_\theta. \end{cases} \tag{21}$$

Since a base for $L^2(S^1)$ is given by the functions $e^{in\theta}$ with integer n, we consider solutions of the type $\omega = f_x(x)e^{im\theta}dx + f_\theta(x)e^{in\theta}d\theta$, with integers m and n. Substitution in equation (21) gives

$$\begin{cases} -\partial_x^2 f_x e^{im\theta} - \dfrac{1}{x}\partial_x f_x e^{im\theta} + \dfrac{(\nu m)^2+1}{x^2} f_x e^{im\theta} + \dfrac{2i\nu^2 n}{x^3} f_\theta e^{in\theta} = \lambda^2 f_x e^{im\theta}, \\ -\partial_x^2 f_\theta e^{in\theta} + \dfrac{1}{x}\partial_x f_\theta e^{in\theta} + \dfrac{(\nu n)^2}{x^2} f_\theta e^{in\theta} - \dfrac{2im}{x} f_x e^{im\theta} = \lambda^2 f_\theta e^{in\theta}, \end{cases}$$

that is satisfied if and only if $m = n$. Therefore, it follows that the solutions of equation (20) are of the form $\omega = e^{in\theta}(f_x(x)dx + f_\theta(x)d\theta)$, with $n \in \mathbb{Z}$, or in other words, that the operator $\Delta^{(1)}$ decomposes as

$$\Delta^{(1)} = \sum_{n\in\mathbb{Z}} L_n \Pi_n,$$

where

$$L_n = \begin{pmatrix} -d_x^2 - \frac{1}{x}d_x + \frac{(\nu n)^2+1}{x^2} & \frac{2i\nu^2 n}{x^3} \\ -\frac{2in}{x} & -d_x^2 + \frac{1}{x}d_x + \frac{(\nu n)^2}{x^2} \end{pmatrix},$$

on $(L^2(0,1))^2$, and Π_n is the projection onto the subspace generated by $e^{in\theta}$ of the eigenspace relative to the eigenvalue n^2 of $-d_\theta^2$. Therefore we need to solve the eigenvalues equation

$$L_n u = \lambda_n^2 u,$$

where $u = (f_x, f_\theta)$ are two functions in $L^2(0,l)$. This corresponds to the system

$$\begin{cases} -d_x^2 f_x - \dfrac{1}{x}d_x f_x + \dfrac{(\nu n)^2+1}{x^2} f_x + \dfrac{2i\nu^2 n}{x^3} f_\theta = \lambda_n^2 f_x, \\ -d_x^2 f_\theta + \dfrac{1}{x}d_x f_\theta + \dfrac{(\nu n)^2}{x^2} f_\theta - \dfrac{2in}{x} f_x = \lambda_n^2 f_\theta. \end{cases}$$

With the change of base $(f_x, f_\theta) = (\nu g_x, -ixg_\theta)$, we obtain

$$\begin{cases} \left(-d_x^2 - \dfrac{1}{x}d_x + \dfrac{(\nu n+1)^2}{x^2}\right)(g_x + g_\theta) = \lambda_n^2 (g_x + g_\theta), \\ \left(-d_x^2 - \dfrac{1}{x}d_x + \dfrac{(\nu n-1)^2}{x^2}\right)(g_x - g_\theta) = \lambda_n^2 (g_x - g_\theta). \end{cases} \tag{22}$$

By classical results on the solution of the Bessel equation, and taking only the L^2 solution, we have that a complete set of linearly independent solution is given by the two vectors

$$\begin{cases} (g_x, g_\theta)_{n,+} = (J_{|\nu n+1|}(\lambda_n x), J_{|\nu n+1|}(\lambda_n x)), \\ (g_x, g_\theta)_{n,-} = (J_{|\nu n-1|}(\lambda_n x), -J_{|\nu n-1|}(\lambda_n x)). \end{cases}$$

Therefore, the eigenvalues equation (20) relative to the operator $\Delta^{(1)}$, has the following complete set of linearly independent L^2 solutions with $n \in \mathbb{Z}$

$$\left\{ f_{n,\pm} = J_{|\nu n \pm 1|}(\lambda_n x) e^{in\theta} \left(\nu dx \mp ix d\theta\right) \right\}. \tag{23}$$

Eventually, we apply the boundary conditions. For 0-forms, $\omega_{\rm tan} = \omega$ and $\omega_{\rm norm} = 0$. Relative boundary conditions given in equation (13) applied to the solutions in equation (18), give $\lambda_n = \lambda_{n,k} = \frac{j_{\nu n,k}}{l}$, where $j_{\nu,k}$ are the positive zeros of the Bessel function J_ν, arranged in increasing order, with $k = 1, 2, \ldots$. Since it is known that the set $\{J_\nu(j_{\nu,k}x)\}_{k=1,2,\dots}$ defines an orthogonal basis of the space $L^2(0,1)$, we have proved that the set

$$\left\{\phi_0(\theta)\psi_{0,k}(x) = J_0(\frac{j_{0,k}}{l}x), \phi_{n,\pm}(\theta)\psi_{n,k}(x) = \mathrm{e}^{\pm in\theta}J_{\nu n}(\frac{j_{\nu n,k}}{l}x)\right\}_{n\in\mathbb{N}_0},$$

defines a complete set of orthogonal linear independent solutions of the eigenvalues equation (16) for $\Delta^{(0)}$ with Dirichlet BC at $x = l$ on $L^2(0,l)$, and where $\lambda_n = \lambda_{n,k} = \frac{j_{\nu n,k}}{l}$ for both $\phi_{n,\pm}J_{\nu n}$ when $n \neq 0$. Absolute boundary conditions are given in equation (13). Applying to the solutions in equation (18), we obtain

$$\frac{\partial\omega}{\partial x}(l,\theta) = \lambda_n e^{in\theta}J'_{|\nu n|}(\lambda_n l) = 0,$$

that give $\lambda_n = \lambda'_{n,k} = \frac{j'_{|\nu n|,k}}{l}$, where the $j'_{\nu n,k}$ are the zeros of $J'_{\nu n}(z)$.

The result for 2-forms is the dual of that for 0-forms. Note that, applying the inverse of the previous Liouville transform, we get a complete system for $\Delta^{(2)}$ with absolute boundary conditions:

$$\left\{\frac{j^2_{\nu|n|,k}}{l^2}, \omega^{(2)}_{n,k}(x,\theta) = \phi_n(\theta)\rho_{\nu|n|,k}(x) = \mathrm{e}^{in\theta}xJ_{\nu|n|}(\frac{j_{\nu|n|,k}}{l}x)dx\wedge d\theta\right\}_{n\in\mathbb{Z},k\in\mathbb{N}_0}.$$

For a 1-form $\omega(x,\theta) = \omega_x(x,\theta)dx + \omega_\theta(x,\theta)d\theta$, $\omega_{\tan} = \omega_\theta$ and $\omega_{\rm norm} = \omega_x$. Note that none of the solutions in (23) satisfy the BC (15), for $\lambda_n \neq 0$. So we consider linear combinations $\omega_{n,\pm}(x,\theta) = f_{n,+}(x,\theta) \pm f_{n,-}(x,\theta)$. Applying the absolute BC (15) to $\omega_{n,\pm}(x,\theta)$ we obtain, if $n \neq 0$, the eigenvalues $\lambda^2_{n,k,+} = j^2_{|\nu n|,k}/l^2$, and $\lambda^2_{n,k,-} = (j'_{|\nu n|,k})^2/l^2$. If $n = 0$, we have $\lambda^2_{0,k,+} = j^2_{1,k}/l^2$, and $\lambda^2_{0,k,-} = j^2_{0,k}/l^2$.

Applying the relative BC (15) to $\omega_{n,\pm}(x,\theta)$ we obtain, if $n \neq 0$, the eigenvalues $\lambda^2_{n,k,+} = (j'_{|\nu n|,k})^2/l^2$, and $\lambda^2_{n,k,-} = j^2_{|\nu n|,k}/l^2$. If $n = 0$, we have $\lambda^2_{0,k,+} = j^2_{0,k}/l^2$, and $\lambda^2_{0,k,-} = j^2_{1,k}/l^2$. The eigenforms follow from equation (23). □

Lemma 4. *The spectrum of the (Friedrich extension of the) Laplacian operator* $\Delta^{(q)}_{C_\alpha S^2_{l\sin\alpha}}$ *on q-forms with absolute boundary conditions is:*

$$\begin{aligned}
{\rm Sp}\Delta^{(0)}_{C_\alpha S^2_{l\sin\alpha}} &= \left\{2n+1 : \tilde{j}^2_{\mu_n,k,-}/l^2\right\}_{n,k=1}^{\infty} \cup \left\{j^2_{\frac{3}{2},k}/l^2\right\}_{k=1}^{\infty},\\
{\rm Sp}\Delta^{(1)}_{C_\alpha S^2_{l\sin\alpha}} &= \left\{j^2_{\frac{3}{2},k}/l^2\right\}_{k=1}^{\infty} \cup \left\{2n+1 : j^2_{\mu_n,k}/l^2\right\}_{n,k=1}^{\infty}\\
&\quad \cup \left\{2n+1 : \tilde{j}^2_{\mu_n,k,+}/l^2\right\}_{n,k=1}^{\infty} \cup \left\{2n+1 : \tilde{j}^2_{\mu_n,k,-}/l^2\right\}_{n,k=1}^{\infty},\\
{\rm Sp}\Delta^{(2)}_{C_\alpha S^2_{l\sin\alpha}} &= \left\{j^2_{\frac{1}{2},k}/l^2\right\}_{k=1}^{\infty} \cup \left\{2n+1 : j^2_{\mu_n,k}/l^2\right\}_{n,k=1}^{\infty}\\
&\quad \cup \left\{2n+1 : \tilde{j}^2_{\mu_n,k,+}/l^2\right\}_{n,k=1}^{\infty} \cup \left\{2n+1 : j^2_{\mu_n,k}/l^2\right\}_{n,k=1}^{\infty},\\
{\rm Sp}\Delta^{(3)}_{C_\alpha S^2_{l\sin\alpha}} &= \left\{2n+1 : j^2_{\mu_n,k}/l^2\right\}_{n,k=1}^{\infty} \cup \left\{j^2_{\frac{1}{2},k}/l^2\right\}_{k=1}^{\infty}.
\end{aligned}$$

The spectrum with relative boundary conditions is:

$$\begin{aligned}
\mathrm{Sp}\Delta^{(0)}_{C_\alpha S^2_{l\sin\alpha}} &= \left\{2n+1 : j^2_{\mu_n,k}/l^2\right\}_{n,k=1}^{\infty} \cup \left\{j^2_{\frac{1}{2},k}/l^2\right\}_{k=1}^{\infty},\\
\mathrm{Sp}\Delta^{(1)}_{C_\alpha S^2_{l\sin\alpha}} &= \left\{j^2_{\frac{1}{2},k}/l^2\right\}_{k=1}^{\infty} \cup \left\{2n+1 : j^2_{\mu_n,k}/l^2\right\}_{n,k=1}^{\infty}\\
&\quad \cup \left\{2n+1 : \tilde{j}^2_{\mu_n,k,+}/l^2\right\}_{n,k=1}^{\infty} \cup \left\{2n+1 : j^2_{\mu_n,k}/l^2\right\}_{n,k=1}^{\infty},\\
\mathrm{Sp}\Delta^{(2)}_{C_\alpha S^2_{l\sin\alpha}} &= \left\{j^2_{\frac{3}{2},k}/l^2\right\}_{k=1}^{\infty} \cup \left\{2n+1 : j^2_{\mu_n,k}/l^2\right\}_{n,k=1}^{\infty}\\
&\quad \cup \left\{2n+1 : \tilde{j}^2_{\mu_n,k,+}/l^2\right\}_{n,k=1}^{\infty} \cup \left\{2n+1 : \tilde{j}^2_{\mu_n,k,-}/l^2\right\}_{n,k=1}^{\infty},\\
\mathrm{Sp}\Delta^{(3)}_{C_\alpha S^2_{l\sin\alpha}} &= \left\{2n+1 : \tilde{j}^2_{\mu_n,k,-}/l^2\right\}_{n,k=1}^{\infty} \cup \left\{j^2_{\frac{3}{2},k}/l^2\right\}_{k=1}^{\infty},
\end{aligned}$$

where $\mu_n = \sqrt{\nu^2 n(n+1) + \frac{1}{4}}$*, and where the* $\tilde{j}_{\nu,k,\pm}$ *are the zeros of the function* $G^{\pm}_{\nu}(z) = \pm\frac{1}{2}J_\nu(z) + zJ'_\nu(z)$*.*

Proof. Recall we parameterize $C_\alpha S^2_{l\sin\alpha}$ by

$$C_\alpha S^2_{l\sin\alpha} = \begin{cases} x_1 = x\sin\alpha\sin\theta_2\cos\theta_1\\ x_2 = x\sin\alpha\sin\theta_2\sin\theta_1\\ x_3 = x\sin\alpha\cos\theta_2\\ x_4 = x\cos\alpha \end{cases}$$

where $(x,\theta_1,\theta_2) \in [0,l]\times[0,2\pi]\times[0,\pi]$, α is a fixed positive real number and $0 < a = \frac{1}{\nu} = \sin\alpha \le 1$. The induced metric is

$$g = dx\otimes dx + a^2x^2\sin^2\theta_2 d\theta_1\otimes d\theta_1 + a^2x^2 d\theta_2\otimes d\theta_2.$$

The Hodge star acts as follows

$$\begin{aligned}
&* : 1 \mapsto a^2x^2\sin\theta_2 dx\wedge d\theta_1\wedge d\theta_2;\\
&* : dx \mapsto a^2x^2\sin\theta_2 d\theta_1\wedge d\theta_2, * : d\theta_1 \mapsto \frac{-1}{\sin\theta_2}dx\wedge d\theta_2, * : d\theta_2 \mapsto \sin\theta_2 dx\wedge d\theta_1;\\
&* : dx\wedge d\theta_1 \mapsto \frac{1}{\sin\theta_2}d\theta_2, * : dx\wedge d\theta_2 \mapsto -\sin\theta_2 d\theta_1, * : d\theta_1\wedge d\theta_2 \mapsto \frac{1}{a^2x^2\sin\theta_2}dx;\\
&* : dx\wedge d\theta_1\wedge d\theta_2 \mapsto \frac{1}{a^2x^2\sin\theta_2}.
\end{aligned}$$

The Laplacian on forms reads

$$\Delta^{(0)}(\omega) = -\left(\frac{2}{x}\partial_x\omega + \partial_x^2\omega + \frac{\cos\theta_2}{a^2x^2\sin\theta_2}\partial_{\theta_2}\omega + \frac{\partial^2_{\theta_2}\omega}{a^2x^2} + \frac{\partial^2_{\theta_1}\omega}{a^2x^2\sin^2\theta_2}\right),$$

$$\Delta^{(1)}(\omega_x dx + \omega_{\theta_1} d\theta_1 + \omega_{\theta_2} d\theta_2) =$$

$$\left(-\partial_x^2\omega_x + \frac{2}{x^2}\omega_x - \frac{2}{x}\partial_x\omega_x - \frac{1}{a^2x^2}\partial_{\theta_2}^2\omega_x - \frac{\cos\theta_2}{a^2x^2\sin\theta_2}\partial_{\theta_2}\omega_x\right.$$
$$\left.-\frac{1}{a^2x^2\sin^2\theta_2}\partial_{\theta_1}^2\omega_x + \frac{2}{a^2x^3\sin^2\theta_2}\partial_{\theta_1}\omega_{\theta_1} + \frac{2}{a^2x^3}\partial_{\theta_2}\omega_{\theta_2} + \frac{2\cos\theta_2}{a^2x^3\sin\theta_2}\omega_{\theta_2}\right) dx$$

$$+\left(-\partial_x^2\omega_{\theta_1} - \frac{2}{x}\partial_{\theta_1}\omega_x - \frac{1}{a^2x^2}\partial_{\theta_2}^2\omega_{\theta_1}\right.$$
$$\left.+\frac{\cos\theta_2}{a^2x^2\sin\theta_2}(\partial_{\theta_2}\omega_{\theta_1} - 2\partial_{\theta_1}\omega_{\theta_2}) - \frac{1}{a^2x^2\sin^2\theta_2}\partial_{\theta_1}^2\omega_{\theta_1}\right) d\theta_1$$

$$+\left(-\partial_x^2\omega_{\theta_2} - \frac{2}{x}\partial_{\theta_2}\omega_x - \frac{1}{a^2x^2}\partial_{\theta_2}^2\omega_{\theta_2} - \frac{\cos\theta_2}{a^2x^2\sin\theta_2}\partial_{\theta_2}\omega_{\theta_2}\right.$$
$$\left.+\frac{1}{a^2x^2\sin^2\theta_2}(\omega_{\theta_2} - \partial_{\theta_1}^2\omega_{\theta_2}) + \frac{2\cos\theta_2}{a^2x^2\sin^3\theta_2}\partial_{\theta_1}\omega_{\theta_1}\right) d\theta_2,$$

with the following set of BC. For the 0-forms:

$$\text{abs.}: \partial_x\omega(l,\theta_1,\theta_2) = 0, \qquad \text{rel.}: \omega(l,\theta_1,\theta_2) = 0. \tag{24}$$

For 1-forms:

$$\text{abs.}: \begin{cases} \omega_x(l,\theta_1,\theta_2) = 0, \\ \partial_x\omega_{\theta_1}(l,\theta_1,\theta_2) = 0, \\ \partial_x\omega_{\theta_2}(l,\theta_1,\theta_2) = 0, \end{cases} \qquad \text{rel.}: \begin{cases} \omega_{\theta_1}(l,\theta_1,\theta_2) = 0, \\ \omega_{\theta_2}(l,\theta_1,\theta_2) = 0, \\ \partial_x(x^2\omega_x)(l,\theta_1,\theta_2) = 0. \end{cases} \tag{25}$$

Next we solve the eigenvalues equations. For 0-forms:

$$\Delta^{(0)}(\omega) = -\partial_x^2\omega - \frac{2}{x}\partial_x\omega + \frac{1}{a^2x^2}\Delta_{S^2}^{(0)}(\omega) = \lambda^2\omega. \tag{26}$$

We can decompose the problem in the eigenspaces of $\Delta_{S^2}^{(0)}$. Let $Y_n^k(\theta_1,\theta_2) = \mathrm{e}^{ik\theta_1}P_n^{|k|}(\cos\theta_2)$, where $P_n^{|k|}(\cos\theta_2)$ are the associated Legendre polynomials and $|k| \leq n$. $Y_n^k(\theta_1,\theta_2)$ is a complete system of eigenforms for $\Delta_{S^2}^{(0)}$ and the eigenvalues are $n(n+1)$, with multiplicity $2n+1$ and $n \in \mathbb{Z}$, $n \geq 0$. Thus,

$$\Delta^{(0)} = \sum_{n\geq 0} T_n \Pi_n,$$

with

$$T_n = -\partial_x^2\omega - \frac{2}{x}\partial_x\omega + \frac{\nu^2 n(n+1)}{x^2},$$

and the eigenvalues equation reads

$$T_n(u) = \left(-\partial_x^2\omega - \frac{2}{x}\partial_x\omega + \frac{\nu^2 n(n+1)}{x^2}\right) u = \lambda_n^2 u.$$

This can be solved in terms of Bessel function and the solution is $u_n(x) = x^{-\frac{1}{2}}J_{\mu_n}(\lambda_n x)$. Hence the solution for the 0-Laplacian equation is

$$\alpha_n^{(0)}(x,\theta_1,\theta_2) = x^{-\frac{1}{2}}J_{\mu_n}(\lambda_n x)Y_n^k(\theta_1,\theta_2). \tag{27}$$

For 1-form we have

$$\Delta^{(1)}(\omega) = \lambda^2\omega, \tag{28}$$

with $\omega = \omega_x dx + \omega_{\theta_1} d\theta_1 + \omega_{\theta_2} d\theta_2$. Write $\omega = f_{\theta_1\theta_2}(x)\phi(\theta_1,\theta_2) + f_x(x)h(\theta_1,\theta_2)dx$ where $\phi = f_{\theta_1} d\theta_1 + f_{\theta_2} d\theta_2$ and $h(\theta_1,\theta_2)$ is a 0-form on S^2. Hence replacing in (28) we have the system

$$\begin{cases} (-d_x^2 f_{\theta_1\theta_2})\phi + \dfrac{\Delta^{(1)}_{S^2}(\phi)}{a^2x^2} f_{\theta_1\theta_2} - \dfrac{2f_x d(h)}{x} = \lambda^2 f_{\theta_1\theta_2}\phi, \\ \left((-d_x^2 f_x - \dfrac{2}{x} d_x f_x + \dfrac{2}{x^2} f_x)h + \dfrac{\Delta^{(0)}_{S^2}(h)}{a^2x^2} f_x - \dfrac{2f_{\theta_1\theta_2} d^\dagger_{S^2}(\phi)}{a^2x^3} \right) dx = \lambda^2 f_x h dx. \end{cases} \tag{29}$$

Consider $f_x = 0$ or $h = 0$ and ϕ a coexact eigenform on S^2 with non zero eigenvalue. We have the equation, for $n \geq 1$,

$$(-d_x^2 f_{\theta_1\theta_2})\phi + \frac{\nu^2 n(n+1) f_{\theta_1\theta_2}}{x^2}\phi = \lambda_n^2 f_{\theta_1\theta_2}\phi.$$

Solving this equation in x we find that $f_{\theta_1\theta_2} = x^{\frac{1}{2}} J_{\mu_n}(\lambda_n x)$ and then $\alpha_n^{(1)} = x^{\frac{1}{2}} J_{\mu_n}(\lambda_n x)\phi$. Note that $\phi = d^\dagger_{S^2}(\sin\theta_2 Y_n^k(\theta_1,\theta_2) d\theta_1 \wedge d\theta_2)$.

Now we consider $f_x \neq 0$, $f_{\theta_1\theta_2} \neq 0$, and h a coexact 0-eigenform of S^2 with non zero eigenvalue such that $d(h) = \phi$. Hence, $\Delta^{(1)}_{S^2}(\phi) = n(n+1)\phi$, $d^\dagger_{S^2}(d(h)) = n(n+1)h$, and the system (29) becomes

$$\begin{cases} (-d_x^2 f_{\theta_1\theta_2})\phi + \dfrac{n(n+1) f_{\theta_1\theta_2}}{a^2x^2}\phi - \dfrac{2f_{\theta_1\theta_2}\phi}{x} = \lambda_n^2 f_{\theta_1\theta_2}\phi, \\ (-d_x^2 f_x - \dfrac{2}{x} d_x f_x + \dfrac{2}{x^2} f_x)h + \dfrac{n(n+1)h}{a^2x^2} f_x - \dfrac{2n(n+1) f_{\theta_1\theta_2}(\phi)}{a^2x^3} = \lambda_n^2 f_x h. \end{cases}$$

Changing the base by $(f_{\theta_1\theta_2}, f_x) = (x^{-\frac{1}{2}} g_x, \partial_x(x^{-\frac{1}{2}} g_x))$, we solve the system, and the solution is $g_x = J_{\mu_n}(\lambda_n x)$. Hence the solution for the system (29) is

$$\beta_n^{(1)} = x^{-\frac{1}{2}} J_{\mu_n}(\lambda_n x)\phi(\theta_1,\theta_2) + \partial_x(x^{-\frac{1}{2}} J_{\mu_n}(\lambda_n x)) h(\theta_1,\theta_2) dx,$$

where $h(\theta_1,\theta_2) = Y_n^k(\theta_1,\theta_2)$. Consider now $f_x \neq 0$, $f_{\theta_1\theta_2} \neq 0$, and ψ a coexact 0-eigenform of S^2 with non zero eigenvalue such that $d(\psi) = \phi$ and $d^\dagger_{S^2} d(\psi) = h$. Then $h = n(n+1)\psi$, and the system (29) becomes

$$\begin{cases} (-d_x^2 f_{\theta_1\theta_2})\phi + \dfrac{n(n+1) f_{\theta_1\theta_2}}{a^2x^2}\phi - \dfrac{2f_x n(n+1)}{x}\phi = \lambda_n^2 f_{\theta_1\theta_2}\phi, \\ \left((-d_x^2 f_x - \dfrac{2}{x} d_x f_x + \dfrac{2}{x^2} f_x)h + \dfrac{n(n+1) f_x}{a^2x^2} h - \dfrac{2f_{\theta_1\theta_2}}{a^2x^3} h \right) dx = \lambda_n^2 f_x h dx. \end{cases} \tag{30}$$

Changing the base by $(f_{\theta_1\theta_2}, f_x) = (\partial_x(x^{\frac{1}{2}} g_x), x^{-\frac{3}{2}} g_x)$ we solve the system (30) and the solution is $g_x = J_{\mu_n}(\lambda_n x)$. Hence the solution for the system (29) in this case is

$$\gamma_n^{(1)} = \partial_x(x^{\frac{1}{2}} J_{\mu_n}(\lambda_n x))(\lambda_n x)\phi(\theta_1,\theta_2) + x^{-\frac{3}{2}} J_{\mu_n}(\lambda_n x) h(\theta_1,\theta_2) dx,$$

where $\psi(\theta_1,\theta_2) = Y_n^k(\theta_1,\theta_2)$. In the case $\Delta^{(1)}_{S^2}(\phi) = \Delta^{(0)}_{S^2}(h) = 0$ we have the equation

$$\left((-d_x^2 f_x - \frac{2}{x} d_x f_x + \frac{2}{x^2} f_x)h \right) dx = \lambda_n^2 f_x h dx.$$

Hence changing the base by $f_x = \partial_x(x^{-\frac{1}{2}} g_x)$ we find $g_x = J_{\frac{1}{2}}(\lambda_0 x)$ and the solution is $D^{(1)} = \partial_x(x^{-\frac{1}{2}} J_{\frac{1}{2}}(\lambda_0 x))h$.

Since we know that the Hodge decomposition of square integrable forms into exact and coexact forms holds as in the smooth case [7], we conclude that equation (28) has the following complete set of linearly independent L^2 solutions

$$\begin{aligned}
\alpha_n^{(1)} &= x^{\frac{1}{2}} J_{\mu_n}(\lambda_n x) d^{\dagger}_{S^2}(\sin\theta_2 Y_n^k(\theta_1,\theta_2) d\theta_1 \wedge d\theta_2),\\
\beta_n^{(1)} &= \partial_x(x^{-\frac{1}{2}} J_{\mu_n}(\lambda_n x)) Y_n^k(\theta_1,\theta_2) dx + x^{-\frac{1}{2}} J_{\mu_n}(\lambda_n x) d(Y_n^k(\theta_1,\theta_2)),\\
\gamma_n^{(1)} &= x^{-\frac{3}{2}} J_{\mu_n}(\lambda_n x)(n(n+1))\nu^2 Y_n^k(\theta_1,\theta_2) dx\\
&\quad + \partial_x(x^{\frac{1}{2}} J_{\mu_n}(\lambda_n x)) d(Y_n^k(\theta_1,\theta_2)),\\
D^{(1)} &= \partial_x(x^{-\frac{1}{2}} J_{\frac{1}{2}}(\lambda_0 x)) dx.
\end{aligned} \tag{31}$$

Next we determine the eigenvalues.

0-forms. We have only two type of forms in (27), that are,

$$\alpha_n^{(0)} = x^{-\frac{1}{2}} J_{\mu_n}(\lambda_n x)\phi_n^{(0)}, \qquad E_n^{(0)} = x^{-\frac{1}{2}} J_{\frac{1}{2}}(\lambda_0 x)\phi_n^{(0)}.$$

Using the absolute BC in (24) we have

$$\begin{aligned}
\partial_x(\alpha_{1,n}^{(0)})(l,\theta_1,\theta_2) &= \partial_x(x^{-\frac{1}{2}} J_{\mu_n}(\lambda_n x)\phi_n^{(0)})(l,\theta_1,\theta_2) = 0,\\
\partial_x(E_n^{(0)})(l,\theta_1,\theta_2) &= \partial_x(x^{-\frac{1}{2}} J_{\frac{1}{2}}(\lambda_0 x)\phi_n^{(0)})(l,\theta_1,\theta_2) = 0,
\end{aligned}$$

and so we need the square of the solutions of $-\frac{1}{2} l^{-\frac{3}{2}} J_{\mu_n}(l\lambda_n) + l^{-\frac{1}{2}} \lambda J'_{\mu_n}(l\lambda_n) = 0$ and $-\lambda_0^{-\frac{1}{2}} J_{\frac{3}{2}}(l\lambda_0) = 0$ that are $\tilde{\lambda}_{n,k}^2 = \frac{\tilde{j}^2_{\mu_n,k,-}}{l^2}$ and $\lambda_{0,k}^2 = \frac{j^2_{\frac{3}{2},k}}{l^2}$.

Using the relative BC in (24) we have $\alpha_n^{(0)}(l,\theta_1,\theta_2) = E_n^{(0)}(l,\theta_1,\theta_2) = 0$ and the eigenvalues are $\lambda_{n,k}^2 = \frac{j^2_{\mu_n,k}}{l^2}$, and $\lambda_{0,k}^2 = \frac{j^2_{\frac{1}{2},k}}{l^2}$.

1-forms. In this case we have the four types of forms in (31) ($s = 1, 2$):

$$\begin{aligned}
\alpha_n^{(1)} &= x^{\frac{1}{2}} J_{\mu_n}(\lambda_n x)\phi_n^{(1)},\\
\beta_n^{(1)} &= \partial_x(x^{-\frac{1}{2}} J_{\mu_n}(\lambda_n x))\phi_n^{(0)} dx + x^{-\frac{1}{2}} J_{\mu_n}(\lambda_n x) d\phi_n^{(0)},\\
\gamma_n^{(1)} &= x^{-\frac{3}{2}} J_{\mu_n}(\lambda_n x)(n(n+1))\nu^2 \phi_n^{(0)} dx + \partial_x(x^{\frac{1}{2}} J_{\mu_n}(\lambda_n x)) d\phi_n^{(0)},\\
D_n^{(1)} &= \partial_x(x^{-\frac{1}{2}} J_{\frac{1}{2}}(\lambda_0 x))\phi_n^{(0)} dx.
\end{aligned}$$

Using the absolute BC in (25) we have, for the four types,

$$\begin{aligned}
\partial_x((\alpha_n^{(1)})_{\theta_s})(l,\theta_1,\theta_2) &= \partial_x(x^{\frac{1}{2}} J_{\mu_n}(\lambda_n x))(l) = 0,\\
(\beta_n^{(1)})_x(l,\theta_1,\theta_2) = \partial_x((\beta_n^{(1)})_{\theta_s})(l,\theta_1,\theta_2) &= \partial_x(x^{-\frac{1}{2}} J_{\mu_n}(\lambda_n x))(l) = 0,\\
(\gamma_n^{(1)})_x(l,\theta_1,\theta_2) &= J_{\mu_n}(l\lambda_n) = 0,\\
\partial_x(\gamma_n^{(1)})_{\theta_s}(l,\theta_1,\theta_2) &= -\frac{1}{4} J_{\mu_n}(l\lambda_n) + \lambda_n J'_{\mu_n}(l\lambda_n) + \lambda_n^2 J''_{\mu_n}(l\lambda_n) = 0,\\
\partial_x(D_n^{(1)}) &= \partial_x(x^{-\frac{1}{2}} J_{\frac{1}{2}}(\lambda_0 x))(l) = 0,
\end{aligned}$$

and so we obtain the square of the zeros of $\frac{1}{2}J_{\mu_n}(l\lambda_n)+l\lambda J'_{\mu_n}(l\lambda_n)=0$, $-\frac{1}{2}J_{\mu_n}(l\lambda_n)+l\lambda J'_{\mu_n}(l\lambda_n)=0$, $J_{\mu_n}(l\lambda_n)=0$, and $-\lambda_0^{-\frac{1}{2}}J_{\frac{3}{2}}(l\lambda_0)=0$, that are

$$\tilde\lambda_{n,k}^2=\frac{\tilde j^2_{\mu_n,k,+}}{l^2},\quad \tilde\lambda_{n,k}^2=\frac{\tilde j^2_{\mu_n,k,-}}{l^2},\quad \lambda_{n,k}^2=\frac{j^2_{\mu_n,k}}{l^2}\quad\text{and}\quad \lambda_{0,k}^2=\frac{j^2_{\frac{3}{2},k}}{l^2}.$$

Using the relative BC in (25) we have, for the five types,

$$\begin{aligned}
(\alpha_n^{(1)})_{\theta_s}(l,\theta_1,\theta_2)&=(x^{\frac{1}{2}}J_{\mu_n}(\lambda_n x))(l)=0,\\
(\beta_n^{(1)})_{\theta_s}(l,\theta_1,\theta_2)&=J_{\mu_n}(l\lambda_n)=0,\\
\partial_x(x^2(\beta_n^{(1)})_x)(l,\theta_1,\theta_2)&=-\frac{1}{4}J_{\mu_n}(l\lambda_n)+l\lambda_n J'_{\mu_n}(l\lambda_n)+(l\lambda_n)^2J''_{\mu_n}(l\lambda_n)=0,\\
\partial_x((\gamma_n^{(1)})_{\theta_s})(l,\theta_1,\theta_2)&=\frac{1}{2}J_{\mu_n}(l\lambda_n)+l\lambda_n J'_{\mu_n}(l\lambda_n)=0,\\
\partial_x(x^2(\gamma_n^{(1)})_x)(l,\theta_1,\theta_2)&=\frac{1}{2}J_{\mu_n}(l\lambda_n)+l\lambda J'_{\mu_n}(l\lambda_n)=0,\\
\partial_x(x^2(D_n^{(1)})_x)(l,\theta_1,\theta_2)&=\partial_x(x^2\partial_x(x^{-\frac{1}{2}}J_{\frac{1}{2}}(\lambda_0 x)))(l)=0,
\end{aligned}$$

and so we need the square of the zeros of $J_{\mu_n}(l\lambda_n)=0$, $-\frac{1}{4}J_{\mu_n}(l\lambda_n)+l\lambda_n J'_{\mu_n}(l\lambda_n)+(l\lambda_n)^2J''_{\mu_n}(l\lambda_n)=0$, $\frac{1}{2}J_{\mu_n}(l\lambda_n)+l\lambda_n J'_{\mu_n}(l\lambda_n)=0$ and $J_{\frac{1}{2}}(l\lambda_0)=0$, that are

$$\lambda_{n,k}^2=\frac{j^2_{\mu_n,k}}{l^2}\ \text{(twice)},\quad \tilde\lambda_{n,k}^2=\frac{\tilde j^2_{\mu_n,k,+}}{l^2}\quad\text{and}\quad \lambda_{0,k}^2=\frac{j^2_{\frac{1}{2},k}}{l^2}.$$

This concludes the proof for 0-forms and 1-forms. The result for 2-forms and 3-forms follows by duality. □

5.2. **The analytic torsion of a cone over the circle.** This case is now a particular instance of the general case covered in [12] and [13]. We will recall the main points, see the on line file for a complete account [14]. Consider absolute B.C.. By the analysis in Section 5.1, the relevant zeta functions are

$$\begin{aligned}
\zeta(s,\Delta^{(1)})&=\sum_{k=1}^\infty\frac{j_{0,k}^{-2s}}{l^{-2s}}+\sum_{k=1}^\infty\frac{j_{1,k}^{-2s}}{l^{-2s}}+2\sum_{n,k=1}^\infty\frac{j_{\nu n,k}^{-2s}}{l^{-2s}}+2\sum_{n,k=1}^\infty\frac{(j'_{\nu n,k})^{-2s}}{l^{-2s}},\\
\zeta(s,\Delta^{(2)})&=\sum_{k=1}^\infty\frac{j_{0,k}^{-2s}}{l^{-2s}}+2\sum_{n,k=1}^\infty\frac{j_{\nu n,k}^{-2s}}{l^{-2s}},
\end{aligned}$$

and by equation (4), the torsion is ($a=\sin\alpha=\frac{1}{\nu}$) $\log T_{\rm abs}((C_\alpha S^1_{la},g_E);\rho)=-\frac{1}{2}\zeta'(0,\Delta^{(1)})+\zeta'(0,\Delta^{(2)})$. The torsion zeta function is

$$\begin{aligned}
t(s)&=-\frac{1}{2}\zeta(s,\Delta^{(1)})+\zeta(s,\Delta^{(2)})=l^{2s}\left(\frac{1}{2}z_0(s)-\frac{1}{2}z_1(s)+Z(s)-\hat Z(s)\right)\\
&=\frac{1}{2}l^{2s}\sum_{k=1}^\infty j_{0,k}^{-2s}-\frac{1}{2}l^{2s}\sum_{k=1}^\infty j_{1,k}^{-2s}+l^{2s}\sum_{n,k=1}^\infty j_{\nu n,k}^{-2s}-l^{2s}\sum_{n,k=1}^\infty (j'_{\nu n,k})^{-2s},
\end{aligned}$$

and $\log T_{\rm abs}((C_\alpha S^1_{la},g_E);\rho)=t'(0)$. Using equation (14) of [12], we compute $z_{0/1}(0)$ and $z'_{0/1}(0)$. It remains to deal with the differences $Z(0)-\hat Z(0)$ and $Z'(0)-\hat Z'(0)$. For we use Theorem 3 of [12]. The relevant sequences are the double sequences $S=\{j^2_{\nu n,k}\}$ and $\hat S=\{(j'_{\nu n,k})^2\}$, and the simple sequence $U=\{\nu n\}_{n=1}^\infty=\nu^{-s}\zeta_R(s)$, and $Z(s)=\zeta(s,S)$, $\hat Z(s)=\zeta(s,\hat S)$. It is possible to show that the results of [12]

Section 4 apply. U, S_n, and $\hat{S}_n$ are totally regular sequences of spectral type with genus 1, and the relative genus of S and $\hat{S}$ are $(1,0,0)$. Also, S and $\hat{S}$ are spectrally decomposable over U with power $\kappa=2$ and length $\ell=2$, as in Definition 1 of [12]. This follows using the uniform expansions for the Bessel functions given for example in [19] (7.18), and Ex. 7.2. The final expansions read

$$\begin{aligned}\log\Gamma(-\lambda,S_n/u_n^2)=&\sum_{h=0}^{\infty}\phi_{h-1}(\lambda)u_n^{1-h}=\left(1-\log 2-\sqrt{1-\lambda}+\log(1+\sqrt{1-\lambda})\right)\nu n\\&+\frac{1}{4}\log(1-\lambda)-\left(U_1(\sqrt{-\lambda})+\frac{1}{12}\right)\frac{1}{\nu n}+O\left(\frac{1}{(\nu n)^2}\right),\\\log\Gamma(-\lambda,\hat{S}_n/u_n^2)=&\sum_{h=0}^{\infty}\hat{\phi}_{h-1}(\lambda)u_n^{1-h}=\left(1-\log 2-\sqrt{1-\lambda}+\log(1+\sqrt{1-\lambda})\right)\nu n\\&-\frac{1}{4}\log(1-\lambda)-\left(V_1(\sqrt{-\lambda})+\frac{1}{12}\right)\frac{1}{\nu n}+O\left(\frac{1}{(\nu n)^2}\right).\end{aligned}$$

Applying Theorem 1 of [12], since the unique pole of $\zeta(s,U)$ is at $s=1$,

$$\begin{aligned}\zeta(0,S)-\zeta(0,\hat{S})=&-A_{0,1}(0)+\hat{A}_{0,1}(0)+\frac{1}{2\nu}\operatorname*{Res}_{s=0}{}_1(\Phi_1(s)-\hat{\Phi}_1(s)),\\\zeta'(0,S)-\zeta'(0,\hat{S})=&-A_{0,0}(0)-A'_{0,1}(0)+\hat{A}_{0,0}(0)+\hat{A}'_{0,1}(0)\\&+\frac{1}{2\nu}\operatorname*{Res}_{s=0}{}_0(\Phi_1(s)-\hat{\Phi}_1(s))+\frac{1}{\nu}\left(\frac{3}{2}\gamma+\log\nu\right)\operatorname*{Res}_{s=0}{}_1(\Phi_1(s)-\hat{\Phi}_1(s)),\end{aligned}$$

where, by definition in equation (11) of [12], and the calculations in Appendix 6,

$$\Phi_1(s)=\frac{\Gamma\left(s+\frac{1}{2}\right)}{12\sqrt{\pi}s}(1+5s),\qquad\hat{\Phi}_1(s)=\frac{\Gamma\left(s+\frac{1}{2}\right)}{12\sqrt{\pi}s}(1-7s).$$

Whence

$$\begin{aligned}Z(0)-\hat{Z}(0)=&-(A_{0,1}(0)-\hat{A}_{0,1}(0)),\\Z'(0)-\hat{Z}'(0)=&-\left(A_{0,0}(0)+A'_{0,1}(0)-\hat{A}_{0,0}(0)-\hat{A}'_{0,1}(0)\right)+\frac{1}{2\nu}.\end{aligned}\tag{32}$$

Recalling the definition in a equation (13) of [12] of the terms $A_{0,0}(0)$ and $A'_{0,1}(0)$, and computing the expansion for large λ of the functions $\log\Gamma(-\lambda,S_n/u_n^2)$ and $\phi_1(\lambda)$ (using classical expansions for the Bessel functions and their derivative and the formulas in equation (12) of [12]) we obtain that $A_{0,0}(0)=\hat{A}_{0,0}(0)$, and that

$$A_{0,1}(s)-\hat{A}_{0,1}(s)=\frac{1}{2}\sum_{n=1}^{\infty}u_n^{-2s}=\frac{1}{2}\zeta(2s,U)=\frac{1}{2}\nu^{-2s}\zeta(2s).$$

Substitution in equation (32) gives $Z(0)-\hat{Z}(0)=\frac{1}{4}$, $Z'(0)-\hat{Z}'(0)=-\frac{1}{2}\log\nu+\frac{1}{2}\log 2\pi+\frac{1}{2\nu}$, and and hence (compare with [28])

$$\log T_{\rm abs}((C_\alpha S^1_{l\sin\alpha},g_E);\rho)=\frac{1}{2}\log\frac{\pi}{\nu}l^2+\frac{1}{2\nu}.$$

5.3. **The analytic torsion of a cone over the sphere.** We consider absolute B.C.. By the analysis in Section 5.1, the relevant zeta functions are

$$\zeta(s,\Delta^{(1)})=\sum_{k=1}^\infty \frac{j_{\frac{3}{2},k}^{-2s}}{l^{-2s}}+\sum_{n,k=1}^\infty (2n+1)\frac{j_{\mu_n,k}^{-2s}}{l^{-2s}}+\sum_{n,k=1}^\infty (2n+1)\frac{\tilde{j}_{\mu_n,k,\pm}^{-2s}}{l^{-2s}},$$

$$\zeta(s,\Delta^{(2)})=\sum_{k=1}^\infty \frac{j_{\frac{1}{2},k}^{-2s}}{l^{-2s}}+2\sum_{n,k=1}^\infty (2n+1)\frac{j_{\mu_n,k}^{-2s}}{l^{-2s}}+\sum_{n,k=1}^\infty (2n+1)\frac{\tilde{j}_{\mu_n,k,+}^{-2s}}{l^{-2s}},$$

$$\zeta(s,\Delta^{(3)})=\sum_{k=1}^\infty \frac{j_{\frac{1}{2},k}^{-2s}}{l^{-2s}}+\sum_{n,k=1}^\infty (2n+1)\frac{j_{\mu_n,k}^{-2s}}{l^{-2s}},$$

and by equation (4), the torsion is

$$\log T_{\rm abs}((C_\alpha S^2_{l\sin\alpha},g_E);\rho)=-\frac{1}{2}\zeta'(0,\Delta^{(1)})+\zeta'(0,\Delta^{(2)})-\frac{3}{2}\zeta'(0,\Delta^{(3)}).$$

Define the torsion zeta function

$$\begin{aligned}t(s)&=-\frac{1}{2}\zeta(s,\Delta^{(1)})+\zeta(s,\Delta^{(2)})-\frac{3}{2}\zeta(s,\Delta^{(3)})\\&=-\frac{1}{2}\sum_{k=1}^\infty \frac{j_{\frac{1}{2},k}^{-2s}}{l^{-2s}}-\frac{1}{2}\sum_{k=1}^\infty \frac{j_{\frac{3}{2},k}^{-2s}}{l^{-2s}}+\frac{1}{2}\sum_{n,k=1}^\infty (2n+1)\frac{\tilde{j}_{\mu_n,k,+}^{-2s}}{l^{-2s}}-\frac{1}{2}\sum_{n,k=1}^\infty (2n+1)\frac{\tilde{j}_{\mu_n,k,-}^{-2s}}{l^{-2s}}\\&=l^{2s}\left(-\frac{1}{2}z_{\frac{1}{2}}(s)-\frac{1}{2}z_{\frac{3}{2}}(s)+\frac{1}{2}Z_+(s)-\frac{1}{2}Z_-(s)\right),\end{aligned}$$

then

$$\begin{aligned}\log T_{\rm abs}((C_\alpha S^2_{la},g_E);\rho)=t'(0)=&\log l^2\left(-\frac{1}{2}z_{\frac{1}{2}}(0)-\frac{1}{2}z_{\frac{3}{2}}(0)+\frac{1}{2}Z_+(0)-\frac{1}{2}Z_-(0)\right)\\&-\frac{1}{2}z'_{\frac{1}{2}}(0)-\frac{1}{2}z'_{\frac{3}{2}}(0)+\frac{1}{2}Z'_+(0)-\frac{1}{2}Z'_-(0).\end{aligned}$$

Using equations (14) of [12], we compute

$$\begin{aligned}\log T_{\rm abs}((C_\alpha S^2_{l\sin\alpha},g_E);\rho)=&\left(\frac{3}{4}+\frac{1}{2}Z_+(0)-\frac{1}{2}Z_-(0)\right)\log l^2\\&+\frac{1}{2}Z'_+(0)-\frac{1}{2}Z'_-(0)+\frac{1}{2}\log\frac{4}{3}.\end{aligned}\qquad(33)$$

It remains to deal with the differences $Z_+(0)-Z_-(0)$ and $Z'_+(0)-Z'_-(0)$. For we use Theorem 3 of [12], in the form given in the corollary. The relevant sequences are the double sequences $S_\pm=\{\tilde{j}^2_{\mu_n,k,\pm}\}$, and the simple sequence $U=\{2n+1:\mu_n\}_{n=1}^\infty$, where $\mu_n=\sqrt{\nu^2 n(n+1)+\frac{1}{4}}$, and $Z_\pm(s)=\zeta(s,S_\pm)$. Using classical estimates for the zeros of Bessel function [29], the genus of $S_\pm$ is 0, the genus of U is 2, and the relative genus of S are $(1,0,1)$. This only differs from the case of the circle by $\mathsf{g}(S_{\pm,k})$, with fixed k. Using classical estimates for the zeros of the Bessel function, the behavior of this sequence is given by the behavior of the sequence of the eigenvalues of the Laplacian on the sphere S^2, that is known. In particular, we recall the main features here below. We check that U, and $S_{\pm,n}$ are totally regular sequences of spectral type. By definition of the sequence U, $\zeta(s,U)=\nu^{-s}\zeta(s,L_{\frac{1}{4\nu^2}})$, where $L_q=\{2n+1:\sqrt{n(n+1)+q}\}_{n=1}^\infty$. Hence, U is related to the sequence of the eigenvalues of the Laplacian on the 2 sphere shifted by some positive constant q. More precisely, $\zeta(2s, L_0) = \zeta(s, \mathrm{Sp}_+\Delta^{(0)}_{S^2})$. The zeta function $\zeta(s, \mathrm{Sp}_+\Delta^{(0)}_{S^2})$ has been studied in [25], Section 3.3, where it was proved that $\mathsf{e}(\mathrm{Sp}_+\Delta^{(0)}_{S^2}) = \mathsf{g}(\mathrm{Sp}_+\Delta^{(0)}_{S^2}) = 1$, and that $\mathrm{Sp}_+\Delta^{(0)}_{S^2}$ is a totally regular sequence of spectral type with infinite order, by giving the explicit formula for the associated Gamma function $\Gamma(-\lambda, U)$ in terms of the Barnes G function. It follows that $\mathsf{e}(U) = \mathsf{g}(U) = 2$, and that U is a totally regular sequence of spectral type with infinite order. Also, $\zeta(s, \mathrm{Sp}_+\Delta^{(0)}_{S^2})$ has one simple pole at $s = 1$, with residues

$$\operatorname*{Res_0}_{s=1} \zeta(s, \mathrm{Sp}_+\Delta^{(0)}_{S^2}) = 2\gamma, \qquad \operatorname*{Res_1}_{s=1} \zeta(s, \mathrm{Sp}_+\Delta^{(0)}_{S^2}) = 1,$$

and hence, $\zeta(s, L_0)$ has one simple pole at $s = 2$, with the same finite part and double residue. Expanding the power of the binomial, we have that $\zeta(s, L_q) = \zeta(s, L_0) + f(s)$, where $f(s)$ is a regular function at $s = 2$. Therefore,

$$\operatorname*{Res_0}_{s=2} \zeta(s, L_q) = 2\gamma + f(2), \qquad \operatorname*{Res_1}_{s=2} \zeta(s, L_q) = 2,$$

and

$$\zeta(s, U) = \nu^{-s}\zeta(s, L_q) = \frac{2}{\nu^2}\frac{1}{s-2} + f(s),$$

near $s = 2$. For $S_\pm$, we proceed as in Section 5.2 of [12]. Introducing the functions

$$G^\pm_\nu(z) = \pm\frac{1}{2}J_\nu(z) + zJ'_\nu(z),$$

we have the product representation, where $H^\pm_\nu(z) = \mathrm{e}^{-\frac{\pi}{2}i\nu}G^\pm_\nu(iz)$,

$$H^\pm_\nu(z) = \pm\frac{1}{2}I_\nu(z) + zI'_\nu(z) = \left(1 \pm \frac{1}{2\nu}\right)\frac{z^\nu}{2^\nu\Gamma(\nu)}\prod_{k=1}^\infty\left(1 + \frac{z^2}{z^2_{\nu,k,\pm}}\right). \tag{34}$$

Using this representation, we obtain a product representations for the Gamma functions associated to the sequences $S_{\pm,n}$, and hence a complete asymptotic expansion of $\log\Gamma(-\lambda, S_{\pm,n})$, proving that therefore S_n and $\hat{S}_n$ are sequences of spectral type. Considering the expansions, it follows that they are both totally regular sequences of infinite order. Next, we prove that $S_\pm$ are spectrally decomposable over U with power $\kappa = 2$ and length $\ell = 3$, as in Definition 1 of [12]. We have to show that the functions $\log\Gamma(-\lambda, S_{\pm,n}/u_n^2)$, have the appropriate uniform expansions for large n. Using equation (34) and the uniform expansions for the Bessel functions and their derivatives [19] (7.18), and Ex. 7.2, we obtain

$$\begin{aligned}
\log\Gamma(-\lambda, S_{n,\pm}/\mu_n^2) &= \sum_{h=0}^\infty \phi_{h-1,\pm}(\lambda)\mu_n^{1-h} \\
&= \left(1 - \sqrt{1-\lambda} + \log(1 + \sqrt{1-\lambda}) - \log 2\right)\mu_n \\
&\quad - \frac{1}{4}\log(1-\lambda) + \left(-W_{1,\pm}(\sqrt{-\lambda}) \pm \frac{1}{2} - \frac{1}{12}\right)\frac{1}{\mu_n} \\
&\quad + \left(-W_{2,\pm}(\sqrt{-\lambda}) + \frac{1}{2}W^2_{1,\pm}(\sqrt{-\lambda}) - \frac{1}{8}\right)\frac{1}{\mu_n^2} + O\left(\frac{1}{\mu_n^3}\right),
\end{aligned}$$

(see [12] pg. 429, for the explicit formulas of the $W_{k,\pm}$), and hence

$$\begin{aligned}
\phi_{1,+}(\lambda) &= -\frac{1}{8}\frac{1}{(1-\lambda)^{\frac{1}{2}}} - \frac{7}{24}\frac{1}{(1-\lambda)^{\frac{3}{2}}} + \frac{5}{12},\\
\phi_{1,-}(\lambda) &= \frac{7}{8}\frac{1}{(1-\lambda)^{\frac{1}{2}}} - \frac{7}{24}\frac{1}{(1-\lambda)^{\frac{3}{2}}} - \frac{7}{12},
\end{aligned}$$

$$\begin{aligned}
\phi_{2,+}(\lambda) &= \frac{1}{16}\frac{1}{1-\lambda} - \frac{3}{8}\frac{1}{(1-\lambda)^2} + \frac{7}{16}\frac{1}{(1-\lambda)^3} - \frac{1}{8},\\
\phi_{2,-}(\lambda) &= \frac{9}{16}\frac{1}{1-\lambda} - \frac{7}{8}\frac{1}{(1-\lambda)^2} + \frac{7}{16}\frac{1}{(1-\lambda)^3} - \frac{1}{8}.
\end{aligned} \tag{35}$$

The length ℓ of the decomposition is precisely 3. For the $\mathsf{e}(U) = 2$, and therefore the larger integer such that $h - 1 = \sigma_h \leq 2$ is 3, since $\sigma_0 = -1$, $\sigma_1 = 0$, $\sigma_2 = 1$, $\sigma_3 = 2$. However, only the term with $\sigma_h = 2$, namely $h = 3$, appears in the formula of Theorem 3 of [12], since the unique pole of $\zeta(s, U)$ is at $s = 2$.

We now apply the formula in that theorem. First, since $\kappa = 2$, $\mathrm{Res}_{0\,s=2}\,\zeta(s,U) = K$, and $\mathrm{Res}_{1\,s=2}\,\zeta(s,U) = \frac{2}{\nu^2}$, we obtain

$$\begin{aligned}
\zeta(0,S_+) - \zeta(0,S_-) &= -A_{0,1,+}(0) + A_{0,1,-}(0) + \frac{1}{\nu^2}\operatorname*{Res_1}_{s=0}(\Phi_{2,+}(s) - \Phi_{2,-}(s)),\\
\zeta'(0,S_+) - \zeta'(s,S_-) &= -\left(A_{0,0,+}(0) + A'_{0,1,+}(0) - A_{0,0,-}(0) - A'_{0,1,-}(0)\right)\\
&\quad + \frac{1}{\nu^2}\operatorname*{Res_0}_{s=0}(\Phi_{2,+}(s) - \Phi_{2,-}(s))\\
&\quad + \left(\frac{\gamma}{\nu^2} + K\right)\operatorname*{Res_1}_{s=0}(\Phi_{2,+}(s) - \Phi_{2,-}(s)).
\end{aligned}$$

Second, using the definition in equation (11) of [12], by equation (35) and the formula in Appendix 6, we obtain

$$\Phi_{2,+}(s) - \Phi_{2,-}(s) = \frac{1}{2}\Gamma(s+1),$$

and hence

$$\operatorname*{Res_0}_{s=0}(\Phi_{2,+}(s) - \Phi_{2,-}(s)) = \frac{1}{2}, \qquad \operatorname*{Res_1}_{s=0}(\Phi_{2,+}(s) - \Phi_{2,-}(s)) = 0.$$

This gives

$$\begin{aligned}
Z_+(0) - Z_-(0) &= -A_{0,1,+}(0) + A_{0,1,-}(0),\\
Z'_+(0) - Z'_-(0) &= \zeta'(0,S_+) - \zeta'(s,S_-)\\
&= -\left(A_{0,0,+}(0) + A'_{0,1,+}(0) - A_{0,0,-}(0) - A'_{0,1,-}(0)\right) + \frac{1}{2\nu^2}.
\end{aligned} \tag{36}$$

Third, by equation (13) of [12], the terms $A_{0,0}(s)$ and $A_{0,1}(s)$, are

$$\begin{aligned}
A_{0,0,\pm}(s) &= \sum_{n=1}^{\infty}\left(a_{0,0,n,\pm} - b_{2,0,0,\pm}u_n^{-1}\right)u_n^{-2s},\\
A_{0,1,\pm}(s) &= \sum_{n=1}^{\infty}\left(a_{0,1,n,\pm} - b_{2,0,1,\pm}u_n^{-1}\right)u_n^{-2s}.
\end{aligned}$$

Hence, we need the expansion for large λ of the functions $\log\Gamma(-\lambda, S_{n,\pm}/u_n^2)$ and $\phi_{2,\pm}(\lambda)$. This comes from the expansion

$$H_\nu^\pm(z) \sim \frac{\sqrt{z}\mathrm{e}^z}{\sqrt{2\pi}}\left(1+\sum_{k=1}^\infty b_k z^{-k}\right) + O(\mathrm{e}^{-z}),$$

for large z. After some calculations (see [12] pg. 429 for details), we find that $A_{0,1,+}(s) = A_{0,1,-}(s)$, and therefore $Z_+(0) - Z_-(0) = 0$, and that

$$\begin{aligned} A_{0,0,+}(s) - A_{0,0,-}(s) &= \sum_{n=1}^\infty (2n+1)\mu_n^{-2s}\left(\log\left(1+\frac{1}{2\mu_n}\right) - \log\left(1-\frac{1}{2\mu_n}\right)\right) \\ &= F(s,\nu). \end{aligned}$$

This series converges uniformly for $\mathrm{Re}(s) > 2$, and using the analytic extension of the zeta function $\zeta(s,U)$, has an analytic extension that is regular at $s = 0$. Substitution in equation (36) gives

$$Z_+'(0) - Z_-'(0) = -F(0,\nu) + \frac{1}{2\nu^2}.$$

Substitution in equation (33) gives

$$\log T_{\mathrm{abs}}((C_\alpha S^2_{l\sin\alpha}, g_E);\rho) = \frac{1}{2}\log\frac{4l^3}{3} - \frac{1}{2}F(0,\csc\alpha) + \frac{1}{4}\sin^2\alpha. \tag{37}$$

We give in the Appendix 7 a series representation for the $F(0,\nu)$ for $\nu > 1$. Consider here the case $\nu = 1$. Then, $\mu_n = \sqrt{n(n+1)+\frac{1}{4}} = n + \frac{1}{2}$, and hence

$$F(s,1) = 2^{2s}\sum_{n=1}^\infty (2n+1)^{1-2s}(\log(n+1) - \log n).$$

For $\mathrm{Re}(s) > 2$, due to absolute convergence, we can rearrange the terms in the sum. We obtain

$$\begin{aligned} F(s,1) &= -2^{2s}\sum_{n=1}^\infty \left((2n+1)^{1-2s} - (2n-1)^{1-2s}\right)\log n \\ &= \sum_{j=0}^\infty \binom{1-2s}{j}\frac{(1-(-1)^j)}{2^j}\zeta'(2s+j-1) \\ &= (1-2s)\zeta'(2s) + \sum_{k=1}^\infty \binom{1-2s}{2k+1}\frac{\zeta'(2s+2k)}{2^{2k+1}}, \end{aligned}$$

and hence, by substitution in equation (37),

$$\log T_{\mathrm{abs}}((C_{\frac{\pi}{2}} S_l^2, g_E);\rho) = \log T((D_l^3, g_E),\rho) = \frac{1}{2}\log\frac{4\pi l^3}{3} + \frac{1}{2}\log 2 + \frac{1}{4}.$$

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6. Appendix A

We give here a formula for a contour integral appearing in the text. The proof is in [24] Section 4.2. Let $\Lambda_{\theta,c} = \{\lambda \in \mathbb{C} \mid |\arg(\lambda - c)| = \theta\}$, $0 < \theta < \pi$, $0 < c < 1$, a real, then

$$\int_0^\infty t^{s-1} \frac{1}{2\pi i} \int_{\Lambda_{\theta,c}} \frac{\mathrm{e}^{-\lambda t}}{-\lambda} \frac{1}{(1-\lambda)^a} d\lambda dt = \frac{\Gamma(s+a)}{\Gamma(a)s}.$$

7. Appendix B

We give a power series representation for the function $F(0,\nu)$ for $\nu > 1$. Assume $\mathrm{Re}(s) > 2$, then

$$\begin{aligned} F(s,\nu) &= \sum_{n=1}^\infty (2n+1)\mu_n^{-2s} \left(\log\left(1 + \frac{1}{2\mu_n}\right) - \log\left(1 - \frac{1}{2\mu_n}\right)\right) \\ &= \sum_{n=1}^\infty (2n+1)\mu_n^{-2s} \sum_{k=0}^\infty \frac{2^{-2k}}{2k+1} \mu^{-2k-1} \\ &= \sum_{k=0}^\infty \frac{1}{(2k+1)2^{2k}} \sum_{n=1}^\infty (2n+1)\mu_n^{-2s-2k-1}. \end{aligned}$$

Now,

$$\mu_n^{2x} = \left(\nu^2 n(n+1) + \frac{1}{4}\right)^x = \sum_{j=0}^\infty \frac{1}{2^{2j}} \binom{x}{j} (n(n+1))^{x-j} \nu^{2x-2j},$$

and therefore

$$F(s,\nu) = \sum_{k=0}^\infty \frac{1}{(2k+1)2^{2k}} \sum_{j=0}^\infty \frac{1}{2^{2j}} \binom{-s-k-\frac{1}{2}}{j} \frac{\zeta(s+k+j+\frac{1}{2}, \mathrm{Sp}_+ \Delta_{S^2}^{(0)})}{\nu^{2s+2k+2j+1}},$$

where

$$\zeta(s, \mathrm{Sp}_+ \Delta_{S^2}^{(0)}) = \sum_{n=1}^\infty (2n+1)(n(n+1))^{-s}.$$

Since the unique pole of the meromorphic extension of $\zeta(s, \mathrm{Sp}_+ \Delta_{S^2}^{(0)})$ is at $s = 1$, writing

$$\begin{aligned} F(s,\nu) &= \zeta(s, \mathrm{Sp}_+ \Delta_{S^2}^{(0)}) \nu^{-2s-1} \\ &\quad + \sum_{\substack{j,k=0,\\ j+k\neq 0}}^\infty \frac{1}{(2k+1)2^{2k}} \frac{1}{2^{2j}} \binom{-s-k-\frac{1}{2}}{j} \frac{\zeta(s+k+j+\frac{1}{2}, \mathrm{Sp}_+ \Delta_{S^2}^{(0)})}{\nu^{2s+2k+2j+1}}, \end{aligned}$$

and using the analytic extension of $\zeta(s, \mathrm{Sp}_+ \Delta_{S^2}^{(0)})$, we obtain

$$F(0,\nu) = \zeta(\frac{1}{2}, \mathrm{Sp}_+ \Delta_{S^2}^{(0)}) \frac{1}{\nu} + \sum_{\substack{j,k=0,\\ j+k\neq 0}}^\infty \frac{1}{(2k+1)2^{2k}} \frac{1}{2^{2j}} \binom{-k-\frac{1}{2}}{j} \frac{\zeta(k+j+\frac{1}{2}, \mathrm{Sp}_+ \Delta_{S^2}^{(0)})}{\nu^{2k+2j+1}}.$$

It is easy to see that the coefficient in the power series above are all convergent series, and can be evaluated numerically. The leading term requires independent treatment. Using the theorem of Plana as in [23], we obtain

$$\zeta(\frac{1}{2}, \mathrm{Sp}_+ \Delta^{(0)}_{S^2}) = -\frac{5}{4}\sqrt{2} + 6\int_0^\infty \frac{(y^4+y^2+4)^{-\frac{1}{4}}}{\mathrm{e}^{2\pi y}-1} \sin\left(\frac{1}{2}\arctan\frac{3y}{2-y^2}\right) dy$$
$$- 4\int_0^\infty \frac{(y^4+y^2+4)^{-\frac{1}{4}}}{\mathrm{e}^{2\pi y}-1} \cos\left(\frac{1}{2}\arctan\frac{3y}{2-y^2}\right) dy.$$

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