On norm resolvent and quadratic form convergences in asymptotic thin spatial waveguides

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Abstract

A quantum particle is restricted to Dirichlet three-dimensional tubes built over a smooth curve $r(x) \subset \mathbb{R}^3$ through a bounded cross section that rotates along $r(x)$. Then the confining limit as the diameter of the tube cross section tends to zero is studied, and special attention is paid to the interplay between uniform quadratic form convergence and norm resolvent convergence of the respective Hamiltonians. In particular, it is shown a norm resolvent convergence to an effective Hamiltonian in case of null curvature and unbounded tubes, and, by means of an example, it is concluded that just norm resolvent convergence does not imply the quadratic form convergence.

1 Introduction

The question of dimensional reduction in quantum systems restricted to planar strips and spatial tubes (i.e., Dirichlet Laplacian) is as interesting as difficult to be mathematically analyzed. However, it has been of great interest in recent years, and many results are related to the confinement of motion from such strips and tubes to curves; see, for instance, [1, 3, 4, 6, 8, 11, 12, 13, 15, 16, 18]. The main questions to be addressed are the action of effective operators and how to approach them, the behavior of the essential spectrum and eigenvalue expansions in terms of the small diameter (with respect to some directions) of the regions.

It should be noted that such theoretical studies are not isolated from other areas of research and applications; without mentioning that it is an alternative approach to quantize constrained systems. For instance, electronic motion in nanostructures and some periodic solids present confining characteristics due to large potential barriers in some directions, particularly in quantum wires and carbon nanotubes (see, for instance, [19]). Dimensional reductions also appear in the quantum network model used to study the motion of valence electrons in aromatic molecules [2], as well as in studies of quantum graphs (see [14] and references therein), in which the limiting dynamics to a graph must define an effective Hamiltonian operator. Under some conditions it is also related to adiabatic dynamics and the Born-Oppenheimer approximation, since the presence of strong barriers induces different scales in the system [20].

In [11, 1] the authors considered the dynamics of a planar particle as the region approaches a broken line, a particular case of a quantum graph, as the limit $\varepsilon \to 0$ of the dynamics in $\mathbb{R}^2$ under a positive potential $W(x,y)/2\varepsilon^2$ that vanishes exactly on the graph. In [13] the hydrogen atom was restricted between two parallel planes in $\mathbb{R}^3$ and the planar limit was studied, whereas in [10] the confinement of the hydrogen atom in a space tube to the real line was investigated, and the main question was about which boundary condition at the origin should be selected.

A possible technique to prove the norm resolvent convergence of a sequence of self-adjoint operators is to consider the uniform convergence of the corresponding sesquilinear forms. We then present a detailed description of this in Subsection 4.1, although it has appeared in some arguments in [17]; we will show that the converse of such result does not necessarily holds true, for we discuss an example of norm resolvent convergence with no form convergence. For this example we consider a quantum particle initially restricted to Dirichlet tubes in $\mathbb{R}^3$ and take the squeezing limit of the particle motion from tubes to lines in space; the known results are in the sense of norm resolvent convergence to effective operators in case of bounded tubes [4, 8], whereas for unbounded tubes the results are restricted to strong resolvent sense [8]. In any event, herein we present a proof of norm resolvent convergence to unbounded tubes in case the curvature of the curve vanishes, that is, if the curve is a straight line (the tube cross section may has a nontrivial rotation; see ahead); the norm resolvent convergence for the general case of nonvanishing curvature remains an open question.

We think that at this point it is worth illustrating the general problem with a well-known simple example. Consider a quantum particle restricted to the box $B_\varepsilon = \{(x, y_1, y_2) \in \mathbb{R}^3 : 0 \le x \le L,\ 0 \le y_1, y_2 \le \varepsilon\}$, and we are interested in the limit $\varepsilon \to 0$. We use units so that the particle mass is $1/2$ and Planck constant $\hbar = 1$; hence the energy of such particle is described by the Laplacian $-\Delta_\varepsilon = -(\partial^2/\partial x^2 + \partial^2/\partial y_1^2 + \partial^2/\partial y_2^2)$ acting in $\mathrm{L}^2(B_\varepsilon)$ with Dirichlet (i.e., vanishing of the wavefunctions) boundary condition at the border ∂B_ε, thus with domain $\mathcal{H}_0^1(B_\varepsilon) \cap \mathcal{H}^2(B_\varepsilon)$. Its eigenvalues are

$$\lambda_{n,l_1,l_2} = \frac{1}{4}\left(n^2/L^2 + (l_1^2 + l_2^2)/\varepsilon^2\right),$$

where $n, l_1, l_2 \in \mathbb{N} = \{1, 2, 3, \cdots\}$ are the quantum numbers related to directions x, y_1, y_2, respectively. Since $\varepsilon \ll L$, the bottom of its spectrum is regulated by the ground states in the transversal directions y_1, y_2, that is, $l_1 = l_2 = 1$. As $\varepsilon \to 0$, the excited state in the transversal directions will play no role in the description of the spectrum of the system. If one looks at the corresponding normalized eigenfunctions

$$\psi_{n,l_1,l_2}(x,y_1,y_2) = \left(\frac{2}{L}\right)^{\frac{1}{2}} \sin\left(\frac{\pi n x}{L}\right) \times \left[\frac{2}{\varepsilon} \sin\left(\frac{\pi l_1 y_1}{\varepsilon}\right) \sin\left(\frac{\pi l_2 y_2}{\varepsilon}\right)\right],$$

one notices a clear decoupling of variables due to the particular symmetry of the box, and in the limit as the box is squeezed to the interval $[0, L]$, that is, $\varepsilon \to 0$, the particle motion is essentially one-dimensional and thus one simply disregards the transversal motion and works with the space $\mathrm{L}^2[0, L]$ generated by $\psi_n(x) = (2/L)^{1/2} \sin(\pi n x/L)$, $n \in \mathbb{N}$, and effective energy operator $H_{\text{eff}} = -d^2/dx^2$ (just the restriction of the Laplacian to one coordinate) with Dirichlet boundary condition and eigenvalues $n^2/(4L^2)$.

In case the interval $[0, L]$ is replaced by a more general curve $r(x)$ in $\mathbb{R}^3$ and the square cross section by a bounded open planar set S with diameter ε, thus getting a tube Ω_ε, the asymptotic situation is not so clear. But the example above indicates that, in the limit $\varepsilon \to 0$, one should remove from the initial Laplacian the ground state energy (which depends on ε^{-2}) of the Dirichlet Laplacian in S in order to get finite expectation values. It is known that now the effective operator H_{eff} will depend on geometric properties of the reference curve $r(x)$ (see [4, 12] and references therein), and the techniques to study the problem have to be carefully selected, as well as suitable identifications of operators acting in different spaces. In what follows we are going to be more specific in the description of our setting.

Let $r(x)$ be a curve in $\mathbb{R}^3$, $x \in I$, with I denoting either $\mathbb{R}$ or $x \in [0, L]$, is its arc-length parameter, and denote by $k(x)$ and $\tau(x)$ its curvature and torsion at the point $r(x)$, respectively. Let S be an open, bounded, simply connected and nonempty subset of $\mathbb{R}^2$. We build a tube Ω in $\mathbb{R}^3$ by moving the region S along $r(x)$. At each point the region may present a rotation angle which is denoted by $\alpha(x)$ (assume that $\alpha(0) = 0$). We suppose that the functions $k, (\tau + \alpha') \in \mathrm{L}^\infty(I)$, the Dirichlet condition at the boundary $\partial\Omega$, and we study its behavior as the tube is squeezed to $r(x)$.

The self-adjoint operators associated with this problem are

$$\psi \mapsto -\Delta_\varepsilon \psi, \qquad \psi \in \mathcal{H}_0^1(\Omega_\varepsilon) \cap \mathcal{H}^2(\Omega_\varepsilon),$$

where Ω_ε is the tube generated by the cross-section εS, and Δ_ε denotes the Laplacian in Ω_ε. Observe that, as $\varepsilon \to 0$, the sequence of the tubes Ω_ε approaches the curve r. It is expected that there is an effective operator, which should be identified with a one-dimensional operator in $\mathrm{L}^2(I)$, describing such singular limit $\varepsilon \to 0$. However, as $\varepsilon \to 0$, the region Ω_ε becomes narrower and the transverse oscillations of the particle become faster. Let λ_0 be the first (i.e., the lowest) eigenvalue of the Dirichlet Laplacian restricted to S and u_0 the (positive) associated normalized eigenfunction, that is,

$$-\Delta u_0 = \lambda_0 u_0, \quad u_0 \in \mathcal{H}_0^1(S), \quad u_0 \geq 0, \quad \int_S |u_0|^2 dy = 1.$$

Write [4]

$$C(S) := \int_S |\left(\nabla_y u_0, Ry\right)|^2 dy \quad \text{and} \quad R = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

where ∇_y denotes the gradient in the cartesian variables $y = (y_1, y_2)$ in S. Motivated by the particular box example above, one considers the sequence of operators

$$H_\varepsilon \psi = -\Delta_\varepsilon \psi - \frac{\lambda_0}{\varepsilon^2}\psi, \qquad \operatorname{dom} H_\varepsilon = \mathcal{H}_0^1(\Omega_\varepsilon) \cap \mathcal{H}^2(\Omega_\varepsilon). \tag{1}$$

The term λ_0/ε^2 is intended to renormalize the divergence of the transverse oscillations.

After an appropriate change of variables and regularizations, in case the curve is finite (i.e., $I = [0, L]$), so that the tubes are bounded, in [4] the asymptotic behavior of the eigenvalues λ_j^ε of (1) (with $j \in \mathbb{N}$) were found to be ruled by the eigenvalues of the effective one-dimensional self-adjoint operator

$$Tw := -w'' + C(S)(\tau + \alpha')(x)w - \frac{k(x)^2}{4}w, \tag{2}$$

with $\operatorname{dom} T = \mathcal{H}_0^1(0, L) \cap \mathcal{H}^2(0, L)$. More precisely, if μ_j are the eigenvalues of T, it was shown that, for each $j \in \mathbb{N}$,

$$\lambda_j^\varepsilon \to \mu_j, \quad \varepsilon \to 0. \tag{3}$$

Observe that the operator (2) depends on geometric features of the tube, despite the initial three-dimensional problem had no explicitly potential.

The case of unbounded tubes (more precisely, $I = \mathbb{R}$) was studied in [8] through the variational technique of strong and weak Γ-convergences, and it was found a strong resolvent operator convergence to

$$Tw := -w'' + C(S)(\tau + \alpha')(x)w - \frac{k(x)^2}{4}w, \quad \operatorname{dom} T = \mathcal{H}^2(\mathbb{R}). \tag{4}$$

As a byproduct of [8], the eigenvalue convergence (3) (in case of bounded tubes) was justified as a result of an operator norm resolvent convergence to the effective operator T in (2). It will be convenient to use the same notation for the operators (2) and (4), and it is expected to cause no confusion.

The proof of a kind of norm resolvent convergence also in the case of unbounded tubes $I = \mathbb{R}$, with T in (4) playing the role of an effective operator, thus improving some results of [4, 8], was implicitly conjectured in Remark 4.8 in [4] and proposed as an open problem at the end of [18]. We then tried to get such result by way of uniform quadratic form convergence as employed in [17, 16, 9] (see Subsection 4.1 in this work); however, such technique only works in case $k(x) \equiv 0$, that is, $r(x)$ is a straight line (see Theorem 3). For general nonvanishing curvatures, the analysis of the quadratic form convergence has revealed a counterexample to the natural question if the norm resolvent convergence implies in the uniform form convergence, as discussed in Subsection 4.3.2.

Shortly, this is the content of this work. In Section 2 we describe in detail the tube we work with. In Section 3 we introduce the relevant quadratic forms. In Section 4 we state our main results, whereas part of their proofs appear in Section 5.

2 Construction of the tubes

By following [4], let $r : I \to \mathbb{R}^3$ be a simple C^2 curve in $\mathbb{R}^3$ parametrized by its arc-length parameter x, and $k(x)$ and $\tau(x)$ denote its curvature and torsion, respectively. The vectors

$$T(x) = r'(x), \qquad N(x) = \frac{1}{k(x)} T'(x), \qquad B(x) = T(x) \times N(x),$$

denote, respectively, the tangent, normal and binormal vectors of the curve. We assume that Frenet equations are satisfied, that is,

$$\begin{pmatrix} T' \\ N' \\ B' \end{pmatrix} = \begin{pmatrix} 0 & k & 0 \\ -k & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix} \begin{pmatrix} T \\ N \\ B \end{pmatrix}.$$

Let S be an open, bounded, simply connected and nonempty subset of $\mathbb{R}^2$. The set

$$\Omega = \left\{ \mathbf{x} \in \mathbb{R}^3 : \mathbf{x} = r(x) + y_1 N(x) + y_2 B(x),\, x \in I, y = (y_1, y_2) \in S \right\}$$

is obtained by translating the region S along the curve r. At each point $r(x)$ we allow a rotation of the region S by an angle $\alpha(x)$ with respect to $\alpha(0) = 0$, so that the new region is given by

$$\Omega^\alpha = \left\{ \mathbf{x} \in \mathbb{R}^3 : \mathbf{x} = r(x) + y_1 N_\alpha(x) + y_2 B_\alpha(x),\, x \in I, (y_1, y_2) \in S \right\},$$

with

$$\begin{aligned} N_\alpha(x) &:= \cos\alpha(x) N(x) + \sin\alpha(x) B(x), \\ B_\alpha(x) &:= -\sin\alpha(x) N(x) + \cos\alpha(x) B(x). \end{aligned}$$

Next, for each $0 < \varepsilon < 1$, we "squeeze" the cross sections of the above region, that is, consider

$$\Omega^\alpha_\varepsilon = \left\{ \mathbf{x} \in \mathbb{R}^3 : \mathbf{x} = r(x) + \varepsilon y_1 N_\alpha(x) + \varepsilon y_2 B_\alpha(x),\, x \in I, (y_1, y_2) \in S \right\}.$$

From now on we will omit the symbol α in most notations and write $d\mathbf{x} = dx dy_1 dy_2$ and $dy = dy_1 dy_2$. The symbol $\nabla = (\partial_x, \nabla_y)$ denotes the gradient in the coordinates (x, y_1, y_2) in $\mathbb{R}^3$.

In this work we study the behavior of a quantum particle that moves in Ω_ε, under Dirichlet condition at the boundary $\partial\Omega_\varepsilon$, in the singular limit $\varepsilon \to 0$, that is, when Ω_ε approaches the curve $r(x)$ as $\varepsilon \to 0$. Thus, we initially consider the family of quadratic forms

$$g_\varepsilon(\psi) := \int_{\Omega_\varepsilon} |\nabla\psi|^2 d\mathbf{x}, \qquad \operatorname{dom} g_\varepsilon = \mathcal{H}_0^1(\Omega_\varepsilon), \tag{5}$$

which is associated with the Dirichlet Laplacian operator $-\Delta_\varepsilon$ in Ω_ε.

Remark 1. *For each closed lower bounded quadratic form, say b with domain* $\operatorname{dom} b$ *in a Hilbert space $\mathcal{H}$, it will tacitly be supposed that $b(\xi) = +\infty$ if $\xi \in \mathcal{H} \backslash \operatorname{dom} b$. It is a convenient way to work in the larger space $\mathcal{H}$ instead of only in the closure of* $\operatorname{dom} b$*; moreover, with such praxis, b becomes explicitly lower semicontinuous. See, for instance, Section 9.3 and Chapter 10 in [7].*

3 Quadratic forms

As usual, we perform a change of variables so that the integration region in (5), become independent of $\varepsilon > 0$. For the singular limit $\varepsilon \to 0$, customary "regularizations" will be employed.

Consider the mapping

$$\begin{array}{rccl} f_\varepsilon : & I \times S & \to & \Lambda_\varepsilon \\ & (x, y_1, y_2) & \mapsto & r(x) + \varepsilon\,(y_1 N_\alpha(x) + y_2 B_\alpha(x)), \end{array}$$

and suppose the boundedness $\|k\|_\infty, \|\tau\|_\infty, \|\alpha'\|_\infty < \infty$. These conditions are to guarantee that f_ε will be a diffeomorphism. With this change of variables we work with a fixed region for all $\varepsilon > 0$; more precisely, the domain of the quadratic form (5) turns out to be $\mathcal{H}_0^1(I \times S)$. On the other hand, the price to be paid is a nontrivial Riemannian metric $G = G_\varepsilon$ which is induced by f_ε, i.e.,

$$G = (G_{ij}), \qquad G_{ij} = \langle e_i, e_j \rangle = G_{ji}, \qquad 1 \le i, j \le 3,$$

where

$$e_1 = \frac{\partial f_\varepsilon}{\partial s}, \qquad e_2 = \frac{\partial f_\varepsilon}{\partial y_1}, \qquad e_3 = \frac{\partial f_\varepsilon}{\partial y_2}.$$

Some calculations show that in the Frenet frame

$$\begin{aligned} J &= \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} \\ &= \begin{pmatrix} \beta_\varepsilon & -\varepsilon(\tau + \alpha')\langle z_\alpha^\perp, y\rangle & \varepsilon(\tau + \alpha')\langle z_\alpha, y\rangle \\ 0 & \varepsilon\cos\alpha & \varepsilon\sin\alpha \\ 0 & -\varepsilon\sin\alpha & \varepsilon\cos\alpha \end{pmatrix}, \end{aligned}$$

where

$$\beta_\varepsilon(x,y) = 1 - \varepsilon k(x)\langle z_\alpha, y\rangle, \quad z_\alpha := (\cos\alpha, -\sin\alpha), \quad z_\alpha^\perp := (\sin\alpha, \cos\alpha).$$

The inverse matrix of J is given by

$$J^{-1} = \begin{pmatrix} \frac{1}{\beta_\varepsilon} & \frac{(\tau+\alpha')y_2}{\beta_\varepsilon} & -\frac{(\tau+\alpha')y_1}{\beta_\varepsilon} \\ 0 & \frac{\cos\alpha}{\varepsilon} & \frac{-\sin\alpha}{\varepsilon} \\ 0 & \frac{\sin\alpha}{\varepsilon} & \frac{\cos\alpha}{\varepsilon} \end{pmatrix}.$$

Note that $JJ^t = G$ and $\det J = |\det G|^{1/2} = \varepsilon^2\beta_\varepsilon(x,y)$. Since k is a bounded function, for ε small enough β_ε does not vanish in $I\times S$. Thus, $\beta_\varepsilon > 0$ and f_ε is a local diffeomorphism. By requiring that f_ε is injective (that is, the tube is not self-intersecting), a global diffeomorphism is obtained.

Introducing the notation

$$||\psi||_G^2 := \int_{I\times S} |\psi(x,y)|^2 \varepsilon^2 \beta_\varepsilon(x,y)\, dxdy,$$

we obtain a sequence of quadratic forms

$$\tilde{t}_\varepsilon(\psi) := \left|\left|J^{-1}\nabla\psi\right|\right|_G, \qquad \operatorname{dom}\tilde{t}_\varepsilon = \mathcal{H}_0^1(I\times S, G).$$

More precisely, the above change of coordinates was obtained by a unitary transformation

$$U_\varepsilon : \mathrm{L}^2(\Omega_\varepsilon) \rightarrow \mathrm{L}^2(I\times S, G) \tag{6}$$
$$\phi \mapsto \phi\circ f_\varepsilon \tag{7}$$

However, we still denote $U_\varepsilon\psi$ by ψ.

Recall that λ_0 is the lowest eigenvalue of the negative Laplacian with Dirichlet boundary conditions in the cross-section region S, and $u_0 \geq 0$ the corresponding eigenfunction of this restricted problem, and we remove the diverging energy λ_0/ε^2 from their quadratic forms. Therefore, we turn to the study of the sequence of quadratic forms

$$\hat{t}_\varepsilon(\psi) := \varepsilon^{-2}\left(\left|\left|J^{-1}\nabla\psi\right|\right|_G^2 - \frac{\lambda_0}{\varepsilon^2}\|\psi\|_G^2 + c\|\psi\|_G^2\right),$$

where c is a positive constant to be chosen later, and its role is to turn $\hat{t}_\varepsilon$, and t_ε below as well, into strictly positive quadratic forms for all $\varepsilon > 0$. In order to properly deal with the dimensional reduction, we have introduced a convenient global multiplicative factor ε^{-2}. After the norms are written out, we obtain

$$\begin{aligned} \hat{t}_\varepsilon(\psi) &= \int_{I\times S}\Big[\frac{1}{\beta_\varepsilon(x,y)}\left|\psi' + \nabla_y\psi\cdot Ry(\tau+\alpha')(x)\right|^2 \\ &+ \frac{\beta_\varepsilon(x,y)}{\varepsilon^2}\left(|\nabla_y\psi|^2 - \lambda_0|\psi|^2\right) + \beta_\varepsilon(x,y)c|\psi|^2\Big]\, dxdy\,. \end{aligned}$$

Note that $\operatorname{dom}\hat{t}_\varepsilon = \mathcal{H}_0^1(I \times S)$ is a subspace of $\mathrm{L}^2(I \times S, \beta_\varepsilon(x, y))$. Since $\beta_\varepsilon(x, y) \to 1$ uniformly, as $\varepsilon \to 0$, the spaces $\mathrm{L}^2(I \times S, \beta_\varepsilon(x, y))$ and $\mathrm{L}^2(I \times S)$ are algebraically equivalent (see Remark 5 in [8]). Due to this fact, we assume that $\operatorname{dom}\hat{t}_\varepsilon$ is a subspace of $\mathrm{L}^2(I \times S)$.

We denote by $\hat{T}_\varepsilon$ the self-adjoint operator associated with the quadratic form $\hat{t}_\varepsilon$, so that

$$\hat{T}_\varepsilon = \varepsilon^{-2} U_\varepsilon^{-1} H_\varepsilon U_\varepsilon + \beta_\varepsilon(x, y)c,$$

where H_ε and U_ε were introduced in Equations (1) and (6), respectively.

Finally, introduce the quadratic form $\operatorname{dom} t_\varepsilon = \mathcal{H}_0^1(I \times S)$ with action

$$\begin{aligned} t_\varepsilon(\psi) \;:=\; & \int_{I\times S} dydx \\ & \left[|\psi' + \nabla_y \psi \cdot Ry(\tau + \alpha')|^2 + \frac{\beta_\varepsilon}{\varepsilon^2}(|\nabla_y \psi|^2 - \lambda_0 |\psi|^2) + c|\psi|^2 \right]. \end{aligned}$$

The form t_ε is obtained from $\hat{t}_\varepsilon$ after replacing both, the multiplication by $1/\beta_\varepsilon$ in the first term inside the integral and the multiplication by β_ε in the last term inside the integral, simply by the constant value 1.

Let T_ε be the self-adjoint operator associated with t_ε. By taking into account the discussion just after Lemma 1, it follows that $\hat{T}_\varepsilon$ and T_ε are strictly positive operators, thus for each $\mu > 0$, we have that $-\mu$ is a common element of the resolvent sets of $\hat{T}_\varepsilon$ and T_ε. See Proposition 1.

4 Main results

The goal here is an analysis of the convergence, or not, of the sequence of quadratic forms $\hat{t}_\varepsilon$ as $\varepsilon \to 0$. It will be shown that one may consider t_ε instead of $\hat{t}_\varepsilon$, and the principal conclusions appear in Subsection 4.3. The proofs of the propositions in this section are postponed to Section 5.

4.1 Uniform quadratic form convergence

In this subsection we describe how the uniform convergence of forms (understood in the sense of (8) below and, in a more general context, in terms of the set of relations (9)–(13)) implies the norm resolvent convergence of the associated operators. We state two theorems and they are based on references [17, 16]; however, we only present the proof of one of them, since their proofs are in fact similar .

Theorem 1. *Let $(a_\varepsilon)_\varepsilon, (q_\varepsilon)_\varepsilon$ be two sequences of positive closed sesquilinear quadratic forms in a separable Hilbert space $\mathcal{H}$ with $\operatorname{dom} a_\varepsilon = \operatorname{dom} q_\varepsilon = \mathcal{D}$, for all $\varepsilon > 0$, and $A_\varepsilon, Q_\varepsilon$ the self-adjoint operators associated with $(a_\varepsilon)_\varepsilon$*

and $(q_\varepsilon)_\varepsilon$*, respectively. Suppose that there is* $\lambda > 0$ *so that* $a_\varepsilon, q_\varepsilon \geq \lambda$*, for all* $\varepsilon > 0$*, and*

$$|a_\varepsilon(\psi) - q_\varepsilon(\psi)| \leq q(\varepsilon)\, q_\varepsilon(\psi), \qquad \forall \psi \in \mathcal{D}, \tag{8}$$

with $q(\varepsilon) \to 0$ *as* $\varepsilon \to 0$*. Then, there exists* $\tilde{C} > 0$ *so that, for* $\varepsilon > 0$ *small enough,*

$$\|A_\varepsilon^{-1} - Q_\varepsilon^{-1}\| \leq \tilde{C}\, q(\varepsilon).$$

Now we discuss a generalization of such theorem, that it is related to the following setting. For each $\varepsilon > 0$, let a_ε be a sequence of positive closed sesquilinear forms in a separable Hilbert space $\mathcal{H}$, and A_ε the corresponding associated positive self-adjoint operators. Let $\mathcal{H}_\varepsilon$ be a closed subspace of $\mathcal{H}$ and $\mathcal{H}^\varepsilon$ the orthogonal complement of $\mathcal{H}_\varepsilon$ in $\mathcal{H}$. Then, through the decomposition $\mathcal{H} = \mathcal{H}_\varepsilon \oplus \mathcal{H}^\varepsilon$, each $\psi \in \mathcal{H}$ can be uniquely written as

$$\psi = \psi_\varepsilon + \psi^\varepsilon, \qquad \psi_\varepsilon \in \mathcal{H}_\varepsilon,\, \psi^\varepsilon \in \mathcal{H}^\varepsilon.$$

Suppose that $\psi \in \operatorname{dom} a_\varepsilon$ implies $\psi_\varepsilon \in \operatorname{dom} a_\varepsilon$; consequently, $\psi^\varepsilon \in \operatorname{dom} a_\varepsilon$ as well. Thus,

$$d_\varepsilon := \{\psi_\varepsilon : \psi \in \operatorname{dom} a_\varepsilon\} \quad \text{and} \quad d^\varepsilon := \{\psi^\varepsilon : \psi \in \operatorname{dom} a_\varepsilon\},$$

are dense subsets in $\mathcal{H}_\varepsilon$ and $\mathcal{H}^\varepsilon$ respectively.

Consider the family of restrictions

$$q_\varepsilon := a_\varepsilon|_{d_\varepsilon}, \qquad b_\varepsilon := a_\varepsilon|_{d^\varepsilon},$$

of quadratic forms in $\mathcal{H}_\varepsilon$ and $\mathcal{H}^\varepsilon$ respectively. Both q_ε and b_ε are positive closed quadratic forms. We denote by Q_ε and B_ε the corresponding associated self-adjoint operators. Thus, the quadratic form a_ε can be written as

$$a_\varepsilon(\psi) = q_\varepsilon(\psi_\varepsilon) + b_\varepsilon(\psi^\varepsilon) + 2\, m_\varepsilon(\psi_\varepsilon, \psi^\varepsilon), \tag{9}$$

and for parameters $p(\varepsilon) > 0, q(\varepsilon) > 0$, suppose that the following conditions hold true:

$$q_\varepsilon(\psi_\varepsilon) \geq c(\varepsilon)||\psi_\varepsilon||^2, \quad \forall \psi_\varepsilon \in d_\varepsilon, \quad c(\varepsilon) \geq c_0 > 0; \tag{10}$$

$$b_\varepsilon(\psi^\varepsilon) \geq p(\varepsilon)||\psi^\varepsilon||^2, \quad \forall \psi^\varepsilon \in d^\varepsilon; \tag{11}$$

$$|m_\varepsilon(\psi_\varepsilon, \psi^\varepsilon)|^2 \leq q(\varepsilon)^2\, q_\varepsilon(\psi_\varepsilon)\, b_\varepsilon(\psi^\varepsilon), \quad \forall \psi \in \operatorname{dom} a_\varepsilon; \tag{12}$$

$$q(\varepsilon) \to 0,\ p(\varepsilon) \to \infty,\ c(\varepsilon) = \mathrm{O}(p(\varepsilon)) \quad \text{as} \quad \varepsilon \to 0. \tag{13}$$

Theorem 2. *For* ε *small enough, there exists* $\tilde{D} > 0$*, so that,*

$$\big|\big|A_\varepsilon^{-1} - Q_\varepsilon^{-1} \oplus 0\big|\big| \leq p(\varepsilon)^{-1} + \tilde{D}\, q(\varepsilon) c(\varepsilon)^{-1}, \tag{14}$$

where 0 *is the null operator on the subspace* $\mathcal{H}^\varepsilon$*.*

Proof. Initially we are going to analyze the quadratic form

$$l_\varepsilon(\psi) := q_\varepsilon(\psi_\varepsilon) + b_\varepsilon(\psi^\varepsilon), \quad \psi \in \operatorname{dom} a_\varepsilon.$$

For each $\varepsilon > 0$, l_ε is a positive closed quadratic form. Let L_ε be the self-adjoint operator associated with l_ε. By conditions (10), (11) and (13), for ε small enough,

$$\left|\left|L_\varepsilon^{-1}\right|\right| \le D_1\, c(\varepsilon)^{-1},$$

for some $D_1 > 0$. The condition (12) and the definition of l_ε implies that

$$|m_\varepsilon(\psi_\varepsilon, \psi^\varepsilon)| \le q(\varepsilon)\, l_\varepsilon(\psi).$$

Consequently

$$|a_\varepsilon(\psi) - l_\varepsilon(\psi)| = 2\,|m_\varepsilon(\psi_\varepsilon, \psi^\varepsilon)| \le 2\, q(\varepsilon)\, l_\varepsilon(\psi).$$

For ε small enough, so that $q(\varepsilon) \le 1/4$, the above inequality implies that

$$1/2\, l_\varepsilon(\psi) \le a_\varepsilon(\psi) \le 3/2\, l_\varepsilon(\psi), \quad \forall \psi \in \operatorname{dom} a_\varepsilon.$$

Thus,

$$\left|\left|A_\varepsilon^{-1}\right|\right| \le 2\, D_1\, c(\varepsilon)^{-1}.$$

Condition (11) implies that

$$\left|\left|B_\varepsilon^{-1}\right|\right| \le p(\varepsilon)^{-1}. \tag{15}$$

For $\psi_1, \psi_2 \in \operatorname{dom} a_\varepsilon$ and ε small enough, we have

$$\left|\langle A_\varepsilon^{1/2}\psi_1, A_\varepsilon^{1/2}\psi_2\rangle - \langle L_\varepsilon^{1/2}\psi_1, L_\varepsilon^{1/2}\psi_2\rangle\right| = |a_\varepsilon(\psi_1,\psi_2) - l_\varepsilon(\psi_1,\psi_2)|$$

$$\le 2q(\varepsilon)\, (l_\varepsilon(\psi_1) l_\varepsilon(\psi_2))^{1/2} \le 2\sqrt{2}\, q(\varepsilon)\, (l_\varepsilon(\psi_1) a_\varepsilon(\psi_2))^{1/2}.$$

Picking $\psi_1 = L_\varepsilon^{-1}\xi$, $\psi_2 = A_\varepsilon^{-1}\zeta$, where ξ, ζ are arbitrary elements of $\mathcal{H}$, we have

$$\begin{aligned} \left|\langle A_\varepsilon^{-1}\xi, \zeta\rangle - \langle L_\varepsilon^{-1}\xi, \zeta\rangle\right| &\le 2\sqrt{2}\, q(\varepsilon) \left(\langle A_\varepsilon^{-1}\zeta, \zeta\rangle\, \langle L_\varepsilon^{-1}\xi, \xi\rangle\right)^{1/2} \\ &\le 4\, D_1\, q(\varepsilon)\, c(\varepsilon)^{-1} ||\xi||\, ||\zeta||. \end{aligned}$$

Therefore,

$$\left|\left|A_\varepsilon^{-1} - L_\varepsilon^{-1}\right|\right| \le 4\, D_1\, q(\varepsilon)\, c(\varepsilon)^{-1}. \tag{16}$$

Since $L_\varepsilon^{-1} = Q_\varepsilon^{-1} \oplus B_\varepsilon^{-1}$, we conclude that

$$\left|\left|L_\varepsilon^{-1} - Q_\varepsilon^{-1} \oplus 0\right|\right| = \left|\left|B_\varepsilon^{-1}\right|\right|,$$

where 0 is the null operator on the subspace $\mathcal{H}^\varepsilon$. Together with (15) and (16), this leads to (14). □

Remark 2. *There are versions of Theorems 1 and 2 for which the subspace $\mathcal{H}_\varepsilon$ and/or the operator Q_ε in fact do/does not depend on ε, say they are replaced by $\mathcal{H}_0$ and Q_0, respectively. The corresponding proofs are actually simpler than the one presented above.*

4.2 More about spatial tubes

The proposition below is a justification for considering the simpler action of t_ε instead of $\hat{t}_\varepsilon$, and it is a consequence of the fact that β_ε converges uniformly to 1 as $\varepsilon \to 0$. Its proof, along with the proofs of the other propositions in this section, will be the subject of Section 5.

Proposition 1. *Given $\mu > 0$, then for $\varepsilon > 0$ small enough, there exist $E_3, \tilde{E} > 0$ so that*

$$| \hat{t}_\varepsilon(\psi) - t_\varepsilon(\psi) | \le E_3 \, \varepsilon \, (t_\varepsilon + \mu)(\psi), \quad \psi \in \operatorname{dom} t_\varepsilon,$$

and

$$\left\| (\hat{T}_\varepsilon + \mu \mathbf{1})^{-1} - (T_\varepsilon + \mu \mathbf{1})^{-1} \right\|_{\mathrm{L}^2(I \times S)} \le \tilde{E} \, \varepsilon.$$

Now we recall some results of [4] we shall use ahead. For each $\xi \in \mathbb{R}^2$, consider the problem

$$-\mathrm{div}[(1 - \xi \cdot y)\nabla_y u] = \lambda(1 - \xi \cdot y)u, \qquad u \in \mathcal{H}_0^1(S).$$

By picking $\xi = \varepsilon k(x) z_\alpha$, for ε small enough this operator is positive and with compact resolvent. Denote by $\lambda(\xi) > 0$ its first eigenvalue, i.e.,

$$\lambda(\xi) = \inf_{\{u \in \mathcal{H}_0^1(S) \colon u \ne 0\}} \frac{\int_S (1 - \xi \cdot y) |\nabla_y u|^2 dy}{\int_S (1 - \xi \cdot y)|u|^2 dy}.$$

Thus, for $v \in \mathcal{H}_0^1(I \times S)$, for a.e.$[x]$ we have

$$\frac{1}{\varepsilon^2} \int_S \beta_\varepsilon(x, y)(|\nabla_y v|^2 - \lambda_0 |v|^2) dy \ge \gamma_\varepsilon(x) \int_S \beta_\varepsilon(x, y)|v|^2 dy, \qquad (17)$$

where

$$\gamma_\varepsilon(x) := \frac{\lambda\left(\varepsilon k(x) z_\alpha(x)\right) - \lambda_0}{\varepsilon^2}.$$

By taking into account that $k(x)$ is a bounded function, the following lemma was proven in [4]:

Lemma 1. $\gamma_\varepsilon(x) \to -k(x)^2/4$ *uniformly as* $\varepsilon \to 0$.

By (17), it is the clear that if we pick $c > \left\| k(x)^2/4 \right\|_\infty$, then the quadratic forms t_ε and $\hat{t}_\varepsilon$ are (strictly) positive for $\varepsilon > 0$ small enough, and from now on we suppose that this inequality holds. Hence, $(-\infty, 0]$ is contained in the resolvent sets of both T_ε and $\hat{T}_\varepsilon$.

Again, for $\xi \in \mathbb{R}^2$ fixed, consider the problem [4]

$$-\Delta u_\xi - \lambda_0 u_\xi = -\xi \cdot \nabla_y u_0, \qquad u_\xi \perp u_0. \qquad (18)$$

We recall that u_0 is the eigenfunction corresponding to the first eigenvalue λ_0 of the Dirichlet Laplacian in $\mathcal{H}_0^1(S)$. Denote by χ_1 and χ_2 the solutions to (18) for $\xi = (1,0)$ and $\xi = (0,1)$, respectively. By linearity, for $\xi = (\xi_1, \xi_2) \in \mathbb{R}^2$, the solution to (18) is given by

$$u_\xi = \xi_1 \chi_1 + \xi_2 \chi_2.$$

The following lemma will be useful ahead and it was also proven in [4].

Lemma 2. *For every $\xi \in \mathbb{R}^2$, we have*

$$\inf_{v \in H_0^1(S)} \int_S \Big[|\nabla_y v|^2 - \lambda_0 |v|^2 + 2(\xi \cdot \nabla_y u_0) v \Big] dy = -\frac{|\xi|^2}{4}. \tag{19}$$

Furthermore, the above infimum is reached for u_ξ given by (18).

Since (19) is a convex variational problem, its minimizer point u_ξ is unique. The function

$$\varphi_0(x,y) := k(x) \cos \alpha(x)\, \chi_1(y) - k(x) \sin \alpha(x)\, \chi_2(y) \tag{20}$$

is the solution to (18) for $\xi = k(x) z_\alpha$. We underline that both χ_1 and χ_2 are orthogonal to u_0 in $\mathrm{L}^2(S)$, thus φ_0 is orthogonal to u_0 as well. Such results from [4], that we have just reviewed, will be important for our considerations of the singular limit $\varepsilon \to 0$.

Therefore, for $\xi = k(x) z_\alpha$ the infimum in (19) is reached exactly at $\varphi_0(x,y)$ and such minimum value is $-k(x)^2/4$. Note that this function of curvature $k(x)$ is present in the operator (4) and also in the associated form (27), and this motivates our introduction of the specific subspaces $\mathcal{L}_0$ and $\mathcal{L}_\varepsilon$ below (and the different quadratic forms ahead as well).

First introduce the closed subspace $\mathcal{L}_0 := \{ w u_0 : w \in \mathrm{L}^2(I) \}$ of $\mathrm{L}^2(I \times S)$ and the orthogonal decomposition

$$\mathrm{L}^2(I \times S) = \mathcal{L}_0 \oplus \mathcal{L}_0^\perp. \tag{21}$$

Thus, given $\psi \in \mathrm{L}^2(I \times S)$, we can write

$$\psi(x,y) = w(x) u_0(y) + \eta(x,y), \tag{22}$$

for some unique $w \in \mathrm{L}^2(I)$ and unique $\eta \in \mathcal{L}_0^\perp$. Now, for each $\varepsilon > 0$, we introduce the subspace $\mathcal{L}_\varepsilon \subset \mathrm{L}^2(I \times S)$ of functions of the form

$$w(x) u_0(y) + \varepsilon w(x) \varphi_0(x,y), \qquad w \in \mathrm{L}^2(\mathbb{R}). \tag{23}$$

Using the characterization (22), we write each $\psi \in \mathrm{L}^2(I \times S)$ in the following form

$$\psi = (w u_0 + \varepsilon w \varphi_0) + (\eta - \varepsilon w \varphi_0), \tag{24}$$

with $(wu_0 + \varepsilon w\varphi_0) \in \mathcal{L}_\varepsilon$ and $(\eta - \varepsilon w\varphi_0) \in \mathcal{L}_0^\perp$; since $\mathcal{L}_\varepsilon \cap \mathcal{L}_0^\perp = \{0\}$, such decomposition is unique and so the Hilbert space $\mathrm{L}^2(I \times S)$ can be written as a (nonorthogonal) direct sum of the subspaces $\mathcal{L}_\varepsilon$ and $\mathcal{L}_0^\perp$.

For each $\varepsilon > 0$ define the quadratic form

$$l_\varepsilon(\psi) := t_\varepsilon(wu_0 + \varepsilon w\varphi_0) + t_\varepsilon(\eta - \varepsilon w\varphi_0), \tag{25}$$

$\operatorname{dom} l_\varepsilon = \mathcal{H}_0^1(I \times S)$, and denote by L_ε the associated self-adjoint operator.

Proposition 2. *Given $\mu > 0$, suppose that the derivatives of the functions $k(x), \tau(x)$ and $\alpha(x)$, up to second order, are continuous and bounded. Then, there exist $F_5, \tilde{F} > 0$ so that, for $\varepsilon > 0$ small enough,*

$$|t_\varepsilon(\psi) - l_\varepsilon(\psi)| \leq F_5\, \varepsilon\, (l_\varepsilon + \mu)(\psi), \quad \psi \in \operatorname{dom} t_\varepsilon,$$

and

$$\left\| (T_\varepsilon + \mu \mathbf{1})^{-1} - (L_\varepsilon + \mu \mathbf{1})^{-1} \right\|_{\mathrm{L}^2(I \times S)} \leq \tilde{F}\, \varepsilon.$$

Now each space $\mathcal{L}_\varepsilon$ can be identified with $\mathcal{L}_0$ through the mapping

$$\omega(x)\,(u_0(y) + \varepsilon\varphi_0(x,y)) \longmapsto \omega(x)u_0(x),$$

and $\mathcal{L}_0$ with $\mathrm{L}^2(\mathbb{R})$ through

$$\omega(x)u_0(y) \mapsto \omega(x).$$

We have

$$\begin{aligned} \|\omega\,(u_0 + \varepsilon\varphi_0)\,\|_{\mathcal{L}_\varepsilon}^2 &= \int_{\mathbb{R}\times S} |\omega(x)\,(u_0(y) + \varepsilon\varphi_0(x,y))\,|^2\, dydx \\ &= \int_{\mathbb{R}} (1 + \varepsilon^2 k(x)^2)|w(x)|^2 dx \\ &= \int_{\mathbb{R}\times S} (1 + \varepsilon^2 k(x)^2)|w(x)u_0(y)|^2 dydx, \end{aligned}$$

and thus

$$\|wu_0\|_{\mathcal{L}_0}^2 \leq \|\omega\,(u_0 + \varepsilon\varphi_0)\,\|_{\mathcal{L}_\varepsilon}^2 \leq (1 + \varepsilon^2\|k\|_\infty^2)\,\|wu_0\|_{\mathcal{L}_0}^2,$$

as well as

$$\|w\|_{\mathrm{L}^2(I)}^2 \leq \|\omega\,(u_0 + \varepsilon\varphi_0)\,\|_{\mathcal{L}_\varepsilon}^2 \leq (1 + \varepsilon^2\|k\|_\infty^2)\,\|w\|_{\mathrm{L}^2(I)}^2,$$

and by direct calculation,

$$\|w\|_{\mathrm{L}^2(I)} = \|wu_0\|_{\mathcal{L}_0}.$$

The subspaces $\mathcal{L}_\varepsilon$, $\mathcal{L}_0$ and $\mathrm{L}^2(I)$ are algebraically equivalent to each other, and also with equivalent norms; in addition, $(1 \pm \varepsilon^2\|k\|_\infty^2) \to 1$ uniformly as $\varepsilon \to 0$. Therefore it is possible to identify operators acting in $\mathcal{L}_\varepsilon$ with operators acting in $\mathcal{L}_0$, and then with operators acting in $\mathrm{L}^2(I)$. This is important in our dimensional reduction from the tube to the reference curve.

Restricted to $\mathcal{L}_\varepsilon$, that is, $\eta = \varepsilon w \varphi_0 \in \mathcal{L}_0^\perp$ (see (24)), we have

$$l_\varepsilon(\psi) = t_\varepsilon(wu_0 + \varepsilon w \varphi_0).$$

Due to the many identifications involved, it will be convenient to introduce the the following sequence s_ε of quadratic forms in $\mathcal{L}_0$,

$$s_\varepsilon(wu_0) := t_\varepsilon(wu_0 + \varepsilon w \varphi_0), \quad w \in \mathcal{H}_0^1(I), \tag{26}$$

as well as the form t in $\mathcal{L}_0$, $\operatorname{dom} t = \{wu_0 : w \in \mathcal{H}_0^1(I)\}$,

$$\begin{aligned} t(wu_0) &:= \int_{I\times S} \left[|w' u_0|^2 + \left(C(S)(\tau + \alpha')(x) - \frac{k(x)^2}{4} + c \right) |wu_0|^2 \right] dy dx \\ &= \int_I \left[|w'|^2 + \left(C(S)(\tau + \alpha')(x) - \frac{k(x)^2}{4} + c \right) |w|^2 \right] dx, \end{aligned} \tag{27}$$

whose associated self-adjoint operator is T described in (2) and (4), in case $I = [0, L]$ and $I = \mathbb{R}$, respectively, that can be thought of acting in either $\mathcal{L}_0$ or $\mathrm{L}^2(I)$.

We finish this subsection with a technical result that will be used ahead.

Lemma 3. For $\eta \in \mathcal{H}_0^1(I \times S) \cap \mathcal{L}_0^\perp$, there exists $\tilde{G} > 0$ so that, for ε small enough,

$$t_\varepsilon(\eta) \geq \frac{\tilde{G}}{\varepsilon^2} \|\eta\|^2.$$

Proof. Let λ_1 be the second eigenvalue of the Laplacian in $\mathcal{H}_0^1(S)$ and $\eta \in \mathcal{H}_0^1(I \times S) \cap \mathcal{L}_0^\perp$. It then follows that

$$\int_S \beta_\varepsilon(x,y) \left(\frac{|\nabla_y \eta|^2}{\varepsilon^2} - \lambda_1 \frac{|\eta|^2}{\varepsilon^2} \right) dy \geq \gamma_\varepsilon(x) \int_S \beta_\varepsilon(x,y) |\eta|^2 dy.$$

Since $\gamma_\varepsilon(x)$ converges uniformly to a bounded function as $\varepsilon \to 0$ (see Lemma 1), there exists $G_1 \in \mathbb{R}$ so that, for ε small enough,

$$\gamma_\varepsilon(x) \geq G_1, \qquad \forall x \in \mathbb{R}.$$

Thus,

$$\gamma_\varepsilon(x) \int_S \beta_\varepsilon(x,y)|\eta|^2 dy \geq G_1 \int_S \beta_\varepsilon(x,y)|\eta|^2 dy,$$

and so

$$\int_{I\times S} \beta_\varepsilon(x,y) \left(\frac{|\nabla_y \eta|^2}{\varepsilon^2} - \lambda_1 \frac{|\eta|^2}{\varepsilon^2} \right) dy dx \geq G_1 \int_{I\times S} \beta_\varepsilon(x,y)|\eta|^2 dy dx.$$

Adding and subtracting the term $\frac{\lambda_0}{\varepsilon^2}\int_{I\times S}\beta_\varepsilon(x,y)|\eta|^2 dydx$ on the left hand side of the inequality above, we obtain

$$\begin{aligned}\int_{I\times S} & \beta_\varepsilon(x,y) \left(\frac{|\nabla_y\eta|^2}{\varepsilon^2} - \lambda_0\frac{|\eta|^2}{\varepsilon^2}\right) dydx \\ & \geq \quad G_1\int_{I\times S}\beta_\varepsilon(x,y)|\eta|^2 dydx + \frac{(\lambda_1-\lambda_0)}{\varepsilon^2}\int_{I\times S}\beta_\varepsilon(x,y)|\eta|^2 dydx.\end{aligned}$$

Since $\beta_\varepsilon(x,y)\to 1$ uniformly as $\varepsilon\to 0$, for $\varepsilon>0$ small enough, there exist numbers $\delta_1,\delta_2\in\mathbb{R}$ so that $G_1\beta_\varepsilon(x,y)\geq\delta_1$ and $\beta_\varepsilon(x,y)\geq\delta_2>0$ for all $(x,y)\in I\times S$.

Therefore, for ε small enough, there exists $G_2>0$ so that

$$\begin{aligned} t_\varepsilon(\eta) & \geq \int_{I\times S}\beta_\varepsilon(x,y)\left(\frac{|\nabla_y\eta|^2}{\varepsilon^2}-\lambda_0\frac{|\eta|^2}{\varepsilon^2}\right)dydx + c\int_{\mathbb{R}\times S}|\eta|^2 dydx \\ & \geq \delta_1\int_{I\times S}|\eta|^2 dydx + \frac{(\lambda_1-\lambda_0)}{\varepsilon^2}\delta_2\int_{I\times S}|\eta|^2 dydx + c\int_{I\times S}|\eta|^2 dydx \\ & \geq \frac{G_2}{\varepsilon^2}\int_{I\times S}|\eta|^2 dydx.\end{aligned}$$

Lastly, it is enough to take $\tilde{G}=G_2$ to complete the proof of the lemma. □

In the next subsection a discussion of the main consequences of the above preparation is presented; in our opinion it is to be considered the main conclusions of this work.

4.3 Discussions

With respect to the results presented up to now, we discuss two points we have found compelling. The first one is a proof that if the reference curve has zero curvature, then the norm resolvent convergence of operators is in effect. The second one is a kind of counterexample to the converse of Theorems 1 and 2, a point of particular mathematical interest.

4.3.1 Norm convergence in case of zero curvature

The goal here is to apply Theorem 2 in order to prove norm resolvent convergence in case the curvature of the reference curve vanishes; it holds for bounded or unbounded tubes, but for definiteness we suppose that $I=\mathbb{R}$; in fact, for bounded tubes it was proven in [8] that such norm resolvent convergence holds for any curvature such that $k(x)\in \mathrm{L}^\infty[0,L]$. As already mentioned, a proof of norm convergence in case of unbounded tubes and nonvanishing curvature is still lacking.

Observe that if $k(x)\equiv 0$, then automatically $r(x)$ is a straight line (here it is then supposed to coincide with the x-axis) and the torsion $\tau(x)$

also vanishes. For fixed $\varepsilon > 0$, the asymptotic terms of the sequence of eigenvalues of the Laplacian in such tube was considered in [5].

Our norm resolvent convergence is as follows.

Theorem 3. *Suppose that $k(x) = 0$ for all $x \in \mathbb{R}$. Then, there exists $\tilde{J} > 0$ so that, for ε small enough,*

$$\left\|T_\varepsilon^{-1} - T^{-1} \oplus 0\right\| \leq \tilde{J}\varepsilon,$$

where 0 denotes the null operator on the subspace $\mathcal{L}_0^\perp$.

Proof. By the orthogonal decomposition (21)–(22), $\psi \in \operatorname{dom} t_\varepsilon = \mathcal{H}_0^1(\mathbb{R}\times S)$ may be written in the form

$$\psi(s, y) = w(s)u_0(y) + \eta(s, y),$$

with $w \in \mathcal{H}^1(\mathbb{R})$ and $\eta \in \mathcal{H}_0^1(\mathbb{R} \times S) \cap \mathcal{L}_0^\perp$. Thus, the quadratic form $t_\varepsilon(\psi)$ can be rewritten as

$$t_\varepsilon(\psi) = t_\varepsilon(wu_0) + t_\varepsilon(\eta) + 2\, m_\varepsilon(wu_0, \eta),$$

where

$$m_\varepsilon(wu_0, \eta) = \int_{\mathbb{R}\times S} \Big[w' u_0 + w\,(\nabla_y u_0 \cdot Ry)\,\alpha'(x)\Big]\Big[\eta' + (\nabla_y \eta \cdot Ry)\,\alpha'(x)\Big] dxdy.$$

We are going to show that $t_\varepsilon(wu_0)$, $t_\varepsilon(\eta)$ and $m_\varepsilon(wu_0, \eta)$ satisfy the conditions (10)–(13), and then we apply Theorem 2.

A direct substitution shows that

$$\begin{aligned} t_\varepsilon(wu_0) &= \int_{\mathbb{R}} \Big[|w'|^2 + C(S)\alpha'(x)^2|w|^2 + c|w|^2\Big] dx \\ &\geq c\int_{\mathbb{R}} |w|^2 dx = c\int_{\mathbb{R}\times S} |wu_0|^2 dxdy. \end{aligned}$$

Lemma 3 guarantees that there exists $\tilde{G} > 0$ so that

$$t_\varepsilon(\eta) \geq \frac{\tilde{G}}{\varepsilon^2}\int_{\mathbb{R}\times S} |\eta|^2 dxdy.$$

Thus, the conditions (10), (11), (13) are satisfied. Now, we are going to show the condition (12). Since

$$\int_S \eta'(x, y)u_0(y)\, dy = 0, \qquad \text{a.e.[x]},$$

we have

$$\begin{aligned} m_\varepsilon(wu_0, \eta) &= \int_{\mathbb{R}\times S} w' u_0\,(\nabla_y \eta \cdot Ry)\,\alpha'(x)\, dxdy \\ &+ \int_{\mathbb{R}\times S} w\,(\nabla_y u_0 \cdot Ry)\,\eta'\alpha'(x)\, dxdy \\ &+ \int_{\mathbb{R}\times S} w\,(\nabla_y u_0 \cdot Ry)\,(\nabla_y \eta \cdot Ry)\,(\alpha'(x))^2\, dxdy \end{aligned}$$

Upon integration by parts and Lemma 3, it follows that

$$\begin{aligned}
&\left|\int_{\mathbb{R}\times S} w' u_0 \, (\nabla_y \eta \cdot Ry)\, \alpha'(x)\, dxdy\right| = \left|\int_{\mathbb{R}\times S} w' \, (\nabla_y u_0 \cdot Ry)\, \eta\, \alpha'(x)\, dxdy\right| \\
\leq\; & C_1 \left(\int_{\mathbb{R}\times S} |w' u_0|^2 dxdy\right)^2 \left(\int_{\mathbb{R}\times S} |\eta|^2\, dxdy\right)^2 \\
\leq\; & J_1\, \varepsilon\, t_\varepsilon(w u_0)^{1/2}\, t_\varepsilon(\eta)^{1/2},
\end{aligned}$$

for some $J_1 > 0$. Repeating this idea with the remaining integrals, we obtain

$$|m_\varepsilon(w u_0, \eta)| \leq J_2\, \varepsilon\, t_\varepsilon(w u_0)^{1/2}\, t_\varepsilon(\eta)^{1/2},$$

for some number $J_2 > 0$. Hence, the result follows by Theorem 2. □

4.3.2 Counterexample

In roughly terms, Theorems 1 and 2 tell us that a uniform convergence of positive quadratic forms implies a norm resolvent convergence of the corresponding (lower bounded) self-adjoint operators. A natural question is if the norm resolvent convergence of a sequence of positive (or uniformly lower bounded) operators implies the uniform convergence of the respective quadratic forms; we answer such question in the negative through an example.

For bounded tubes, that is, $I = [0, L]$, it was proven in [8] (see Theorem 7 and its proof therein) that $\hat{T}_\varepsilon$ converges to T (see (4) and (27)) in the norm resolvent sense; combined with Proposition 1, it follows that T_ε converges to T in the norm resolvent sense. Recall that t_ε and t denote the respective sesquilinear forms. Nevertheless, now we perform a study of the convergence of $t_\varepsilon(\psi)$ in the case of nonzero curvature of the reference curve; we will make use the quadratic form s_ε, introduced in (26), and we have the following results:

Proposition 3. *For each $\psi \in \mathcal{H}_0^1(I \times S)$, take account of the decomposition* (24)*.*

(i) If ψ is so that $\eta \neq 0$ and $\eta \neq \varepsilon w \varphi_0$, for all $\varepsilon > 0$ small enough, that is, ψ has a nonzero component in $\mathcal{L}_\varepsilon^\perp$, then there is $\tilde{K} > 0$ with

$$t_\varepsilon(\eta - \varepsilon \omega \varphi_0) \geq \frac{\tilde{K}}{\varepsilon^2} \|\eta - \varepsilon \omega \varphi_0\|^2,$$

and thus $\lim_{\varepsilon\to 0} t_\varepsilon(\psi) = +\infty$.

(ii) If ψ is so that $\eta = 0$, that is $\psi = \omega u_0 \in \mathcal{L}_0$, then

$$\begin{aligned}
\lim_{\varepsilon\to 0} t_\varepsilon(\psi) &= \int_{I\times S} \left[|w' u_0|^2 + \big(C(S)(\tau + \alpha') + c\big)\, |w u_0|^2\right] dydx \\
&= \int_I \left[|w'|^2 + \big(C(S)(\tau + \alpha') + c\big)\, |w|^2\right] dx \\
&\neq t(\psi).
\end{aligned}$$

Now we discuss a missing case in Proposition 3, that is, for nonzero curvature what happens with $t_\varepsilon(\psi_\varepsilon)$, for $\psi_\varepsilon \in \mathcal{L}_\varepsilon$, as $\varepsilon \to 0$.

Proposition 4. *Let $\mu > 0$ and suppose that the derivatives of the functions $k(x), \tau(x)$ and $\alpha(x)$, up to second order, are continuous and bounded. Then, there is $\tilde{M} > 0$ so that for all $w \in \mathcal{H}_0^1(I)$ one has*

$$|s_\varepsilon(wu_0) - t(wu_0)| \le \tilde{M}\,\varepsilon\,(t+\mu)(wu_0).$$

Therefore (see (27)), for any vector $(wu_0+\varepsilon w\varphi_0) \in \mathcal{L}_\varepsilon$ one has $\lim_{\varepsilon\to 0}(wu_0+\varepsilon w\varphi_0) = wu_0$ in $\mathrm{L}^2(I\times S)$ and

$$\lim_{\varepsilon\to 0} t_\varepsilon(wu_0+\varepsilon w\varphi_0) = t(wu_0).$$

Hence, in this example the situation is more severe than just nonuniform convergence of quadratic forms, since by Propositions 3 and 4, there is a kind of discontinuity in the convergence to vectors in $\mathcal{L}_0$, in the sense that

$$t(\omega u_0) = \lim_{\varepsilon\to 0} t_\varepsilon(wu_0+\varepsilon w\varphi_0) \neq \lim_{\varepsilon\to 0} t_\varepsilon(wu_0), \quad \forall w \in \mathcal{H}_0^1[0,L].$$

The missing term in $\lim_{\varepsilon\to 0} t_\varepsilon(wu_0)$, with respect to $t(\omega u_0)$, is directly related to the curvature $k(x)$, and when it is zero we have uniform convergence as previously discussed in Theorem 3. We think the possibility of this kind of phenomenon should not be discarded due to the singular nature of the involved limits, and it is also a way to take into account some subtle contributions of the "discarded dimensions." Otherwise it would be merely enough to write down the relevant ε-dependent quadratic form ($\hat{t}_\varepsilon$ in our case) and apply it to elements of $\mathcal{L}_0$ before taking $\varepsilon \to 0$, but sometimes this is not the case.

5 Proofs of the propositions

Proof of Proposition 1

Initially, observe that

$$(\hat{t}_\varepsilon+\mu)(\psi) \ge \mu\|\psi\|^2 \qquad \text{and} \qquad (t_\varepsilon+\mu)(\psi) \ge \mu\|\psi\|^2.$$

Consequently,

$$\|(\hat{T}_\varepsilon+\mu\mathbf{1})^{-1}\| \le \frac{1}{\mu} \qquad \text{and} \qquad \|(T_\varepsilon+\mu\mathbf{1})^{-1}\| \le \frac{1}{\mu}.$$

Since $\beta_\varepsilon \to 1$ uniformly as $\varepsilon \to 0$, for $\varepsilon > 0$ small enough, there exists $\sigma_1 > 0$ so that $\sigma_1 \le \beta_\varepsilon$. Thus, since $k \in \mathrm{L}^\infty(\mathbb{R})$, $y \in S$ and S is a bounded region, there exist $E_1, E_2 > 0$ so that

$$\left|\left(\frac{1}{\beta_\varepsilon}-1\right)\right| = \left|\frac{\varepsilon k(x)(y\cdot z_\alpha(x))}{\beta_\varepsilon}\right| \le E_1\varepsilon,$$

and

$$c|(\beta_\varepsilon - 1)| \le E_2\varepsilon,$$

for $\varepsilon > 0$ small enough. Under these conditions

$$\begin{aligned}
|\,\hat{t}_\varepsilon(\psi) &- t_\varepsilon(\psi)\,| = |\,(\hat{t}_\varepsilon + \mu)(\psi) - (t_\varepsilon + \mu)(\psi)\,| \\
&\le \int_{\mathbb{R}\times S} \left|\left(\frac{1}{\beta_\varepsilon} - 1\right)\right| |\psi' + \nabla_y\psi \cdot Ry(\tau + \alpha')|^2 dydx \\
&+ \int_{\mathbb{R}\times S} c|(\beta_\varepsilon - 1)||\psi|^2 dydx \\
&\le E_1\varepsilon \int_{\mathbb{R}\times S} |\psi' + \nabla_y\psi \cdot Ry(\tau + \alpha')|^2 dydx + E_2\varepsilon \int_{\mathbb{R}\times S} |\psi|^2 dydx \\
&\le E_3\,\varepsilon\,(t_\varepsilon + \mu)(\psi)
\end{aligned}$$

for some $E_3 > 0$. Apply Theorem 1 to complete the proof of the proposition.

Proof of Proposition 2

Due to decomposition (21), each $\psi \in \mathcal{H}_0^1(\mathbb{R}\times S)$ can be written in the form

$$\psi = wu_0 + \eta, \qquad w \in \mathcal{H}^1(\mathbb{R}),\ \eta \in \mathcal{H}_0^1(\mathbb{R}\times S) \cap \mathcal{L}_0^\perp,$$

and consequently

$$\psi = wu_0 + \varepsilon w\varphi_0 + \eta_\varepsilon, \tag{28}$$

where $\eta_\varepsilon = \eta - \varepsilon w\varphi_0$.

By using (28), the quadratic form $(t_\varepsilon + \mu)(\psi)$ can be rewritten as

$$(t_\varepsilon + \mu)(\psi) = (t_\varepsilon + \mu)(wu_0 + \varepsilon w\varphi_0) + (t_\varepsilon + \mu)(\eta_\varepsilon) + 2\,m_\varepsilon(wu_0 + \varepsilon w\varphi_0, \eta_\varepsilon),$$

where

$$\begin{aligned}
m_\varepsilon(wu_0 &+ \varepsilon w\varphi_0, \eta_\varepsilon) \\
&= \int_{\mathbb{R}\times S} dydx \Big\{ \left[(wu_0 + \varepsilon w\varphi_0)' + \nabla_y(wu_0 + \varepsilon w\varphi_0) \cdot Ry(\tau + \alpha')\right] \\
&\quad \times \left[\eta_\varepsilon' + \nabla_y\eta_\varepsilon \cdot Ry(\tau + \alpha')\right] \Big\} \\
&+ \int_{\mathbb{R}\times S} \frac{\beta_\varepsilon(x,y)}{\varepsilon^2} \left[\nabla_y(wu_0 + \varepsilon w\varphi_0)\nabla_y\eta_\varepsilon - \lambda_0(wu_0 + \varepsilon w\varphi_0)\eta_\varepsilon\right] dydx \\
&+ \int_{\mathbb{R}\times S} (c + \mu)(wu_0 + \varepsilon w\varphi_0)\eta_\varepsilon\, dydx.
\end{aligned}$$

By definition

$$(l_\varepsilon + \mu)(\psi) = (t_\varepsilon + \mu)(wu_0 + \varepsilon w\varphi_0) + (t_\varepsilon + \mu)(\eta_\varepsilon);$$

thus,

$$(t_\varepsilon + \mu)(\psi) - (l_\varepsilon + \mu)(\psi) = 2\,m_\varepsilon(wu_0 + \varepsilon w\varphi_0, \eta_\varepsilon).$$

Using integration by parts and the fact that all the derivatives of the functions $k(x), \tau(x), \alpha(x)$, up to second order, are defined and bounded functions, some calculations (rather long) show that there exist numbers $F_1, F_2, F_3, F_4 > 0$ so that

$$\begin{aligned} m_\varepsilon(wu_0 + \varepsilon w\varphi_0, \eta_\varepsilon) \;&\leq\; F_1 \left(\int_{\mathbb{R}} |w'|^2 dx\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}\times S} |\eta_\varepsilon|^2 dydx\right)^{\frac{1}{2}} \\ &+\; F_2 \left(\int_{\mathbb{R}} |w|^2 dx\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}\times S} |\eta_\varepsilon|^2 dydx\right)^{\frac{1}{2}}, \end{aligned}$$

and, for $\varepsilon > 0$ small enough,

$$(t_\varepsilon + \mu)(wu_0 + \varepsilon w\varphi_0) \geq F_3 \int_{\mathbb{R}} |w|^2 dx, \tag{29}$$

$$(t_\varepsilon + \mu)(wu_0 + \varepsilon w\varphi_0) \geq F_4 \int_{\mathbb{R}} |w'| dx. \tag{30}$$

Since $\eta_\varepsilon \in \mathcal{H}_0^1(\mathbb{R} \times S) \cap \mathcal{L}_0^\perp$, we can apply Lemma 3 to get

$$(t_\varepsilon + \mu)(\eta_\varepsilon) \geq t_\varepsilon(\eta_\varepsilon) \geq \frac{\tilde{G}}{\varepsilon^2} \int_{\mathbb{R}\times S} |\eta_\varepsilon|^2 \, dydx. \tag{31}$$

By (29), (30) and (31), it follows that

$$\begin{aligned} 2\,|\, m_\varepsilon(wu_0 \;&+\; \varepsilon w\varphi_0, \eta_\varepsilon)| \\ &\leq\; F_5\, \varepsilon\, (t_\varepsilon + \mu)(wu_0 + \varepsilon w\varphi_0)^{1/2}\, (t_\varepsilon + \mu)(\eta_\varepsilon)^{1/2}, \end{aligned} \tag{32}$$

for some number $F_5 > 0$.

Now, observe that

$$(t_\varepsilon + \mu)(wu_0 + \varepsilon w\varphi_0) \leq (l_\varepsilon + \mu)(\psi),$$

and

$$(t_\varepsilon + \mu)(\eta_\varepsilon) \leq (l_\varepsilon + \mu)(\psi).$$

Thus, the inequality (32) can be rewritten as

$$\begin{aligned} |(t_\varepsilon + \mu)(\psi) - (l_\varepsilon + \mu)(\psi)| \;&=\; 2\,|\, m_\varepsilon(wu_0 + \varepsilon w\varphi_0, \eta_\varepsilon)| \\ &\leq\; F_5\, \varepsilon\, (l_\varepsilon + \mu)(\psi). \end{aligned} \tag{33}$$

An application of Theorem 1 completes the proof.

Proof of Proposition 3

For definiteness we assume that $I = \mathbb{R}$; the same proof applies if $I = [0, L]$. For $\psi \in \mathcal{H}_0^1(\mathbb{R} \times S) \cap \mathrm{L}^2(\mathbb{R} \times S)$ we shall use the decomposition (24). To prove (i), observe first that

$$l_\varepsilon(\psi) = t_\varepsilon(wu_0 + \varepsilon w\varphi_0) + t_\varepsilon(\eta - \varepsilon w\varphi_0) \geq t_\varepsilon(\eta - \varepsilon w\varphi_0).$$

By Lemma 3,

$$t_\varepsilon(\eta - \varepsilon w\varphi_0) \geq \frac{\tilde{G}}{\varepsilon^2} \int_{\mathbb{R}\times S} |\eta - \varepsilon w\varphi_0|^2 dydx.$$

Since it is supposed that $\eta \neq 0$ and $\eta \neq \varepsilon w\varphi_0$, we obtain

$$\lim_{\varepsilon\to 0} l_\varepsilon(\psi) \geq \lim_{\varepsilon\to 0} t_\varepsilon(\eta - \varepsilon w\varphi_0) = +\infty.$$

Now we are going to prove (ii). By Lemma 2,

$$\int_S \Big[|\nabla_y \varphi_0|^2 - \lambda_0|\varphi_0|^2 + 2k(x)(z_\alpha(x)\cdot\nabla_y u_0)\varphi_0\Big] dy = -\frac{k(x)^2}{4}. \tag{34}$$

This equality and some calculations imply that

$$\lim_{\varepsilon\to 0} t_\varepsilon(wu_0 + \varepsilon w\varphi_0) = t(wu_0).$$

On the other hand, the function φ_0 satisfy

$$-\Delta\varphi_0 - \lambda_0\varphi_0 = -k(x)(z_\alpha(x)\cdot\nabla_y u_0). \tag{35}$$

Equations (34) and (35) imply that

$$\int_S \left(|\nabla_y\varphi_0|^2 - \lambda_0|\varphi_0|^2\right) dydx = \frac{k(x)^2}{4}.$$

Again, this equality and some calculations imply that

$$\lim_{\varepsilon\to 0} t_\varepsilon(-\varepsilon w\varphi_0) = \int_{\mathbb{R}\times S} \frac{k(x)^2}{4}|w|^2|u_0|^2 dydx.$$

Therefore

$$\lim_{\varepsilon\to 0} l_\varepsilon(\psi) = \int_{\mathbb{R}\times S} \left[|w'u_0|^2 + \big(C(S)(\tau+\alpha') + c\big)\,|wu_0|^2\right] dydx.$$

Proof of Proposition 4

Since $\int_S |u_0|^2 dy = 1$ and $C(S) = \int_S |\nabla_y u_0 \cdot Ry|^2 dy$, one can rewrite $t(wu_0)$ in the following form

$$\begin{aligned} t(wu_0) &= \int_{\mathbb{R}} \left[|w'|^2 + \left((\tau(x) + \alpha'(x))^2 C(S) - \frac{k(x)^2}{4} + c \right) |w|^2 \right] dx \\ &= \int_{\mathbb{R}\times S} \left[|w'|^2 |u_0|^2 + |\nabla_y u_0 \cdot Ry|^2 (\tau(x) + \alpha'(x))^2 |w|^2 \right] dy dx \\ &+ \int_{\mathbb{R}\times S} \left(-\frac{k(x)^2}{4} + c \right) |w|^2 |u_0|^2 dy dx. \end{aligned}$$

Step 1. We begin with some observations. The Dirichlet condition at the boundary ∂S implies $\int_S \nabla_y |u_0|^2 dy = 0$. Consequently

$$\int_S u_0 \nabla_y u_0 \cdot Ry \, dy = 0.$$

By definition of $\varphi_0(x,y)$, one sees that

$$\int_S \varphi_0(x,y) u_0(y) dy = 0 \quad \text{and} \quad \int_S \varphi_0'(x,y) u_0(y) dy = 0 \quad \text{a.e.}[x].$$

Now, some calculations show that

$$\begin{aligned} &\Big| \int_{\mathbb{R}\times S} |(wu_0 + \varepsilon w\varphi_0)' + \nabla_y (wu_0 + \varepsilon w \varphi_0) \cdot Ry(\tau + \alpha')|^2 dy dx \\ &- \int_{\mathbb{R}\times S} \left[|w'|^2 |u_0|^2 + |\nabla_y u_0 \cdot Ry|^2 (\tau + \alpha')^2 |w|^2 \right] dy dx \Big| \\ = &\Big| \int_{\mathbb{R}\times S} \varepsilon^2 |w'|^2 |\varphi_0|^2 \, dy dx + \int_{\mathbb{R}\times S} \varepsilon^2 |w|^2 |\varphi_0'|^2 \, dy dx \\ + &\int_{\mathbb{R}\times S} \varepsilon^2 |w|^2 |\nabla_y \varphi_0 \cdot Ry|^2 (\tau + \alpha')^2 \, dy dx \\ + &\int_{\mathbb{R}\times S} 2\varepsilon w' w u_0 (\nabla_y \varphi_0 \cdot Ry)(\tau + \alpha') \, dy dx \\ + &\int_{\mathbb{R}\times S} 2\varepsilon^2 w' w \varphi_0 \varphi_0' \, dy dx + \int_{\mathbb{R}\times S} 2\varepsilon w' w \varphi_0 (\nabla_y u_0 \cdot Ry)(\tau + \alpha') \, dy dx \\ + &\int_{\mathbb{R}\times S} 2\varepsilon^2 w' w \varphi_0 (\nabla_y \varphi_0 \cdot Ry)(\tau + \alpha') \, dy dx \\ + &\int_{\mathbb{R}\times S} 2\varepsilon |w|^2 \varphi_0' (\nabla_y u_0 \cdot Ry)(\tau + \alpha') \, dy dx \\ + &\int_{\mathbb{R}\times S} 2\varepsilon^2 |w|^2 \varphi_0' (\nabla_y \varphi_0 \cdot Ry)(\tau + \alpha') \, dy dx \\ + &\int_{\mathbb{R}\times S} 2\varepsilon |w|^2 (\nabla_y u_0 \cdot Ry)(\nabla_y \varphi_0 \cdot Ry)(\tau + \alpha')^2 \, dy dx \Big| \end{aligned}$$

Observe that all the above integrals in the variable y are bounded functions of the variable x. For instance, the function

$$
\begin{aligned}
g(x) &:= \int_S \varphi_0(x,y)\nabla_y u_0 \cdot Ry\,dy \\
&= k(x)\cos\alpha(x)\int_S \chi_1(y)\left(\frac{\partial u_0}{\partial y_1}y_2 - \frac{\partial u_0}{\partial y_2}y_1\right)dy \\
&- k(x)\sin\alpha(x)\int_S \chi_2(y)\left(\frac{\partial u_0}{\partial y_1}y_2 - \frac{\partial u_0}{\partial y_2}y_1\right)dy
\end{aligned}
$$

is bounded in $\mathbb{R}$ since $\|k\|_\infty < \infty$. Using these properties, there exist numbers $M_1, M_2, M_3 > 0$ so that

$$
\begin{aligned}
&\Bigg| \int_{\mathbb{R}\times S} |(wu_0+\varepsilon w\varphi_0)' + \nabla_y(wu_0+\varepsilon w\varphi_0)\cdot Ry(\tau+\alpha')|^2 dydx \\
&\quad - \int_{\mathbb{R}\times S}\left[|w'|^2|u_0|^2 + |\nabla_y u_0\cdot Ry|^2(\tau+\alpha')^2|w|^2\right]dydx\Bigg| \\
&\leq M_1\varepsilon\int_{\mathbb{R}}|w'|^2dx + M_2\varepsilon\int_{\mathbb{R}}|w|^2dx + M_3\varepsilon\left(\int_{\mathbb{R}}|w|^2dx\right)^{1/2}\left(\int_{\mathbb{R}}|w'|^2dx\right)^{1/2}.
\end{aligned}
$$

Step 2. We shall apply Lemma 2. A direct replacement, using the definition of $\varphi_0(x,y)$, shows that

$$
\begin{aligned}
&\int_{\mathbb{R}\times S}\left[(|\nabla_y\varphi_0|^2 - \lambda_0|\varphi_0|^2) + 2k(x)(z_\alpha(x)\cdot\nabla_y u_0)\varphi_0\right]|w|^2dydx \\
&\qquad = -\int_{\mathbb{R}\times S}\frac{k(x)^2}{4}|w|^2|u_0|^2dydx.
\end{aligned}
$$

Thus, one sees that there exists $M_4 > 0$ so that

$$
\begin{aligned}
&\left|\int_{\mathbb{R}\times S}\frac{\beta_\varepsilon}{\varepsilon^2}(|\nabla_y\psi|^2 - \lambda_0|\psi|^2)dydx + \int_{\mathbb{R}}\frac{k(x)^2}{4}|w|^2|u_0|^2dydx\right| \\
= &\Bigg|\int_{\mathbb{R}\times S}\left[\beta_\varepsilon(x,y)(|\nabla_y\varphi_0|^2 - \lambda_0|\varphi_0|^2) + 2k(x)(z_\alpha(x)\cdot\nabla_y u_0)\varphi_0\right]|w|^2dydx \\
+ &\int_{\mathbb{R}\times S}\frac{k(x)^2}{4}|w|^2|u_0|^2dydx\Bigg| \\
= &\left|\int_{\mathbb{R}\times S}\varepsilon k(x)(y\cdot z_\alpha(x))(|\nabla_y\varphi_0|^2 - \lambda_0|\varphi_0|^2)|w|^2dydx\right| \\
\leq &\ M_4\,\varepsilon\int_{\mathbb{R}}|w|^2dx.
\end{aligned}
$$

Again we have used the fact that

$$
h(x) := k(x)\int_S (y\cdot z_\alpha(x))\,(|\nabla_y\varphi_0|^2 - \lambda_0|\varphi_0|^2)\,dy,
$$

is a bounded function. With this we conclude step 2.

By steps 1 and 2, there exist numbers $M_5, M_6, M_7 > 0$ so that

$$\begin{aligned} |s_\varepsilon(wu_0) \quad &- \quad t(wu_0)| = |(s_\varepsilon + \mu)(wu_0) - (t+\mu)(wu_0)| \\ &\leq \quad M_5\, \varepsilon \int_{\mathbb{R}} |w'|^2 dx + M_6\, \varepsilon \int_{\mathbb{R}} |w|^2 dx \\ &+ \quad M_7\, \varepsilon \left(\int_{\mathbb{R}} |w|^2 dx \right)^{1/2} \left(\int_{\mathbb{R}} |w'|^2 dx \right)^{1/2} . \end{aligned}$$

Observe that $\int_{\mathbb{R}} |w'|^2 dx \leq (t+\mu)(wu_0)$ and

$$\int_{\mathbb{R}} |w|^2 dx = \frac{1}{\mu} \int_{\mathbb{R}} \mu |w|^2 dx \leq \frac{1}{\mu} (t+\mu)(wu_0).$$

Thus, there exists $M_8 > 0$, so that

$$|(s_\varepsilon + \mu)(wu_0) - (t+\mu)(wu_0)| \leq$$

$$M_5\, \varepsilon\, (t+\mu)(wu_0) + \frac{M_6}{\mu}\, \varepsilon\, (t+\mu)(wu_0) + \frac{M_7}{\mu^{1/2}}\, \varepsilon\, ((t+\mu)(wu_0))^{1/2} ((t+\mu)(wu_0))^{1/2}$$

$$\leq M_8\, \varepsilon\, (t+\mu)(wu_0)$$

This completes the proof of Proposition 4.

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