# ANALYTIC VECTORS IN LOCALLY INTEGRABLE STRUCTURES

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Abstract. In this note we introduce the sheaves of Gevrey vectors on a smooth manifold endowed with a locally integrable structure. We discuss the important case when the structure is hypocomplex and show that, in the real-analytic category, if the structure is analytic hypoelliptic then every analytic vector associated to it is a real-analytic function. We also discuss some related questions concerning the global situation as well as the regularity of the $s$-Gevrey vectors associated to the structure when $s > 1$.

## 1. Introduction

Let $P = P(x, D)$ be an analytic linear partial differential operator, of order $m \geq 1$, defined in an open subset $\Omega$ of $\mathbb{R}^N$, and let also $s \geq 1$. A distribution $u$ defined in $\Omega$ is called an *$s$-Gevrey vector for $P$* (or an *analytic vector for $P$* when $s = 1$), if $P^k u \in L^2_{\mathrm{loc}}(\Omega)$ for every $k = 0, 1, \ldots$ and, moreover, the estimates $\|P^k u\|_{L^2(K)} \leq C_K^{k+1} k!^{ms}$, $k = 0, 1, 2, \ldots$, hold for each compact set $K \subset \Omega$.

A similar concept can be introduced for systems of operators.

During the period 1970–1990 the problem of obtaining the precise Gevrey regularity of the $s$-Gevrey vectors for a given operator or system has attracted the attention of a good number of specialists in the field [BG], [BM], [BCM], [Da], [DaH], [G], [HM], [M], [N]. See also the survey [BCR] and the references therein.

After the development of the theory of locally integrable and hypo-analytic structures (cf. [T], [BCH]), new classes of systems of complex vector fields became relevant, and then it is natural to analyze to what extent the known regularity results for Gevrey vectors remain valid in this new situation. A first attempt was done in [CCP], where the authors study such questions for the so-called locally integrable structures of tube type.

One of the main purposes of this note is to present the notion of Gevrey vectors in an arbitrary locally integrable structure. This, of course, requires that such objects be defined intrinsically, a fact that is not completely obvious. Recall that a locally integrable structure on a smooth manifold $\Omega$ is the datum of a subbundle $\mathcal{V}$ of the complexified tangent bundle to $\Omega$ whose orthogonal bundle $\mathcal{V}^\perp$, which is now a subbundle of the complexified *cotangent* bundle to $\Omega$, is locally spanned by the differential of smooth functions. In the bundle $\mathfrak{T}(\mathcal{V}) = \mathbb{C} \otimes \mathrm{T}^*\Omega/\mathcal{V}^\perp$ we introduce a hermitian metric and from it, inspired by the standard construction of the Fock spaces in quantum mechanics [RS, p.53], we build a certain sequence of operators induced by the exterior derivative on $\Omega$, and the Gevrey

1991 *Mathematics Subject Classification.* Primary: 35B65; Secondary: 35F50.
*Key words and phrases.* Analytic vectors, locally integrable structures.
The second and third authors were also partially supported by CNPq.

vectors for $\mathcal{V}$ are then first introduced as the Gevrey vectors associated to such operators. They define sheaves over $\Omega$, and next we show that such sheaves indeed do not depend on the choice of the hermitian metric we started with.

All the preceding discussion is explained in sections 2 and 3. In Section 4 we turn our attention to the analytic vectors associated to $\mathcal{V}$ and obtain an easy representation for them (Theorem 4.1), which is applied in Section 5 to prove their regularity in the case when the structure $\mathcal{V}$ is hypocomplex (Theorem 5.1). Next, in Section 6, we derive the regularity of the analytic vectors when $\Omega$ and $\mathcal{V}$ are real-analytic: we prove that whenever $\mathcal{V}$ is analytic hypoelliptic then every analytic vector for $\mathcal{V}$ is a real-analytic function, which gives, in this set up, the analogous result as in the case [1] of principal type, analytic linear partial differential operators [BM]. Finally we conclude the work by remarking that both the global version of this result and its extension for $s$-Gevrey vectors when $s > 1$ are no longer true. These results certainly open the doors for future investigation, to which we hope to return.

## 2. Preliminaries

Let $\Omega$ be a smooth, paracompact manifold of dimension $n+m$ endowed with a locally integrable structure $\mathcal{V}$ of rank $n$. Thus $\mathcal{V}$ is a vector subbundle of $\mathbb{C}\mathrm{T}\Omega$ of rank $n$ whose orthogonal bundle $\mathcal{V}^\perp \subset \mathbb{C}\mathrm{T}^*\Omega$ is locally spanned by the differential of $m$ smooth functions. We denote by $\mathfrak{T}(\mathcal{V})$ the vector bundle $\mathbb{C}\mathrm{T}^*\Omega/\mathcal{V}^\perp$. Such bundle $\mathfrak{T}(\mathcal{V})$ has rank $n$ and the exterior derivative induces a first order operator

$$\mathrm{d}' : C^\infty(\Omega') \longrightarrow C^\infty(\Omega'; \mathfrak{T}(\mathcal{V})),\ \Omega' \subset \Omega \text{ open},$$

through the composition

$$C^\infty(\Omega') \xrightarrow{\mathrm{d}} C^\infty(\Omega'; \mathbb{C}\mathrm{T}^*\Omega) \longrightarrow C^\infty(\Omega'; \mathfrak{T}(\mathcal{V})),$$

where d stands for the exterior derivative acting on scalar functions, and the last arrow is induced by the projection map.

We recall that if $\mathfrak{E}$ is an arbitrary vector bundle over $\Omega$ then the operator $\mathrm{d}'$ induces an operator

$$\mathrm{d}' \otimes I : C^\infty(\Omega'; \mathfrak{E}) \longrightarrow C^\infty(\Omega'; \mathfrak{T}(\mathcal{V}) \otimes \mathfrak{E}),\ \Omega' \subset \Omega \text{ open}.$$

If we set, for $N = 1, 2, \dots$,

$$\mathfrak{T}_N(\mathcal{V}) = \underbrace{\mathfrak{T}(\mathcal{V}) \otimes \cdots \otimes \mathfrak{T}(\mathcal{V})}_{\text{N-times}},$$

taking $\mathfrak{E} = \mathfrak{T}_N(\mathcal{V})$, gives, for each $N = 1, 2, \dots$, an operator

$$\mathrm{D}_N : C^\infty(\Omega'; \mathfrak{T}_N(\mathcal{V})) \longrightarrow C^\infty(\Omega'; \mathfrak{T}_{N+1}(\mathcal{V})),\ \Omega' \subset \Omega \text{ open}.$$

For completeness we shall also write $\mathrm{D}_0 = \mathrm{d}'$ and $\mathfrak{T}_0(\mathcal{V}) = \mathbb{C}$. Finally, we shall also set $D^{(0)} =$ identity and, for $N \geq 1$,

$$\mathrm{D}^{(N)} : C^\infty(\Omega') \longrightarrow C^\infty(\Omega'; \mathfrak{T}_N(\mathcal{V})), \quad \mathrm{D}^{(N)} = \mathrm{D}_{N-1} \circ \cdots \mathrm{D}_1 \circ \mathrm{D}_0 \,.$$

[1]In general, analytic vectors for analytic hypoelliptic operators are not real-analytic functions ([G], [BCR]).

We now assume that $\mathfrak{T}(\mathcal{V})$ is endowed with a smooth hermitian metric $\mathfrak{h}$. Such hermitian metric induces a smooth hermitian metric $\mathfrak{h}_N$ on each of the bundles $\mathfrak{T}_N(\mathcal{V})$ ($\mathfrak{h}_1 = \mathfrak{h}$). From this we can define, for $u \in C(\Omega', \mathfrak{T}_N(\mathcal{V}))$, and $K \subset \Omega'$ compact, the norms

$$\|u\|_{K,\mathfrak{h}} = \sup_{A \in K} \{\mathfrak{h}_N(u(A), u(A))\}^{1/2} .$$

**Definition 2.1.** *Let $\Omega$ be a smooth manifold over which a locally integrable structure is defined. Assume that $\mathfrak{T}(\mathcal{V})$ is endowed with an hermitian metric $\mathfrak{h}$. Let also $s \geq 1$ and $\Omega' \subset \Omega$ open. We shall denote by $G^s_{\mathfrak{h}}(\Omega'; \mathcal{V})$ the space of all $u \in C(\Omega')$ such that $D^{(N)}u \in C(\Omega', \mathfrak{T}_N(\mathcal{V}))$ for every $N = 0, 1, \ldots$ and, for each $K \subset \Omega'$ compact, there is a constant $C = C(K) > 0$ such that*

$$\|\mathrm{D}^{(N)}u\|_{K,\mathfrak{h}} \leq C^{N+1} N!^s, \quad N = 0, 1, \ldots$$

In the next section we shall show that the sheaves $\Omega' \mapsto G^s_{\mathfrak{h}}(\Omega'; \mathcal{V})$ are indeed independent of the choice of the metric $\mathfrak{h}$.

## 3. Local expressions

We begin by recalling the standard coordinates and generators associated to a locally integrable structure (cf. [T, I.5] and [BCH, I.10]). Each point of $\Omega$ is the center of a coordinate system $(x_1, \ldots, x_m, t_1, \ldots, t_n)$, which can be assumed defined in a product $U = B \times \Theta$, where $B$ (respectively $\Theta$) is an open ball centered in the origin in $\mathbb{R}^m_x$ (respectively $\mathbb{R}^n_t$), over which there is defined a smooth vector-valued function $\Phi(x, t) = (\Phi_1(x, t), \ldots, \Phi_m(x, t))$ satisfying

$$\Phi(0, 0) = 0, \;\; \Phi_x(0, 0) = 0,$$

such that the differential of the functions

$$Z_k(x, t) = x_k + i\Phi_k(x, t), \qquad k = 1, \ldots, m,$$

span $\mathcal{V}^\perp$ over $U$.

If we define the vector fields

$$\mathrm{M}_k = \sum_{k'=1}^{m} \mu_{kk'}(x, t) \frac{\partial}{\partial x_{k'}}, \;\; k = 1, \ldots, m$$

characterized by the rule

$$\mathrm{M}_k Z_{k'} = \delta_{k,k'} \, , \quad k, k' = 1, \ldots, m,$$

then the complex vector fields

$$\mathrm{L}_j = \frac{\partial}{\partial t_j} - i \sum_{k=1}^{m} \frac{\partial \phi_k}{\partial t_j}(x, t) \mathrm{M}_k, \;\; j = 1, \ldots, n,$$

span $\mathcal{V}$ over $U$.

The following properties are easily checked:

(1) $\mathrm{L}_1, \ldots, \mathrm{L}_n, \mathrm{M}_1, \ldots, \mathrm{M}_m$ span $\mathbb{C}\mathrm{T}\Omega$ over $U$ and are pairwise commuting.
(2) $\mathrm{d}Z_1, \ldots, \mathrm{d}Z_m, \mathrm{d}t_1, \ldots, \mathrm{d}t_n$ span $\mathbb{C}\mathrm{T}^*\Omega$ over $U$.

Property (2) allows us to identify $\mathfrak{T}(\mathcal{V})|_U$ to the bundle spanned by the differential forms $\mathrm{d}t_1, \dots, \mathrm{d}t_n$ and the formula

$$\mathrm{d}u = \sum_{j=1}^{n} (\mathrm{L}_j u)\,\mathrm{d}t_j + \sum_{k=1}^{m} (\mathrm{M}_k u)\,\mathrm{d}Z_k\,, \quad u \in C^1(U),$$

allows us to express $\mathrm{d}'u$ as

$$\mathrm{d}'u = \sum_{j=1}^{n} (\mathrm{L}_j u)\,\mathrm{d}t_j\,.$$

Likewise we can identify $\mathfrak{T}_N(\mathcal{V})|_U$ to the bundle spanned by $\mathrm{d}t_{i_1} \otimes \cdots \otimes \mathrm{d}t_{i_N}$, where $i_1, \dots, i_N \in \{1, \dots, n\}$; we have

$$\mathrm{D}_N \left( \sum_{1 \le i_1, \dots, i_N \le n} u_{i_1 \cdots i_N} \mathrm{d}t_{i_1} \otimes \cdots \otimes \mathrm{d}t_{i_N} \right) = \sum_{j=1}^{n} \sum_{1 \le i_1, \dots, i_N \le n} (\mathrm{L}_j u_{i_1 \cdots i_N})\,\mathrm{d}t_j \otimes \mathrm{d}t_{i_1} \otimes \cdots \otimes \mathrm{d}t_{i_N}\,,$$

$$\mathrm{D}^{(N)} u = \sum_{1 \le i_1, \dots, i_N \le n} (\mathrm{L}_{i_1} \cdots \mathrm{L}_{i_N} u)\,\mathrm{d}t_{i_1} \otimes \cdots \otimes \mathrm{d}t_{i_N}\,.$$

We now assume that $\mathfrak{T}(\mathcal{V})$ is endowed with a smooth hermitian metric $\mathfrak{h}$. In the local coordinates just described, if $I, J \in \{1, \dots, n\}^N$, $I = (i_1, \dots, i_N)$, $J = (j_1, \dots, j_N)$, if we set

$$h^{(N)}_{IJ} = \mathfrak{h}_N(\mathrm{d}t_{i_1} \otimes \ldots \otimes \mathrm{d}t_{i_N}, \mathrm{d}t_{j_1} \otimes \ldots \otimes \mathrm{d}t_{j_N})$$

and if we take

$$u = \sum_{I}^{n} u_I \mathrm{d}t_{i_1} \otimes \ldots \otimes \mathrm{d}t_{i_N} \in C(U; \mathfrak{T}_N(\mathcal{V}))$$

we have

$$\|u\|_{K,\mathfrak{h}} = \sup_K \left\{ \sum_{I,J} h^{(N)}_{IJ} u_I \overline{u_J} \right\}^{1/2}.$$

Now by definition we have, if $h_{ij} = \mathfrak{h}(\mathrm{d}t_i, \mathrm{d}t_j)$, $1 \le i, j \le n$,

$$h^{(N)}_{IJ} = h_{i_1, j_1} h_{i_2, j_2} \cdots h_{i_N, j_N}\,.$$

**Lemma 3.1.** *For each $K \subset U$ compact there are constants $b > 0$, $B > 0$ such that*

$$b^N \sup_K \sum_I |u_I|^2 \le \|u\|^2_{K,\mathfrak{h}} \le B^N \sup_K \sum_I |u_I|^2, \quad u \in C(U; \mathfrak{T}_N(\mathcal{V})). \tag{1}$$

**Proof.** We have, by the Cauchy-Schwarz inequality,

$$\left| \sum_{I,J=1}^{n} h^{(N)}_{IJ} u_I \overline{u_J} \right| \le \left\{ \sum_{I,J} \left( h^{(N)}_{IJ} \right)^2 \right\}^{1/2} \left\{ \sum_I |u_I|^2 \right\} = \left\{ \sum_{i,j} h^2_{ij} \right\}^{N/2} \left\{ \sum_I |u_I|^2 \right\},$$

and hence we can take $B^2 = \sup_K \sum_{i,j} h_{ij}^2$.

For the first inequality in (1) we first notice that the matrix that represents the inverse of $\mathfrak{h}_N$ is given by $h^{i_N,j_N}\cdots h^{i_2,j_2}h^{i_1,j_1}$, where $h^{ij}$ denotes the inverse matrix of $h_{ij}$.

We reason pointwise. For this we denote

$$|u|^2 = \sum_I |u_I|^2, \qquad \langle u, v\rangle = \sum_I u_I \overline{v_I}\,.$$

We shall also denote by $H$ the linear operator $Hu = (w_I)_I$, where $w_I = \sum_J h_{IJ}^{(N)} u_J$. Then $H > 0$, which implies

$$|u|^2 = |H^{-1/2}H^{1/2}u|^2 \le \|H^{-1/2}\|^2 \langle Hu, u\rangle\,.$$

Hence, in order to complete the proof, it suffices to notice that

$$\|H^{-1/2}\|^2 = \|H^{-1}\| \le \left\{\sum_{i,j}\left(h^{i,j}\right)^2\right\}^{N/2}. \qquad \blacksquare$$

In the next statement we shall use the following notation: if $\alpha \in \mathbb{Z}_+^n$ is a multi-index we shall write $\mathrm{L}^\alpha = \mathrm{L}_1^{\alpha_1}\ldots\mathrm{L}_n^{\alpha_n}$.

**Corollary 3.1.** *Fix an hermitian metric $\mathfrak{h}$ on $\mathfrak{T}(\mathcal{V})$. Let $u \in C(U)$. Then $u \in G^s_{\mathfrak{h}}(U;\mathcal{V})$ if and only if $\mathrm{L}^\alpha u \in C(U)$ for every $\alpha \in \mathbb{Z}_+^n$ and, for each compact subset $K$ of $U$, there is a constant $A = A(K) > 0$ such that*

$$\sup_K |\mathrm{L}^\alpha u| \le A^{|\alpha|+1}\alpha!^s, \qquad \alpha \in \mathbb{Z}_+^n\,. \tag{2}$$

*In particular it follows that the sheaf $\Omega' \mapsto G^s_{\mathfrak{h}}(\Omega')$ does not depend on the choice of the hermitian metric $\mathfrak{h}$.*

**Proof.** Since the vector fields $\mathrm{L}_j$ are pairwise commuting we can write

$$\sum_{1\le i_1,\ldots,i_N\le n} |\mathrm{L}_{i_1}\ldots\mathrm{L}_{i_N}u|^2 = \sum_{|\alpha|=N}\frac{N!}{\alpha_1!\cdots\alpha_n!}|\mathrm{L}^\alpha u|^2$$

and hence

$$|\mathrm{L}^\alpha u| = \frac{\alpha!^{1/2}}{|\alpha|!^{1/2}}\left[\frac{N!}{\alpha!}|\mathrm{L}^\alpha u|^2\right]^{1/2} \le \left\{\sum_{1\le i_1,\ldots,i_N\le n}|\mathrm{L}_{i_1}\ldots\mathrm{L}_{i_N}u|^2\right\}^{1/2}.$$

Hence, if $u \in G^s_{\mathfrak{h}}(U;\mathcal{V})$ and if $K \subset U$ is compact, Lemma 3.1 implies the existence of $C_\bullet > 0$ such that

$$\sup_K\left\{\sum_{1\le i_1,\ldots,i_N\le n}|\mathrm{L}_{i_1}\ldots\mathrm{L}_{i_N}u|^2\right\}^{1/2} \le C_\bullet^{N+1}N!^s.$$

Then

$$\sup_K |\mathrm{L}^\alpha u| \le C_\bullet^{|\alpha|+1}|\alpha|!^s \le n^{s|\alpha|}C_\bullet^{|\alpha|+1}\alpha!^s, \qquad \alpha \in \mathbb{Z}_+^n\,.$$

Conversely, if estimates (2) hold then the inequalities

$$\sum_{1\le i_1,\dots,i_N\le n} |\mathrm{L}_{i_1}\dots\mathrm{L}_{i_N}u|^2 = \sum_{|\alpha|=N} \frac{N!}{\alpha_1!\cdots\alpha_n!}|\mathrm{L}^\alpha u|^2 \le n^N \sum_{|\alpha|=N}|\mathrm{L}^\alpha u|^2$$

together with (1) imply that $u \in G^s_{\mathfrak{h}}(U;\mathcal{V})$. ■

We shall denote by $\mathfrak{G}^s_{\mathcal{V}}$ the sheaf $\Omega' \mapsto G^s_{\mathfrak{h}}(\Omega')$ over $\Omega$ (we have the right to drop the mention to $\mathfrak{h}$ in this definition). If $\Omega'$ is open the elements of $\Gamma(\Omega';\mathfrak{G}^s_{\mathcal{V}})$ will be called *s-Gevrey vectors for $\mathcal{V}$ on $\Omega'$* (resp. *analytic vectors for $\mathcal{V}$ on $\Omega'$* when $s=1$).

## 4. Local characterization of the analytic vectors for $\mathcal{V}$

We continue to work under the local coordinates and generators as described in the preceding section.

**Theorem 4.1.** *Let $u \in \Gamma(U;\mathfrak{G}^1_{\mathcal{V}})$. Then there are an open neighborhood $V \subset U$ of the origin, an open neighborhood $D$ of the origin in $\mathbb{C}^n$ and a function $v = v(x,t,w)$, defined on $V \times D$, satisfying*

- $v \in L^\infty(V\times D)\cap C(V;\mathcal{O}(D))$;
- $\mathrm{L}_j v(\cdot,\cdot,w) = 0$, $j = 1,\dots,n$, $w \in D$;
- $v(x,t,t) = u(x,t)$.

*Here $\mathcal{O}(D)$ denotes the space of holomorphic functions on $D$.*

**Proof.** Contracting $U = B\times\Theta$ if necessary we can assume that $\mathrm{L}^\alpha u \in C(U)$ for every $\alpha$ and that $\sup_U |\mathrm{L}^\alpha u| \le C^{|\alpha|+1}\alpha!$ for every $\alpha$ and some $C>0$.

We set $\Theta_\bullet = \{t\in\Theta : |t| < 1/(4C)\}$ and $D = \{w \in \mathbb{C}^n : |w| < 1/(4C)\}$. The series

$$v(x,t,w) = \sum_{\alpha\in\mathbb{Z}^n_+} (-1)^{|\alpha|}\frac{(\mathrm{L}^\alpha u)(x,t)}{\alpha!}(t-w)^\alpha$$

defines an element $v \in C(B\times\Theta_\bullet;\mathcal{O}(D)) \cap L^\infty(B\times\Theta_\bullet\times D)$ which satisfies the required properties. ■

Given a locally integrable structure $\mathcal{V}$ over a smooth manifold $\Omega$, a distribution $u \in \mathcal{D}'(\Omega')$, $\Omega'\subset\Omega$ open, is a *solution* for $\mathcal{V}$ if $\mathrm{L}u = 0$, whatever smooth section L of $\mathcal{V}$ defined in an open subset of $\Omega'$. Denote by $\mathfrak{S}_{\mathcal{V}}$ the sheaf of germs of solutions for $\mathcal{V}$; denote also by $\mathfrak{S}_{\mathfrak{o}\mathcal{V}}$ the subsheaf of $\mathfrak{S}_{\mathcal{V}}$ formed by all solutions that are defined by continuous functions. Finally, let $\mathcal{O}_{(n)}$ denote the ring of germs of holomorphic functions at the origin in $\mathbb{C}^n$. We return to the situation described in Theorem 4.1 and consider the inverse limit

$$(\mathfrak{S}_{\mathfrak{o}\mathcal{V}})_0 \,\hat{\otimes}\, \mathcal{O}_{(n)} = \lim_{U\times D\to(0,0)} \mathfrak{S}_{\mathfrak{o}\mathcal{V}}(U)\hat{\otimes}\mathcal{O}(D),$$

where $\mathfrak{S}_{\mathfrak{o}\mathcal{V}}(U)\hat{\otimes}\mathcal{O}(D)$ stands for the completion of the tensor product between the Fréchet space $\mathfrak{S}_{\mathfrak{o}\mathcal{V}}(U)$ and the Fréchet-nuclear space $\mathcal{O}(D)$. There is a homomorphism between stalks at the origin

$$\mu_0 : (\mathfrak{S}_{\mathfrak{o}\mathcal{V}})_0 \,\hat{\otimes}\, \mathcal{O}_{(n)} \longrightarrow (\mathfrak{G}^1_{\mathcal{V}})_0$$

defined as follows: if $\mathbf{v} \in (\mathfrak{S}_{\mathfrak{o}\mathcal{V}})_0 \,\hat{\otimes}\, \mathcal{O}_{(n)}$ is represented by $(x,t,\zeta)\mapsto v(x,t,\zeta)$ we set $\mu_0(\mathbf{v})$ as being the germ of analytic vector for $\mathcal{V}$ at the origin defined by $(x,t)\mapsto v(x,t,t)$.

The conclusion of Theorem 4.1 implies that $\mu_0$ is an isomorphism, which provides another invariant characterization for the analytic vectors for $\mathcal{V}$.

## 5. Analytic vectors in hypocomplex structures

We begin by recalling the following definition ([T, III.5]).

**Definition 5.1.** *Let $\Omega$ be a smooth manifold over which a locally integrable structure is defined. We say that $\mathcal{V}$ is hypocomplex at a point $A \in \Omega$ if there are an open neighborhood $W$ of $A$ in $\Omega$ and smooth functions $Z_j : W \to \mathbb{C}$, $Z_j(A) = 0$, $j = 1, \dots, m$, whose differentials span $\mathcal{V}^\perp$ over $W$, and such that the following is true: given any solution $u$, defined near $A$ there is a holomorphic function $H$, defined in an open neighborhood of $0 \in \mathbb{C}^m$, such that $u = H \circ Z$ near the origin.*

We shall now return to the local coordinates and notation described in Section 3. We shall assume that $A$ is the origin for the coordinate system $(x_1, \dots, x_m, t_1, \dots, t_n)$.

**Lemma 5.1.** *Assume that $\mathcal{V}$ is hypocomplex at the origin. Let $V \subset U$ be an open neighborhood of the origin. If $F$ is a Banach space continuously contained in $\{u \in \mathcal{D}'(V) : \mathrm{L}_j u = 0,\ j = 1, \dots, n\}$, with norm denoted by $\|\cdot\|_F$, there exist a complex neighborhood $\mathcal{W}$ of $0 \in \mathbb{C}^m$ and a constant $C > 0$ satisfying the following property:*

- *Given $u \in F$ there is $h \in \mathcal{O}(\mathcal{W})$ such that $u = h \circ Z$ in $Z^{-1}(\mathcal{W}) \cap V$ and*
$$\sup_{\mathcal{W}} |h| \leq C \|u\|_F.$$

In the proof we shall use the following notation: if $\mathcal{W}$ is an open subset of $\mathbb{C}^m$ we shall denote by $\mathcal{O}_\infty(\mathcal{W})$ the Banach space of all bounded, holomorphic functions on $\mathcal{W}$ endowed with the supremum norm. Also we shall denote by $\mathcal{B}_\delta$ the open ball in $\mathbb{C}^m$ centered at the origin and with radius $\delta > 0$.

**Proof:** Let $p$ be a large positive integer. Denote by $E_p$ the Banach space of all pairs $(u, h) \in F \times \mathcal{O}_\infty(\mathcal{B}_{1/p})$ satisfying $u = h \circ Z$ on $Z^{-1}(\mathcal{B}_{1/p})$. Denote also by $T_p$ the continuous linear map $E_p \to F$, $T_p(u, h) = u$. Since $\mathcal{V}$ is hypocomplex at the origin we must have $\cup_p T_p(E_p) = F$. By Baire's theorem some $T_{p_0}(E_{p_0})$ must be of second category in $F$ and the open mapping theorem then implies that $T_{p_0} : E_{p_0} \to F$ is surjective and open. In particular there must exist $\epsilon > 0$ such that
$$\{u \in F : \|u\|_F \leq \epsilon\} \subset T_{p_0}\left(\left\{(u, h) \in E_{p_0} : \sup_{\mathcal{B}_{1/p_0}} |h| < 1\right\}\right),$$
from which the result follows after taking $C = 1/\epsilon$ and $\mathcal{W} = \mathcal{B}_{1/p_0}$. ■

**Theorem 5.1.** *Assume that the system $\mathcal{V}$ is hypocomplex at the origin and let $u \in \Gamma(U; \mathfrak{G}^1_{\mathcal{V}})$. Then there are an open set $\mathcal{W} \times W$ in $\mathbb{C}^m \times \mathbb{C}^n$ containing the origin and $G \in \mathcal{O}(\mathcal{W} \times W)$ satisfying $u(x, t) = G(Z(x, t), t)$ in $\{(x, t) : (Z(x, t), t) \in \mathcal{W} \times W\}$.*

**Proof.** Take $V$, $D$ and $v$ as in Proposition 4.1. If we set $v_\alpha(x, t) \doteq (\partial_w^\alpha v)(x, t, 0)$ we also have $\mathrm{L}_j v_\alpha = 0$, $j = 1, \dots, n$, $\alpha \in \mathbb{Z}^n_+$ and the Cauchy estimates give
$$\sup_V |v_\alpha| \leq A^{|\alpha|+1} \alpha!\,. \tag{3}$$

We apply Lemma 5.1 with $F = \{u \in C(V) \cap L^\infty(V) : \mathrm{L}_j u = 0,\, j = 1, \dots, n\}$. We conclude the existence of an open neighborhood $\mathcal{W}$ of the origin in $\mathbb{C}^m$, of a constant $C > 0$ and of holomorphic functions $g_\alpha \in \mathcal{O}(\mathcal{W})$ such that $v_\alpha = g_\alpha \circ Z$ in $Z^{-1}(\mathcal{W}) \cap V$ and

$$\sup_{\mathcal{W}} |g_\alpha| \leq C \sup_V |v_\alpha|, \qquad \alpha \in \mathbb{Z}_+^n. \tag{4}$$

Define

$$G(z, w) = \sum_{\alpha \in \mathbb{Z}_+^n} \frac{g_\alpha(z)}{\alpha!} w^\alpha .$$

Thanks to (3) and (4) there is a neighborhood $W$ of the origin in $\mathbb{C}^n$ such that $G$ defines a holomorphic function in $\mathcal{W} \times W$. Finally,

$$G(Z(x,t), t) = \sum_{\alpha \in \mathbb{Z}_+^n} \frac{g_\alpha(Z(x,t))}{\alpha!} t^\alpha = \sum_{\alpha \in \mathbb{Z}_+^n} \frac{v_\alpha(x,t)}{\alpha!} t^\alpha = \sum_{\alpha \in \mathbb{Z}_+^n} \frac{(\partial_w^\alpha v)(x,t,0)}{\alpha!} t^\alpha = v(x,t,t) = u(x,t),$$

which concludes the proof. ■

**Remark 5.1.** Fix a locally integrable structure $\mathcal{V}$ and consider the local coordinates $(x, t)$ and generators $\mathrm{L}_j$, $\mathrm{M}_k$ as before. According to [T, Proposition II.4.2], a smooth function $f(x, t)$ can be written near the origin as $f(x,t) = \tilde{f}(Z(x,t), t)$, with $\tilde{f}$ holomorphic in $\mathbb{C}^m \times \mathbb{C}^n$, if and only if $f$ is an analytic vector for the complete system $\{\mathrm{M}_1, \dots, \mathrm{M}_m, \mathrm{L}_1, \dots, \mathrm{L}_n\}$. Thus Theorem 5.1 just says that, in a hypocomplex structure, the concepts of being an analytic vector for $\{\mathrm{L}_1, \dots, \mathrm{L}_n\}$ and for $\{\mathrm{M}_1, \dots, \mathrm{M}_m, \mathrm{L}_1, \dots, \mathrm{L}_n\}$ are equivalent.

Theorem 5.1 in connection with the arguments in [CCP, Section 9] allow us to state:

**Corollary 5.1.** *Assume that $\mathcal{V}$ is hypocomplex at the origin and let $u$ be a $C^1$ function defined in an open neighborhood of the origin in $U$ and satisfying the system*

$$(\mathrm{L}_j u)(x,t) = f_j(Z(x,t), t, u(x,t)), \quad j = 1, \dots, n, \tag{5}$$

*where the functions $f_j(z, w, \zeta)$ are holomorphic near the origin in $\mathbb{C}^{m+n+1}$. Then near the origin we can write $u(x,t) = G(Z(x,t), t)$, where $G(z, w)$ is holomorphic in a neighborhood of the origin in $\mathbb{C}^{m+n}$.*

## 6. Real-analytic locally integrable structures

In this section we assume that both the manifold $\Omega$ and the vector bundle are real-analytic. Notice that in such situation, returning to the general set up described in Section 3, all the vector fields $\mathrm{L}_j$ and the functions $Z_k$ can be assumed real-analytic.

The proof of the next result follows after an elementary argument ([CCP, Proposition 2.1], cf. also [BCR, Proposition 1.1.3]).

**Proposition 6.1.** *Given $\Omega' \subset \Omega$ open and $s \geq 1$ we have the inclusion*

$$\{u \in C(\Omega') : \mathrm{d}' u \in G^s(\Omega', \mathfrak{T}(\mathcal{V}))\} \subset \Gamma(\Omega'; \mathfrak{G}_{\mathcal{V}}^s) .$$

We shall now discuss the analytic regularity of the analytic vectors for $\mathcal{V}$. For this we first recall some standard concepts (cf. [T, III.5]).

**Definition 6.1.** *We say that*

- *$\mathcal{V}$ is analytic hypoelliptic if given $\Omega' \subset \Omega$ open and $u \in \mathcal{D}'(\Omega')$ then*
$$\mathrm{d}'u \in C^\omega(\Omega'; \mathfrak{T}(\mathcal{V})) \implies u \in C^\omega(\Omega').$$
- *$\mathcal{V}$ is globally analytic-hypoelliptic if given $u \in \mathcal{D}'(\Omega)$ then*
$$\mathrm{d}'u \in C^\omega(\Omega; \mathfrak{T}(\mathcal{V})) \implies u \in C^\omega(\Omega).$$

By Proposition 6.1 it follows that when $\mathcal{V}$ is not analytic hypoelliptic then there are analytic vectors which are not real analytic. On the other hand, according to [T, Proposition III.5.3], $\mathcal{V}$ is analytic hypoelliptic if and only if $\mathcal{V}$ is hypocomplex at each point of $\Omega$ and then, as a consequence of Theorem 5.1, we can state:

**Corollary 6.1.** *Assume that $\mathcal{V}$ is analytic hypoelliptic. If $\Omega' \subset \Omega$ is open then $\Gamma(\Omega'; \mathfrak{G}^1_{\mathcal{V}}) \subset C^\omega(\Omega')$.*

A similar result for global analytic hypoellipticity is no longer true. To give an example we take as $\Omega$ the two dimensional torus $\Omega = S^1 \times S^1$, with coordinates written as $(x, t)$, and let $\mathcal{V} \subset \mathbb{C}\mathrm{T}(S^1 \times S^1)$ be the vector bundle spanned by the real vector field
$$\mathrm{L} = \frac{\partial}{\partial t} - \alpha \frac{\partial}{\partial x},$$
where $\alpha \in \mathbb{R}$. It is clear that $\mathcal{V}$ defines a real-analytic, locally integrable structure over $S^1 \times S^1$ of rank one, which is never analytic hypoelliptic; on the other hand there are values of $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ for which $\mathcal{V}$ is globally analytic hypoelliptic (cf. [G]).

**Proposition 6.2.** *For any $\alpha \in \mathbb{R}$ and any $s \geq 1$ there is $u \in \Gamma(S^1 \times S^1; \mathfrak{G}^1_{\mathcal{V}})$ such that $u \notin G^s(S^1 \times S^1)$.*

**Proof.** Since when $\alpha \in \mathbb{Q}$ it is easy to construct global continuous solutions to the equation $\mathrm{L}u = 0$ which are not smooth, we can restrict ourselves to the case $\alpha \in \mathbb{R} \setminus \mathbb{Q}$. Changing $x$ to $-x$ if necessary allows us to assume that $\alpha > 0$ and then there are sequences $\{p_k\}$, $\{q_k\}$ of natural numbers such that $q_k \to \infty$ and $\{p_k - \alpha q_k\}$ is bounded. We set
$$\tau_k = 1/\log(\log q_k).$$
Then $\tau_k \to 0$; moreover, we can find $k_0$ so that $0 < \tau_k < 1/(2s)$ if $k \geq k_0$. We then set, for $k \geq k_0$,
$$A_k \doteq \exp\left\{-(p_k + q_k)^{-\tau_k + 1/s} - |p_k - \alpha q_k|\right\}.$$
Since
$$A_k \leq \exp\left\{-(p_k + q_k)^{1/(2s)}\right\}$$
it follows that
$$u(x,t) \doteq \sum_{k \geq k_0} A_k \exp\{i(p_k t + q_k x)\}$$
defines a smooth function on $S^1 \times S^1$ (indeed, $u \in G^{2s}(S^1 \times S^1)$). Moreover, for $N = 0, 1, \ldots$,
$$\mathrm{L}^N u(x,t) = \sum_{k \geq k_0} i^N (p_k - \alpha q_k)^N A_k \exp\{i(p_k t + q_k x)\}$$
and thus, noticing that
$$|p_k - \alpha q_k|^N A_k \leq N! \exp\left\{-(p_k + q_k)^{-\tau_k + 1/s}\right\},$$

we obtain

$$|\mathrm{L}^N u(x,t)| \leq N! \sum_{k\geq k_0} \exp\left\{-(p_k+q_k)^{1/(2s)}\right\}.$$

In particular $u \in \Gamma(S^1\times S^1;\mathfrak{G}^1_{\mathcal{V}})$. It remains to prove that $u \notin G^s(S^1\times S^1)$. Indeed, if this were true we would obtain constants $B>0$, $\epsilon>0$, such that

$$A_k \leq B\exp\left\{-\epsilon(p_k+q_k)^{1/s}\right\}, \quad k\geq k_0.$$

Hence

$$\exp\left\{\epsilon(p_k+q_k)^{1/s}\right\} \leq B\exp\left\{(p_k+q_k)^{-\tau_k+1/s} + |p_k-\alpha q_k|\right\}, \quad k\geq k_0,$$

or, equivalently,

$$\epsilon(p_k+q_k)^{1/s} \leq \log B + (p_k+q_k)^{-\tau_k+1/s} + |p_k-\alpha q_k|, \quad k\geq k_0.$$

Dividing by $(p_k+q_k)^{1/s}$ gives

$$\epsilon \leq \frac{\log B + |p_k-\alpha q_k|}{(p_k+q_k)^{1/s}} + \frac{1}{(p_k+q_k)^{\tau_k}}, \quad k\geq k_0. \tag{6}$$

If we now notice that

$$\frac{\log B + |p_k-\alpha q_k|}{(p_k+q_k)^{1/s}} \leq \frac{\text{const.}}{(p_k+q_k)^{1/s}} \longrightarrow 0$$

and that

$$\frac{1}{(p_k+q_k)^{\tau_k}} \leq q_k^{-\tau_k} = e^{-\tau_k\log q_k} \longrightarrow 0,$$

since $\tau_k\log q_k\to\infty$, we conclude that the right end side of (6) converges to 0 as $k\to\infty$, which gives the sought contradiction. ■

**Regularity of Gevrey vectors**. Still in the case when $\Omega$ and $\mathcal{V}$ are real-analytic, it is a natural question to ask about the Gevrey regularity of the elements in $\Gamma(\Omega';\mathfrak{G}^s_{\mathcal{V}})$, $s>1$. When $\mathcal{V}$ is elliptic, that is when $\mathcal{V}^\perp\cap \mathrm{T}^*\Omega=0$, then $\Gamma(\Omega';\mathfrak{G}^s_{\mathcal{V}}) = G^s(\Omega')$ for every $\Omega'\subset\Omega$ open and $s\geq 1$. Indeed, when $\mathcal{V}$ is elliptic we have $m\leq n$ and we can take, in the local coordinates decribed in Section 3, $\Phi_j(x,t)=t_j$, $j=1,\dots,m$. The vector fields $\mathrm{L}_j$ now read

$$\mathrm{L}_j = \frac{\partial}{\partial t_j} - i\frac{\partial}{\partial x_j}, \quad j=1,\dots,m, \quad \mathrm{L}_j = \frac{\partial}{\partial t_j}, \quad j=m+1,\dots,n.$$

If $u\in\Gamma(U;\mathfrak{G}^s_{\mathcal{V}})$ then $u$ is an $s$-Gevrey vector for each $\mathrm{L}_j$, in the following sense: for every $K\subset U$ compact there is a constant $C=C(K)>0$ such that

$$\sup_K |\mathrm{L}_j^p u| \leq C^{p+1}p!, \quad p=0,1,\dots, \quad j=1,\dots,n.$$

It then follows from the results in [BCM] that the $G^s$-wave-front of $u$ is contained in $\sigma(\mathrm{L}_j)$, the characteristic set of $\mathrm{L}_j$ over $U$, $j = 1, \dots, n$. Since $\cap_{j=1}^n \sigma(\mathrm{L}_j) = \emptyset$ it follows that $u \in G^s(U)$.

We conclude this work by presenting a partial converse of this statement, which is inspired by a result on scalar operators due to G. Métivier [M]:

**Proposition 6.3.** *Assume that $\mathcal{V}$ is not elliptic at $A \in \Omega$, that is, assume that $\mathcal{V}_A^\perp \cap \mathrm{T}_A^*\Omega \neq 0$. Then given $s, s'$ satisfying*

$$1 < s \le s' < 2s - 1$$

*there is an $s$-Gevrey vector for $\mathcal{V}$ near $A$ which is not a Gevrey function of order $s'$.*

**Proof.** We can select the local coordinates $(x,t)$ and the local generators $\mathrm{L}_j$, $\mathrm{d}Z_k$ as in section 3 in such a way that, in these coordinates, $A = (0,0)$.

By hypohesis there is $\zeta_0 = (\zeta_{01}, \dots, \zeta_{0m}) \in \mathbb{C}^m$ not zero such that $\sum_k \zeta_{0k}\mathrm{d}Z_k(0,0)$ is a real covector. Since $\mathrm{d}Z_k(0,0) = \mathrm{d}x_k + i\mathrm{d}_t\Phi_k(0,0)$ it follows that $\zeta_0 = \xi_0 \in \mathbb{R}^m$ and that $\sum_k \xi_{0k}\mathrm{d}_t\Phi_k(0,0) = 0$. Consequently $\mathrm{d}(\Phi \cdot \xi_0) = 0$ at $(0,0)$ and hence there is a constant $C > 0$ such that $|\Phi(x,t)\cdot\xi_0| \le C(|x|^2 + |t|^2)$ when $(x,t) \in U$.

For convenience, we will write the local generators $\mathrm{L}_j$ as

$$\mathrm{L}_j = \frac{\partial}{\partial t_j} + \sum_{k=1}^m a_{jk}(x,t)\frac{\partial}{\partial x_k}, \quad j = 1, \dots, n,$$

where the coefficients $a_{jk}(x,t)$ are assumed to be real-analytic functions in a neighborhood of the closure of $U$.

Let $\alpha \in ]0,1[$ be such that $s' < 1/\alpha < 2s - 1$ and define $\sigma \doteq s - (1-\alpha)/(2\alpha) > 1$. Next we select $\zeta \in G_c^\sigma(\mathbb{R}^n)$ such that $\zeta(0) = 1$ and with support contained in the ball $|t| \le \rho/\sqrt{m+1}$, where $\rho < \min\{r, 1/\sqrt{C}\}$ and $r$ is the radius of $\Theta$. We can assume that $]-r, r[^m \subset B$. We also take a cut-off function $\psi \in G_c^\sigma(\mathbb{R})$, satisfying $\psi \equiv 1$ in a neighborhood of the origin and supported in the interval $\left[-\rho/\sqrt{m+1}, \rho/\sqrt{m+1}\right]$.

We then set

$$u(x,t) = \int_1^\infty e^{i\lambda Z(x,t)\cdot\xi_0 - \lambda^\alpha}\zeta\left(\lambda^{(1-\alpha)/2}t\right)\psi\left(\lambda^{(1-\alpha)/2}x_1\right)\dots\psi\left(\lambda^{(1-\alpha)/2}x_m\right)d\lambda.$$

If $\lambda^{(1-\alpha)/2}t \in \operatorname{supp}\zeta$ and $\lambda^{(1-\alpha)/2}x_1, \dots, \lambda^{(1-\alpha)/2}x_m \in \operatorname{supp}\psi$ then

$$\lambda|\Phi(x,t)\cdot\xi_0| \le C\lambda(|x|^2 + |t|^2) \le C\rho^2\lambda^\alpha$$

and, consequently, $u$ is well defined and smooth in $U$.

Since the derivatives of the function $\psi$ vanish identically in a neighborhood of the origin, setting $\xi_0 \cdot \mathrm{M} = \sum_{j=1}^m \xi_{0j}\mathrm{M}_j$ gives

$$\{(\xi_0 \cdot \mathrm{M})^k u\}(0,0) = \int_1^\infty \lambda^k e^{-\lambda^\alpha}d\lambda = \frac{1}{\alpha}\Gamma\left(\frac{k+1}{\alpha}\right) - \int_0^1 \lambda^k e^{-\lambda^\alpha}d\lambda.$$

Noticing that the last term at the right end side is bounded in $k$ it follows from the asymptotic behaviour of the Gamma function that $u$ is not of Gevrey class $\tau$ near the origin, for any $\tau < 1/\alpha$.

Summing up, in order to complete the argument we must show that $u$ is an $s$-Gevrey vector for $\mathcal{V}$. For $\beta \in \mathbb{Z}_+^n$ we have

$$\mathrm{L}^\beta u(x,t) = \sum_{\gamma\leq\beta}\binom{\beta}{\gamma}\int_1^\infty \lambda^{(1-\alpha)(|\beta|-|\gamma|)/2} e^{i\lambda Z(x,t)\cdot\xi_0-\lambda^\alpha}\zeta^{(\beta-\gamma)}(\lambda^{(1-\alpha)/2}t)\mathrm{L}^\gamma\left(\Psi(\lambda,x)\right)d\lambda, \tag{7}$$

where $\Psi(\lambda,x) = \psi\left(\lambda^{(1-\alpha)/2}x_1\right)\dots\psi\left(\lambda^{(1-\alpha)/2}x_m\right)$. We shall prove that, for some constant $A>0$, the estimates

$$\sup_U |\mathrm{L}^\gamma(\Psi(\lambda,x))| \leq A^{|\gamma|+1}|\gamma|!^\sigma, \quad \gamma\in\mathbb{Z}_+^n \tag{8}$$

hold. Assuming this for a moment we obtain, from (7),

$$\begin{aligned}\sup_U|\mathrm{L}^\beta u| &\leq A_1^{|\beta|+1}|\beta|!^\sigma\int_1^\infty \lambda^{(1-\alpha)|\beta|/2}e^{-(1-C\rho^2)\lambda^\alpha}\mathrm{d}\lambda\\ &\leq A_2^{|\beta|+1}\beta!^{\sigma+(1-\alpha)/(2\alpha)}\\ &= A_2^{|\beta|+1}\beta!^s,\end{aligned}$$

which shows that indeed $u$ is an $s$-Gevrey vector for $\mathcal{V}$.

The remainder of the argument will be devoted to the proof of estimates (8).

Write $r=\lambda^{(1-\alpha)/2}(\geq 1)$ and $w(x_\ell) = rx_\ell$, $\ell=1,\dots,m$. In order to estimate $\mathrm{L}^\gamma(\Psi(\lambda,x))$ we must first analyze the terms $\mathrm{L}^\eta\psi(w(x_\ell))$, when $\eta\in\mathbb{Z}_+^n$ and $\ell=1,\dots,m$. For this, we first apply the Faà di Bruno's formula

$$\mathrm{L}^\eta\psi(w(x_\ell)) = \sum_{\substack{\theta_1+\dots+\theta_q=\eta\\ |\theta_j|>0}} C_{q,\theta}^\eta\psi^{(q)}(w(x_\ell))\mathrm{L}^{\theta_1}w(x_\ell)\dots\mathrm{L}^{\theta_q}w(x_\ell).$$

Here $C_{q,\theta}^\eta$ are universal constants. Since the vector fields $\mathrm{L}_j$ commute pairwise, and since also $|\theta_j|>0$, we can choose integers $1\leq i_j\leq n$ such that $\mathrm{L}^{\theta_j}w(x_\ell) = r\mathrm{L}^{\theta_j-e_{i_j}}a_{i_j\ell}(x,t)$ (here $e_k=(\delta_{kp})_{1\leq p\leq k}\in\mathbb{Z}_+^n$). The last equality can then be written as

$$\mathrm{L}^\eta\psi(w(x_\ell)) = \sum_{\substack{\theta_1+\dots+\theta_q=\eta\\ |\theta_j|>0}} C_{q,\theta}^\eta r^q\psi^{(q)}(w(x_\ell))\mathrm{L}^{\theta_1-e_{i_1}}a_{i_1\ell}(x,t)\dots\mathrm{L}^{\theta_q-e_{i_q}}a_{i_q\ell}(x,t).$$

Since the coefficients $a_{jk}$ are real-analytic in $\bar{U}$ and since $\psi\in G_c^\sigma(\mathbb{R})$ we obtain the existence of a constant $B>0$ such that

$$\sup_U|\mathrm{L}^\eta\psi(w(x_\ell))| \leq r^{|\eta|}B^{|\eta|+1}\sum_{\substack{\theta_1+\dots+\theta_q=\eta\\ |\theta_j|>0}} C_{q,\theta}^\eta B^q q!^\sigma|\theta_1|!\dots|\theta_q|!. \tag{9}$$

For $0<\varepsilon<1/B$ and $c>0$ we set

$$N_k = \frac{ck!}{(k+1)^{n+1}\varepsilon^k}.$$

Here $c > 0$ is chosen in such a way that

$$\sum_{\beta' \leq \beta} \binom{\beta}{\beta'} N_{|\beta'|} N_{|\beta-\beta'|} \leq N_{|\beta|}, \quad \beta \in \mathbb{Z}^n_+.$$

Then we can write

$$|\theta_j|! = N_{|\theta_j|} \frac{(|\theta_j|+1)^{n+1}\varepsilon^{|\theta_j|}}{c} \leq N_{|\theta_j|} \frac{(n+1)! e^{|\theta_j|+1} \varepsilon^{|\theta_j|}}{c}, \quad \forall j = 1, \dots, q$$

and therefore we have

$$\begin{aligned} |\theta_1|! \dots |\theta_q|! &\leq N_{|\theta_1|} \dots N_{|\theta_q|} (\varepsilon e)^{|\eta|} \left( \frac{(n+1)! e}{c} \right)^q \\ &\leq N_{|\theta_1|} \dots N_{|\theta_q|} (\varepsilon e)^{|\eta|} \left( 1 + \frac{(n+1)! e}{c} \right)^{|\eta|}. \end{aligned} \tag{10}$$

Inserting (10) into (9) gives, with a new constant $B_1 > 0$,

$$\sup_U |\mathrm{L}^\eta \psi(w(x_\ell))| \leq B_1^{|\eta|+1} \varepsilon^{|\eta|} |\eta|!^{\sigma-1} r^{|\eta|} \sum_{\substack{\theta_1 + \dots + \theta_q = \eta \\ |\theta_j| > 0}} C^\eta_{q,\theta} B^q q! N_{|\theta_1|} \dots N_{|\theta_q|}. \tag{11}$$

If we now introduce the formal power series

$$\phi(w) = \sum_{\ell=1}^\infty B^\ell w^\ell, \quad w \in \mathbb{R},$$

$$\tau(y) = \sum_{|\alpha|>0} \frac{N_{|\alpha|}}{\alpha!} y^\alpha, \quad y \in \mathbb{R}^n$$

again the Faà di Bruno's formula shows that (11) can be written as

$$\sup_U |\mathrm{L}^\eta \psi(w(x_\ell))| \leq B_1^{|\eta|+1} \varepsilon^{|\eta|} |\eta|!^{\sigma-1} r^{|\eta|} \partial^\eta (\phi \circ \tau)(0). \tag{12}$$

Arguing as in [AM, p. 197], (see also [CCP, Section 9]), we conclude that

$$|\partial^\eta (\phi \circ \tau)(0)| \leq \frac{B}{1 - B\varepsilon} N_{|\eta|}$$

and consequently, with another constant $B_2 > 0$,

$$\begin{aligned} \sup |\mathrm{L}^\eta \psi(w(x_\ell))| &\leq B_1^{|\eta|+1} \varepsilon^{|\eta|} |\eta|!^{\sigma-1} r^{|\eta|} \frac{B}{1-B\varepsilon} N_{|\eta|} \\ &= B_1^{|\eta|+1} \varepsilon^{|\eta|} |\eta|!^{\sigma-1} r^{|\eta|} \frac{B}{1-B\varepsilon} \frac{c|\eta|!}{(|\eta|+1)^{n+1} \varepsilon^{|\eta|}} \\ &\leq r^{|\eta|} B_2^{|\eta|+1} |\eta|!^\sigma. \end{aligned} \tag{13}$$

Finally, if we observe that

$$\mathrm{L}^\gamma(\Psi(\lambda, x)) =$$

$$\sum_{\theta_1 \leq \gamma} \sum_{\theta_2 \leq \theta_1} \cdots \sum_{\theta_{m-1} \leq \theta_{m-2}} \binom{\gamma}{\theta_1} \binom{\theta_1}{\theta_2} \cdots \binom{\theta_{m-2}}{\theta_{m-1}} \mathrm{L}^{\gamma-\theta_1} \psi(w(x_1)) \cdots \mathrm{L}^{\theta_{m-1}} \psi(w(x_m)),$$

inequality (13) allows us to estimate

$$\sup_U |\mathrm{L}^\gamma(\Psi(\lambda, x))| \leq 2^{(m-1)|\gamma|} r^{|\gamma|} B_2^{|\gamma|+m} |\gamma|!^\sigma = 2^{(m-1)|\gamma|} \lambda^{(1-\alpha)|\gamma|/2} B_2^{|\gamma|+m} |\gamma|!^\sigma,$$

from which (8) follows. The proof of Proposition 6.3 is complete. ■

## References


[AM] S. Alinhac and G. Metivier, *Propagation de l'analyticité des solutions de systèmes hyperboliques non-linéaires.* Invent. Math. **75** (1984), 189–204.

[BG] M.S. Baouendi and C. Goulaouic, *Régularité analytique et itères d'operateurs elliptiques dégénérés; applications.* J. Funct. Analysis **9** (1972), 208–248.

[BM] M.S. Baouendi and G. Metivier, *Analytic vectors of hypoelliptic operators of principal type.* American J. Math. **104** (1982), 287–319.

[BCH] S. Berhanu, P.D. Cordaro and J. Hounie, An introduction to involutive structures. Cambridge University Press, 2008.

[BCM] P. Bolley, J. Camus, C. Mattera, *Analyticité microlocale et itères d'operateurs.* Seminaire Goulaouic–Schwartz 1978–1979, Exposé XIII. École Polytechnique, France.

[BCR] P. Bolley, J. Camus and L. Rodino, *Hypoellipticite analytique-Gevrey et itères d'operateurs.* Rend. Sem. Mat. Univers. Politecn. Torino **45** (1987), 1–61.

[CCP] J.E. Castellanos, P.D. Cordaro and G. Petronilho, *Gevrey vectors in involutive tube structures and Gevrey regularity for the solutions to certain classes of semilinear systems,* (2010), to appear.

[Da] M. Damlakhi, *Analyticité et itères d'operateus pseudo-différentiels.* J. Math. Pures et Appl. **58** (1979), 63–74.

[DaH] M. Damlakhi and B. Helffer, *Analyticité et itères d'un système de champs non elliptique.* Annales scient. Éc. Norm. Sup. 4e. série, **13** (1980), 397–403.

[G] C. Goulaouic, *Interpolation entre des espaces localement convexes définis à l'aide de semi-groupes.* Ann. Inst. Fourier Grenoble **19** (1969), 269–278.

[HM] B. Hellfer and C. Mattera, *Analyticité et itères reduits d'un système de champs de vecteurs.* Comm. PDE **5(10)** (1980), 1065–1072.

[M] G. Metivier, *Propriete des itères et ellipticite.* Comm. PDE **3(9)** (1978), 827–876.

[N] E. Nelson, *Analytic vectors.* Ann. Math. **70** (1959), 572–615.

[RS] M. Reed and B. Simon, Methods of modern mathematical physics I: functional analysis. Academic Press, 1972.

[T] F. Treves, Hypo-analytic structures: local theory. Princeton University Press, 1992.



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