Approximate Solutions and Micro-Regularity in the Denjoy-Carleman Classes

Ziad Adwan
University of Texas at Brownsville

Gustavo Hoepfner
Federal University of São Carlos

March 2, 2010

Abstract

We begin with a sequence M of positive real numbers and we consider the Denjoy-Carleman class C^M. We show how to construct M-approximate solutions for complex vector fields with C^M coefficients. We then use our construction to study micro-local properties of boundary values of approximate solutions in general $M-$involutive structures of codimension one, where the approximate solution is defined in a wedge whose edge (where the boundary value exists) is a maximally real submanifold. We also obtain a C^M version of the Edge-of-the-Wedge Theorem.

1 Introduction

Let $M = (M_j)$ be a sequence of positive real numbers satisfying some properties. Also, let (z_j) be a sequence of complex numbers satisfying the estimate $|z_j| \leq A^{j+1}M_j$ $(j = 1, 2, 3, ...)$, where $A > 0$ is a constant independent of j. Carleman's problem can be stated as follows:

Carleman's Problem: Construct a function $f = f(x) \in C^\infty([-1, 1])$ such that for all $j \in \mathbb{N}$, $f^{(j)}(0) = z_j$, and such that for some constant $C > 0$, independent of $x \in [-1, 1]$ and $j \in \mathbb{N}$, $\left|f^{(j)}(x)\right| \leq C^{j+1}M_j$, for all $x \in [-1, 1]$ and all $j \in \mathbb{N}$.

Assuming that the sequence $M_j = (j!)^s$, $s > 1$, Mityagin (see (Mi)) proved the existence of the required function f, while Džanašija (see (Dz)) constructed f explicitly. In this paper, assuming that the sequence M satisfies some properties (see Conditions $(P1) - (P4)$ in Definition 3), we solve the following problem:

Problem 1: Given a complex vector field $L = \frac{\partial}{\partial t} + \sum_{k=1}^{m} a_k(x, t) \frac{\partial}{\partial x_k}$, defined in a neighborhood $\Omega = U \times (-\delta, \delta)$ of the origin in $\mathbb{R}^{m+1} = \mathbb{R}^m_x \times \mathbb{R}_t$, where the coefficients $a_k(x, t)$ are in the class $C^M(\Omega)$ (see Definition 1), and a function

$f = f(x)$ in the class $C^M(U)$, construct (explicitly) a C^M function $u = u(x,t)$ which is an *approximate solution* of $Lu = 0$ (see Definition 19 in section 4) and such that $u(x,0) = f(x)$.

In general, it is not possible to construct homogeneous solutions of an overdetermined system with given initial data defined on an (appropriate) initial submanifold, but the existence of "approximate solutions", i.e., functions that satisfy the initial condition and are mapped by the vector fields of the system into functions that, instead of vanishing identically, just vanish to infinite order at the initial manifold, is a useful substitute. For instance, when the system is the Cauchy-Riemann equation $\overline{\partial} u = 0$ in $\mathbb{C}^n$ and the initial manifold is maximally real, approximate solutions correspond to almost analytic extensions of the initial data. The existence of approximate solutions proved to be useful when investigating the regularity of solutions of first order nonlinear pde, as shown, for example, in the papers (A) and (BP). In these papers, it was *crucial* to show that approximate solutions in the *right class* exist for the linearized operator.

In the special case when $M_j = (j!)^s$ so that the class C^M is the Gevrey class G^s of exponent $s > 1$, we solved the above problem (see (AH)) with u in $G^{s'}$ and s' is any real number satisfying $s' > s + 1$. Building up on our method and on the explicit solution of the Carleman problem in (Dz), Barostichi and Petronilho (see (BP)) improved our result by constructing an approximate solution u in the same Gevrey space G^s. Their construction of the approximate solution in the same Gevrey space was crucial to the proof of their main theorem on the Gevrey regularity of solutions of first order nonlinear pdes.

After we solve Problem 1, we use the (explicitly constructed) approximate solution to show the existence of almost analytic extensions for C^M functions, and to obtain a C^M version of the so called Edge-of-the-Wedge theorem. It is to be noted that our second main theorem in this paper (see Theorem 24) improves our previous Gevrey result in (AH) (see Theorem 5.1 on page 2858).

A natural question to arise is about the existence of solutions of $Lu = f$ in the Denjoy-Carleman classes. Recently, P. Caetano and P.Cordaro (see (CC)) investigated the Gevrey solvability for first order nonlinear pdes. A consequence of their main result (see Theorem 6.1 in (CC)) can be stated as follows: "Let $s > 1$. Then any first order linear pde with analytic coefficients $Lu = f$ satisfying the Nirenberg-Treves condition $(\mathcal{P})$ has local solutions $u \in G^s$ for every $f \in G^s$". The authors believe that the above result can be generalized to the Denjoy-Carleman classes and it is in fact the topic of an ongoing investigation.

This paper is organized as follows: In section 2, we give all the background material about C^M spaces which is necessary for stating our results. In section 3, we solve a Carleman problem in the C^M class and we also prove that any C^M function defined in an open set in $\mathbb{R}^m$ has a C^M almost analytic extension in an open set in $\mathbb{C}^m$. Then, in section 4, we use our solution of the Carleman problem in section 3 to solve Problem 1 (mentioned above). In section 5, we make a brief detour to review involutive structures and some basic geometric constructions which will be needed in section 6. In section 6, we prove our second main result about the $M-$wavefront set (see Definition 13) of boundary values of M-approximate solutions defined in wedges in M-involutive structures of rank 1 (which are not necessarily locally integrable), where the boundary value exists on a maximally real submanifold. We show that the M-wavefront set of the boundary value is located in the polar of a certain open convex cone in the tangent space of the maximally real submanifold (this cone was constructed in the original paper (EG)). We use this second main result to obtain a C^M version of the Edge-of-the-Wedge theorem.

For other earlier versions of the Edge-of-the-Wedge Theorem in the Denjoy-Carleman classes, we refer the reader to the papers of H. Fourlinnie (F) and J.P. Rosay (R).

2 An Overview of C^M Spaces

2.1 C^M Spaces

Generalizations of Gevrey classes have been proposed by many authors during the past decades. A natural extension is obtained by considering a sequence of positive real numbers $M = (M_j)$ satisfying some properties (see below). If U is an open set in $\mathbb{R}^m$, then we define the class $C^M(U)$ as follows:

Definition 1 *$C^M(U)$ is the space of all C^∞ functions $f = f(x)$ defined in U with the property that for any fixed compact subset $K \Subset U$, there exists a constant $C > 0$ and independent of $x \in K$ and $\alpha \in \mathbb{Z}_+^m$ such that $|\partial_x^\alpha f(x)| \leq C^{|\alpha|+1} M_{|\alpha|}$. $C^M(U)$ is called the space of C^M-**ultradifferentiable functions** in U.*

Remark 2 *Fix a real number $s \geq 1$. Then the choice $M_j = (j!)^s$ gives the standard Gevrey class $G^s(U)$.*

Under suitable assumptions on the sequence M, one obtains for $C^M(U)$ results similar to those valid for $G^s(U)$; see for instance (Mo), (Mt), (Ku), (Rud). A good reference for analysis in Gevrey spaces is (Rod). The properties listed in the definition below will be used throughout this paper; in some results we will only need some of them or some of their consequences, but we list all of them in the definition below.

Definition 3 *Let $M = (M_j)$ be a sequence of positive real numbers satisfying the following properties:*

(**1**) (***Initial Conditions***)

$$M_0 = M_1 = 1. \qquad \textbf{(P1)}$$

(**2**) (***Strong non-quasianalyticity***) *There exists a constant $A > 1$ such that for all $p = 1, 2, ...,$ we have*

$$\sum_{j=p}^{\infty} \frac{M_j}{M_{j+1}} \leq Ap \frac{M_p}{M_{p+1}}. \qquad \textbf{(P2)}$$

(**3**) (***Strong logarithmic convexity***) *For some fixed* $A > 0$ *and for any* r, *with* $0 \leq r < 1/A$, *if we set* $P_j = M_j / (j!)^r$, *then*

$$\text{the sequence } \left(\frac{P_j}{jP_{j-1}}\right) \text{ is increasing.} \qquad (\mathbf{P3})$$

(**4**) (***Stability under ultradifferential operators***) *There are constants* $A > 1$ *and* $H > 1$, *independent of* n, *such that for all* $n = 1, 2, 3, ...$, *we have*

$$M_n \leq AH^n \min_{0 \leq j \leq n} M_j M_{n-j}. \qquad (\mathbf{P4})$$

2.2 Some Remarks

Remark 4 (**1**) *Fix a real number* $s > 1$. *Then the choice* $M_j = (j!)^s$ *satisfies conditions* $(P1) - (P4)$.

(**2**) *The condition* $(P2)$ *implies the (usual)* ***non-quasianalyticity*** *condition*

$$\sum_{j=1}^{\infty} \frac{M_j}{M_{j+1}} < +\infty. \qquad (\mathbf{P2}; 1)$$

This condition insures the existence of nontrivial C^M *functions with compact support.*

(**3**) (**i**) *The condition* $(P3)$ *implies the following condition (also known as* ***strong logarithmic convexity****): For all* $j = 1, 2, 3, ...$

$$\left(\frac{M_j}{j!}\right)^2 \leq \frac{M_{j+1}}{(j+1)!} \frac{M_{j-1}}{(j-1)!}. \qquad (\mathbf{P3}; 1)$$

Also, condition $(P3; 1)$ *implies the (usual)* ***logarithmic convexity*** *condition: For all* $j = 1, 2, 3, ...$

$$M_j^2 \leq M_{j-1} M_{j+1}. \qquad (\mathbf{P3}; 1')$$

(**ii**) *In the paper (RSW) (see corollary 6.2 on page 772), it is shown that the condition* $(P3; 1)$ *implies the following condition*

$$\text{the sequence } \left(\frac{M_j}{j!}\right)^{1/j} \text{ is increasing.} \qquad (\mathbf{P3}; 2)$$

Condition $(P3; 2)$ *insures that the class* $C^M(U)$ *is* ***inverse-closed****; i.e., if* $f \in C^M(U)$ *and* $\inf_{x \in U} |f(x)| > 0$, *then* $1/f \in C^M(U)$. *Condition* $(P3; 2)$ *implies the following condition*

$$\text{the sequence } \left(M_j^{1/j}\right) \text{ is increasing.} \qquad (\mathbf{P3}; 2')$$

(**iii**) *The condition* $(P3; 2)$ *implies the following condition: For all* $0 \leq j \leq n$,

$$\binom{n}{j} M_j M_{n-j} \leq M_n. \qquad (\mathbf{P3}; 3)$$

Condition $(P3;3)$ *insures that the class* $C^M(U)$ *is invariant under composition. It also implies, in particular, that for all* $0 \leq j \leq n$,

$$M_j M_{n-j} \leq M_n \qquad (\mathbf{P3};3')$$

$(\mathbf{4})$ *The condition* $(P4)$ *implies the (usual)* ***Stability under differential operators*** *condition; i.e., There are constants* $A > 1$ *and* $H > 1$*, independent of* n *and* j*, such that for all* $1 \leq j \leq n$*, we have*

$$M_n \leq AH^{n-1} M_j M_{n-j}. \qquad (\mathbf{P4};1)$$

We will often replace AH^{n-1} *with* C^n *(for instance by taking* $C = AH$*); hence the condition* $(P4;1)$ *will take the form*

$$M_n \leq C^n M_j M_{n-j}. \qquad (\mathbf{P4};1)$$

$(\mathbf{5})$ $(\mathbf{i})$ *If the sequence* (M_j) *satisfies conditions* $(P1)$ *and* $(P3;3)$*, then it satisfies the following condition: For all* $n = 1, 2, 3, \ldots$

$$M_n \geq n! \qquad (\mathbf{A}1)$$

Condition $(A1)$ *insures that every analytic function belongs to the class* C^M*.*

$(\mathbf{ii})$ *If the sequence* (M_j) *satisfies conditions* $(P3;2')$ *and* $(P4;1)$*, then it satisfies the following useful condition: For* C *the same constant as in condition* $(P4;1)$ *and for all* $0 \leq j \leq n$*,*

$$M_n^{1/n} \leq C^{n/j} M_j^{1/j} \qquad (\mathbf{A}2)$$

$(\mathbf{iii})$ *If the sequence* (M_j) *satisfies conditions* $(P1)$*,* $(P3;3')$ *and* $(P4;1)$*, then it satisfies the following condition: For all* $j, k \in \mathbb{N}$*, if we set* $n = jk$*, there is a constant* $C > 1$*, independent of* n*, such that*

$$M_j^k \leq C^n M_{n-k} \qquad (\mathbf{A}3)$$

2.3 Associated Functions

Definition 5 *For each sequence* (M_j) *of positive numbers we define its* ***associated function*** $M(t)$ *on* $(0, \infty)$ *by* $M(t) = \sup_j \log \frac{t^j}{M_j}$*.*

For the reader who is interested in learning more about associated functions and how each of the conditions which we impose on the sequence can be written in terms of the associated function, we recommend the paper by H. Komatsu (Ku) .

Lemma 6 *Let* (M_j) *be a sequence of positive numbers satisfying conditions* $(P1)$ *and* $(A1)$*, and let* $M(t)$ *be its associated function. Then*

(1) *For all* $t>0$,

$$\log t \leq M(t) \leq t. \tag{1}$$

(2) $M(t)$ *is an increasing convex function in* $\log t$ *which vanishes for sufficiently small* $t>0$ *and increases more rapidly than* $\log t^p$ *for any* p *as* $t \to +\infty$.

(3) *Suppose that the sequence* (M_j) *satisfies condition* $(P4;1)$. *Then for any* $k>0$, *and for all* $t>0$

$$M(kt) - M(t) \geq \frac{\log(t/A)\log k}{\log H}, \tag{2}$$

where A *and* H *are as in condition* $(P4;1)$.

(4) *The sequence* (M_j) *satisfies condition* $(P4)$ *if and only if for all* $t>0$

$$M(t/H) \leq \frac{1}{2}M(t) + \log\sqrt{A}, \tag{3}$$

where A *and* H *are as in condition* $(P4)$.

(5) *Suppose that the sequence* (M_j) *satisfies conditions* $(P1)$, $(A1)$, *and* $(P4;1)$. *Fix* $0<k<1$. *Then for* t *large enough (*t *depends on* k*):*

$$\frac{3}{2}M(kt) - M(t) \geq 0. \tag{4}$$

(6) *Suppose that the sequence* (M_j) *satisfies conditions* $(P1)$, $(A1)$, *and* $(P4)$. *Then for a fixed* $k>0$ *and for* t *large enough,*

$$M\left(t/H^3\right) \leq \frac{1}{4}M(kt) + \frac{7}{4}\log\sqrt{A}, \tag{5}$$

where H *and* A *are as in condition* $(P4)$.

Proof. (1) We have $\log t \overset{(P1)}{=} \log \frac{t^1}{M_1} \leq M(t) \overset{(A1)}{\leq} \sup_j \log \frac{t^j}{j!} \leq t$.

(2), (3) & (4): See (Ku) (Page 49, and Proposition 3.4 and Proposition 3.6 on pages 50-51).

(5) Since (M_j) satisfies condition $(P4;1)$, we can use (2) to obtain

$$\frac{3}{2}M(kt) - M(t) \overset{(P4;1)}{\geq} \frac{1}{2}M(kt) + \frac{\log(t/A)\log k}{\log H}.$$

Now, since the sequence satisfies $(P1)$ and $(A1)$, we can use part (2) of this lemma to conclude that for $p \in \mathbb{N}$ and $t>0$ large enough

$$\frac{1}{2}M(kt) + \frac{\log(t/A)\log k}{\log H} \geq \frac{1}{2}\log(kt)^p + \frac{\log(t/A)\log k}{\log H}.$$

The term on the RHS is ≥ 0 for t large enough. Hence, the inequality (4) follows.

(6) Using (3), we obtain

$$M\left(t/H^{3}\right) \leq \frac{1}{8}M\left(t\right)+\frac{7}{4}\log\sqrt{A}.$$

Now, using (2), we obtain for t large

$$\frac{1}{8}M\left(t\right) \leq \frac{1}{4}M\left(kt\right),$$

where we proceed as in the proof of part (5) of this lemma. This gives the desired inequality in (5). ■

In the following lemma, $\Omega \subseteq \mathbb{R}^{m+1}=\mathbb{R}_{x}^{m}\times\mathbb{R}_{t}$ will denote an open subset containing the origin, $x=(x_1,...,x_m)\in\mathbb{R}_{x}^{m}$ and $t\in\mathbb{R}_t$. Also, for $z=(z_1,...,z_m)\in\mathbb{C}^m$ we use the notation

$$\langle z\rangle^{2}=z_1^2+\cdots+z_m^2.$$

Lemma 7 *Fix $y\in\mathbb{R}^m$ close to the origin, and $\xi\in\mathbb{R}^m\backslash\{0\}$. Let $M=(M_j)$ be a sequence of positive numbers satisfying conditions $(P3;3')$, $(P4)$, and $(A3)$. For $(x,t)\in\Omega$, let $f(x,t)=(f_1(x,t),...,f_m(x,t))\in C^M(\Omega)$ with $f(0,0)=0$. Set*

$$Q\left(x,t\right)=-i\xi\cdot f\left(x,t\right)-\frac{1}{2}\left|\xi\right|\left\langle y-f\left(x,t\right)\right\rangle^{2}.$$

There is a constant $E>1$ (independent of x,t,y,ξ and α) such that for all multi-indices $\alpha\in\mathbb{Z}_{+}^{m}$ and all $(x,t)\in\Omega$,

$$\left|\partial_{x}^{\alpha}\left(e^{Q(x,t)}\right)\right| \leq e^{\operatorname{Re}Q(x,t)}E^{|\alpha|+1}M_{|\alpha|}e^{\frac{1}{2}M(|\xi|)}, \tag{6}$$

where $M(|\xi|)$ is the associated function, to the sequence (M_j).

Proof. Since $f(x,t)\in C^M(\Omega)$, there is a constant $C>0$ such that for all $(x,t)\in\Omega$ (possibly after shrinking Ω) and for all $\alpha\in\mathbb{Z}_{+}^{m}$, $|\partial_x^\alpha f(x)|\leq C^{|\alpha|+1}M_{|\alpha|}$. We will use the multivariate Faa di Bruno's formula (see (CS) Theorem 2.1 on page 505 and Corollary 2.10 on page 512). Applying this formula to our situation, and keeping in mind the hypothses of the lemma, we get (for C larger than the one appearing above)

$$\left|\partial_{x}^{\alpha}\left(e^{Q(x,t)}\right)\right| \leq \sum_{r=1}^{|\alpha|}e^{\operatorname{Re}Q(x,t)}\sum_{p(\alpha,r)}(\alpha!)\prod_{j=1}^{|\alpha|}\frac{1}{k_j!}\left(\frac{|\xi|\,C^{|\beta_j|+1}M_{|\beta_j|}}{\beta_j!}\right)^{k_j}, \tag{7}$$

where $p(\alpha,r)=\{\left(k_1,...,k_{|\alpha|};\beta_1,...,\beta_{|\alpha|}\right)$: for some $1\leq s\leq|\alpha|$, $k_i=0$ and $\beta_i=0$ for $1\leq i\leq|\alpha|-s$; $k_i>0$ for $|\alpha|-s+1\leq i\leq|\alpha|$; and

$0 \prec \beta_{|\alpha|-s+1} \prec \cdots \prec \beta_{|\alpha|}$ are such that $\sum_{i=1}^{|\alpha|} k_i = r$ and $\sum_{i=1}^{|\alpha|} k_i \beta_i = \alpha\}$. Notice that

$$\prod_{j=1}^{|\alpha|} \left(\frac{|\xi| \, C^{|\beta_j|+1}}{\beta_j!} \right)^{k_j} \leq \frac{|\xi|^r \, C^{|\alpha|+r} 4^{|\alpha|}}{|\alpha|!},$$

and

$$\prod_{j=1}^{|\alpha|} M_{|\beta_j|}^{k_j} \overset{(A3)}{\leq} \prod_{j=1}^{|\alpha|} M_{k_j|\beta_j|-k_j} \overset{(P3;3')}{\leq} M_{|\alpha|-r}.$$

Now, with A and H as in condition $(P4)$, we have

$$|\xi|^r \, M_{|\alpha|-r} = \frac{H^r \left(|\xi| / H \right)^r}{M_r} M_{|\alpha|-r} M_r \overset{(P3;3')}{\leq} H^r M_{|\alpha|} \frac{\left(|\xi| / H \right)^r}{M_r} \leq H^{|\alpha|} M_{|\alpha|} e^{M(|\xi|/H)}.$$

Making use of (3), this last ineqality becomes

$$|\xi|^r \, M_{|\alpha|-r} \leq \sqrt{A} H^{|\alpha|} M_{|\alpha|} e^{\frac{1}{2} M(|\xi|)}.$$

Hence, with D a large constant, (7) becomes

$$\begin{aligned} \left| \partial_x^\alpha \left(e^{Q(x,t)} \right) \right| &\leq e^{\operatorname{Re} Q(x,t)} D^{|\alpha|+1} M_{|\alpha|} e^{\frac{1}{2} M(|\xi|)} \sum_{r=1}^{|\alpha|} \left(\sum_{p(\alpha,r)} \prod_{j=1}^{|\alpha|} \frac{1}{k_j!} \right) \\ &\leq e^{\operatorname{Re} Q(x,t)} E^{|\alpha|+1} M_{|\alpha|} e^{\frac{1}{2} M(|\xi|)}, \end{aligned}$$

where the last inequality follows by definition of $p(\alpha, r)$ (see (CS) page 515) and the constant E is larger than D. This completes the proof of the lemma. ■

We end this subsection with the following elementary lemma which will be used later in this paper.

Lemma 8 (1) *For all multi-indices $\alpha, \beta \in \mathbb{Z}_+^m$ with $\beta \leq \alpha$, we have*

$$\binom{\alpha}{\beta} \leq \binom{|\alpha|}{|\beta|}. \tag{8}$$

(2) *Let $A > 0$ be a fixed constant. Given $\alpha \in \mathbb{Z}_+^m$, there exist constants $L > 1$ and $G > 1$, independent of α, such that*

$$\frac{A}{L-1} \sum_{\beta \leq \alpha} G^{1-|\alpha|+|\beta|} \leq 1. \tag{9}$$

Proof. (1) is by induction and for a proof of (2), see Lemma 4.2 in (BP). ■

2.4 The spaces C_0^M and $\mathcal{D}'_M$

Assuming that the sequence M satisfies properties $(P1)$, $(P2)$, $(P3)$, and $(P4;1)$, A. Lambert (see (L) pages 69-70) was able to prove the existence of C^M functions with compact support *satisfying* some specified properties. Seventeen years prior to (L), Džanašija (see (Dz)) had proved a similar result in the Gevrey class and used it to solve Carleman's problem in the Gevrey class. Before we state Lambert's result, we recall the following notation from (L): set $M_j^{(0)} = 1$ and for $k = 1, 2, ...$ set

$$M_j^{(k)} = \frac{M_j}{(M_k/k!)^{j/k}}. \tag{10}$$

Lemma 9 *For each $k \in \mathbb{N}$ there exists a function $a_k \in C^M(\mathbb{R})$ such that*

(i) supp$(a_k) \subset [-1,1]$, (ii) $a_k(0) = 1$, (iii) For each $j = 1, 2, ...,$ we have $\left(\partial^j a_k/\partial t^j\right)(0) = 0$, and (iv) There are constants $A_1, B_1, C_1 > 1$ and independent of j, k such that for all $j, k \in \mathbb{N}$, $\left|\left(\partial^j a_k/\partial t^j\right)(t)\right| \leq C_1 B_1^k A_1^j M_j^{(k)}$, where $M_j^{(k)}$ is defined in (10).

Definition 10 *Let $U \subset \mathbb{R}^m$ be an open set. We shall denote by $C_0^M(U)$ the vector space of all $\varphi \in C^M(U)$ with compact support in U. The space $\mathcal{D}'_M(U)$ of M-**ultradistributions** is defined to be the dual of $C_0^M(U)$; more precisely, $\mathcal{D}'_M(U)$ is the space of all linear forms u on $C_0^M(U)$ such that for every $K \Subset U$ and for all $\epsilon > 0$ there is a constant $C_\epsilon > 0$ such that*

$$|u(\varphi)| \leq C_\epsilon \sup_{\alpha \in \mathbb{Z}_+^m} \left\{ \epsilon^{|\alpha|} M_{|\alpha|}^{-1} \sup_{x \in K} |\partial^\alpha u(x)| \right\},$$

for all $\varphi \in C_0^M(K) = C^M(U) \cap C_0^\infty(K)$.

2.5 FBI Transform and the $M-$Wavefront Set

Following (CK), we define the FBI transform of an $M-$ultradistribution:

Definition 11 *Let $u \in \mathcal{D}'_M(U)$, $\varphi \in C_0^M(U)$, and $(y, \xi) \in \mathbb{R}^m \times \mathbb{R}^m$. The **FBI transform** of φu, denoted $\mathcal{F}_{\varphi u}(y, \xi)$, is the integral (which, in reality, is a duality bracket)*

$$\mathcal{F}_{\varphi u}(y, \xi) = \int_U e^{-i\xi \cdot x - \frac{1}{2}|\xi||y-x|^2} \varphi(x) u(x)\, dx.$$

In the paper (CK), assuming that the sequence $M = (M_j)$ satisfies conditions $(P1)$, $(P3;1')$, $(P4;1)$, and $(A1)$, Chung and Kim proved the following FBI transform characterization of C^M spaces (Here, $M(t)$ is the associated function to the sequence (M_j)).

Proposition 12 (Theorem 2.1 in (CK)) *Let $u \in \mathcal{D}'_M(\mathbb{R}^m)$ and $x_0 \in \mathbb{R}^m$. The following are equivalent:* (1) *There is a neighborhood U of x_0 such that $u \in C^M(U)$; and* (2) *There are constants $A_1,\ A_2,\ A_3 > 0$ and a neighborhood V of x_0 such that for all $\varphi \in C_0^M(U)$, with $\varphi \equiv 1$ near x_0, we have*

$$|\mathcal{F}_{\varphi u}(y,\xi)| \leq A_1 e^{-M(A_2|\xi|)}$$

for all $y \in V$ and $|\xi| \geq A_3$.

In case $u \in \mathcal{D}'_M(U)$ is non-C^M at x_0, we can obtain additional information about the structure of the singularities at x_0 by examining the directions in which the above inequalities break down.

Definition 13 *For fixed $x_0 \in U$ and $\xi_0 \in \mathbb{R}^m \backslash \{0\}$, we say that $u \in \mathcal{D}'_M(U)$ is $M-$**micro-regular** at (x_0,ξ_0) if there exists $\varphi \in C_0^M(U)$, with $\varphi \equiv 1$ near x_0, a neighborhood V of x_0 in $\mathbb{R}^m$, and a conic neighborhood Γ of ξ_0 in $\mathbb{R}^m \backslash \{0\}$ such that the FBI estimate in Proposition (12) holds for all $y \in V$, $\xi \in \Gamma$, $|\xi| \geq A_3$. The $M-$**wave-front set** of u, denoted $WF_M(u)$, is the complement in $U \times \mathbb{R}^m \backslash \{0\}$ of the set of all (x_0,ξ_0) where u is $M-$micro-regular.*

3 Carleman's Problem and Almost Analytic Extensions

3.1 Carleman's Problem

We begin this section by extending lemma 3.1 in (BP) to the C^M class. Here, we assume that our sequence M satisfies conditions $(P1)$, $(P2)$, $(P3)$, and $(P4;1)$.

Lemma 14 *Let $\{v_k(x)\}_{k=0}^{\infty}$ be a sequence of C^∞ functions defined on an open neighborhood, $U \subset \mathbb{R}^m$ of the origin, so that given $K \Subset U$, there exists $B > 0$ such that*

$$|\partial_x^\alpha v_k(x)| \leq B^{|\alpha|+k+1} M_{|\alpha|} M_k \qquad \forall x \in K,\ k = 0,1,2,...,\ \alpha \in \mathbb{Z}_+^m. \tag{11}$$

Then, shrinking U if necessary, there exists $f = f(x,t) \in C^M(U \times (-1,1))$ such that for each $n = 0,1,2,...$

$$\frac{\partial^n f}{\partial t^n}(x,0) = v_n(x) \qquad \forall x \in U. \tag{12}$$

Proof. For $k \in \mathbb{N}$, let $a_k(t)$ be as in Lemma 9, and set $\sigma_k = D^{-1} M_1^{(k)} = D^{-1}(k!/M_k)^{1/k}$, where $D > 0$ is to be determined at the end of this proof. Take an open neighborhood U of the origin in $\mathbb{R}^m$. For $(x,t) \in U \times [-1,1]$ we consider the formal series

$$\sum_{k=1}^{\infty} \frac{v_k(x)}{k!} a_k(t/\sigma_k)\, t^k. \tag{13}$$

Note that by definition, $a_k(t/\sigma_k)$ vanishes outside $[-\sigma_k, \sigma_k]$. Let $K \Subset U$. It follows, from (11) and from Lemma 9, that the series in (13) converges uniformly on $K \times [-1,1]$. Shrinking U if necessary, we set for $(x,t) \in U \times [-1,1]$

$$f(x,t) = \sum_{k=0}^{\infty} \frac{v_k(x)}{k!} a_k(t/\sigma_k)\, t^k. \tag{14}$$

In order to show that f satisfies the conditions of Lemma 14, it suffices to prove that $g(x,t) = \sum_{k=1}^{\infty} \frac{v_k(x)}{k!} a_k(t/\sigma_k)\, t^k$ belongs to $C^M(U \times (-1,1))$ and satisfies $(\partial^n g/\partial t^n)(x,0) = v_n(x)$ for all $n \in \mathbb{N},\ x \in U$. Let $\alpha \in \mathbb{Z}_+^m$ and $n \in \mathbb{N}$ be given. Then $\partial_x^\alpha \partial_t^n g(x,t) = \sum_{k=1}^{\infty} \partial_x^\alpha v_k(x) \sum_{i=0}^{n} \binom{n}{i} \left[\frac{(\partial_t^{n-i} a_k)(t/\sigma_k)}{\sigma_k^{n-i}}\right] \left[\frac{t^{k-i}}{(k-i)!}\right]$. Note that if $k < n$, then the sum, in i, above vanishes for $i > k$. We now have

$$|\partial_x^\alpha \partial_t^n g(x,t)| \le \sum_{k=1}^{\infty} \left(B^{|\alpha|+k+1} M_{|\alpha|} M_k\right) \sum_{i=0}^{n} \binom{n}{i} \left(\frac{C_1 B_1^k A_1^{n-i} M_{n-i}^{(k)}}{\sigma_k^{n-i}}\right)\left(\frac{\sigma_k^{k-i}}{(k-i)!}\right).$$

Note that

$$\frac{M_{n-i}^{(k)} \sigma_k^{k-i}}{\sigma_k^{n-i}(k-i)!} \le \frac{D^{n-k}}{M_k} \left(M_{n-i} M_k^{i/k}\right) \left(\frac{k^{k-i}}{(k-i)!}\right) \le \frac{D^{n-k}}{M_k} \left(C^k M_{n-i} M_i\right)\left(e^k\right),$$

where we have used condition $(A2)$ in the last inequality. Hence,

$$\binom{n}{i} \frac{M_{n-i}^{(k)} \sigma_k^{k-i}}{\sigma_k^{n-i}(k-i)!} M_k \le D^n \left(\frac{Ce}{D}\right)^k \left(\binom{n}{i} M_{n-i} M_i\right) \le D^n \left(\frac{Ce}{D}\right)^k (C' M_n),$$

where we have used condition $(P3;3)$ in the last inequality. Using this last estimate, we return to our estimatation of $|\partial_x^\alpha \partial_t^n g(x,t)|$:

$$\begin{aligned} |\partial_x^\alpha \partial_t^n g(x,t)| &\le \sum_{k=1}^{\infty} B^{|\alpha|+k+1} M_{|\alpha|} \sum_{i=0}^{n} C_1 B_1^k A_1^{n-i} D^n \left(\frac{Ce}{D}\right)^k (C' M_n) \\ &\le \sum_{k=1}^{\infty} 2 C_1 C' B^{|\alpha|+1} (D A_1)^n M_{|\alpha|} M_n \left(\frac{B B_1 C e}{D}\right)^k, \end{aligned}$$

where we are assuming, and we could, that $A_1 \ge 2$. Choose $D > 0$ large enough so that

$$|\partial_x^\alpha \partial_t^n g(x,t)| \le B^{|\alpha|+1} (D A_1)^n M_{|\alpha|} M_n \le A^{|\alpha|+n+1} M_{|\alpha|} M_n,$$

where $A = \max\{D A_1, B\}$ is independent of α and n. Hence, $g \in C^M(U \times (-1,1))$ and satisfies $(\partial^n g/\partial t^n)(x,0) = v_n(x)$ for all $n \in \mathbb{N},\ x \in U$. This proves that $f \in C^M(U \times (-1,1))$ and satisfies condition (12). ∎

We now extend Lemma 14 to a multi-sequence $\{v_\beta(x)\}_{\beta \in \mathbb{Z}_+^n}$ and $t \in (-1,1)^n$. This generalizes Lemma 3.3 in (BP) to the situation $m \ne n$ and their Gevrey result to the class C^M.

Lemma 15 *Fix $n \in \mathbb{N}$. Let $\{v_\beta(x)\}_{\beta \in \mathbb{Z}_+^n}$ be a multi-sequence of C^∞ functions, defined in a neighborhood $U \subset \mathbb{R}^m$ of the origin, such that given $K \Subset U$, there exists a constant $B > 1$ such that*

$$|\partial_x^\alpha v_\beta(x)| \leq B^{|\alpha|+|\beta|+1} M_{|\alpha|} M_{|\beta|} \qquad \forall x \in K,\ \alpha \in \mathbb{Z}_+^m,\ \beta \in \mathbb{Z}_+^n. \tag{15}$$

Then, shrinking U if necessary, there exists $F \in C^M(U \times (-1,1)^n)$ such that for each $\gamma \in \mathbb{Z}_+^n$

$$\partial_t^\gamma F(x,0) = v_\gamma(x) \qquad \forall x \in U. \tag{16}$$

Proof. For $x \in U$ and $t \in [-1,1]^n$, we define

$$F(x,t) = \sum_{\beta \in \mathbb{Z}_+^n} \frac{v_\beta(x)}{\beta!} A_\beta(t/\sigma_\beta) t^\beta, \tag{17}$$

where we set

$$A_\beta(t/\sigma_\beta) = a_{\beta_1}\left(t_1/\sigma_{\beta_1}\right) \cdot \ldots \cdot a_{\beta_n}\left(t_n/\sigma_{\beta_n}\right),$$

and a_{β_j} and σ_{β_j} are as defined in Lemma 14. Applying $\partial_x^\alpha \partial_t^\gamma$ to F we obtain a series whose general term is given by

$$W_\beta(x,t) = \frac{\partial_x^\alpha v_\beta(x)}{\beta!} \sum_{\gamma' \leq \gamma} \binom{\gamma}{\gamma'} \partial_t^{\gamma-\gamma'} A_\beta(t/\sigma_\beta)\, \partial_t^{\gamma'}\left(t^\beta\right).$$

It follows from (16) that for $(x,t) \in K \times [-1,1]^n$, there exists a constant $B > 1$ such that

$$|W_\beta(x,t)| \leq \frac{B^{|\alpha|+|\beta|+1}}{\beta!} M_{|\alpha|} M_{|\beta|} \left|G_1 \cdot \ldots \cdot G_n\right|,$$

where $G_j = G_j(\gamma,\beta,t) = \sum_{\gamma_j' \leq \gamma_j} \binom{\gamma_j}{\gamma_j'} \partial_{t_j}^{\gamma_j-\gamma_j'} a_{\beta_j}\left(t_j/\sigma_{\beta_j}\right) \partial_{t_j}^{\gamma_j'}\left(t_j^{\beta_j}\right)$. We first consider the case $n \leq m$. For convenience, we use the notation $\alpha = (\alpha', \alpha'') \in \mathbb{Z}_+^n \times \mathbb{Z}_+^{m-n}$. We obtain

$$\begin{aligned} |W_\beta(x,t)| &\leq \left(\prod_{j=1}^n \frac{B^{\alpha_j+\beta_j+1}}{\beta_j!} M_{\alpha_j} M_{\beta_j} |G_j|\right) \left(\prod_{j=n+1}^m B^{\alpha_j} M_{\alpha_j}\right) \\ &\leq \left(B^{|\alpha|-|\alpha'|} M_{|\alpha|-|\alpha'|}\right) \prod_{j=1}^n \frac{B^{\alpha_j+\beta_j+1}}{\beta_j!} M_{\alpha_j} M_{\beta_j} |G_j|, \end{aligned} \tag{18}$$

where the constant B in the second inequality is larger than in the first. For the product in the above estimate, we can apply the techniques used in the proof of Lemma 14 (with $m = n = 1$). We conclude that for each $j = 1,..,n$ there exists a constant $A_j > 1$ independent of α_j and γ_j such that for all $(x,t) \in K \times [-1,1]^n$, we have

$$\begin{aligned}\sum_{\beta \in \mathbb{Z}_+^n} \prod_{j=1}^{n} \frac{B^{\alpha_j+\beta_j+1}}{\beta_j!} M_{\alpha_j} M_{\beta_j} |G_j| &\leq \prod_{j=1}^{n} \sum_{\beta_j=0}^{\infty} \frac{B^{\alpha_j+\beta_j+1}}{\beta_j!} M_{\alpha_j} M_{\beta_j} |G_j| \\ &\leq \prod_{j=1}^{n} A_j^{\alpha_j+\beta_j+1} M_{\alpha_j} M_{\beta_j} \\ &\leq A^{|\alpha'|+|\beta|+1} M_{|\alpha|} M_{|\beta|},\end{aligned}$$

where $A = B \cdot A_1 \cdot ... \cdot A_n$ and we have used property $(P3; 3')$ to justify the last inequality. Using this last estimate, combined with the estimate in (18), we get

$$\sum_{\beta \in \mathbb{Z}_+^n} |W_\beta (x,t)| \leq A^{|\alpha|+|\beta|+1} M_{|\alpha|} M_{|\beta|}.$$

Hence, shrinking U, we see that $F \in C^M (U \times (-1,1)^n)$ and $\partial_t^\gamma F(x,0) = v_\gamma (x)$ $\forall x \in U$, $\gamma \in \mathbb{Z}_+^n$, as desired. The case $n > m$ is similar. ■

3.2 Existence of Almost Analytic Extensions

Definition 16 *Let $U \subset \mathbb{R}^m$ be an open set and $f = f(x) \in C^M(U)$. We say that a C^∞ function $\widetilde{f} = \widetilde{f}(x,t) \in C^\infty (U \times (-1,1)^m)$ is an M-**almost analytic extension** of f if the following is true: (i) $\widetilde{f} \in C^M (U \times (-1,1)^m)$; (ii) $\widetilde{f}(x,0) = f(x)$ for all $x \in U$; and (iii) For every $(x,t) \in U \times (-1,1)^m$ and for all $N = 1,2,...$, there exists a constant $C > 0$ independent of N such that*

$$\left| \frac{\partial \widetilde{f}}{\partial \overline{z}_j} (x,t) \right| \leq \frac{C^{N+1}}{N!} M_N |t|^N. \tag{19}$$

Here, we write $z_j = x_j + it_j$ and $\partial/\partial\overline{z}_j = 1/2 (\partial/\partial x_j + i\partial/\partial t_j)$ for $j = 1,...,m$. Notice that when $M_N = (N!)^s$, $s \geq 1$, the above definition agrees with the definition for almost analytic extensions of exponent s.

The following lemma generalizes Lemma 3.4 in (BP) to the C^M case:

Lemma 17 *Every $f \in C^M(U)$ has an $M-$almost analytic extension.*

Proof. Define, for $(x,t) \in U \times (-1,1)^m$, $\widetilde{f}(x,t) = \sum_{\alpha \in \mathbb{Z}_+^m} \frac{\partial_x^\alpha f(x)}{\alpha!} A_\alpha (t/\sigma_\alpha) (it)^\alpha$,
where $A_\alpha (t/\sigma_\alpha)$ is defined as in the proof of Lemma 15. The proof now is a consequence of Lemma 15. ■

4 Existence of $M-$Approximate Solutions

Throughout this section, unless otherwise stated, $\Omega \subset \mathbb{R}^m \times \mathbb{R}$ and $U \subset \mathbb{R}^m$ will denote open neighborhoods of the origin (in thier respective Euclidean spaces) such that $U \times \{0\} \subset \Omega$. Also, the sequence M will be assumed to satisfy conditions $(P1), (P2), (P3)$, and $(P4;1)$ so that we can use the results proved in the previous section.

Lemma 18 *Let $f(x) \in C^M(U)$ and $a_k(x,t) \in C^M(\Omega)$, $k = 1, ..., m$. Set $u_0(x) = f(x)$ and for $j \geq 1$,*

$$u_j(x) = -\frac{1}{j} \sum_{p+q=j-1} \frac{1}{q!} \sum_{k=1}^{m} \frac{\partial u_p}{\partial x_k}(x)\, \partial_t^q a_k(x,0). \tag{20}$$

Then, given $K \Subset U$, there exist constants $B, D > 0$ such that

$$|\partial_x^\alpha u_j(x)| \leq \frac{B^j}{j!} D^{|\alpha|+1} M_{|\alpha|+j} \qquad \forall x \in K,\ j = 0,1,2,...,\ \alpha \in \mathbb{Z}_+^m. \tag{21}$$

Proof. Let $K \Subset U$. Since $f \in C^M(U)$ and $a_k \in C^M(\Omega)$, there exists a constant $A > 1$ such that for all $n \in \mathbb{Z}_+,\ \alpha \in \mathbb{Z}_+^m,\ x \in K$, and $k = 1, .., m$, we have

$$|\partial_x^\alpha f(x)| \leq A^{|\alpha|+1} M_{|\alpha|} \qquad \text{and} \qquad |\partial_x^\alpha \partial_t^n a_k(x,0)| \leq A^{|\alpha|+n+1} M_{|\alpha|+n}. \tag{22}$$

We now choose $L, G > 1$ such that the inequality (9) holds and we define $B = mAL$ and $D = AG$. We will prove (21) using induction on j. The case $j = 0$ is trivial:

$$|\partial_x^\alpha u_0(x)| = |\partial_x^\alpha f(x)| \leq A^{|\alpha|+1} M_{|\alpha|} \leq \frac{B^0}{0!} D^{|\alpha|+1} M_{|\alpha|+0}.$$

Suppose now that (21) holds for $j-1,\ j \geq 1$. Then it follows from (20) that

$$|\partial_x^\alpha u_j(x)| \leq \frac{1}{j} \sum_{p+q=j-1} \frac{1}{q!} \sum_{k=1}^{m} \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} \left|\partial_x^{\beta+e_k} u_p(x)\right| \left|\partial_x^{\alpha-\beta} \partial_t^q a_k(x,0)\right| \tag{23}$$

where $\{e_k\}_{k=1}^m$ is the standard basis of $\mathbb{R}^m$. By our induction hypothesis, we have for all $x \in K$:

$$\left|\partial_x^{\beta+e_k} u_p(x)\right| \leq \frac{B^p}{p!} D^{|\beta|+2} M_{|\beta|+p+1} \tag{24}$$

and from (22) we have

$$\left|\partial_x^{\alpha-\beta} \partial_t^q a_k(x,0)\right| \leq A^{|\alpha|-|\beta|+q+1} M_{|\alpha|-|\beta|+q}. \tag{25}$$

Using $(P3;3)$ and the inequality in (8), we obtain

$$\binom{\alpha}{\beta} \frac{M_{|\beta|+p+1} M_{|\alpha|-|\beta|+q}}{p!q!} \leq \binom{|\alpha|+j}{|\beta|+p+1} \frac{M_{|\beta|+p+1} M_{|\alpha|-|\beta|+q}}{(j-1)!} \leq \frac{M_{|\alpha|+j}}{(j-1)!}.$$

It follows from this last estimate, together with (24), (25) and (9) that

$$|\partial_x^\alpha u_j(x)| \le \frac{B^j}{j!} D^{|\alpha|+1} M_{|\alpha|+j}.$$

This completes the proof. ■

Definition 19 *Consider the complex vector field* $L = \partial/\partial t + \sum_{k=1}^{m} a_k(x,t)\,\partial/\partial x_k$ *where the coefficients* $a_k \in C^M(\Omega)$. *We say that* $u = u(x,t) \in C^M(\Omega)$ *is an* M-***approximate solution*** *of* L *if there exists a constant* $A > 0$ *such that for all* $(x,t) \in \Omega$, *we have*

$$|Lu(x,t)| \le \frac{A^{N+1}}{N!} M_N |t|^N \qquad \forall N \in \mathbb{N}. \tag{26}$$

We are now in position to state and prove our main result in this section: L and Ω are as in the above definition.

Theorem 20 *Let* $f = f(x) \in C^M(U)$. *Then, shrinking* Ω, *there exists a function* $u = u(x,t) \in C^M(\Omega)$ *which is a* $M-$*approximate solution of* L *and such that* $u(x,0) = f(x)$. *Moreover, there exists a constant* $C > 1$ *such that for all* $N \in \mathbb{N}$, *all* $\alpha \in \mathbb{Z}_+^m$, *and all* $(x,t) \in \Omega$:

$$|D_x^\alpha Lu(x,t)| \le \frac{C^{|\alpha|+N+1}}{N!} M_{|\alpha|+N} |t|^N. \tag{27}$$

Proof. The conditions that u has to satisfy determine the Taylor coefficients of the formal power series $u(x,t) = \sum_{j=0}^{\infty} u_j(x) t^j$, where $u_j(x) = \partial_t^j u(x,0)/j!$. Set $u_0(x) = f(x)$. For each j, since $Lu(x,t) = O\left(t^{j+1}\right)$, we have $\partial_t^{j-1}(Lu)(x,0) = 0$. This then leads to $u_j(x) = -\frac{1}{j} \sum_{p+q=j-1} \frac{1}{q!} \left[\sum_{k=1}^{m} \frac{\partial u_p}{\partial x_k}(x) \frac{\partial^q a_k}{\partial t^q}(x,0)\right]$. It follows from our hypothesis that for each fixed j, $u_j \in C^M(U)$. It now follows from Lemma (18), that for every $K \Subset U$ there exist constants $B, D > 1$ such that $|\partial_x^\alpha u_j(x)| \le \frac{B^j}{j!} D^{|\alpha|+1} M_{|\alpha|+j}$ $\forall x \in K$, $\alpha \in \mathbb{Z}_+^m$. We now define $v_j(x) = j! u_j(x)$. Then $\forall x \in K$,and $\alpha \in \mathbb{Z}_+^m$, we have

$$\begin{aligned} |\partial_x^\alpha v_j(x)| &\le B^j D^{|\alpha|+1} M_{|\alpha|+j} \\ &\le B^j D^{|\alpha|+1} \left(C^{|\alpha|+j+1} M_{|\alpha|} M_j\right) \\ &\le E^{|\alpha|+j+1} M_{|\alpha|} M_j, \end{aligned}$$

where $E = \max\{BD, C\}$ and the second inequality follows from condition $(P4;1)$. It follows from Lemma 14 that, shrinking U, there exists $u \in C^M(\Omega)$ such that for each j, $\frac{\partial^j u}{\partial t^j}(x,0) = v_j(x) = j!u_j(x)\ \ \forall x \in U$, and so $u_j(x) = \frac{1}{j!}\frac{\partial^j u}{\partial t^j}(x,0)\ \ \forall x \in U$. In particular, we have $u(x,0) = u_0(x) = f(x)$. It is now easy to see that u is our desired $M-$approximate solution of L. Finally, using Taylor's theorem we obtain the estimate in (27). ■

5 Preliminaries on Involutive Structures

In this section we will briefly recall some of the geometric notions and results we will need about involutive structures. For a good reference on Involutive structures, we refer the reader to the book (BCH). We also point out that many of the constructions in this section are due to Eastwood and Graham (see (EG) for more details). We assume $(\mathcal{M}, \mathcal{V})$ is an $M-$**involutive structure** (i.e., $\mathcal{M}$ is a C^M manifold and $\mathcal{V}$ is a complex subbundle of $\mathbb{C}T\mathcal{M}$ which is closed under the bracket operation) and the fiber dimension of $\mathcal{V}$ equals n. A smooth submanifold X of $\mathcal{M}$ is called **maximally real** if $\mathbb{C}T_p\mathcal{M} = \mathcal{V}_p \oplus \mathbb{C}T_pX$ for each $p \in X$. If X is a maximally real submanifold and $p \in X$, define $\mathcal{V}_p^X = \{L \in \mathcal{V}_p : \Re L \in T_pX\}$. We recall the following result from (EG) which is also valid for a general involutive structure.

Proposition 21 *(Lemma II.1 in (EG))* $\mathcal{V}^X$ *is a real subbundle of* $\mathcal{V}|_X$ *of rank* n. *The map* $\Im : \mathcal{V}|_X \to T\mathcal{M}$ *which takes the imaginary part induces an isomorphism* $\mathcal{V}^X \cong T\mathcal{M}|_X \diagup TX$.

Proposition 21 shows that when X is maximally real, for $p \in X$, $\Im$ defines an isomorphism from $\mathcal{V}_p^X$ to an $n-$dimensional subspace N_p of $T_p\mathcal{M}$ which is a canonical complement to T_pX in the sense that $T_p\mathcal{M} = T_pX \oplus N_p$.

Definition 22 *Let* E *be a submanifold of* $\mathcal{M}$, $\dim_{\mathbb{R}} E = k$. *We say an open set* $\mathcal{W}$ *is a* ***wedge in*** $\mathcal{M}$ ***at*** $p \in E$ ***with edge*** E *if the following holds: there exists a* C^M*-diffeomorphism* F *of a neighborhood* V *of* 0 *in* $\mathbb{R}^N$ $(N = \dim_{\mathbb{R}} \mathcal{M})$ *onto a neighborhood* U *of* p *in* $\mathcal{M}$ *with* $F(0) = p$ *and a set* $B \times \Gamma \subseteq V$ *with* B *a ball centered at* $0 \in \mathbb{R}^k$ *and* Γ *a truncated, open convex cone in* $\mathbb{R}^{N-k}$ *with vertex at* 0 *such that*

$$F(B \times \Gamma) = \mathcal{W} \quad \text{and} \quad F(B \times \{0\}) = E \cap U.$$

Let E, $\mathcal{W}$ and $p \in E$ be as in the previous definition. The **direction wedge** $\Gamma_p(\mathcal{W}) \subseteq T_p\mathcal{M}$ is defined as the interior of the set

$$\{c'(0)|\, c : [0,1) \to \mathcal{M} \text{ is } C^\infty,\ c(0) = p,\ c(t) \in \mathcal{W}\ \forall t > 0\}.$$

It is easy to see that $\Gamma_p(\mathcal{W})$ is a linear wedge in $T_p\mathcal{M}$ with edge T_pE. Set $\Gamma(\mathcal{W}) = \bigcup_{p \in E} \Gamma_p(\mathcal{W})$. Suppose $\mathcal{W}$ is a wedge in $\mathcal{M}$ with a maximally real edge X. As observed in (EG), since $\Gamma_p(\mathcal{W})$ is determined by its image in $T_p\mathcal{M} \diagup T_pX$, the isomorphism $\Im$ can be used to define a corresponding wedge in $\mathcal{V}_p^X$ by setting $\Gamma_p^{\mathcal{V}}(\mathcal{W}) = \left\{L \in \mathcal{V}_p^X : \Im L \in \Gamma_p(\mathcal{W})\right\}$. $\Gamma_p^{\mathcal{V}}(\mathcal{W})$ is a linear wedge in $\mathcal{V}_p^X$ with edge $\{0\}$, that is, it is a cone. Define $\Gamma_p^T(\mathcal{W}) = \left\{\Re L : L \in \Gamma_p^{\mathcal{V}}(\mathcal{W})\right\}$. $\Gamma_p^T(\mathcal{W})$ is an open cone in $(\Re\mathcal{V}_p) \cap T_pX$ (see (EG)). Set $\Gamma^T(\mathcal{W}) = \bigcup_{p\in X} \Gamma_p^T(\mathcal{W})$.

Definition 23 *Let $\mathcal{W}$ be a wedge in $\mathcal{M}$ with edge a maximally real submanifold X. We say a distribution $f \in \mathcal{D}'_M(\mathcal{W})$ is an $M-$**approximate solution** if $Lf \in L^1_{loc}(\mathcal{W})$ and for all sections L of $\mathcal{V}$, $Lf(p) = \frac{C^{N+1}}{N!} M_N(dist(p,X))^N$ $\forall N = 1,2,3,...$, where the constant $C > 1$ is independent of the section L and the integer N.*

6 The M-Wavefront Set and Edge-of-the-Wegde Theorem

In this section, we will assume that the sequence M satisfies conditions $(P1) - (P4)$. The following main result and its corollary are inspired by the results in (EG) and (AB). This result also improves our previous Gevrey result in (AH) (see Theorem 5.1 on page 2858).

Theorem 24 *Let $(\mathcal{M},\mathcal{V})$ be an $M-$involutive structure,* $\dim_{\mathbb{R}} \mathcal{M} = m+1$, *rank of $\mathcal{V} = 1$, $X \subset \mathcal{M}$ a C^M maximally real submanifold, and $\mathcal{W}$ a wedge in $\mathcal{M}$ with edge X. Suppose that $u \in \mathcal{D}'_M(X)$ is the boundary value of an M-approximate solution $f \in \mathcal{D}'_M(\mathcal{W})$. Then $WF_M(u) \subset \left(\Gamma^T(\mathcal{W})\right)^0$.*

Proof. Since $\mathcal{W}$ is a wedge in $\mathcal{M}$ with edge X, in a neighborhood Ω (in $\mathcal{M}$) of a point $p \in X$, there are C^M coordinates $(x,t) = (x_1,\dots,x_m,t)$ vanishing at p so that in Ω, $X = \{(x,0) : |x| < r\} = B_r(0)$, and $\mathcal{W} = X \times (0,\lambda)$ for some $\lambda > 0$. Since X is maximally real, there exists a section L of $\mathcal{V}$ (near 0) and C^M functions $a_k(x,t)$ such that $L = \partial/\partial t + \sum_{k=1}^{m} a_k(x,t)\,\partial/\partial x_k$. Observe that, near 0, $\mathcal{V} = span_{\mathbb{C}}\{L\}$. Using Theorem 20, let $\{Z_1(x,t),...,Z_m(x,t)\}$ be a complete set of $M-$approximate first integrals for L in $\Omega = B_r(0) \times (-\lambda,\lambda)$ such that $Z_l(x,0) = x_l$, for $1 \le l \le m$, and for all $N \in \mathbb{N}$ there exists a constant $C > 1$, independent of N, α, and $(x,t) \in \Omega$, such that for all $\alpha \in \mathbb{Z}_+^m$ and all $(x,t) \in \Omega$:

$$|D_x^\alpha L Z_l(x,t)| \le \frac{C^{|\alpha|+N+1}}{N!} M_{|\alpha|+N} |t|^N, \qquad N = 1,2,3,\cdots. \tag{28}$$

Set $Z(x,t) = (Z_1(x,t),\dots,Z_m(x,t))$ and write $Z(x,t) = x + A(x,t)t$. It follows that $\left(\Gamma_0^T(\mathcal{W})\right)^0 = \{\xi \in \mathbb{R}^m\backslash\{0\} : \xi\cdot\Im A(0,0)b \ge 0 \;\; \forall b > 0\}$. Thus

$$\xi^0 \notin \left(\Gamma_0^T(\mathcal{W})\right)^0 \Leftrightarrow \xi^0\cdot\Im A(0,0)\mathbb{R}^+ < 0. \tag{29}$$

Define the vector field $L' = L - \sum_{k=1}^m LZ_k(x,t)\mathfrak{M}_k$, where for each k, $\mathfrak{M}_k = \frac{1}{\partial_{x_k} Z_k(x,t)}\left(\frac{\partial}{\partial x_k} - \sum\limits_{p\neq k, 1\leq p\leq m} \partial_{x_k} Z_p\left(x,t\right) \frac{\partial}{\partial x_p}|_{(x,0)}\right)$. Notice that condition $(P3;2)$ implies that $\mathfrak{M}_k$ is a C^M vector field. Also, $\mathfrak{M}_k Z_l = \delta_{kl}$ and so $L' Z_l = 0$ for all $1 \leq l \leq m$. Since $f(x,t)$ is an M-approximate solution of $\mathcal{V}$ in $\mathcal{W}$, $Lf \in L^1_{loc}(\mathcal{W})$ and there is a constant $C > 1$ such that for all $N = 1, 2, ...,$ we have $|Lf(x,t)| \leq \frac{C^{N+1}}{N!} M_N |t|^N$ for all $(x,t) \in \mathcal{W}$. We also know that $\lim\limits_{t\to 0^+} \int_X f(x,t)\varphi(x)\,dx = \langle u, \varphi\rangle$ exists for all $\varphi \in C_0^M(X)$. Let $\eta(x) \in C_0^M(\mathbb{R}^m)$, $\eta(x) \equiv 1$ for $|x| \leq r$, and $\eta(x) \equiv 0$ when $|x| \geq 2r$ (r small). We will consider the following FBI transform of ηf:

$$\mathcal{F}_{\eta f}(t;y,\xi) = \int_X e^{-i\xi\cdot Z(x,t) - \frac{1}{2}|\xi|\langle y - Z(x,t)\rangle^2} \eta(x) f(x,t) \left(\det Z_x(x,t)\right)\, dx.$$

Note that $\mathcal{F}_{\eta f}(0;y,\xi) = \int_X e^{-i\xi\cdot x - \frac{1}{2}|\xi|\langle y-x\rangle^2}\eta(x)u(x)\,dx = \mathcal{F}_{\eta u}(y,\xi)$. Let $\xi^0 \in \mathbb{R}^m\backslash\{0\}$ be such that $\xi^0 \notin \left(\Gamma_0^T(\mathcal{W})\right)^0$. Then, by (29), we see that $\xi^0 \cdot \Im A(0,0)\mathbb{R}^+ < 0$. Fix $T \in \mathbb{R}^+$ and let $\gamma(s) = sT$ for $0 \leq s \leq 1$. Consider the m-form $\omega = g\,dZ$, where $g(x,t) = e^{-i\xi\cdot Z(x,t) - \frac{1}{2}|\xi|\langle y - Z(x,t)\rangle^2}\eta(x) f(x,t)$. Then Stokes' theorem implies

$$\begin{aligned} |\mathcal{F}_{\eta u}(y,\xi)| \quad &\leq \quad \left|\int_X e^{Q(x,T,y,\xi)}\eta(x) f(x,T)\left(\det Z_x(x,T)\right)\, dx\right| \\ &\quad + \left|\int_\gamma\int_X e^{Q(x,t,y,\xi)} f(x,t) L'\eta(x)\left(\det Z_x\left(x,t\right)\right) dxdt\right| \\ &\quad + \left|\int_\gamma\int_X e^{Q(x,t,y,\xi)}\eta(x) L' f(x,t) (\det Z_x(x,t))\, dxdt\right|, \quad (30) \end{aligned}$$

where $Q(x,t,y,\xi) = -i\xi\cdot Z(x,t) - \frac{1}{2}|\xi|\left\langle y - Z(x,t)\right\rangle^2$. Following (AB), there are $\delta > 0$, $C_0 > 0$, an open neighborhood $V \subset \mathbb{R}^m$ of the origin and an open conic neighborhood $\mathcal{C} \subset \mathbb{R}^m\backslash\{0\}$ of ξ^0 such that for all $t \in \gamma$ and all $(y,\xi) \in V \times \mathcal{C}$:

$$\Re Q(x,t,y,\xi) \leq -\frac{1}{4} C_0 \left|t\right| \left|\xi\right| - \left|y - x\right|^2.$$

This immediately implies that the first two terms on the RHS of (30) have an exponential decay for y near 0 and ξ in a conic neighborhood of ξ_0. To estimate the third term, for N a positive integer,

$$|\xi|^N \int_\gamma \left|\int_X e^{Q(x,t,y,\xi)}\eta(x) L' f(x,t) dx\right| dt \leq A\left|\xi\right|^N \mathbf{I}_1 + A\left|\xi\right|^N \mathbf{I}_2,$$

where $\mathbf{I}_1 = \int_\gamma \left|\int_X e^Q \eta\left(Lf\right) dx\right| dt$, $\mathbf{I}_2 = \sum_{k=1}^m \int_\gamma \left|\int_X e^Q \eta\left(LZ_k\right)\left(\mathfrak{M}_k f\right) dx\right| dt$. Since f is an approximate solution of L, we obtain

$$A\left|\xi\right|^N \mathbf{I}_1 \leq D^{N+1} M_N \quad \text{for all } (y,\xi) \in V \times \mathcal{C} \text{ and all } N \in \mathbb{N},$$

and so for all $(y,\xi)\in V\times\mathcal{C}$ and for a suitable constant $A>0$,

$$\mathbf{I}_1 \leq Ce^{-M(A|\xi|)} \tag{31}$$

Since $bf=u$ exists in $\mathcal{D}'_M(X)$, so does $b\left(\mathfrak{M}_k f\right)$ for all $k=1,...,m$. Hence, after decreasing δ, we get (see Definition 10) that $\forall\epsilon>0\ \exists C_\epsilon>0:$

$$A\left|\xi\right|^N \mathbf{I}_2 \leq A'\left|\xi\right|^N \sum_{k=1}^{m}\int_\gamma C_\epsilon \sup_{\alpha\in\mathbb{Z}_+^m}\left\{\frac{\epsilon^{|\alpha|}}{M_{|\alpha|}}\sup_{x\in supp(\eta)}\left|\partial_x^\alpha\left\{\left(e^Q\right)(\eta)\left(LZ_k\right)\right\}\right|\right\}dt.$$

We have $\partial_x^\alpha\left\{\left(e^Q\right)(\eta)\left(LZ_k\right)\right\} = \sum_{\beta+\vartheta+\upsilon=\alpha} C^\alpha_{\beta,\vartheta,\upsilon}\left(\partial_x^\beta e^Q\right)\left(\partial_x^\vartheta\eta\right)\left(\partial_x^\upsilon LZ_k\right)$. Making use of the inequality in (6), the fact that $\eta\in C_0^M(\mathbb{R}^m)$, and the estimate in (27), we obtain for some constant $D>1$, $\left|\partial_x^\beta e^Q\right|\leq e^{-c|t||\xi|}D^{|\beta|+1}M_{|\beta|}e^{\frac{1}{2}M(|\xi|)}$, $\left|\partial_x^\vartheta\eta\right|\leq D^{|\vartheta|+1}M_{|\vartheta|}$, and $\left|\partial_x^\upsilon LZ_k\right|\leq\frac{D^{|\upsilon|+N+1}}{N!}M_{|\upsilon|+N}\left|t\right|^N$. Hence, $\exists C>1$ large so that

$$A'\left|\xi\right|^N\left|\partial_x^\alpha\left\{\left(e^Q\right)(\eta)\left(LZ_k\right)\right\}\right|\leq C^{|\alpha|+N+1}M_{|\alpha|}M_Ne^{\frac{1}{2}M(|\xi|)}.$$

Thus,

$$A\left|\xi\right|^N\mathbf{I}_2\leq C^{N+1}M_Ne^{\frac{1}{2}M(|\xi|)}\sum_{k=1}^m\int_\gamma C_\epsilon\sup_{\alpha\in\mathbb{Z}_+^m}\left\{\epsilon^{|\alpha|}C^{|\alpha|}\right\}dt.$$

Since the above inequality holds for each positive integer N, if we choose $\epsilon<\frac{1}{2C}$, we obtain for some constant $D>1$,

$$\mathbf{I}_2\leq De^{-M(\delta|\xi|)+\frac{1}{2}M(|\xi|)},\qquad(\delta=1/D).$$

Making use of the inequality (4), we have for some large $B=B(\delta)>0$,

$$\frac{3}{2}M\left(\delta\left|\xi\right|\right)-M\left(\left|\xi\right|\right)\geq0,\quad\text{for }\left|\xi\right|\geq B.$$

Thus, for $|\xi|\geq A_3=\max(1,B)$, and for some constant $C>1$,

$$\mathbf{I}_2\leq Ce^{-\frac{1}{4}M(\delta|\xi|)}.$$

We can now use the inequality (5) to get constants B' and E so that

$$\mathbf{I}_2\leq B'e^{-M(E|\xi|)}. \tag{32}$$

Looking back at (30), using proposition 12, and using the inequalities in (31) and (32), we obtain that there are constants $A_1,\ A_2,\ A_3>0$ such that $\left|\mathcal{F}_{\eta u}(y,\xi)\right|\leq A_1e^{-M(A_2|\xi|)}$ for all $(y,\xi)\in V\times\mathcal{C}$ with $|\xi|\geq A_3$. Hence, $\left(0,\xi^0\right)\notin WF_M(u)$, as we desired. ∎

A corollary to Theorem 24 is the following "Edge-of-the-Wedge" Theorem:

Corollary 25 *(Edge-of-the-Wedge Theorem) Let $\mathcal{W}^+$ and $\mathcal{W}^-$ be wedges in Ω with edge X whose directions are opposite: $\Gamma_p(\mathcal{W}^+)=-\Gamma_p(\mathcal{W}^-)$. If $u\in\mathcal{D}'_M(X)$ is the boundary value of an M-approximate solution f^+ of $\mathcal{V}$ on $\mathcal{W}^+$ and also the boundary value of an M-approximate solution f^- of $\mathcal{V}$ on $\mathcal{W}^-$, then $WF_M(u)|_p\subset i_X^*(T_p^0)$. Hence, If $(\mathcal{M},\mathcal{V})$ is an elliptic M-involutive structure, then u is C^M in X.*

References

(**A**) C. Asano, *on the C^∞ wave-front set of solutions of first order non-linear pde's*, Proc. Amer. Math. Soc. **123** (1995), 3009–3019.

(**AB**) Z. Adwan and S. Berhanu, *Edge of the Wedge Theory in Involutive Structures*, Asian J. Math., Vol. 11, No. 1, pp. 001-018, March 2007.

(**AH**) Z. Adwan and G. Hoepfner, *A Generalization of Borel's Theorem and Microlocal Gevrey Regularity in Involutive Structures*, J. Differential Equations 245 (2008) 2846-2870.

(**BCH**) S. Berhanu, P. Cordaro, and J. Hounie, *An Introduction to Involutive Structures*, Cambridge University Press, UK, 2008.

(**BP**) R. F. Barostichi and G. Petronilho, Gevrey Micro-Regularity for Solutions to First Order Nonlinear PDE, J. Differential Equations 247 (2009) 1899-1914.

(**CC**) P. Caetano and P. Cordaro, *Gevrey Solvability and Gevrey Regularity in Differential Complexes Associated to Locally Integrable Structures*, Trans. Amer. Math. Soc., To appear.

(**CK**) S.-Y. Chung and D. Kim, *A Quasianalytic Singular Spectrum with Respect to the Denjoy-Carleman Class*, Nagoya Math. J., Vol. 148 (1997), 137-149.

(**CS**) G. M. Constantine and T. H. Savits, *A multivariate Faa di Bruno formula with applications*, Trans. Amer. Math. Soc., Volume 348, Number 2, 503-520, February 1996.

(**Dz**) G. A. Džanašija, *Carleman's Problem for Functions of the Gevrey Class*, Soviet Math. Dokl., 3 (1962), 969-972.

(**EG**) M. G. Eastwood and C. R. Graham, *Edge of the Wedge Theory in Hypo-Analytic Manifolds*, Comm. Partial Differential Equations 28 (2003), 2003–2028.

(**F**) H. Fourlinnie, *Carleman Version of the "Edge of the Wedge Theorem"*, C. R. Acad. Sci. Paris, Series I, Volume 326, Issue 5, p. 561-566, 1998.

(**Ku**) H. Komatsu, *Ultradistributions I (Structure Theorems and a Characterization)*, J. Fac. Sci. Univ. Tokyo, Sect IA, 20 (1973), 25-105.

(**L**) A. Lambert, *Quelques Theoremes de Decomposition des Ultra-Distributions*, Ann. Inst. Fourier, 29 (1979), 57-100.

(**Mi**) B. S. Mityagin, *An infinitely differentiable function with the values of its derivatives given at a point*, Soviet Math. Dokl., 2 (1961), 594-597.

(**Mt**) S. Mandelbrojt, *Séries Adhérentes, Régularisation des Suites, Applications*, Gauthier-Villars, Paris 1952.

(**Mo**) W. Matsumoto, *Theory of Pseudo-Differntial Operators of Ultra-Differentiable Class*, J. Math. Kyoto Univ., 27 (1987), 453-500.

(**R**) J.P. Rosay, *The C^∞ Version of the Edge of the Wedge for Non-Smooth Edges*, J. Geom. Analysis 3 (1993) 335-341.

(**Rod**) L. Rodino, *Linear Partial Differential Operators in Gevrey Spaces*, World Scientific Publishing Co. Pte. Ltd., (1993).

(**RSW**) J. -P. Rolin, P. Speisseger, and A. J. Wilkie, *Quasianalytic Denjoy-Carleman classes and o-minimality*, J. Amer. Math. Soc. 16 (2003), 751-777.

(**Rud**) W. Rudin, *Division in Algebras of Infinitely Differentiable Functions*, J. Math. Mech., 11 (1962), 797–809.