# THE ROLE OF DIFFUSIVITY ON PATTERNS FORMATION IN A NONLINEAR BOUNDARY FLUX PROBLEM.

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Abstract. We address the question of existence of nonconstant, sometimes stable, stationary solution to the simple scalar diffusion equation with variable diffusivity and nonlinear boundary flux. The goal is to give necessary as well as sufficient conditions on the diffusivity function for existence of such solutions.

**AMS subject classifications. ???35B25, 35B35, 35K57, 35R35.**

## Table of contents



## 1. Introduction

The subject of this work is the following nonlinear boundary value evolution problem

$$\begin{cases} \dfrac{\partial u}{\partial t} = \operatorname{div}(a(x)\nabla u), & (t,x) \in \mathbb{R}^+ \times \Omega \\ a(x)\partial_\nu u = g(u), & (t,x) \in \mathbb{R}^+ \times \partial\Omega. \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ $(N = 2, 3)$ is a smooth bounded domain $\nu$ the exterior unit normal vector to $\partial\Omega$, $g$ is a bistable type nonlinearity and $a$ a positive function in $C^{2,\theta}(\overline{\Omega})$.

Typically (1.1) models the time evolution of the concentration of a diffusing substance in a medium whose diffusivity is given by $a(\cdot)$

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and the flux on the boundary being proportional to the product of a prescribed function of the concentration and $1/a(\cdot)$.

Roughly speaking, once $g$ is fixed, nonconstant stable stationary solutions to (1.1) (occasionally referred to as patterns, for short) arise from specific properties of the geometry of the domain and/or of the diffusivity function $a(\cdot)$.

This work should be seen as an attempt to understand the role played by $a(\cdot)$ on existence and nonexistence of nontrivial equilibria as well as patterns to (1.1).

Our first result states that if $u$ is a stationary solution to (1.1), i.e., $u$ satisfies

$$\begin{cases} \operatorname{div}(a(x)\nabla u) = 0, & x \in \Omega \\ a(x)\partial_\nu u = g(u), & x \in \partial\Omega. \end{cases} \tag{1.2}$$

and $||a(\cdot)-\bar{a}||_{C^{2,\theta}(\overline{\Omega})}$ is sufficiently small for $\bar{a}$ large enough then $u$ must be a constant function.

For the specific case that $\Omega$ is a $N$-dimensional ball the requirement that $\bar{a}$ be large enough can be dropped; more specifically, we show that if $a(\cdot)$ is near to any constant positive function (in the topology of $C^{2,\theta}(\overline{\Omega})$) then the only stable stationary solutions to (1.1) are the constant ones.

And finally we prove existence of a stable nonconstant stationary solution to (1.1) provided that $a(\cdot)$ is large in two disjoint regions of $\overline{\Omega}$ (both intersecting $\partial\Omega$) and uniformly small in a thin tubular region which disconnect $\Omega$ in two sets each of which containing one of these two regions. See Figure 1 in the last Section for an illustration.

Regarding nonexistence the main tool utilized is the Implicit Function Theorem in a special setting. Then, for the case $a(\cdot)$ uniformly large, we resort also to [5] where existence of patterns for (1.2) has been studied for $a(\cdot) =$ constant and $\Omega$ a convex domain.

Existence of patterns for the no-flux boundary value problem

$$\begin{cases} \dfrac{\partial u}{\partial t} = \operatorname{div}(a(x)\nabla u) + g(u), & (t,x) \in \mathbb{R}^+ \times \Omega \\ \partial_\nu u = 0, & (t,x) \in \mathbb{R}^+ \times \partial\Omega. \end{cases} \tag{1.3}$$

has been proved in [6] for a class of diffusivity function which basically carries the same geometric profile.

## 2. Nonexistence of nontrivial stationary solutions

We suppose $\Omega \subset \mathbb{R}^N$ is a $C^{2,\theta}\,(0<\theta<1)$ bounded domain. By technical reasons $\theta$ will be chosen later on in order for weak solutions to (1.2) be $C^{2,\theta}(\bar{\Omega})$ solutions.

We define the set of bi-stable functions $\mathcal{G}$ as the class of sufficiently smooth (at least $C^1$) functions $g:\mathbb{R}\to\mathbb{R}$ such that

- $\exists\,\alpha,\beta\in\mathbb{R},\ \alpha<0<\beta:\ g(\alpha)=g(\beta)=g(0)=0,$
- $g(s)\neq 0$ in $(\alpha,0)\cup(0,\beta)$,
- $g'(\alpha)<0,\ g'(\beta)<0,\ g'(0)>0.$

For $g\in\mathcal{G}$ the only constant solutions to (1.2) satisfying $\alpha\le u(x)\le \beta$ are $u_0=\alpha,\ u_0=\beta$ or $u_0=0$. Therefore otherwise mentioned whenever referring to constant solutions to (1.2) we mean these solutions.

For future reference let us consider the Steklov eigenvalue problem

$$\left.\begin{array}{ll}\triangle\varphi=0, & x\in\Omega\\ \partial_\nu\varphi=\mu\,\varphi, & x\in\partial\Omega\end{array}\right\}\tag{2.1}$$

whose sequence of eigenvalues $\{\mu_j\}_{j=0}^{\infty}$ satisfies $0=\mu_0<\mu_1\le\mu_2\le\ldots$, counting multiplicity.

Before establishing our main result in this section let us present some preliminary results.

**Lemma 2.1.** *Suppose $g\in\mathcal{G}$, $a(\cdot)\in C^{2,\theta}(\overline{\Omega})$ and $u_0$ is a constant solution to (1.2). The following conclusions hold:*

(2.1.$i$) *If $u_0=\alpha$ or $u_0=\beta$ then, for any positive real number $\bar{a}$, there are neighborhoods $\mathcal{V}_{\bar{a}}$ of $\bar{a}$ in $C^{2,\theta}(\overline{\Omega})$ and $\mathcal{U}_{u_0}$ of $u_0$ in $W^{2,p}(\Omega)\,(p>N)$ such that if $a\in\mathcal{V}_{\bar{a}}$ then $u_0$ is the only solution to (1.2) in $\mathcal{U}_{u_0}$.*

(2.1.$ii$) *If $u_0=0$ the same conclusion holds true as long as*

$$\bar{a}\neq\frac{g'(0)}{\mu_j}\ (j=1,2\ldots)\tag{2.2}$$

*where $\mu_j$ are the eigenvalues of (2.1).*

**Proof:** Let us define

$$E_p\stackrel{\text{def}}{=}\left\{(v,w)\in L^p(\Omega)\times W^{1-\frac{1}{p},p}(\partial\Omega);\ \int_\Omega v=\int_{\partial\Omega}w\right\}$$

and the operator $F:C^{2,\theta}(\overline{\Omega})\times W^{2,p}(\Omega)\to E^p\times R$ by

$$F(a,u)=\big(\mathrm{div}(a(x)\nabla u),\ \ a(x)\partial_\nu u-g(u)+⨍_{\partial\Omega}g(u),\ \ ⨍_{\partial\Omega}g(u)\big)\tag{2.3}$$

where $⨍_{\partial\Omega} g(u) = \frac{1}{|\partial\Omega|}\int_{\partial\Omega} g(u)d\sigma$ and $d\sigma$ stands for the $(N-1)-$dimensional volume measure. Note that $F$ is a $C^1$ function and $F(a,u) = (0,0,0)$ if and only if $u$ is a solution of (1.2).

In particular for any $a(\cdot) \in C^{2,\theta}(\overline{\Omega})$ and any constant solutions $u_0$ to (1.2) we have $F(a,u_0) = (0,0,0)$.

Claim: $D_u(\bar{a},u_0) : W^{2,p}(\Omega) \longmapsto E_p \times R$ is an isomorfism.

Note that this will be the case if for each $(v,w,t) \in E^p \times R$ there is only one solution $\phi \in W^{2,p}(\Omega)$ to

$$\begin{cases} \Delta\phi = \dfrac{1}{\bar{a}}v, \ x \in \Omega \\ \bar{a}\partial_\nu\phi = g'(u_0)\phi - ⨍_{\partial\Omega} g'(u_0)\phi + w, \ x \in \partial\Omega \\ ⨍_{\partial\Omega} g'(u_0)\phi = t \end{cases} \tag{2.4}$$

In order to prove that the application above is isomorfism it suffices to show that

$$\begin{cases} \Delta\varphi = \dfrac{1}{\bar{a}}v, \ x \in \Omega \\ \bar{a}\partial_\nu\varphi = g'(u_0)\varphi + w, \ x \in \partial\Omega \end{cases} \tag{2.5}$$

has a unique solution $\varphi$. Indeed if this is the case then, keeping in mind that $(v,w) \in E_p$ and as such $⨍_{\partial\Omega}\varphi = 0$, the function $\phi = \varphi + \dfrac{t}{g'(u_0)}$ will be the only solution to (2.5).

In order to prove existence and uniqueness of solution to (2.5) we start by defining the operator $T : W^{2,p}(\Omega) \to L^p(\Omega) \times W^{1-1/p,p}(\partial\Omega)$ by

$$T(\varphi) = (\bar{a}\triangle\varphi, g'(u_0)\varphi - \bar{a}\,\partial_\nu\varphi).$$

It is well known that $T$ is a Fredholm operator with index zero. Hence $T$ will be an isomorphism provided $g'(u_0)/\bar{a}$ is not an eigenvalue of the Steklov eigenvalue problem. Indeed this will be the case if $g'(u_0) < 0$, i.e., either $u_0 \equiv \alpha$ or $u_0 \equiv \beta$, since the eigenvalues are non-negatives. The case $g'(u_0) > 0$ is ruled out by hypothesis (2.2).

Hence in any case $D_u(\bar{a},u_0)$ is an isomorphism from $W^{2,p}(\Omega)$ to $E_p \times R$.

Finally we conclude from the Implicit Function Theorem (see [3] for instance) the existence of neighborhoods $\mathcal{U}_{u_0}$ and $\mathcal{V}_{\bar{a}}$ such that if $a \in \mathcal{V}_{\bar{a}}$, $u \in \mathcal{U}_{u_0}$ and $F(a,u) = (0,0,0)$ then $u = u_0$, i.e., $u_0$ is the only solution to (1.2) in $\mathcal{U}_{u_0}$. ■

The next result regarding regularity of solutions to (1.2) will play a important role in the sequel.

**Lemma 2.2.** *Let $u \in H^1(\Omega)$ be a solution to (1.2) where $g \in \mathcal{G}$ is a $C^2$ function and $a(\cdot)$ is a positive function in $C^{2,\theta}(\overline{\Omega})$. Let*

- *$\theta = 1/2$ if $N = 1$ and*

- $\theta = 1 - N/6$ *in case* $N = 2, 3$.

*Then* $u \in C^{2,\theta}(\overline{\Omega})$.

**Proof:** By setting $\psi(x) = \dfrac{g(u)}{a(x)} + u$ we see that $u \in H^1(\Omega)$ is also a solution to

$$\begin{cases} \operatorname{div}(a(x)\nabla u) = 0, & x \in \Omega \\ \partial_\nu u + u = \psi(x), & x \in \partial\Omega. \end{cases} \tag{2.6}$$

As in Lemma 2.1, defining the map $\widetilde{T} : H^2(\Omega) \longmapsto L^2(\Omega) \times H^{1/2}(\partial\Omega)$ by

$$\widetilde{T}(\varphi) = (\operatorname{div}(a\,\nabla\varphi), \varphi - \partial_\nu\varphi)$$

we have that $\widetilde{T}$ is an isomorphism and hence $u$ is the only solution to (2.6) in $H^2(\Omega)$.

If $N = 3$ and $q = 6$, $H^2(\Omega) \subset W^{1,6}(\Omega)$ continuously. Then $u$ is a solution to

$$\begin{cases} \operatorname{div}(a(x)\nabla u) = 0, & x \in \Omega \\ \partial_\nu u = w, & x \in \partial\Omega. \end{cases} \tag{2.7}$$

for $w(x) = \dfrac{g(u)}{a(x)} \in W^{1,6}(\Omega)$. By Grisvard Lemma (see for instance [9], Lemma 2.4.1.4) $u \in W^{2,6}(\Omega)$ which is continuously imbedded in $C^{1,\theta}(\overline{\Omega})$ for $\theta = 1/2$.

If $N = 2$, in particular $u \in W^{2,q}(\Omega)$ for $q = 3/2$, for instance. Again using Grisvard Lemma we conclude $u \in W^{2,6}(\Omega)$ which also is continuously imbedded in $C^{1,\theta}(\overline{\Omega})$ for $\theta = \dfrac{1}{3}$.

And finally if $N = 1$ by Sobolev imbedding $H^2(\Omega) \subset C^{1,\theta}(\overline{\Omega})$ continuously for $\theta = \dfrac{1}{2}$.

So in all these cases we conclude $u \in C^{1,\theta}(\overline{\Omega})$ and hence $u$ is a solution to (2.6) with $\psi \in C^{1,\theta}(\partial\Omega)$. Thus $u \in C^{2,\theta}(\overline{\Omega})$ (cf. [8], Chapter 6, for instance). ■

Now we are ready to show

**Theorem 2.3.** *Let* $\theta$ *be as in Lemma 2.2,* $\Omega$ *a* $C^{2,\theta}$ *bounded domain in* $\mathbb{R}^N$ *(*$N \leq 3$*) and* $g \in \mathcal{G} \cap C^2(\mathbb{R})$.

*Given* $\bar{a}$ *large enough then there is* $\rho > 0$ *such that whenever* $||a(x) - \bar{a}||_{C^{2,\theta}(\overline{\Omega})} < \rho$*, any solution* $u$ *to (1.2) satisfying* $\alpha \leq u(x) \leq \beta$ *is constant.*

**Proof:** Arguing by contradiction we obtain a sequence $\{a_j\}$ satisfying $a_j \to \bar{a}$ in $C^{2,\theta}(\overline{\Omega})$ and a sequence of corresponding nonconstant solutions $\{u_j\}$, $\alpha \le u_j(x) \le \beta$, to the problem

$$\begin{cases} \operatorname{div}(a_j(x)\nabla u) = 0, \ x \in \Omega \\ a_j(x)\partial_\nu u = g(u), \ x \in \partial\Omega \end{cases} \tag{2.8}$$

which, by Lemma 2.2, satisfies $u_j \in C^{2,\theta}(\overline{\Omega})$.

But for all $v \in H^1(\Omega)$ we have

$$\int_\Omega a_j(x)\nabla v \nabla u_j dx - \int_{\partial\Omega} v g(u_j) d\mathcal{H}^{N-1} = 0 \tag{2.9}$$

For $j$ large enough, $a_j(x) \ge \frac{\bar{a}}{2} \, \forall x \in \overline{\Omega}$. Thus

$$\frac{\bar{a}}{2}\int_\Omega |\nabla u_j|^2 dx \le \int_\Omega a_j(x)|\nabla u_j|^2 dx = \int_{\partial\Omega} u_j g(u_j)\, d\sigma \le \int_{\partial\Omega} c\, u_j^2\, d\sigma$$

and given that the sequence $u_j$ is bounded in $L^\infty(\Omega)$ it also bounded in $H^1(\Omega)$. Extracting a subsequence, still denoted by $\{u_j\}$, there is a function $\overline{u} \in H^1(\Omega)$ such that $u_j \rightharpoonup \bar{u}$ weakly in $H^1(\Omega)$ and strongly in $L^2(\Omega)$ and $L^2(\partial\Omega)$. Given the uniform convergence of $\{a_j\}$ in $\overline{\Omega}$ we conclude from (2.9) that $\bar{u}$ is a weak solution to (2.8) with $a_j = \bar{a}$ and therefore $\bar{u} \in C^{2,\theta}(\overline{\Omega})$ by Lemma 2.2.

Since that $u_j$ and $\bar{u}$ are in $C^2(\overline{\Omega})$ we utilize the classical Amann inequality

$$\begin{gathered} ||u_j - \bar{u}||_{H^1(\Omega)} \le \\ C\left(||\Delta(u_j - \bar{u})||_{L^2(\Omega)} + ||\partial_\nu(u_j - \bar{u}) + (u_j - \bar{u})||_{L^2(\partial\Omega)}\right) \le \\ C\left(||\frac{1}{a_j}\nabla a_j \nabla u_j||_{L^2(\Omega)} + ||\frac{g(u_j)}{a_j} - \frac{g(\bar{u})}{\bar{a}}||_{L^2(\partial\Omega)} + ||u_j - \bar{u}||_{L^2(\partial\Omega)}\right) \end{gathered} \tag{2.10}$$

and then we conclude strong convergence in $H^1(\Omega)$. Now by Agmon-Douglis-Nirenberg inequality and previous convergence we conclude that $u_j \to \bar{u}$ in $H^2(\Omega)$.

Having in mind an application of Lemma 2.1 we now aim at proving that actually $u_j \to \bar{u}$ in $W^{2,p}(\Omega)$, for $p > N$. Recall the continuous inclusion of $H^2(\Omega) \subset W^{1,p}(\Omega)$ for $p > N$ if $N \le 2$, and for $4 < p \le 6$ if $N = 3$.

Then in any of these situations we conclude that $u_j \to \bar{u}$ in $W^{1,p}(\Omega)$ and resorting to the same argument used to obtain $H^2-$convergence we conclude that

$$u_j \to \bar{u} \text{ in } W^{2,p}(\Omega).$$

Also $\bar{u} \in C^{2,\theta}(\overline{\Omega})$, $\alpha \leq \bar{u} \leq \beta$ and $\bar{u}$ satisfies

$$\begin{cases} \triangle \bar{u} = 0, \ x \in \Omega \\ \partial_\nu \bar{u} = \bar{a}^{-1} \, g(\bar{u}), \ x \in \partial\Omega. \end{cases} \tag{2.11}$$

But this problem has been studied in [5] where it is shown that for $\bar{a}$ large enough any solution to (2.13) must be constant in $\overline{\Omega}$ and equal to $\alpha$, $\beta$ or 0. Summing up: we have $u_j \to \bar{u}$ in $W^{2,p}(\Omega)$ where $\bar{u} \in \{\alpha, 0, \beta\}$, $a_j \to \bar{a}$ in $C^{2,\theta}(\overline{\Omega})$ with $\bar{a}$ large enough. According to Lemma 2.1, for $j$ large enough, this cannot happen given that each $u_j$, from the contraction hypothesis, is a nonconstant function. ■

**Remark** One might raise the question; would Theorem 2.3 hold true if $\bar{a}$ were not large enough? The answer is negative at least for a class of $C^1$ domains. Actually by setting $a(\cdot) = \bar{a}$ =constant and $\lambda = 1/\bar{a}$ then (1.1) may be written as

$$\begin{cases} u_t = \lambda^{-1} \triangle u, \ x \in \Omega \\ \partial_\nu u = \lambda \, g(u), \ x \in \partial\Omega \end{cases} \tag{2.12}$$

and it was shown in [5] that if $\lambda$ is large enough then there is a class of $C^1$ convex domains depending on $\lambda$, which is obtained by smoothing out the corner and edges of a cube, such that (2.12) has some nonconstant stable stationary solution.

## 3. **Nonexistence of nontrivial stable stationary solutions in balls**

In the conclusion of Theorem 2.3 we can get rid of the requirement that $\bar{a} > 0$ be large enough provided the domain is a ball and only stable constant stationary solutions to (1.1) are considered.

**Theorem 3.1.** *Let $\theta$ be as in Lemma 2.2 and assume*

- *$\Omega$ is any $N$-dimensional ball and*
- *$\bar{a}$ any positive real number.*

*Then there is $\rho > 0$ such that, whenever $||a(x) - \bar{a}||_{C^{2,\theta}(\overline{\Omega})} < \rho$ and $u$ is a stable stationary solution to the corresponding problem (1.1), it holds that $u$ is constant. Hence either $u \equiv \alpha$ or $u \equiv \beta$.*

**Proof:** As in the proof of Theorem 2.3 arguing by contradiction we suppose the existence of a sequence $\{a_j\}$ satisfying $a_j \to \bar{a} > 0$ in $C^{2,\theta}(\overline{\Omega})$ and a corresponding sequence of nonconstant stable stationary solutions $\{u_j\}$, $\alpha \leq u_j \leq \beta$, to the problem

$$\begin{cases} u_t = \operatorname{div}(a_j(x)\nabla u), \ x \in \Omega \\ a_j(x)\dfrac{\partial u}{\partial \nu} = g(u), \ x \in \partial\Omega \end{cases} \tag{3.1}$$

By the same argument we find a subsequence (still denoted by $\{u_j\}$) and $\bar{u} \in C^{2,\theta}(\overline{\Omega})$ satisfying $u_j \to \bar{u}$ in $W^{2,p}(\Omega)$ where $\bar{u}$ is solution to (1.2) with $a(x) = \bar{a}$.

Let us consider the eigenvalue problem

$$\left.\begin{array}{ll} \operatorname{div}(a_j(x)\nabla\varphi) = \lambda\,\varphi, & x \in \Omega \\ a_j(x)\dfrac{\partial\varphi}{\partial\nu} - g'(u_j)\varphi = 0 & x \in \partial\Omega \end{array}\right\} \tag{3.2}$$

Since $\{u_j\}\,(j = 1, 2, ...)$ are stable stationary solutions to (3.1) it is well known that the first eigenvalue (which depends on $u_j$ and $a_j$) $\lambda_1(a_j, u_j)\,(j = 1, 2, \ldots)$ of (3.2) satisfies

$$\lambda_1(a_j, u_j) \geq 0\,(j = 1, 2, \ldots).$$

Hence taking into account that $u_j \to \bar{u}$ in $W^{2,p}(\Omega)$ and $a_j \to \bar{a}$ in $C^{2,\theta}(\overline{\Omega})$ one immediately verifies passing to the limit in the variational characterization of the first eigenvalue of (3.2) that $\lambda_1(\bar{a}, \bar{u}) \geq 0$. It turns out that in this case (scalar equation with $\lambda_1$ being a simple eigenvalue) this is also a sufficient condition for $\bar{u}$ to be a stable stationary solution to (3.1) with $a_j$ replaced with $\bar{a}$(see [4], for instance).

But now it can be easily proved (see [4], for instance) that $\bar{u}$ must be constant. Hence, since $u \equiv 0$ is a unstable equilibrium, we conclude $\bar{u} = \alpha$ or $\bar{u} = \beta$, which again contradicts Lemma 2.1 for $u_j \to \bar{u}$ in $W^{2,p}(\Omega)$ with each $u_j$ nonconstant. ∎

## 4. **Existence of nontrivial stable stationary solutions**

Our goal in this section is to give sufficient condition on the diffusivity function $a(\cdot)$ for existence of stable nonconstant stationary solution to (1.1). It will be clear that the diffusivity function $a$ must be sufficiently far (in the $C^{2,\theta}(\overline{\Omega})$ topology) from any constant function. See Fig. 1 for an illustration of such a function.

Physically it means ...........

Let $g \in \mathcal{G}$ satisfies

(H) $0 \leq sg(s) \leq cs^2$ for $\alpha \leq s \leq \beta$ and some $c > 0$.

and set $G(u) = \int_0^u g$. Assume without loss of generality that $G(\alpha) \leq G(\beta)$. Also define the twice continuously differentiable energy functional $E : W^{1,p}(\Omega) \longmapsto \mathbb{R}$ by

$$E(u) = \frac{1}{2} \int_{\Omega} a(x) \left| \nabla u \right|^2 \, dx - \int_{\partial\Omega} G(u) \, d\mathcal{H}^{N-1},$$

where $p > N$. One can prove that $E$ is ????(see [4], for instance). For $g$ in this conditions we have,

**Lemma 4.1.** *Let $\Omega \subset \mathbb{R}^N$ $(N \geq 2)$ be a smooth bounded domain, $\Omega_l$ and $\Omega_r$ two disjoint sub-domains of $\Omega$ with smooth boundaries and $S_j = \partial\Omega \cap \partial\Omega_j$, $\mathcal{H}^{N-1}(S_j) > 0$ $(j = l, r)$. For $p > N$, we define the set*

$$\Lambda = \left\{ \begin{array}{l} v \in W^{1,p}(\Omega) : \ \alpha \leq v(x) \leq \beta, \quad x \in \overline{\Omega}, \\ \displaystyle\int_{S_l} v \, d\mathcal{H}^{N-1} < 0, \int_{S_r} v \, d\mathcal{H}^{N-1} > 0, \\ E(v) < \varepsilon_0 - G(\beta)\mathcal{H}^{N-1}(\partial\Omega) \end{array} \right\}$$

*where*

$$\varepsilon_0 = G(\beta) \min \left\{ \mathcal{H}^{N-1}(S_l) \min \left\{ 1, \mu_1(\Omega_l) a_m^{\Omega_l} \right\}, \mathcal{H}^{N-1}(S_r) \min \left\{ 1, \mu_1(\Omega_r) a_m^{\Omega_r} \right\} \right\},$$

*$a_m^{\Omega_j} = \min\limits_{x \in \Omega_j} a(x)$ $(j = l, r)$ and $\mu_1(\Omega_j)$ is the first positive eigenvalue of Steklov Problem (2.1) defined in $\Omega_j$ $(j = l, r)$.*

*If $\Lambda \neq \emptyset$ then problem (1.1) has at least one nonconstant stationary solution $u \in \Lambda$ which is stable in $W^{1,p}(\Omega)$.*

For a proof we refer the reader to [5], Lemma 3.1 and Theorem 3.2 where the case $a \equiv 1$ was treated.

As mentioned before the goal in this section is to give sufficient conditions for the existence of patterns for (1.1), and this will be accomplished by giving conditions on $a(x)$ so that $\Lambda$ is not empty.

**Lemma 4.2.** *Let $\Omega \subset \mathbb{R}^N$ be a smooth bounded domain and suppose the equal-area condition $G(\alpha) = G(\beta)$ holds. Then there is a positive smooth function $a : \overline{\Omega} \longmapsto \mathbb{R}$ such that (1.1) has a nonconstant stable equilibrium solution.*

**Proof:** According to Lemma 4.1 it suffices to show that $\Lambda \neq \emptyset$. Let us take two separate balls $B_l$ and $B_r$, centered at points of $\partial\Omega$, such that $\Omega_l = B_l \cap \Omega, \Omega_r = B_r \cap \Omega$ are nonempty connected smooth open sets in $\Omega$ satisfying $\overline{\Omega}_l \cap \overline{\Omega}_r = \emptyset$, $S_j = \partial\Omega \cap \partial\Omega_j$ and $\mathcal{H}^{N-1}(S_j) \neq 0$ $(j = l, r)$.

Then there is an hyperplane $S$ which separates $R^N$ in two disjoint regions, denoted by $R_l^N$ and $R_r^N$, with the following properties:

i) $B_l \subset R_l^N$ and $B_r \subset R_r^N$,

ii) $\exists\, m > 0$ such that $\text{dist}(\Omega_j, S) \geq m$ $(j = l, r)$.

We define the signed distance function in $R^N$ by

$$d(x,S) = \begin{cases} \operatorname{dist}(x,S) & \text{if } x \in R^N_r, \\ -\operatorname{dist}(x,S) & \text{if } x \in R^N_l. \end{cases}$$

and, for $\delta > 0$, the tubular neighborhood of $S$ by

$$Q_\delta = \left\{ x \in \overline{\Omega} \mid |d(x,S)| < \delta \right\}.$$

For $S_l, S_r$ as in Lemma 4.1 we suppose $\mathcal{H}^{N-1}(S_l) \leq \mathcal{H}^{N-1}(S_r)$ and choose $\delta < m$ small enough such that

$$\mathcal{H}^{N-1}(\partial Q_\delta \cap \partial\Omega) < \mathcal{H}^{N-1}(S_l) \tag{4.1}$$

Consider a function $\xi : \mathbb{R} \longrightarrow \mathbb{R}$ defined by

$$\xi(t) = \begin{cases} \alpha, & \text{if } t \leq -\delta \\ \alpha + \beta + \dfrac{(\beta-\alpha)}{\delta} t, & \text{if } -\delta < t < \delta \\ \beta, & \text{if } t \geq \delta. \end{cases}$$

Then $w_0(x) = \xi(d(x,S))$ is a Lipschtizian function in $R^N$ and consequently its restriction to $\Omega$ is in $W^{1,p}(\Omega)$. We will show that under certain conditions on $a(x)$ we have $w_0 \in \Lambda$, with $\Lambda$ defined as in Lemma 4.1.

Clearly $\alpha \leq w_0 \leq \beta$, $\int_{S_l} w_0 \, d\mathcal{H}^{N-1} < 0$ and $\int_{S_r} w_0 \, d\mathcal{H}^{N-1} > 0$.

Let $S^-$ and $S^+$ be portions of $\partial\Omega$ defined by $\partial\Omega \backslash (\partial Q_\delta \cap \partial\Omega) = S^- \cup S^+$. Then $S^- \cap S^+ = \emptyset$, $S^- \cap S_l \neq \emptyset \neq S^+ \cap S_r$.

Since $w_0$ is constant on each connected component of $\Omega \backslash Q_\delta$ and $G(\alpha) = G(\beta)$ we obtain

$$\begin{aligned} E(w_0) &= \frac{1}{2} \int_{Q_\delta} a(x) |\nabla w_0|^2 \, dx - \int_{\partial Q_\delta \cap \partial\Omega} G(w_0) \, d\mathcal{H}^{N-1} \\ &- G(\beta) \mathcal{H}^{N-1}(\partial\Omega \backslash (\partial Q_\delta \cap \partial\Omega)). \end{aligned}$$

Given that $\displaystyle\int_{\partial Q_\delta \cap \partial\Omega} G(w_0) d\mathcal{H}^{N-1} \geq 0$ in order to have

$$E(w_0) < \varepsilon_0 - G(\beta) \mathcal{H}^{N-1}(\partial\Omega), \tag{4.2}$$

($\varepsilon_0$ as in Lemma 4.1) it suffices to require

$$\varepsilon_0 > \frac{1}{2} \int_{Q_\delta} a(x) |\nabla w_0|^2 \, dx + G(\beta) \mathcal{H}^{N-1}(\partial Q_\delta \cap \partial\Omega). \tag{4.3}$$

Since the diffusivity function $a$ is the parameter to be chosen let $a_m^{\Omega_j} = \min_{x\in\Omega_j} a(x)$ $(j = l, r)$ and take

$$a_m^{\Omega_l} > \frac{1}{\mu_1(\Omega_l)} \quad \text{and} \quad a_m^{\Omega_r} > \frac{1}{\mu_1(\Omega_r)} \tag{4.4}$$

Hence

$$\varepsilon_0 = G(\beta)\mathcal{H}^{N-1}(S_l). \tag{4.5}$$

Moreover setting

$$a_M^\delta = \max_{x\in Q_\delta} a(x)$$

we have

$$\frac{1}{2}\int_{Q_\delta} a(x)|\nabla w_0|^2\, dx \le \frac{a_M^\delta}{2}\frac{(\beta-\alpha)^2}{\delta^2}\mathcal{H}^N(Q_\delta)$$

Therefore (4.3), and consequently (4.2), will be realized provided

$$0 < a_M^\delta < \frac{2\delta^2 G(\beta)}{(\beta-\alpha)^2\mathcal{H}^N(Q_\delta)}\left[\mathcal{H}^{N-1}(S_l) - \mathcal{H}^{N-1}(\partial Q_\delta \cap \partial\Omega)\right] \tag{4.6}$$

Note that in view of (4.1) the righthand side of (4.6) is positive and does not depend on $a$. Therefore (4.6) can clearly be satisfied by taking $a_M^\delta$ small enough. ■

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