

On the instability of a class of periodic travelling wave solutions of the modified Boussinesq equation ^{*†}

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Abstract

This paper is concerned with instability of periodic travelling wave solutions of the modified Boussinesq equation. Periodic travelling wave solutions with a fixed fundamental period L will be constructed by using Jacobi's elliptic functions. It will be shown that these solutions, called *dnoidal waves*, are nonlinearly unstable in the energy space for a range of their speeds of propagation and periods.

1 Introduction

The original Boussinesq equations are among the classical models for the propagation of small amplitude, planar long waves on the surface of water [8, 9]. These equations possess special travelling wave solutions known as Scott Russel's solitary waves or *solitons* [6, 13], *cnoidal waves* [3] and *dnoidal waves* ([4], Section 3 below). The cnoidal and dnoidal wave solutions are periodic travelling waves written in terms of the Jacobi elliptic functions.

Our purpose is to investigate the nonlinear stability of periodic travelling wave solutions $\phi(x - ct)$ of the modified Boussinesq equation

$$u_{tt} - u_{xx} + (u^3 + u_{xx})_x = 0. \quad (1)$$

The above equation (1), has the following equivalent form as a Hamiltonian system

$$\begin{cases} u_t = v_x \\ v_t = (u - u_{xx} - u^3)_x \end{cases} \quad (2)$$

for $x \in \mathbb{R}$, $t > 0$. Here subscripts t and x denote partial differentiation with respect to t and x .

The above equation conserves energy, namely the integral

$$H(u, v) = \frac{1}{2} \int_0^L (u^2 + v^2 + u_x^2 - \frac{u^4}{2}) dx, \quad (3)$$

does not depend on the time t . Another conservation law is the momentum

$$I(u, v) = \int_0^L uv dx \quad (4)$$

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which turns out to be a relevant quantity in the investigation of stability properties of travelling waves.

To make precise the notion of stability we use, let τ_s be the translation by s , $\tau_s\phi(x) = \phi(x + s)$ for $x \in \mathbb{R}$ and let $\vec{\phi}_c = (\phi_c(x - ct), \psi_c(x - ct))$ be an L -periodic travelling wave solution to system (2), where $\phi_c : \mathbb{R} \rightarrow \mathbb{R}$, $\psi_c : \mathbb{R} \rightarrow \mathbb{R}$, $L > 0$ is the period of ϕ_c and ψ_c and c is the wave's speed of propagation. If we define the $\vec{\phi}_c$ -orbit to be the set $\Omega_{\vec{\phi}_c} = \{\vec{\phi}_c(\cdot + s), s \in \mathbb{R}\}$, $\vec{\phi}_c$ is called orbitally stable if profiles near its orbit remain near the orbit for as long as it exists.

So, we have the following definition. Let X be a Hilbert space.

Definition 1.1 (Orbital Stability) Let $\vec{\phi}_c = (\phi_c(x - ct), \psi_c(x - ct)) \in X$ be an L -periodic travelling wave solution to system (2). We say that the orbit $\Omega_{\vec{\phi}_c}$ is stable in the X -sense by the flow of system (2) if for each $\epsilon > 0$ there exists $\delta = \delta(\epsilon) > 0$ such that if $\vec{u}_0 \in X$ and $\inf_{s \in \mathbb{R}} \|\vec{u}_0 - \tau_s(\vec{\phi}_c)\|_X < \delta$ then the solution $\vec{u}(t)$ of (2) with $\vec{u}(0) = \vec{u}_0$ satisfies, for all t for which $\vec{u} = (u, v)$ exists,

$$\inf_{s \in \mathbb{R}} \|\vec{u}(t) - \tau_s(\vec{\phi}_c)\|_X < \epsilon. \quad (5)$$

Otherwise, we say that $\Omega_{\vec{\phi}_c}$ is X -unstable.

Here, $X := H_{per}^1([0, L]) \times L_{per}^2([0, L])$. (The choice of norm in (5) is dictated by the form of the Hessian or "linearized Hamiltonian" $H''(\vec{\phi}_c) + cI''(\vec{\phi}_c)$ and varies from problem to problem.)

Inserting the L -periodic travelling wave solution $\vec{\phi}_c = (\phi_c(x - ct), \psi_c(x - ct))$ in (2) leads to the system

$$\begin{cases} -c\phi'_c(\xi) = \psi'_c(\xi) \\ -c\psi'_c(\xi) = (\phi_c - \phi_c'' - \phi_c^3)'(\xi), \end{cases}$$

where $'$ connotes $\frac{d}{d\xi}$ and $\xi = x - ct$. Integrating the latter system, we obtain the nonlinear system

$$\begin{cases} -c\phi(\xi) = \psi(\xi) + K_1 \\ -c\psi(\xi) = \phi(\xi) - \phi''(\xi) - \phi^3(\xi) + K_2, \end{cases}$$

where K_1, K_2 are integration constants, which will be considered equal to zero here. Then, we obtain

$$(H' + cI')(\vec{\phi}_c) = 0. \quad (6)$$

Next observe that relation (6) characterizes $\vec{\phi}_c = (\phi_c, \psi_c)$ as a critical point of H subject to the constraint $I(u, v) = I(\phi_c, \psi_c)$. In order to prove instability for $\vec{\phi}_c$ we will examine the relation between the concavity properties of the function

$$d(c) = H(\vec{\phi}_c(\cdot)) + cI(\vec{\phi}_c(\cdot)), \quad (7)$$

and the properties of the functional H near the critical point $\vec{\phi}_c$ under the constraint $I = \text{constant}$.

Bona and Sachs in [6] proved that the well known solitary waves $\vec{\phi} = (\phi_c(x - ct), \psi_c(x - ct))$ of the generalized Boussinesq equation

$$\begin{cases} u_t = v_x \\ v_t = (u - u_{xx} - u^p)_x \end{cases} \quad (8)$$

are stable in the $H^1(\mathbb{R}) \times L^2(\mathbb{R})$ norm for speeds c such that $\frac{p-1}{4} < c^2 < 1$ if d in (7) is a convex function of c . The aim of this paper is to prove that the solutions given by Theorem 3.1 below are unstable if $d(c)$ is concave. The proof follows the main ideas of Y. Liu [13] (see also Bona, Souganidis and Strauss in [7]). Differently from the solitary wave solutions case, we do not know explicit periodic travelling wave solutions in the x -variable for the system (8) for every p . For this reason, we will treat here only the case $p = 3$. Stability of dnoidal waves for this case is also treated by the author in a forthcoming paper [4]. Regarding the classical case $p = 2$, in [3] the author proves nonlinear stability properties of a class of

L -periodic travelling wave solutions, called *cnoidal waves*, in the energy space $H_{per}^1([0, L]) \times L_{per}^2([0, L])$, by periodic disturbances with period L .

In this paper, we first show the existence of a smooth curve $c \mapsto \vec{\phi}_c = (\phi_c, \psi_c)$ of dnoidal wave solutions to system (2), with a fixed period L (Theorem 3.1 below). Then, a proof of orbital instability of these solutions is established in X for a certain range of their speeds of propagation and periods, based on a modification of the general procedure of [11]. More precisely, our main result regarding stability of the dnoidal waves $\vec{\phi}_c$, given by Theorem 3.1 below, is the following:

Theorem 1.1 (Instability Theorem) *Let $c \in (-1, 1)$ and $L > \pi\sqrt{2}$. Then the orbit $\Omega_{\vec{\phi}_c}$ is X -unstable with respect to the flow of the modified Boussinesq equation, provided $c^2 < \frac{1}{2}$ and $1 - c^2 > \frac{2\pi^2}{L^2}$.*

The plan of this paper is as follows. A discussion of the evolution equation (1) and its natural invariants is given in Section 2. In section 3 we introduce a smooth family $\{\vec{\phi}_c\}_c$ on the parameter c , of positive dnoidal wave solutions to system (2), with a fixed period L (Theorem 3.1 below) and in section 4 we present a complete study of the spectrum of the operator \mathcal{L}_c . The existence of the smooth curve $c \mapsto \vec{\phi}_c$ will allow us to differentiate the function $d(c)$ and then, in section 5, we prove that it is indeed concave for a certain range of speeds and periods of $\vec{\phi}_c$, which will imply, our result. In Section 6 the Lyapunov functional [11, 12] is constructed and the instability result is proved. In section 7, we give a review of those results about Jacobian elliptic functions which we use throughout the paper.

We remark that orbital instability of $\vec{\phi}$ is established with respect to perturbations of periodic functions of the same period L in X .

The following notation will be used:

$$\begin{aligned} \langle f, g \rangle_0 &= \langle f, g \rangle_{L_{per}^2([0, L])} = \int_0^L f g dx, \\ \langle f, g \rangle_1 &= \langle f, g \rangle_{H_{per}^1([0, L])} = \int_0^L f g dx + \int_0^L f' g' dx, \\ \|f\|_0 &= \|f\|_{L_{per}^2([0, L])} = \left(\int_0^L f^2 dx \right)^{1/2}, \\ \|f\|_1 &= \|f\|_{H_{per}^1([0, L])} = \left(\int_0^L f^2 dx + \int_0^L f'^2 dx \right)^{1/2}, \\ \langle (f, g), (u, v) \rangle &= \langle (f, g), (u, v) \rangle_{L_{per}^2([0, L]) \times L_{per}^2([0, L])} = \int_0^L f u dx + \int_0^L g v dx, \\ \|(f, g)\| &= \|(f, g)\|_{L_{per}^2([0, L]) \times L_{per}^2([0, L])} = \left(\int_0^L f^2 dx + \int_0^L g^2 dx \right)^{1/2}, \\ \|(f, g)\|_X &= \|(f, g)\|_{H_{per}^1([0, L]) \times L_{per}^2([0, L])} = \left(\int_0^L f^2 dx + \int_0^L f'^2 dx + \int_0^L g^2 dx \right)^{1/2}. \end{aligned}$$

2 The evolution equation

The next lemma is the periodic version of a particular case of Lemma 1.1 in [13].

Lemma 2.1 *Let $\vec{u}_0 = (u_0, v_0) \in X \equiv H_{per}^1([0, L]) \times L_{per}^2([0, L])$. Then there exists $T > 0$, and a uniquely weak solution $\vec{u} = (u, v)$ of (2) with $\vec{u}(0) = \vec{u}_0$.*

Proof: In order to obtain the existence of weak solutions for the system (2), we consider the approximate problem

$$\begin{cases} \vec{u}_t + A\vec{u} = F(\vec{u}), \\ \vec{u}(0) = \vec{u}_0^n, \end{cases} \quad (9)$$

with $\vec{u}_0^n \in D(A) = H_{per}^3([0, L]) \times H_{per}^1([0, L])$ and $\vec{u}_0^n \rightarrow \vec{u}_0$ in X , where

$$A = \begin{pmatrix} 0 & -\partial_x \\ -\partial_x + \partial_x^3 & 0 \end{pmatrix}$$

and $-A$ is the infinitesimal generator of a C^0 group of unitary operators in X and $F = F(t, u, v) = \begin{pmatrix} 0 \\ -\partial_x(u^3) \end{pmatrix}$. Since $F \in C^\infty$, the map $(u, v) \rightarrow (0, \partial_x(u^3))$ is locally Lipschitz on X . But then for all $\vec{u}_0^n \in D(A)$, there exists a $T_n > 0$ such that the initial value problem (9) has a unique solution $\vec{u}^n \in C([0, T_n]; D(A)) \cap C^1([0, T_n]; X)$. Moreover, if $T_n < \infty$ then

$$\lim_{t \rightarrow T_n} \|\vec{u}^n(t)\|_X = \infty,$$

by the semigroup theory [15]. By (9), we estimate on $[0, T_n]$

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\vec{u}^n(t)\|_X^2 &= \langle \vec{u}^n(t), \partial_t \vec{u}^n(t) \rangle_X = \langle \vec{u}^n, -A\vec{u}^n + F(\vec{u}^n) \rangle_X \\ &\leq \left| \int_0^L \partial_x(u^n)^3(t) \cdot v^n(t) dx \right| \\ &\leq \|\partial_x(u^n)^3(t)\|_0 \|v^n(t)\|_0 \\ &= 3 \left(\int_0^L |(u^n(t))^2 (u^n(t))_x|^2 dx \right)^{\frac{1}{2}} \|v^n(t)\|_0 \\ &\leq 3 \|u^n(t)\|_\infty^2 \left(\int_0^L (u_x^n(t))^2 dx \right)^{\frac{1}{2}} \|v^n(t)\|_0 \\ &\leq 3S(L) \|u^n(t)\|_1^2 \|u_x^n\|_0 \|v^n(t)\|_0 \\ &\leq 3S(L) \|\vec{u}^n(t)\|_X^2 \|\vec{u}^n\|_X^2, \end{aligned}$$

where we used in the first equality above that $-A\vec{u}^n \in X$.

Consider now $\tilde{f}(s) = 3S(L)s$, which is a continuous, positive and increasing function on \mathbb{R}^+ . Then by Gronwall's inequality, it follows that

$$\|\vec{u}^n(t)\|_X^2 \leq \|\vec{u}_0^n\|_X^2 \exp \left[\int_0^t \tilde{f}(\|\vec{u}^n(\tau)\|_X^2) d\tau \right], \quad \text{on } [0, T_n]. \quad (10)$$

We compare $\|\vec{u}^n(t)\|_X^2$ with the maximal solution $y(t)$

$$\begin{aligned} y(t) &\equiv \frac{1(\sup_n \|\vec{u}_0^n\|_X)^2}{1 - 3St(\sup_n \|\vec{u}_0^n\|_X)^2}, \\ t \in [0, T_0] &\equiv \left[0, \frac{1}{3S(\sup_n \|\vec{u}_0^n\|_X)^2}\right) \end{aligned}$$

of the scalar Cauchy problem

$$\begin{cases} \frac{dy}{dt} = \tilde{f}(y)y, \\ y(0) = y_0 = \sup_n \|\vec{u}_0^n\|_X^2. \end{cases}$$

It follows that

$$\|\vec{u}^n(t)\|_X^2 \leq y(t) \quad \text{on } [0, T_n) \cap [0, T_0]. \quad (11)$$

Let $T < T_0$. Then \vec{u}^n is defined on $[0, T]$ for all n . Moreover,

$$\|\vec{u}^n(t)\|_X^2 \leq C_0 y_0 = K^2 \quad (12)$$

on $[0, T]$, where K is a constant independent of n , since by (10),(11) and the fact that $y(t)$ is bounded on $[0, T]$, we have the following inequality on $[0, T]$

$$\begin{aligned} \|\vec{u}^n(t)\|_X^2 &\leq \|\vec{u}_0^n\|_X^2 \exp \left[\int_0^t \tilde{f}(\|\vec{u}^n(\tau)\|_X^2) d\tau \right] \\ &\leq \|\vec{u}_0^n\|_X^2 \exp \left[\int_0^T \tilde{f}(y(\tau)) d\tau \right] \leq C_0(T) \|\vec{u}_0^n\|_X^2 \\ &\leq C_0(T) y_0. \end{aligned}$$

Finally, from (12) and standard weak limit arguments we have the existence of a unique solution $\vec{u}(t) \in C([0, T]; X)$. □

Proposition 2.1 *The unique solution $\vec{u}(t)$ of (1) with initial data $\vec{u}(0) = \vec{u}_0$, which is given in Lemma 1.1, satisfies*

$$\begin{aligned} H(\vec{u}(t)) &= H(u, v) = \text{constant}, \quad 0 < t < T, \\ I(\vec{u}(t)) &= I(u, v) = \text{constant}, \quad 0 < t < T. \end{aligned}$$

The proof is elementary.

3 Existence of a smooth curve of dnoidal wave solutions with a fixed period L for the system (2)

This section is devoted to establish the existence of a smooth curve of periodic travelling wave solutions for the system (2), which are solutions of the form

$$\vec{u}(x, t) = (u(x, t), v(x, t)) = (\phi(x - ct), \psi(x - ct)). \quad (13)$$

Substituting (13) in (2) leads to the system

$$\begin{cases} -c\phi'(\xi) = \psi'(\xi) \\ -c\psi'(\xi) = (\phi - \phi'' - \phi^3)'(\xi), \end{cases} \quad (14)$$

where $'$ denotes $\frac{d}{d\xi}$ and $\xi = x - ct$. Integrating (14), we obtain the nonlinear system

$$\begin{cases} -c\phi(\xi) = \psi(\xi) + K_1 \\ -c\psi(\xi) = \phi(\xi) - \phi''(\xi) - \phi^3(\xi) + K_2, \end{cases} \quad (15)$$

where K_1, K_2 are integration constants, which will be considered equal to zero here. Then, ϕ must satisfy

$$\phi'' - w\phi + \phi^3 = 0, \quad (16)$$

where $w = w(c) = 1 - c^2$ will be considered positive.

Next, we show how to construct a smooth curve of solutions for (16) with a fixed fundamental period L , and depending on the parameter c . In order to do this we first observe from (16) that ϕ satisfies the first order equation

$$\begin{aligned} (\phi')^2 &= \frac{1}{2}[-\phi^4 + 2w\phi^2 + 4B_\phi] \\ &= \frac{1}{2}(\eta_1^2 - \phi^2)(\phi^2 - \eta_2^2), \end{aligned} \quad (17)$$

where B_ϕ is an integration constant and $-\eta_1, \eta_1, -\eta_2, \eta_2$ are the real zeros of the polynomial $p_\phi(t) = -t^4 + 2wt^2 + 4B_\phi$, which satisfy the relations

$$\begin{cases} 2w &= \eta_1^2 + \eta_2^2 \\ 4B_\phi &= -\eta_1^2\eta_2^2. \end{cases} \quad (18)$$

Moreover, we assume without loss of generality that $\eta_1 > \eta_2 > 0$ and we obtain from (17) that $\eta_2 \leq \phi \leq \eta_1$. By defining $\varphi = \frac{\phi}{\eta_1}$ and $k^2 = \frac{(\eta_1^2 - \eta_2^2)}{\eta_1^2}$, (17) becomes $(\varphi')^2 = \frac{\eta_1^2}{2}(1 - \varphi^2)(\varphi^2 - 1 + k^2)$. We also impose the crest of the wave to be at $\xi = 0$, that is $\phi(0) = 1$. Now, we define a further variable ψ via the relation $\varphi^2 = 1 - k^2 \sin^2 \psi$ and so we get that $(\psi')^2 = \frac{\eta_1^2}{2}(1 - k^2 \sin^2 \psi)$. Then we obtain for $l = \frac{\eta_1}{\sqrt{2}}$ that $\int_0^{\psi(\xi)} \frac{dt}{\sqrt{1 - k^2 \sin^2 t}} = l\xi$. Therefore, from the definition of the Jacobian elliptic function $y = \text{sn}(u; k)$ (see in the Appendix or in Byrd & Friedman [5]) we can write the last equality as $\sin \psi = \text{sn}(l\xi; k)$ and hence $\varphi(\xi) = \sqrt{1 - k^2 \text{sn}^2(l\xi; k)} = \text{dn}(l\xi; k)$. Returning to the initial variable, we obtain the called *dnoidal wave solution* associated to the equation (16),

$$\phi(\xi) \equiv \phi(\xi; \eta_1, \eta_2) = \eta_1 \text{dn}\left(\frac{\eta_1}{\sqrt{2}}\xi; k\right) \quad (19)$$

with

$$k^2 = \frac{\eta_1^2 - \eta_2^2}{\eta_1^2}, \quad \eta_1^2 + \eta_2^2 = 2w, \quad 0 < \eta_2 < \eta_1. \quad (20)$$

Next, dn has fundamental period $2K$, $\text{dn}(u + 2K; k) = \text{dn}(u; k)$, where $K = K(k)$ represents the complete elliptic integral of the first kind (see Appendix); it follows that the dnoidal wave ϕ in (19) has fundamental period, T_ϕ , given by

$$T_\phi \equiv \frac{2\sqrt{2}}{\eta_1} K(k). \quad (21)$$

Now, we show that $T_\phi > \frac{\sqrt{2}\pi}{\sqrt{w}}$. First, we express T_ϕ as a function of η_2 and w . In fact, for every $\eta_2 \in (0, \sqrt{w})$ there is a unique $\eta_1 \in (\sqrt{w}, \sqrt{2w})$ satisfying the first relation in (18), namely, $\eta_1 = \sqrt{2w - \eta_2^2}$. So, from (21) we obtain

$$T_\phi(\eta_2, w) = \frac{2\sqrt{2}}{\sqrt{2w - \eta_2^2}} K(k(\eta_2)) \quad \text{with} \quad k^2(\eta_2, w) = \frac{2w - 2\eta_2^2}{2w - \eta_2^2}. \quad (22)$$

Then, by fixing $w > 0$, we have that $T_\phi \rightarrow +\infty$ as $\eta_2 \rightarrow 0$ and $T_\phi(\eta_2) \rightarrow \frac{\pi\sqrt{2}}{\sqrt{w}}$ as $\eta_2 \rightarrow \sqrt{w}$. So, since the mapping $\eta_2 \mapsto T_{\phi_w}(\eta_2)$ is strictly decreasing (see proof of Proposition 3.1), it follows that $T_\phi > \frac{\sqrt{2}\pi}{\sqrt{w}}$.

Now, we obtain a dnoidal wave solution with period L . For $w_0 > \frac{2\pi^2}{L^2}$ there is a unique $\eta_{2,0} \in (0, \sqrt{w_0})$ such that $T_\phi(\eta_{2,0}, w_0) = L$. So, for $\eta_{1,0}$ such that $\eta_{1,0}^2 + \eta_{2,0}^2 = 2w_0$, the dnoidal wave $\phi(\cdot) = \phi(\cdot, \eta_{1,0}, \eta_{2,0})$ has fundamental period L and satisfies (16) with $w = w_0$.

By the above analysis the dnoidal wave $\phi(\cdot, \eta_1, \eta_2)$ in (19) is completely determined by w and η_2 and will be denoted by $\phi_w(\cdot; \eta_2)$ or ϕ_w .

The next result, which corresponds to Theorem 2.1 and Corollary 2.2 in [2], is concerned with the existence of a smooth curve of dnoidal wave solutions for equation (16).

Proposition 3.1 *Let $L > 0$ be arbitrary but fixed. Consider $w_0 > \frac{2\pi^2}{L^2}$ and the unique $\eta_{2,0} = \eta_2(w_0) \in (0, \sqrt{w_0})$ such that $T_{\phi_{w_0}} = L$. Then,*

- (1) *there exist an interval $\mathcal{I}(w_0)$ around w_0 , an interval $J(\eta_{2,0})$ around $\eta_{2,0}$ and a unique smooth function $\Lambda : \mathcal{I}(w_0) \rightarrow J(\eta_{2,0})$ such that $\Lambda(w_0) = \eta_{2,0}$ and*

$$\frac{2\sqrt{2}}{\sqrt{2w - \eta_2^2}} K(k) = L, \quad (23)$$

where $w \in \mathcal{I}(w_0)$, $\eta_2 = \Lambda(w)$ and $k^2 = k^2(w) \in (0, 1)$ is defined by (22).

- (2) *The positive dnoidal wave solution in (19), $\phi_w(\cdot; \eta_1, \eta_2)$, determined by $\eta_1 \equiv \eta_1(w) = \sqrt{2w - \eta_2^2}$, $\eta_2 \equiv \eta_2(w)$, has fundamental period L and satisfies (16). Moreover, the mapping $w \in \mathcal{I}(w_0) \rightarrow \phi_w \in H_{per}^1([0, L])$ is a smooth function.*

- (3) *$\mathcal{I}(w_0)$ can be chosen as $(\frac{2\pi^2}{L^2}, +\infty)$.*

- (4) *The mapping $\Lambda : (\frac{2\pi^2}{L^2}, +\infty) \rightarrow J(\eta_{2,0})$ is strictly decreasing.*

Proof: See [2]. □

From this result we conclude the following existence theorem.

Theorem 3.1 *Let $L > \pi\sqrt{2}$. Then there exists a smooth curve of dnoidal wave solutions for the system (2) in $H_{per}^n([0, L]) \times H_{per}^m([0, L])$, $n, m \geq 0$ which satisfy the system (15) with integration constants $K_1 = K_2 = 0$; this curve is given, for $w(c) = 1 - c^2$, by*

$$c \in \left(-\sqrt{1 - \frac{2\pi^2}{L^2}}, \sqrt{1 - \frac{2\pi^2}{L^2}}\right) \rightarrow (\phi_{w(c)}, \psi_{w(c)}). \quad (24)$$

Moreover, $\phi_c(\xi) := \phi_{w(c)}(\xi) = \sqrt{2w - \eta_2^2} \operatorname{dn} \left[\frac{\sqrt{2w - \eta_2^2}}{\sqrt{2}} \xi; k \right]$, $\psi_{w(c)} = -c\phi_{w(c)}$, where the smooth function $\eta_2 \equiv \eta_2(w(c))$ is given by Proposition 3.1 and $k = k(w(c))$ by (22).

Remark 3.1 $\partial \vec{\phi}_c / \partial c$ is in $H_{per}^\infty([0, L]) \times H_{per}^\infty([0, L])$ as soon as in $H_{per}^1([0, L]) \times H_{per}^1([0, L])$. This follows from the equation and a bootstrap argument.

4 Spectral Analysis

In this section we study the spectral properties associated to the linear operator

$$\mathcal{L}_c = H''(\phi_{w(c)}, \psi_{w(c)}) + cI''(\phi_{w(c)}, \psi_{w(c)}) \quad (25)$$

determined by the periodic solutions $(\phi_{w(c)}, \psi_{w(c)})$ found in Theorem 3.1. We compute the Hessian operator \mathcal{L}_c by calculating the associated quadratic form, which is denoted by \mathcal{Q}_c . By definition, $\mathcal{Q}_c(g, h)$ is the coefficient of ϵ^2 in

$$H(\phi_{w(c)} + \epsilon g, \psi_{w(c)} + \epsilon h) + cI(\phi_{w(c)} + \epsilon g, \psi_{w(c)} + \epsilon h),$$

and so is given by

$$\begin{aligned} \mathcal{Q}_c(g, h) &= \int_0^L \left\{ \frac{1}{2}(g^2 + g_x^2 + h^2) - \frac{3}{2}\phi_{w(c)}^2 g^2 + cgh \right\} dx \\ &= \int_0^L \left\{ \frac{1}{2} \left[(1 - c^2)g^2 + g'^2 - 3\phi_{w(c)}^2 g^2 \right] + \frac{1}{2}(h + cg)^2 \right\} dx \\ &:= \mathcal{Q}_c^1(g) + \frac{1}{2} \|h + cg\|_0^2. \end{aligned} \quad (26)$$

Note that \mathcal{Q}_c is the sum of the quadratic form \mathcal{Q}_c^1 associated to the operator $-\frac{d^2}{dx^2} + 1 - c^2 - 3\phi_w^2$ and the non-negative term $\frac{1}{2}\|h + cg\|_0^2$. From the equations (14) for the dnoidal wave $(\phi_{w(c)}, \psi_{w(c)})$, it follows that $g = \phi'_{w(c)}$, $h = \psi'_{w(c)}$ satisfy $\mathcal{L}_c(g, h) = 0$. To see that this is the only eigenfunction corresponding to the eigenvalue zero and the other expected properties of the operator \mathcal{L}_c , we will first consider the following result about the periodic eigenvalue problem

$$\begin{cases} \mathcal{L}_{dn}\xi := \left(-\frac{d^2}{dx^2} + w - 3\phi_w^2\right)\xi = \lambda\xi \\ \xi(0) = \xi(L), \xi'(0) = \xi'(L), \end{cases} \quad (27)$$

where ϕ_w is given by Proposition 3.1.

The following result is a consequence of the Floquet theory (Magnus & Winkler [14]) and can be found in [2].

Theorem 4.1 *Let \mathcal{L}_{dn} be the linear operator defined on $H_{per}^2([0, L])$ by (27). Then the first three eigenvalues β_1, β_2 and β_3 of \mathcal{L}_{dn} are simple, and satisfy $\beta_1 < 0 = \beta_2 < \beta_3$; and ϕ' is the eigenfunction of β_2 . Moreover, the rest of the spectrum consists of a discrete set of eigenvalues which are double.*

To prove that the kernel of \mathcal{L}_c is spanned by $\frac{d}{dx}(\phi_{w(c)}, \psi_{w(c)})$, consider the quadratic form $\mathcal{Q}_c(g, h)$ as the pairing of (g, h) against (\tilde{g}, \tilde{h}) in the $H_{per}^1([0, L]) \times L_{per}^2([0, L]) - H_{per}^{-1}([0, L]) \times L_{per}^2([0, L])$ duality, where $(\tilde{g}, \tilde{h})^t$ is the unbounded operator

$$\tilde{\mathcal{L}}_c := \begin{pmatrix} 1 - \partial_{xx} - 3\phi_{w(c)}^2 & c \\ c & 1 \end{pmatrix}$$

applied to $(g, h)^t$. Then $\tilde{\mathcal{L}}_c(g, h)^t = 0$ implies

$$\begin{cases} -g'' + (1 - c^2)g - 3\phi_{w(c)}^2 g & = & 0 \\ h & = & -cg. \end{cases}$$

Now, from the properties of the operator $\mathcal{L}_{dn} = -\partial_x^2 + w - 3\phi_w^2$ established in Theorem 4.1, it follows that $g = \lambda\phi'_{w(c)}$ and $h = -cg = -c\lambda\phi'_{w(c)} = \lambda\psi'_{w(c)}$, where $\lambda \neq 0 \in \mathbb{R}$.

To show that there is a single, simple, negative eigenvalue, consider \mathcal{Q}_c^1 defined in (26) above. By Theorem 4.1, the operator \mathcal{L}_{dn} has exactly one negative eigenvalue which is simple, say λ_0 , with associated eigenfunction $\zeta > 0$. Thus, \mathcal{Q}_c^1 achieves a negative value and so does \mathcal{Q}_c . In fact, considering $\vec{\zeta} = (\zeta, -c\zeta)$, we have $\mathcal{L}_c(\vec{\zeta}) = \mathcal{Q}_c^1(\zeta) + \frac{1}{2}\|c\zeta - c\zeta\|^2 = \mathcal{Q}_c^1(\zeta) = \frac{1}{2}\lambda_0 < 0$. Denoting by β_0 the lowest eigenvalue of \mathcal{L}_c , we will show that the next eigenvalue β_1 is 0, which is known to be simple, and consequently β_2 is in fact strictly positive. These results are proved using the (min-max) Rayley-Ritz characterization of eigenvalues (see [10, 16]), namely

$$\beta_1 = \max_{(\phi_1, \psi_1) \in X} \min_{\substack{(g, h) \in X \setminus \{0\} \\ (g, \phi_1)_1 + (h, \psi_1)_0 = 0}} \frac{Q_c(g, h)}{\|g\|_1^2 + \|h\|_0^2}.$$

Choosing $\phi_1 = \zeta$, $\psi_1 = 0$ we obtain the lower estimate

$$\beta_1 \geq \min_{\substack{(g, h) \in X \setminus \{0\} \\ (g, \zeta)_1 = 0}} \frac{Q_c(g, h)}{\|g\|_1^2 + \|h\|_0^2}. \quad (28)$$

The right hand side of (28) is non-negative on the subspace $\{(g, h) \in X \setminus \{0\}; (g, \zeta)_1 = 0\}$, since $Q_c^1(g) \geq 0$ by Theorem 4.1. Thus, $\beta_1 = 0$ and, from earlier considerations, β_1 is simple and $\beta_2 > 0$.

The above analysis can be summarized in the form of the following theorem:

Theorem 4.2 *Let \mathcal{L}_c be the linear operator defined on $H_{per}^2([0, L]) \times H_{per}^1([0, L])$ by (25). Then the first two eigenvalues β_0 and β_1 of \mathcal{L}_c are simple, and satisfy $\beta_0 < \beta_1 = 0$; and $\vec{\zeta}_c = (\zeta_{1,c}, \zeta_{2,c})$, with $\zeta_{1,c} > 0$ and $\vec{\phi}'_c$ are the eigenfunctions of β_0 and β_1 , respectively. Moreover, the rest of the spectrum consists of a discrete set of eigenvalues and the mapping $c \rightarrow \vec{\zeta}_c$ is continuous with values in $H_{per}^2([0, L]) \times H_{per}^1([0, L])$.*

5 Concavity of $d(c)$

Lemma 5.1 *Let $c \in (-1, 1)$ and $L > \pi\sqrt{2}$. Then the function $d(c)$ is concave, provided $c^2 < \frac{1}{2}$ and $1 - c^2 > \frac{2\pi^2}{L^2}$.*

Remark 5.1 *Relation (6) implies that $d''(c) < 0$ is equivalent to the condition*

$$\frac{d}{dc}I(\phi_{w(c)}, \psi_{w(c)}) < 0.$$

Proof: (of Lemma 5.1) Note that

$$\begin{aligned} \frac{d}{dc}I(\phi_w, \psi_w) &= \frac{d}{dc} \int_0^L \phi_w \psi_w = - \int_0^L \phi_w^2 dx - c \frac{d}{dw} \left[\int_0^L \phi_w^2 dx \right] \frac{dw}{dc} \\ &= - \int_0^L \phi_w^2 dx + 2c^2 \frac{d}{dw} \left[\int_0^L \phi_w^2 dx \right]. \end{aligned} \quad (29)$$

Now,

$$\frac{d}{dw} \frac{1}{2} \int_0^L \phi_w^2 dx = \frac{4}{L} \frac{d}{dk} [K(k)E(k)] \frac{dk}{dw} > 0. \quad (30)$$

Indeed, we observe from (19), (20) and (23) that

$$\|\phi_w\|^2 = \sqrt{2}\eta_1 \int_0^{\frac{\eta_1 L}{\sqrt{2}}} dn^2(x; k) dx = \frac{8K(k)}{L} \int_0^K dn^2(x; k) dx, \quad (31)$$

where we used that the Jacobi elliptic function dn has fundamental period $2K$ and is an even function. Now, by using that $\int_0^K cn^2(x; k) dx = \frac{1}{k^2} [E(k) - (k')^2 K(k)]$ and $dn^2(x; k) = 1 - k^2 + k^2 cn^2(x; k)$, it follows from (31) that

$$\frac{1}{2} \int_0^L \phi_w^2 dx = \frac{4}{L} K(k)E(k). \quad (32)$$

Now, Proposition 3.1 and Theorem 3.1 implies that the map $w \rightarrow \Lambda(w) \equiv \eta_2(w)$ is strictly decreasing and from (22), with $\eta_2 = \eta_2(w)$, we have that

$$\frac{dk}{dw} = \frac{1}{2k} \left[\frac{2\eta_2^2 - 4w\eta_2\eta_2'}{(2w - \eta_2^2)^2} \right] > 0. \quad (33)$$

Thus, since $k \in (0, 1) \rightarrow K(k)E(k)$ is strictly decreasing (see Appendix), the claim (30) follows from (32) and (33).

So, from (29), (30) and (32), we get

$$\frac{d}{dc}I(\phi_w, \psi_w) = -\frac{8}{L} K(k)E(k) + \frac{16c^2}{L} \frac{d}{dk} [K(k)E(k)] \frac{dk}{dw}. \quad (34)$$

Now, considering the function Ψ defined by (2.12) in [2] and using (33), we obtain

$$\begin{aligned} \frac{\partial \Psi}{\partial w} &= \frac{2\sqrt{2}\sqrt{2w - \eta_2^2} \frac{dK}{dw}(k(\eta_2, w)) - 2\sqrt{2}K(k(\eta_2, w))(2w - \eta_2^2)^{-\frac{1}{2}}}{(2w - \eta_2^2)} \\ &= \frac{2\sqrt{2}\sqrt{2w - \eta_2^2} \frac{dK}{dk}(k(\eta_2, w)) \frac{\eta_2^2}{k(2w - \eta_2^2)^2} - 2\sqrt{2}K(k(\eta_2, w))(2w - \eta_2^2)^{-\frac{1}{2}}}{(2w - \eta_2^2)}, \end{aligned}$$

hence

$$\frac{dk}{dw} = \frac{1}{2k(2w - \eta_2^2)^2} \left\{ 2\eta_2^2 - 4w \left[\frac{k(2w - \eta_2^2)K - \eta_2^2 \frac{dK}{dk}}{k(2w - \eta_2^2)K - 2w \frac{dK}{dk}} \right] \right\} > 0. \quad (35)$$

From (34), (35) and using that $2w - \eta_2^2 = \frac{\eta_2^2}{(k')^2}$, we obtain $\theta \frac{L}{8} \frac{dI(\phi_w, \psi_w)}{dc} = \theta \left\{ -EK + 2c^2 \frac{E^2 - k'^2 K^2}{kk'^2} \frac{dk}{dw} \right\} = -\theta EK + 2c^2(E^2 - k'^2 K^2)K(\eta_2^2 - 2w)(k')^2 = K \left\{ -\eta_2^2 E(k^2 \eta_2^2 K - 2wE + 2wk'^2 K) \right\} + K \left\{ 2c^2(E^2 - k'^2 K^2)(k')^2(\eta_2^2 - 2w) \right\}$, or equivalently,

$$\begin{aligned} \theta \frac{L}{8K} \frac{dI(\phi_w, \psi_w)}{dc} &= -\eta_2^2 E(k^2 \eta_2^2 K - 2wE + 2wk'^2 K) \\ &+ 2c^2(k')^2(E^2 - k'^2 K^2)(\eta_2^2 - 2w) \\ &= \eta_2^2(-2wk'^2 - \eta_2^2 k^2)EK \\ &+ (2c^2(k')^2 \eta_2^2 - 4c^2 w(k')^2 + 2w\eta_2^2)E^2 + 2c^2(k')^2 \eta_2^2 K^2, \end{aligned} \quad (36)$$

where $\theta = \eta_2^2(k^2 \eta_2^2 K - 2wE + 2wk'^2 K) < 0$.

Now, given that $(k')^2 = \frac{\eta_2^2}{2w - \eta_2^2}$, we rewrite the coefficient of E^2 in (36) as

$$\begin{aligned} 2c^2(k')^2 \eta_2^2 - 4c^2 w(k')^2 + 2w\eta_2^2 &= 2c^2 \frac{\eta_2^4}{2w - \eta_2^2} - 4c^2 w \frac{\eta_2^2}{2w - \eta_2^2} + 2w\eta_2^2 \\ &= \frac{2c^2 \eta_2^4 - 4c^2 w \eta_2^2 + 4w^2 \eta_2^2 - 2w\eta_2^4}{2w - \eta_2^2} = \frac{\eta_2^2(2w - \eta_2^2)(2w - 2c^2)}{2w - \eta_2^2} = \eta_2^2(2w - 2c^2). \end{aligned}$$

Also, the coefficient of EK can be rewritten as $\eta_2^2(-2wk'^2 - \eta_2^2 k^2) = 2\eta_2^4$. Thus,

$$\frac{L}{8K} \frac{dI(\phi_w, \psi_w)}{dc} = \frac{-2\eta_2^4 EK + 2\eta_2^2(w - c^2)E^2 + 2c^2(k')^2 \eta_2^2 K^2}{\eta_2^2(k^2 \eta_2^2 K - 2wE + 2wk'^2 K)}. \quad (37)$$

We remark that we can write w as a function of complete elliptic integrals. In fact, by integrating the equation (16) from 0 to L we obtain

$$w = w(c) = \frac{\int_0^L \phi_w^3(\xi) d\xi}{\int_0^L \phi_w(\xi) d\xi}, \quad (38)$$

which is well defined, since the solution ϕ_w is positive.

Now, using (19), the expression 314.01 in [5] and the fact that $F(\frac{\pi}{2}; k) = K(k)$ (see Appendix), we obtain

$$\begin{aligned} \int_0^L \phi_w(\xi) d\xi &= \int_0^L \eta_1 \operatorname{dn}\left(\frac{\eta_1}{\sqrt{2}}\xi; k\right) d\xi = \sqrt{2} \int_0^{\frac{\eta_1 L}{\sqrt{2}}} \operatorname{dn}(y; k) dy \\ &= \sqrt{2} \int_0^{2K} \operatorname{dn}(y; k) dy = 2\sqrt{2} \int_0^K \operatorname{dn}(y; k) dy = \pi\sqrt{2}. \end{aligned} \quad (39)$$

Similarly using (19), the expression 314.03 in [5], and the special values $\operatorname{sn}0 = 0$, $\operatorname{sn}K = 1$, $\operatorname{cn}K = 0$ (see Appendix), it follows that

$$\begin{aligned} \int_0^L \phi_w^3(\xi) d\xi &= \int_0^L \eta_1^3 \operatorname{dn}^3\left(\frac{\eta_1}{\sqrt{2}}\xi; k\right) d\xi = \sqrt{2}\eta_1^2 \int_0^{2K} \operatorname{dn}^3(y; k) dy \\ &= 2\sqrt{2}\eta_1^2 \int_0^K \operatorname{dn}^3(y; k) dy \\ &= 16\sqrt{2} \frac{K^2}{L^2} \frac{1}{2} \left[(1 + (k')^2) \frac{\pi}{2} + k^2 \operatorname{sn}K \operatorname{cn}K \right] \\ &= 4\pi\sqrt{2} (1 + (k')^2) \frac{K^2}{L^2}. \end{aligned} \quad (40)$$

Substituting (39) and (40) in (38), we deduce that

$$w(c) = 1 - c^2 = 4(1 + (k')^2) \frac{K^2}{L^2}. \quad (41)$$

Using (41) and $\eta_2^2 = \frac{2w(k')^2}{1+(k')^2}$, the numerator of (37) will be positive if and only if $c^2(k')^2K^2 > (c^2 - w)E^2 + \eta_2^2EK \Leftrightarrow (1-w)(k')^2K^2 > (1-2w)E^2 + \frac{2w(k')^2}{1+(k')^2}EK \Leftrightarrow w \left[2E^2 - (k')^2K^2 - \frac{2(k')^2}{1+k'^2}EK \right] > E^2 - (k')^2K^2 \Leftrightarrow w > \frac{E^2 - (k')^2K^2}{2E^2 - (k')^2K^2 - \frac{2(k')^2}{1+k'^2}EK}$.

Remark 5.2 $2E^2 - (k')^2K^2 - \frac{2(k')^2}{1+k'^2}EK > 0$ since the functions EK and $E + K$ are strictly increasing (see Appendix).

Claim:

$$\lim_{k \rightarrow 0} \frac{E^2 - (k')^2K^2}{2E^2 - (k')^2K^2 - \frac{2(k')^2}{1+k'^2}EK} = \frac{2}{5}. \quad (42)$$

Proof: (of claim) Indeed, denoting by $f(k) := E^2 - (k')^2K^2$ and by $g(k) := E^2 - \frac{2(k')^2}{1+k'^2}EK$, we use L'Hospital's rule to find the limit (42). Specifically, we show using (48)-(57) that

$$\begin{aligned} \lim_{k \rightarrow 0} f^{(j)}(k) &= \lim_{k \rightarrow 0} g^{(j)}(k) = 0 \quad (j = 0, 1, 2, 3), \\ \lim_{k \rightarrow 0} f^{(4)}(k) &= 3\pi^2/4 \quad \text{and} \quad \lim_{k \rightarrow 0} g^{(4)}(k) = 9\pi^2/8, \end{aligned}$$

which implies our claim. □

Note that, by $\lim_{k \rightarrow 1} k'^2K^2 = 0$, we have that

$$\lim_{k \rightarrow 1} \frac{E^2 - (k')^2K^2}{E^2 - (k')^2K^2 + E^2 - \frac{2(k')^2}{1+k'^2}EK} = \frac{1}{2}. \quad (43)$$

Moreover, $0 < \frac{E^2 - (k')^2K^2}{E^2 - (k')^2K^2 + E^2 - \frac{2(k')^2}{1+k'^2}EK} < \frac{1}{2} \quad \forall k \in (0, 1)$, since $E^2 - \frac{2k'^2}{1+k'^2}K^2 > E^2 - k'^2K^2 \quad \forall k \in (0, 1)$. In addition we get $(1+k'^2)K > 2E$, since the function $m(k) := (1+k'^2)K - 2E$ has the following properties: $m(0) = 0$ and $m'(k) > 0 \quad \forall k \in (0, 1)$. We conclude that the function

$$\frac{f(k)}{f(k) + g(k)} = \frac{E^2 - (k')^2K^2}{E^2 - (k')^2K^2 + E^2 - \frac{2(k')^2}{1+k'^2}EK} \quad (44)$$

is strictly positive on $[0, 1]$. Now, continuity plus (42) and (43) implies that $c_0 := \max_{0 \leq k \leq 1} \frac{f(k)}{f(k) + g(k)}$ satisfies $0 < c_0 \leq \frac{1}{2}$.

This concludes the lemma. □

6 Instability

Consider the function $d(c)$ defined by (7). We now examine the relation between concavity properties of d and the properties of the functional H near the critical point $\vec{\phi}_c$ subject to the constraint $I(\vec{u}) = I(\vec{\phi}_c)$.

Theorem 6.1 *Let $c \neq 0$ be fixed. If $d''(c) < 0$, then there exists a curve $w \rightarrow \vec{\Phi}_w$ which satisfies $I(\vec{\Phi}_w) = I(\vec{\phi}_c)$, $\vec{\Phi}_c = \vec{\phi}_c$, and on which $H(\vec{u})$ has a strict local maximum at $\vec{u} = \vec{\phi}_c$.*

Proof: We follow the ideas of [6, 11, 12]. Let $\vec{\zeta}_c$ be the unique, negative eigenfunction of \mathcal{L}_c . Define $\vec{\Phi}_w := \vec{\phi}_w + s(w)\vec{\zeta}_c$, for w near c , where $s(w)$ satisfies $s(c) = 0$ and $I(\vec{\Phi}_w) = I(\vec{\phi}_c)$. The function $s(w)$ can be defined by the implicit function theorem, since

$$\frac{\partial}{\partial s} I(\vec{\phi}_w + s\vec{\zeta}_c) |_{\{s=0, w=c\}} = \int_0^L (\phi_c \zeta_{2,c} + \psi_c \zeta_{1,c}) dx,$$

where $\vec{\zeta}_c = (\zeta_{1,c}, \zeta_{2,c})$ with $\zeta_{2,c} = \frac{c}{\beta_0 - 1} \zeta_{1,c}$, and β_0 is the unique negative eigenvalue of \mathcal{L}_c and $\psi_c = -c\phi_c$, $\phi_c > 0$. Thus

$$\frac{\partial}{\partial s} I(\vec{\phi}_w + s\vec{\zeta}_c) |_{\{s=0, w=c\}} = -c \left(1 + \frac{1}{1 - \beta_0}\right) \int_0^L \phi_c \zeta_{1,c} dx \neq 0.$$

It is easy to see that

$$\frac{d^2}{dw^2} I(\vec{\Phi}_w) |_{\{w=c\}} = \langle \mathcal{L}_c \vec{y}, \vec{y} \rangle,$$

where $\vec{y} = \frac{\partial \vec{\Phi}_w}{\partial w} |_{\{w=c\}} = \frac{\partial \vec{\phi}_c}{\partial c} + s'(c)\vec{\zeta}_c$. In fact, by some calculations we have that

$$\begin{aligned} d''(c) &= -s'(c) \langle I'(\vec{\phi}_c), \vec{\zeta}_c \rangle \\ \mathcal{L}_c \vec{y} &= -I'(\vec{\phi}_c) + s'(c) \mathcal{L}_c \vec{\zeta}_c \end{aligned}$$

so that

$$\langle \mathcal{L}_c \vec{y}, \vec{y} \rangle = -s'(c) \langle I'(\vec{\phi}_c), \vec{\zeta}_c \rangle + s'(c)^2 \langle \vec{\zeta}_c, \mathcal{L}_c \vec{\zeta}_c \rangle < 0$$

in view of the fact that $d''(c) < 0$. □

To prove the instability, we need the following lemmas which are proved in [11] and as in the analogous case of [7]; therefore we state them without proof.

Lemma 6.1 *There exists $\epsilon > 0$ and a unique C^1 map $\alpha : U_\epsilon \rightarrow \mathbb{R}$, such that, for any $\vec{u} = (u, v) \in U_\epsilon$ and $r \in \mathbb{R}$,*

$$\begin{aligned} \langle \vec{u}(\cdot + \alpha(\vec{u})), \partial_x \vec{\phi}_c \rangle &= 0 \\ \alpha(\vec{u}(\cdot + r)) &= \alpha(\vec{u}) - r, \text{ modulo the period} \\ \alpha'(\vec{u}) &= \frac{\partial_x \vec{\phi}_c(\cdot - \alpha(\vec{u}))}{\langle \vec{u}, \partial_x^2 \vec{\phi}_c(\cdot - \alpha(\vec{u})) \rangle}, \end{aligned}$$

where U_ϵ is the "tube"

$$U_\epsilon = \{ \vec{u} \in X : \inf_{s \in \mathbb{R}} \| \vec{u} - \tau_s(\vec{\phi}_c) \|_X < \epsilon \}.$$

Definition 6.1 *For $\vec{u} \in U_\epsilon$, define $B(\vec{u})$ by the formula*

$$B(\vec{u}) = \vec{y}(\cdot - \alpha(\vec{u})) - K \partial_x \alpha'(\vec{u}),$$

where $K = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

By Lemma 6.1, B may also be expressed as

$$B(\vec{u}) = \vec{y}(\cdot - \alpha(\vec{u})) - \frac{\langle K \vec{y}(\cdot - \alpha(\vec{u})), \vec{u} \rangle}{\langle \partial_x^2 \vec{\phi}_c(\cdot - \alpha(\vec{u})), \vec{u} \rangle} K \partial_x^2 \vec{\phi}_c(\cdot - \alpha(\vec{u})).$$

Lemma 6.2 B is a C^1 function from U_ϵ into X , B commutes with translations, $B(\vec{\phi}_c) = \vec{y}$, and $\langle B(\vec{u}), K\vec{u} \rangle = 0$, for any $\vec{u} \in U_\epsilon$.

Lemma 6.3 There exists a C^1 function

$$\Pi : \{\vec{v} \in U_\epsilon : I(\vec{v}) = I(\vec{\phi}_c)\} \rightarrow \mathbb{R}$$

which is invariant under translations, such that if $\vec{v} \in U_\epsilon$ with $I(\vec{v}) = I(\vec{\phi}_c)$ and \vec{v} not a translate of $\vec{\phi}_c$, we have

$$H(\vec{\phi}_c) < H(\vec{v}) + \Pi(\vec{v}) \langle H'(\vec{v}), B(\vec{v}) \rangle.$$

Lemma 6.4 The curve $w \rightarrow \vec{\Phi}_w$ constructed in Theorem 6.1 satisfies $H(\vec{\Phi}_w) < H(\vec{\phi}_c)$ for $w \neq c$, and $\langle H'(\vec{\Phi}_w), B(\vec{\Phi}_w) \rangle$ changes sign as w passes through c , with $c \neq 0$.

6.1 Proof of Theorem 1.1

First, consider $c \neq 0$ with $|c| < 1$. Let $\epsilon > 0$ given small enough. By Lemma 6.4, we can choose $\vec{u}_0 \in X$ arbitrarily close to $\vec{\phi}_c$ such that $I(\vec{u}_0) = I(\vec{\phi}_c)$, $H(\vec{u}_0) < H(\vec{\phi}_c)$ e $\langle H'(\vec{u}_0), B(\vec{u}_0) \rangle > 0$. To prove the instability of $\vec{\phi}_c$, it suffices to show that there are some elements $\vec{u}_0 \in X$ which are close to $\vec{\phi}_c$ but for which the solution \vec{u} of (2) with initial data \vec{u}_0 exits from U_ϵ in finite time. Let $[0, t_1)$ denote the maximal interval for which $\vec{u}(\cdot, t)$ lies continuously in U_ϵ . By Lemma 2.1, $t_1 > 0$. Let T be the maximum existence time of solution \vec{u} in (1) with initial data \vec{u}_0 . If T is finite, then we have the X -instability for $\vec{\phi}_c$ by definition. So we may assume that $T = +\infty$ and it suffices to show that $t_1 < +\infty$.

In view of Lemma 2.1 and Proposition 2.1, \vec{u} has the following properties: $\vec{u} \in C([0, t_1]; X)$, $\vec{u}(0, x) = \vec{u}_0(x)$ and $I(\vec{u}(t))$, $H(\vec{u}(t))$ are constant for $t \in [0, t_1)$. Let $\rho(t) = \alpha(\vec{u}(t))$, where α is defined by Lemma 6.1 and define

$$\begin{aligned} \vec{y}(z) &= \frac{d\vec{\Phi}_c}{dc} = \frac{d\vec{\phi}_c}{dc} + s'(c)\vec{\chi}_c \\ \vec{Y}(x) &= \int_0^x K\vec{y}(z)dz \\ A(t) &= \int_0^L \vec{Y}(x - \rho(t)) \cdot \vec{u}(x, t) dx \text{ for } 0 \leq t < t_1 \\ \vec{\gamma} &= \int_0^L K\vec{y}(x)dx. \end{aligned}$$

The function $A(t)$ serves as a Lyapunov function in our argument.

Now we estimate dA/dt . By differentiation,

$$\frac{dA}{dt} = -\rho'(t) \int_0^L K\vec{y}(x - \rho(t)) \cdot \vec{u}(x, t) dx + \int_0^L \vec{Y}(x - \rho(t)) \cdot \frac{\partial \vec{u}}{\partial t}(x, t) dx,$$

where $\partial \vec{u} / \partial t$ is in distribution sense. Since $\rho'(t) = d\alpha(\vec{u}(t))/dt = \langle \alpha'(\vec{u}), (\partial \vec{u} / \partial t) \rangle$,

$$\frac{dA}{dt} = \left\langle -\langle K\vec{y}(\cdot - \rho), \vec{u} \rangle \alpha'(\vec{u}) + \vec{Y}(\cdot - \rho), \frac{\partial \vec{u}}{\partial t} \right\rangle.$$

Since

$$\frac{d\vec{u}}{dt} = JH'(\vec{u}), \text{ where } J = \partial_x K, \text{ and } \partial_x \vec{Y} = K\vec{y},$$

it follows that

$$\begin{aligned}\frac{dA}{dt} &= \langle -\langle K\vec{y}(\cdot - \rho(t)), \vec{u} \rangle \partial_x \alpha'(\vec{u}) - K\vec{y}(\cdot - \rho(t)), KH'(\vec{u}) \rangle \\ &= \langle -\langle K\vec{y}(\cdot - \rho(t)), \vec{u} \rangle K\partial_x \alpha'(\vec{u}) - \vec{y}(\cdot - \rho(t)), H'(\vec{u}) \rangle \\ &= -\langle B(\vec{u}), H'(\vec{u}) \rangle.\end{aligned}$$

Since

$$0 < H(\vec{\phi}_c) - H(\vec{u}_0) = H(\vec{\phi}_c) - H(\vec{u}(t)),$$

Lemma 6.3 implies that

$$0 < \Pi(\vec{u}(t)) \langle B(\vec{u}(t)), H'(\vec{u}(t)) \rangle.$$

By $\langle B(\vec{u}_0), H'(\vec{u}_0) \rangle > 0$ and continuity we obtain $\langle B(\vec{u}(t)), H'(\vec{u}(t)) \rangle > 0$ for $0 < t < t_1$. Moreover, since $\vec{u}(t) \in U_\epsilon$ and $\Pi(\vec{\phi}_c) = 0$, we may assume that $0 < \Pi(\vec{u}(t)) < 1$ for $0 < t < t_1$, by choosing ϵ smaller if necessary. Therefore, for all $t \in [0, t_1)$, by Lemma 6.3,

$$\begin{aligned}\langle B(\vec{u}(t)), H'(\vec{u}(t)) \rangle &\geq \Pi(\vec{u}(t)) \langle B(\vec{u}(t)), H'(\vec{u}(t)) \rangle \\ &> H(\vec{\phi}_c) - H(\vec{u}(t)) = H(\vec{\phi}_c) - H(\vec{u}_0) > 0.\end{aligned}$$

Hence,

$$-\frac{dA}{dt} \geq H(\vec{\phi}_c) - H(\vec{u}_0) > 0.$$

So we conclude that $t_1 < +\infty$.

Finally, consider the case $c = 0$ and $d''(0) < 0$.

Let $N = \{c \mid \text{the dnoidal wave } \vec{\phi}_c \text{ is } X\text{-unstable}\}$. The curve $c \rightarrow \vec{\phi}_c$ is continuous in X . From the definition of stability, the set N is closed. Since

$$d''(c) = \left\langle I'(\vec{\phi}_c), \frac{\partial \vec{\phi}_c}{\partial c} \right\rangle$$

by Theorem 3.1, $d''(c)$ is continuous on $(-\sqrt{1 - \frac{2\pi^2}{L^2}}, \sqrt{1 - \frac{2\pi^2}{L^2}}) \subset (-1, 1)$. For any nonzero sequence $c_n \rightarrow 0$, we have $d''(c_n) \rightarrow d''(0) < 0$. Hence $d''(c_n) < 0$ for large n . Thus $c_n \in N$, for large n . It follows that $c = 0 \in N$.

7 Appendix

In this Appendix we recall some properties of the Jacobian elliptic integrals that have been used in this work (see [5]).

First, we define *the normal elliptic integral of the first and second kinds*,

$$F(\varphi, k) := \int_0^\varphi \frac{d\theta}{\sqrt{1 - k^2 \sin^2 \theta}} = \int_0^y \frac{dt}{\sqrt{(1 - t^2)(1 - k^2 t^2)}}$$

and

$$E(\varphi, k) := \int_0^\varphi \sqrt{1 - k^2 \sin^2 \theta} d\theta = \int_0^y \sqrt{\frac{1 - k^2 t^2}{1 - t^2}} dt,$$

respectively, where $y = \sin \varphi$.

In their algebraic forms, these two integrals possess the following properties: the first is finite for all real (or complex) values of y , including infinity; the second has a simple pole of order 1 for $y = \infty$. The number k is called the *modulus*. This number may take any real or imaginary value. Here we wish to take $0 < k < 1$. The number k' is called the *complementary modulus* and is related to k by $k' = \sqrt{1 - k^2}$.

The variable φ is the *argument* of the normal elliptic integrals. When $y = 1$, the integrals above are said to be *complete*. In this case, one writes: $F(\pi/2, k) \equiv K(k) \equiv K$, and $E(\pi/2, k) \equiv E(k) \equiv E$.

Some special values of K and E are: $K(0) = E(0) = \pi/2$, $E(1) = 1$ and $K(1) = +\infty$. For $k \in (0, 1)$, one has $K'(k) > 0$, $K''(k) > 0$, $E'(k) < 0$, $E''(k) < 0$ and $E(k) < K(k)$. Moreover, $E(k) + K(k)$ and $E(k)K(k)$ are strictly increasing functions in $(0, 1)$.

Now, we give some derivatives of the complete elliptic integrals K and E and some important limits involving these functions, that we used in this work (cf. [1] ou [5]):

$$\frac{dK}{dk} = \frac{E - k'^2 K}{kk'^2}; \quad (45)$$

$$\frac{dE}{dk} = \frac{E - K}{k}; \quad (46)$$

$$\frac{d^2 E}{dk^2} = -\frac{1}{k} \frac{dK}{dk} = -\frac{E - k'^2 K}{k^2 k'^2}; \quad (47)$$

$$\lim_{k \rightarrow 0} \frac{E - k'^2 K}{k^2} = \lim_{k \rightarrow 0} \frac{K - E}{k^2} = \frac{\pi}{4}; \quad (48)$$

$$\lim_{k \rightarrow 0} \frac{dE}{dk} = \lim_{k \rightarrow 0} \frac{(E - K)}{k^2} k = 0; \quad (49)$$

$$\lim_{k \rightarrow 0} \frac{dK}{dk} = 0; \quad (50)$$

$$\lim_{k \rightarrow 0} \frac{d^2 E}{dk^2} = -\lim_{k \rightarrow 0} \frac{1}{k} \frac{dK}{dk} = -\frac{\pi}{4}; \quad (51)$$

$$\lim_{k \rightarrow 0} \frac{d^2 K}{dk^2} = \lim_{k \rightarrow 0} \left[\frac{1}{k'^2} K + \frac{(3k^2 - 1)}{kk'^2} \frac{dK}{dk} \right] = \frac{\pi}{4}; \quad (52)$$

$$\lim_{k \rightarrow 0} \frac{d^3 E}{dk^3} = 0; \quad (53)$$

$$\lim_{k \rightarrow 0} \frac{d^3 K}{dk^3} = 0; \quad (54)$$

$$\lim_{k \rightarrow 0} \frac{d^4 E}{dk^4} = -\frac{9\pi}{16}; \quad (55)$$

$$\lim_{k \rightarrow 0} \frac{d^4 K}{dk^4} = \frac{27\pi}{16}; \quad (56)$$

$$\lim_{k \rightarrow 0} \frac{1}{k} \left(\frac{dK}{dk} \right)^2 = \lim_{k \rightarrow 0} \frac{1}{k'^4} \frac{(E - k'^2 K)}{k^2} \frac{dK}{dk} = 0. \quad (57)$$

Remark 7.1 To see that $\lim_{k \rightarrow 0} \frac{d^3 K}{dk^3} = 0$, we write $K(k) = \frac{\pi}{2} \left\{ \sum_{r=0}^{\infty} \frac{(2r)!(2r)!}{2^{4r}(r!)^4} k^{2r} \right\}$ (see [1], page 110), and then we get

$$\frac{d^3 K}{dk^3} = \frac{\pi}{2} k \left\{ \sum_{r \geq 2} \frac{(2r)!(2r)!}{2^{4r}(r!)^4} 2r(2r-1)(2r-2)k^{2r-2} \right\}, \quad (58)$$

where the series converges absolutely. Actually, denoting by $a_r := \frac{(2r)!(2r)!}{2^{4r}(r!)^4} 2r(2r-1)(2r-2)k^{2r-2}$, we have $\frac{|a_{r+1}|}{|a_r|} = \frac{(2r+2)^2(2r+1)^3}{2^3(r+1)^3(2r-1)(2r-2)} k$, then we get $\lim_{r \rightarrow \infty} \frac{|a_{r+1}|}{|a_r|} < 1$. So, $\lim_{k \rightarrow 0} \frac{d^3 K}{dk^3} = 0$.

To see that $\lim_{k \rightarrow 0} \frac{d^3 E}{dk^3} = 0$, we write $E(k) = \frac{\pi}{2} \left\{ 1 - \sum_{r=1}^{\infty} \frac{(2r-2)!(2r)!}{2^{4r-1}(r-1)!(r!)^3} k^{2r} \right\}$ (see [1], page 110), from which we get

$$\frac{d^3 E}{dk^3} = -\frac{\pi}{2} \left\{ \sum_{r \geq 2} \frac{(2r-2)!(2r)!}{2^{4r-1}(r-1)!(r!)^3} 2r(2r-1)(2r-2)k^{2r-3} \right\}. \quad (59)$$

Proceeding as before, we obtain the desired limit.

To see that $\lim_{k \rightarrow 0} \frac{d^4 K}{dk^4} = \frac{27\pi}{16}$ and $\lim_{k \rightarrow 0} \frac{d^4 E}{dk^4} = -\frac{9\pi}{16}$, differentiating again the series in (58) and (59), we get, $\frac{d^4 K}{dk^4} = \frac{\pi}{2} \sum_{r=2}^{\infty} b_r 2r(2r-1)(2r-2)(2r-3)k^{2r-4}$ and $\frac{d^4 E}{dk^4} = -\frac{\pi}{2} \sum_{r=2}^{\infty} c_r 2r(2r-1)(2r-2)(2r-3)k^{2r-4}$, where $b_r := \frac{(2r)!(2r)!}{2^{4r}(r!)^4} 2r(2r-1)(2r-2)(2r-3)k^{2r-4}$ and $c_r := \frac{(2r-2)!(2r)!}{2^{4r-1}(r-1)!(r!)^3} 2r(2r-1)(2r-2)(2r-3)k^{2r-4}$. It's easy to see that $\frac{d^4 K}{dk^4} \rightarrow \frac{\pi}{2} b_2 4! = \frac{27\pi}{16}$ and $\frac{d^4 E}{dk^4} \rightarrow -\frac{\pi}{2} c_2 4! = -\frac{9\pi}{16}$, as $k \rightarrow 0$.

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