GLIMM METHOD AND WAVE-FRONT TRACKING FOR THE AW-RASCLE TRAFFIC FLOW MODEL

RICARDO E. FERREIRA[1] AND CEZAR I. KONDO[2]

ABSTRACT. In this paper, we study the model of traffic flow proposed by Aw and Rascle in 2000. We contructed weak global in time solution using the Glimm method, we also contructed weak global in time solution using the wave-front tracking method. In both cases, we assume that the total variation of the initial data is locally bounded and we do not consider the appearance of the vacuum.

1. INTRODUCTION

We consider the following system of equations

$$(1) \qquad \begin{cases} \rho_t + (\rho v)_x = 0, \\ (\rho r)_t + (\rho r v)_x = 0. \end{cases}$$

where ρ is the density, v is the velocity of cars on the roadway and $r = v + p(\rho)$.

We assume that $p(\rho) = \rho^\kappa$, where $\rho > 0$ and $\kappa > 0$ is a constant. For simplicity we write the system as

$$U_t + F(U)_x = 0,$$

where $U = (\rho, \rho v + \rho^{(\kappa+1)})$ and $F(U) = (\rho v, \rho v^2 + \rho^{(\kappa+1)} v)$.

Here we consider the Cauchy problem for (1), in $(x,t) \in \mathbb{R} \times [0, \infty)$, with initial data

$$\rho(x,0) = \rho_0(x), \quad v(x,0) = v_0(x), \quad x \in \mathbb{R},$$

where $\rho_0(x)$, $v_0(x)$ are bounded functions with locally bounded total variation on $\mathbb{R}$ and $\rho_0(x) \geq \underline{\rho} > 0$ and $v_0(x) \geq 0$.

Aw-Rascle [1] solve the Riemann problem for this model, including the appearance of the vaccum. Godvik-Hanche-Olsen [4] has shown the existence of weak solution to Cauchy problem, also including the appearance of the vaccum.

In this paper, we construct weak solutions based in the Glimm method [3]. After, we construct weak solution based in the wave-front tracking scheme [2]. We define a subset appropriate, where the initial data takes values, so that the vacuum does not appear.

The outline of this paper is as follows. After this introduction, we calculate the eigenvalues, eigenvectors and Riemann invariants of the system in Section 2. In Sections 3, we study the elementary waves. In Section 4, we solve the Riemann problem, without the appearance of the vacuum, in a appropriate

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subset. In Section 5, we construct a weak solution to the Cauchy problem using the Glimm method, when the initial data has bounded total variation. In Section 6, we construct a weak solution to the Cauchy problem using the wave-front tracking method, when the initial data has bounded total variation. In Section 7, we construct a weak solution to the Cauchy problem when the initial data has locally bounded total variation.

2. Eigenvalues, eigenvectors and Riemann invariants

We find by direct computation that the Jacobian of U and F are

$$\begin{aligned} DU(\rho, v) &= \begin{pmatrix} 1 & 0 \\ v + (\kappa+1)\rho^{\kappa} & \rho \end{pmatrix}, \\ DF(\rho, v) &= \begin{pmatrix} v & \rho \\ v^2 + (\kappa+1)\rho^{\kappa} v & \rho^{(\kappa+1)} + 2\rho v \end{pmatrix}. \end{aligned} \tag{2}$$

As the determinant of DU is $\rho > 0$, there exists $(DU)^{-1}$, and

$$DU^{-1} = \begin{pmatrix} 1 & 0 \\ -\frac{v}{\rho} - (\kappa+1)\rho^{\kappa-1} & \frac{1}{\rho} \end{pmatrix}.$$

The Jacobian of the system (1) is

$$J \quad \equiv \quad DU^{-1}DF = \begin{pmatrix} v & \rho \\ 0 & v - \kappa\rho^{\kappa} \end{pmatrix}.$$

The eigenvalues λ of the system (1) satisfy the equation

$$|J - \lambda I| = 0,$$

i.e.,

$$(v - \lambda)(v - \kappa\rho^{\kappa} - \lambda) = \lambda^2 - \lambda(2v - \kappa\rho^{\kappa}) + v^2 - \kappa\rho^{\kappa} v = 0.$$

Solving this equation, we obtain

$$\lambda_1 = v - \kappa\rho^{\kappa}, \quad \lambda_2 = v, \tag{3}$$

with corresponding right eigenvectors

$$r_1 = (-1, \kappa\rho^{\kappa}), \quad r_2 = (1, 0).$$

We note that the system (1) is strict hyperbolicity, $\lambda_1 < \lambda_2$, when $\rho > 0$. Moreover, for $\rho = 0$ the eigenvalues coincide and the system is only hyperbolic.

For direct computation, we obtain

$$\begin{aligned} &\langle \nabla\lambda_1, r_1 \rangle = \langle (-\kappa^2\rho^{(\kappa-1)}, 1), (-1, \kappa\rho^{(\kappa-1)}) \rangle = (\kappa^2 + \kappa)\rho^{(\kappa-1)} > 0, \\ &\langle \nabla\lambda_2, r_2 \rangle = \langle (0,1), (1,0) \rangle = 0, \end{aligned}$$

where $\langle .,. \rangle$ denotes the usual inner product in $\mathbb{R}^2$.

Since J has two real and distinct eigevalues, $\lambda_1 < \lambda_2$, when $\rho > 0$, there are two characteristic fields. The 1st characteristic field is genuinely non-linear and the 2nd characteristic field is linearly degenerate.

We can define the two Rieamnn invariants as

$$r \equiv r(\rho, v) = v + \rho^{\kappa}, \quad s \equiv s(\rho, v) = v.$$

Indeed,

$$\begin{aligned}\langle \nabla r, r_1\rangle &= \langle (\kappa\rho^{\kappa-1},1),(-1,\kappa\rho^{(\kappa-1)})\rangle = 0,\\ \langle \nabla s, r_2\rangle &= \langle (0,1),(1,0)\rangle = 0.\end{aligned}$$

3. Elementary waves

The Riemann problem is the Cauchy problem with initial data

$$u_0(x) \equiv \begin{cases} u_l, & x<0,\\ u_r, & x>0,\end{cases} \tag{4}$$

where $u_l \equiv (\rho_l, v_l)$, $u_r \equiv (\rho_r, v_r) \in \{(\rho, v); \rho > 0 \text{ e } v \geq 0\}$ are constant vectors. We solve the Riemann problem using rarefaction wave, shock wave and contact dicontinuity.

A centred 1-rarefaction wave through u_l, is a continuos solution of (1) and (4), that has the form

$$u(x,t) = \begin{cases} u_l, & x<\lambda_1(u_l),\\ \bar u\left(\frac{x}{t}\right), & \frac{x}{t}\in[\lambda_1(u_l),\lambda_1(u_r)],\\ u_r, & x>\lambda_1(u_r),\end{cases}$$

where $t > 0$ and $\bar u\left(\frac{x}{t}\right)$ satisfies

$$\begin{aligned} &\bar u' = \frac{1}{\langle\nabla\lambda_1(\bar u), r_1(\bar u)\rangle} r_1\left(\bar u\left(\tfrac{x}{t}\right)\right), && \tfrac{x}{t} = \lambda_1(\bar u),\\ &\bar u(\lambda_1(u_l)) = u_l, && \bar u(\lambda_1(u_r)) = u_r.\end{aligned}$$

Note that, by genuine non-linearity, the function $\frac{x}{t} \to \lambda_1\left(u\left(\frac{x}{t}\right)\right)$ is strictly increases. We note that $\bar u\left(\frac{x}{t}\right)$ is C^1 when $\frac{x}{t} \in (\lambda_1(u_l), \lambda_1(u_r))$, $t > 0$, $j = 1, 2$.

To find the rarefaction wave, we consider the rarefaction curves. We set $\xi = \frac{x}{t}$, then $(\rho(\xi), v(\xi))$ satisfies the ordinary differential equation

$$\xi U_\xi + F(U)_\xi = 0,$$

or

$$(-\xi DU + DF)\begin{pmatrix}\rho_\xi\\ v_\xi\end{pmatrix} = 0,$$

where DU and DF are defined in (2). Note that $(\rho_\xi, v_\xi) \neq 0$ is an eigenvector of J for the eigenvalue ξ.

First, we consider λ_1. The eigenvector (ρ_ξ, v_ξ) satisfies

$$(-\lambda_1 DU + DF)\begin{pmatrix}\rho_\xi\\ v_\xi\end{pmatrix} = 0. \tag{5}$$

From (2) and (3), we obtain

$$\begin{aligned}(-\lambda_1 DU + DF) &= -(v-\kappa\rho^\kappa)\begin{pmatrix}1 & 0\\ v+(\kappa+1)\rho^\kappa & \rho\end{pmatrix}\\ &\quad + \begin{pmatrix} v & \rho\\ v^2+(\kappa+1)\rho^\kappa v & \rho^{(\kappa+1)}+2\rho v\end{pmatrix}.\\ &= \begin{pmatrix}\kappa\rho^\kappa & \rho\\ \kappa\rho^\kappa(v+(\kappa+1)\rho^\kappa) & \rho v+(\kappa+1)\rho^{(\kappa+1)}\end{pmatrix}.\end{aligned}$$

Thus, from (5), we have

$$\begin{pmatrix} \kappa\rho^\kappa & \rho \\ \kappa\rho^\kappa(v+(\kappa+1)\rho^\kappa) & \rho(v+(\kappa+1)\rho^\kappa) \end{pmatrix}\begin{pmatrix} \rho_\xi \\ v_\xi \end{pmatrix} = 0,$$

which gives

$$\kappa\rho^\kappa\rho_\xi + \rho v_\xi = 0.$$

Since $(\rho_\xi, v_\xi) \neq 0$, we obtain the differential equation

$$\frac{d\rho}{dv} = -\frac{1}{\kappa\rho^\kappa}.$$

We can integrate this to obtain the 1-rarefaction curve

$$R_1 : v - v_l = -(\rho^\kappa - \rho_l^\kappa).$$

The requirement $\lambda_1(u_l) < \lambda_1(u)$ gives

$$\begin{aligned} -(\rho^\kappa - \rho_l^\kappa) = v - v_l &> \kappa(\rho^\kappa - \rho_l^\kappa), \\ 0 &> (\kappa+1)(\rho^\kappa - \rho_l^\kappa), \end{aligned}$$

since $\kappa > 0$, $\rho < \rho_l$ on R_1.

A 1-shock wave is a jump discontinuity that has the form

$$u(x,t) = \begin{cases} u_l, & x < s_1 t, \\ u_r, & x > s_1 t, \end{cases}$$

where s_1 is the speed of the shock. The 1-shock wave satisfies the Rankine-Hugoniot condition

$$\begin{aligned} s_1(\rho_r - \rho_l) &= \rho_r v_r - \rho_l v_l, \\ s_1(\rho_r v_r + \rho^{(\kappa+1)} - \rho_l v_l - \rho_l^{(\kappa+1)}) &= \rho_r v_r^2 + \rho^{(\kappa+1)} v - \rho_l v_l^2 - \rho_l^{(\kappa+1)} v_l. \end{aligned} \tag{6}$$

To ensure that our solution is physically relevent, we impose some restrictions on the solution. These restrictions are called Lax condition. So, the 1-shock wave must satisfy the Lax condition

$$s_1 < \lambda_1(u_l), \quad \lambda_1(u_r) < s_1 < \lambda_2(u_r). \tag{7}$$

Eliminating s_1 from (6), and write $u = u_r$, we obtain

$$\begin{aligned} \frac{\rho v - \rho_l v_l}{\rho - \rho_l} &= \frac{\rho v^2 + \rho^{(\kappa+1)} v - \rho_l v_l^2 - \rho_l^{(\kappa+1)} v_l}{\rho v + \rho^{(\kappa+1)} - \rho_l v_l - \rho_l^{(\kappa+1)}}, \\ (v + v_l)^2 &= (v - v_l)(\rho_l^\kappa - \rho^\kappa), \\ v - v_l &= -(\rho^\kappa - \rho_l^\kappa). \end{aligned}$$

We define the 1-shock curve as

$$S_1 : v - v_l = -(\rho^\kappa - \rho_l^\kappa).$$

From (7), we have that $\lambda_1(u) < \lambda_1(u_l)$, so

$$\begin{aligned} -(\rho^\kappa - \rho_l^\kappa) = v - v_l &< \kappa(\rho^\kappa - \rho_l^\kappa), \\ 0 &< (\kappa+1)(\rho^\kappa - \rho_l^\kappa), \end{aligned}$$

since $\kappa > 0$, $\rho > \rho_l$ on S_1. From (6), we have that

$$s_1 = \frac{\rho v - \rho_l v_l}{\rho - \rho_l}. \tag{8}$$

Lemma 1. *The shock curve S_1 satisfies the Lax condition (7).*

Proof. First, we note that since $\rho > \rho_l$, $v < v_l$ on S_1.
To show that $s_1 < \lambda_1(\rho_l, v_l)$, we define the auxiliary function $\phi : [1, \infty) \to \mathbb{R}$, by

$$\phi(t) \equiv t^{(\kappa+1)} - (\kappa+1)t + \kappa.$$

Since

$$\phi'(t) = (\kappa+1)t^{\kappa} - (\kappa+1) = (\kappa+1)(t^{\kappa} - 1) > 0,$$

the function ϕ is increasing, for $t > 1$, moreover, $\phi(1) = 0$. Since $\rho - \rho_l > 0$ on S_1, we have that

$$\begin{aligned}
0 &< \left(\frac{\rho}{\rho_l}\right)^{(\kappa+1)} + \kappa - (\kappa+1)\frac{\rho}{\rho_l}, \\
0 &< \rho(\rho^{\kappa} - \rho_l^{\kappa}) - \kappa\rho_l^{\kappa}(\rho - \rho_l), \\
0 &< \rho(v_l - v) - \kappa\rho_l^{\kappa}(\rho - \rho_l), \\
\rho v - \rho_l v_l &< \rho v_l - \kappa\rho\rho_l^{\kappa} - \rho_l v_l + \kappa\rho_l^{(\kappa+1)}, \\
\rho v - \rho_l v_l &< (\rho - \rho_l)(v_l - \kappa\rho_l^{\kappa}), \\
\frac{\rho v - \rho_l v_l}{\rho - \rho_l} &< v_l - \kappa\rho_l^{\kappa},
\end{aligned}$$

From (3) and (8), $s_1 < \lambda_1(\rho_l, v_l)$.
Next, we prove that $\lambda_1(\rho, v) < s_1$. We define the auxiliary function $\phi : [1, \infty) \to \mathbb{R}$, by

$$\phi(t) \equiv \kappa t^{(\kappa+1)} - (\kappa+1)t^{\kappa} + 1.$$

Since

$$\phi'(t) = \kappa(\kappa+1)t^{\kappa} - \kappa(\kappa+1)t^{(\kappa-1)} = \kappa(\kappa+1)(t^{\kappa} - t^{(\kappa-1)}) > 0,$$

the function ϕ is increasing, for $t > 1$, moreover, $\phi(1) = 0$. Since $\rho - \rho_l > 0$ on S_1,

$$\begin{aligned}
0 &< \kappa\left(\frac{\rho}{\rho_l}\right)^{(\kappa+1)} - (\kappa+1)\left(\frac{\rho}{\rho_l}\right)^{\kappa} + 1, \\
0 &< -\rho_l(\rho^{\kappa} - \rho_l^{\kappa}) + \kappa\rho^{\kappa}(\rho - \rho_l), \\
0 &< \rho_l(v - v_l) + \kappa\rho^{\kappa}(\rho - \rho_l), \\
\rho v - \kappa\rho^{(\kappa+1)} - \rho_l v + \kappa\rho^{\kappa}\rho_l &< \rho v - \rho_l v_l, \\
(v - \kappa\rho^{\kappa})(\rho - \rho_l) &< \rho v - \rho_l v_l, \\
v - \kappa\rho^{\kappa} &< \frac{\rho v - \rho_l v_l}{\rho - \rho_l}.
\end{aligned}$$

From (3) and (8), we have that $\lambda_1(\rho, v) < s_1$.
Now, we prove that $s_1 < \lambda_2(\rho, v)$. Since $v - v_l < 0$ and $\rho - \rho_l > 0$ on S_1,

$$\begin{aligned}
0 &< \rho_l(v_l - v), \\
\rho v - \rho_l v_l &< \rho v - \rho_l v, \\
\frac{\rho v - \rho_l v_l}{\rho - \rho_l} &< v.
\end{aligned}$$

From (3) and (8), we have that $s < \lambda_2(\rho, v)$. □

Now, let's consider λ_2. Similarly, we obtain the 2-rarefaction curve

$$R_2 : v = v_l.$$

Since the 2nd characteristic field is linearly degenerate, the 2-rarefaction curve generate a type of weak solution called contact discontinuity.

We define $s_2 = v_l$, hence if u_r is any point on this curve, then s_2 satisfies the Rankine-Hugoniot condition, and the Riemann problem for (1) with initial data (4) has a piecewise constant weak solution of the form

$$u(x,t) = \begin{cases} u_l, & x < s_2 t, \\ u_r, & x > s_2 t, \end{cases}$$

this elementary wave is called contact discontinuity.

4. Riemann problem

We solve the Riemann problem for (1), constructing a weak solution that does not include vacuum. This weak solution consists of at most two waves, one of each family. First, we define the subset D_1 by

$$D_1 \equiv \{(\rho, v);\ 0 \le v_- < v < v_+ < \rho_-^\kappa,\ 0 < \rho_-^\kappa < v + \rho^\kappa < \rho_+^\kappa < \infty\}.$$

We assume that $U(x,0)$ is in D_1 for each $x \in \mathbb{R}$.

Equivalent

$$D_2 \equiv \{(r, s);\ 0 \le s_- < s < s_+,\ 0 < r_- < r < r_+\},$$

where $s_\pm \equiv v_\pm$ e $r_\pm \equiv \rho_\pm^\kappa$, Figure 1. We note that $s_+ = v_+ < \rho_-^\kappa = r_-$.

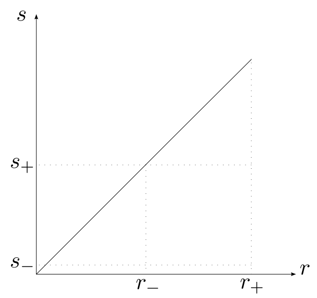

Figure 1.

We define the function

$$\Psi(r, s) \equiv ((r - s)^{\frac{1}{\kappa}}, s) = (\rho, v).$$

Since $r - s > 0$, the Jacobian determinant of this function is $\frac{1}{\kappa}(r-s)^{\frac{1}{\kappa}-1} > 0$. The (r_l, s_l) and (r_r, s_r) are uniquely determined by (ρ_l, v_l) and (ρ_r, s_r). So, the Riemann invariants (r, s) can be used as a coordinate system.

The system (1) is of Temple type, i.e., the shock and rarefaction curves coincide. We note that the Riemann invariant r is constant along the 1-curves and the Riemann invariant s is constant along the 2-curve.

We can parametrize the curves as follows, in $r-s$ plane.

$$\begin{aligned}\gamma &\to R_1(\gamma)(r_l,s_l)=(r_l,\gamma+s_l), \text{ where } \gamma\geq 0,\\ \gamma &\to S_1(\gamma)(r_l,s_l)=(r_l,\gamma+s_l), \text{ where } \gamma<0,\\ \gamma &\to R_2(\gamma)(r_l,s_l)=(\gamma+r_l,s_l), \text{ where } \gamma\in \mathbb{R}.\end{aligned}$$

As consequence, the composite functions

$$\begin{aligned}\Psi_1(\gamma)(r_l,s_l) &\equiv \begin{cases} R_1(\gamma)(r_l,s_l)=(r_l,\gamma+s_l), & \gamma\geq 0,\\ S_1(\gamma)(r_l,s_l)=(r_l,\gamma+s_l), & \gamma<0,\end{cases}\\ \Psi_2(\gamma)(r_l,s_l) &\equiv \quad R_2(\gamma)(r_l,s_l)=(\gamma+r_l,s_l), \quad \gamma\in\mathbb{R},\end{aligned} \tag{9}$$

are smooth.

To solve the Riemann problem, we define the composite function

$$\Lambda(\gamma_1,\gamma_2)(r_l,s_l)\equiv\Psi_2(\gamma_2)\circ\Psi_1(\gamma_1)(r_l,s_l).$$

Theorem 1. *Let u_l, $u_r\in D_2$. So $u_r=\Lambda(\gamma_1,\gamma_2)(u_l)$, where*

$$\begin{aligned}\gamma_1 &= s_r-s_l,\\ \gamma_2 &= r_r-r_l.\end{aligned}$$

Proof. Indeed, without loss of generality, we assume that $\gamma_i>0$, $i=1,2$. By (9), we have

$$\Psi_2(\gamma_2)\circ\Psi_1(\gamma_1)(r_l,s_l)=\Psi_2(\gamma_1)(r_l,\gamma_1+s_l)=(\gamma_2+r_l,\gamma_1+s_l)=(r_r,s_r).$$

□

Since the intermediary state is defined by $u_m=\Psi_1(\gamma_1)(r_l,s_l)$, we can solve the Riemann problem using at most two elementary waves. The form of this weak solution depends of γ_1.

When $\gamma_1<0$, we have

$$u(x,t)\equiv\begin{cases} u_l, & \frac{x}{t}\in(-\infty,s_1),\\ u_m=\Psi_1(\gamma_1)(u_l), & \frac{x}{t}\in[s_1,s_2),\\ u_r, & \frac{x}{t}\in[s_2,\infty),\end{cases}$$

where s_1 is the speed of the 1-shock and s_2 is the speed of the contact discontinuity.

When $\gamma_1>0$, we have

$$u(x,t)\equiv\begin{cases} u_l, & \frac{x}{t}\in(-\infty,\lambda_1(u_l)),\\ R_1(\sigma)(u_l), & \frac{x}{t}\in[\lambda_1(u_l),\lambda_1(u_r)],\\ u_m=\Psi_1(\gamma_1)(u_l), & \frac{x}{t}\in[\lambda_1(u_r),s_2),\\ u_r, & \frac{x}{t}\in[s_2,\infty),\end{cases}$$

where σ is given implicitly by $\frac{x}{t}=\lambda_1(\Psi(R_1(\sigma)(u_l))$, when $\frac{x}{t}\in[\lambda_1(u_l),\lambda_1(u_r)]$.

When u_l, $u_r\in D_2$, we note that $u(x,t)$ is in D_2 because the j-curves are parallel to coodenate axes, $j=1,2$.

5. Existence of solution by Glimm method

We consider the Cauchy problem (1) with initial data $u_0(x) = (r_0(x), s_0(x))$, where $u_0(x)$ is in D_2 for each $x \in \mathbb{R}$. We assume that

$$(\rho_0(x), v_0(x)) = \Psi(r_0(x), s_0(x)),$$

where $\rho_0(x)$, $v_0(x)$ are bounded functions with bounded total variation and $\rho_0(x) \geq \underline{\rho} > 0$ and $v_0(x) \geq 0$. We write $u_0(-\infty) = p_0 \in D_2$.

Let $h > 0$, $k > 0$ be mesh length satisfying the stability condition

$$\frac{h}{k} = K > \sup_{(\rho,v)\in D_1} |\lambda_i(\rho, v)|.$$

$i = 1, 2$, where K is constant.

We define the set

$$Y \equiv \{(m, n) \in \mathbb{Z}; m + n \equiv 0(\text{mod } 2), n \geq 0\}.$$

The half plane $\mathbb{R} \times \mathbb{R}_+$ is divided by a countable number of rectangles shaped domains defined by vertices

$$(m - 1, n), \quad (m + 1, n), \quad (m + 1, n + 1), \quad (m - 1, n + 1).$$

where $(m, n) \in Y$.

Let $\theta = \{\theta_n\}$ be a sequence of random numbers uniformly distributed in $(-1, 1)$, we may assume that θ is an equidistributed sequence in $(-1, 1)$, (see [3], [5]).

We consider an approximation of the initial data by

$$u^{(m,0)}(x, 0) \equiv \begin{cases} u_0((m - 1 + \theta_0)h), & (m - 1)h \leq x < mh, \\ u_0((m + 1 + \theta_0)h), & mh \leq x < (m + 1)h, \end{cases}$$

where $(m, 0) \in Y$.

Now, for each $(m, 0) \in Y$ fixed, we consider the initial data

$$v(x, 0) = \begin{cases} u_0((m - 1 + \theta_0)h), & x < mh, \\ u_0((m + 1 + \theta_0)h), & mh \leq x. \end{cases} \tag{10}$$

Let $v^{(m,0)}$ be the solution of the Riemann problem (1) and (10), constructed as in the previous section.

For each $(m, 0) \in Y$, we define

$$u_1(x, t) \equiv v^{(m,0)}(x, t), \quad (x, t) \in [(m - 1)h, (m + 1)h] \times [0, k).$$

In view of our stability condition $\frac{h}{k} = K$, the waves don't interact with each other across the lines $x = (m - 1)h$, $\forall$ m such that $(m, 0) \in Y$.

Hence u_1 is a function defined in the strip $0 \leq t < k$, constant across the lines $x = (m-1)h$, $\forall$ m such that $(m, 0) \in Y$, and continuous on the right. Thus u_1 is a weak solution in the strip. Moreover, $u_1(x, t)$ is in D_2, $\forall$ $(x, t) \in \mathbb{R} \times [0, k)$

Inductively, we assume that u_n had been defined in the strip $\mathbb{R} \times [(n - 1)h, nk)$. So, for each $(m, n) \in Y$, we consider the inital data

$$v(x, n\Delta t) = \begin{cases} u_n((m - 1 + \theta_n)h, nk^-), & x < mh, \\ u_n((m + 1 + \theta_n)h, nk^-), & mh \leq x, \end{cases} \tag{11}$$

where $nk^- = \lim_{t\to nk^-} t$. Let $v^{(m,n)}$ be the solution of the Riemann problem (1) and (11), constructed as in the previous section.

For each $(m,n) \in Y$, we define

$$u_{n+1}(x,t) \equiv v^{(m,n)}(x,t), \quad (x,t) \in [(m-1)h,(m+1)h] \times [n,(n+1)k).$$

Hence u_{n+1} is a function defined in the strip $n \le t < (n+1)k$, constant across the lines $x = (m-1)h$ and $x = (m+1)h$, $\forall\, m$ such that $(m,n) \in Y$, and continuous on the right. So, u_{n+1} is a weak solution in the strip. Since $u_{n+1}(x,t)$ is in D_2, $\forall\, (x,t) \in \mathbb{R} \times [n,(n+1)k)$, this process can be continued indefinitely.

Thus we can construct a global approximate solution $u_{h,\theta}$ defining

$$u_{h,\theta}(x,t) \equiv u_n(x,t), \quad (x,t) \in \mathbb{R} \times [n,(n+1)k).$$

In order to prove the convergence of the approximate solutions, it is sufficient to show that

$$TV(u_{h,\theta}(.,t)) \le C_0, \quad \forall\, t > 0, \tag{12}$$

where C_0 is a constant chosen independente of h, θ.

To get such a bound, we have to estimate the magnitude of elementary waves in the solution to the Riemann problem (1) and (11). First, for each $(m,n) \in Y$, we define $a_{m,n} \equiv (mh + \theta_n h, nk)_{(m,n)\in Y}$, and we consider the diamond shaped domains $\Diamond_{m,n}$ defined by vertices:

$$N = a_{m,n+1}, \quad E = a_{m+1,n}, \quad S = a_{m,n-1}, \quad W = a_{m-1,n}.$$

which is called the interaction diamond centered at (mh, nk).

The elementary waves issuing from $((m-1)h,(n-1)k)$ and $((m+1)h,(n-1)k)$ and entering $\Diamond_{m,n}$, denoted by $\beta = (\beta_1,\beta_2)$ and $\gamma = (\gamma_1,\gamma_2)$, are called the incoming waves with respect to $\Diamond_{m,n}$. Also the elementary waves issuing from (mh, nk), denoted by $\alpha = (\alpha_1,\alpha_2)$, are called the outgoing waves.

We say that β and γ interact in $\Diamond_{m,n}$ and generate α. Let's denote by ϵ_j, $j = 1,2$, the magnitude of the j-wave in $\epsilon = (\epsilon_1,\epsilon_2)$. We denote by $|\epsilon_j|$ the strength of j-wave, $j = 1,2$.

Lemma 2. *Let β, γ be incoming waves and α outgoing waves with respect to an interaction diamond $\Diamond_{m,n}$. Then it follows that*

$$|\alpha_1| + |\alpha_2| \le |\beta_1| + |\beta_2| + |\gamma_1| + |\gamma_2|.$$

Proof. We note that j-curves are parallel to the coordenate axis, so we have $|\alpha_j| \le |\beta_j| + |\gamma_j|$, $j = 1,2$. □

This lemma says that the variation of the approximate solution does not increase, when t increases. The estimate (12) is an easy consequence of this lemma.

We note that $u_{h,\theta}(-\infty,t) = u_0(-\infty)$, so the function $u_{h,\theta}$ satisfies

$$\begin{cases} ||u_{h,\theta}||_{L^\infty} \le M, & \forall\, t \ge 0, \\ TV(u_{h,\theta}(.,t)) \le TV(u_0), & \forall\, t > 0, \end{cases}$$

where $M > 0$ is independent of h and t.

Lemma 3. *Let s, $t \ge 0$. Then*

$$||u_{h,\theta}(.,t) - u_{h,\theta}(.,s)||_{L^1} \le L(|t-s| + h),$$

where L is independent of s and t.

The proof is analogous as the corresponding result in [6], Corollary 19.8.

Following [6], for θ fixed. we fix a sequence $(h_l)_{l\geq 1}$ decreasing to zero. For each $l \geq 1$, we constructed an appoximate solution u_l using the Glimm scheme, satisfying

$$\begin{cases} u_l(-\infty,t) = p_0, & \forall\ t>0,\\ u_l(x,t)\in D_2, & \forall\ (x,t)\in \mathbb{R}\times[0,\infty),\\ TV(u_l(.,t)) \leq TV(u_0), & \forall\ t>0,\\ ||u_l(.,t)-u_l(.,s)||_{L^1} \leq L(|t-s|+k), & \forall\ s,\ t>0, \end{cases}$$

By convergence of the Glimm scheme, we have that there exist a subsequence $(u_{l_m}) \subset (u_{h_l})$ of approximate solutions satisfying

$$(U_{l_m,\theta})_t + F(U_{l_m,\theta})_x \to 0,$$

when $l_m \to \infty$, in the weak sense, employing Helly's Theorem, we conclude that $u_{l_m,\theta}$ converge in the weak sense to a weak solution u_θ.

6. Wave-front tracking method

For each $h \in \mathbb{N}$, the approximate solution $u^h(x,t)$ is constructed in the following way.

(1) First, we assume that $u_0(x)$ is a bounded function with bounded total variation. We define the function

$$u_0^h \equiv \begin{cases} u_0(-\infty), & x\in(-\infty,h),\\ u_0(-h+2^{-h}i), & x\in[-h+2^{-h}i,-h+2^{-h}(i+1)),\\ u_0(\infty), & x\in[h,\infty), \end{cases}$$

where $i = 0,1,2,...,2^{h+1}h-1$. So we have that

$$||u_0^h||_\infty \leq ||u_0||_\infty, \quad TV(u_0^h) \leq TV(u_0).$$

We note that $u_0^h(x) \to u_0(x)$ a.e. $x\in[-h,h]$. Since $|u_0^h(x)-u_0(x)| \leq 2||u_0||_\infty$, $\forall\ x\in\mathbb{R}$, using the Lebesgue convergence theorem, we obtain

$$||u_0^h - u_0||_{L^1_{loc}} \to 0,$$

when $h\to\infty$.

(2) Let $x_1 < ... < x_N$ be the points of discontinuity of u_0^h. At each x_m, we solve the Riemann problem setting

$$u_l = u_0(x_m^-) \equiv \lim_{x\to x_m^-} u_0(x), \quad u_r = u_0(x_m^+) \equiv \lim_{x\to x_m^+} u_0(x).$$

If the solution is composed only of shock waves, we adopt this piecewise constant solution itself. If it contains a centred rarefaction wave, we approximate it by several small fans consisting of constant states and jump discontinuities separating them. Specifically, if $\gamma_1 > 0$, we consider the integer

$$p_1 \equiv 1 + [\gamma_1 h],$$

where $[w]$ denotes the integer part of w, i.e., the largest integer $\leq w$. For $k = 0, 1, ..., p_1$, we define

$$\omega_{1,k} \equiv \Psi_1 \left(k \frac{\gamma_1}{p_1} \right) (u_l).$$

We approximate the centred j-rarefaction wave by constant states $\omega_{1,k}$, where the states $\omega_{1,k}$ and $\omega_{1,k+1}$ will be separated by a discontinuity moving with velocity $\lambda_1(\Psi(\omega_{1,k+1}))$. We thus construct an approximate solution $u^h(x,t)$ composed of piecewise constant functions.

(3) $u^h(x,t)$ is constructed until a pair of neighbouring jump discontinuities interact. If they interact at $t = t_1$, we construct the approximate solution by solving the Riemann problem with initial data $u^h(x, t_1^-)$. Here, we may assume, changing the speed of shock waves by $O(1)h^{-1}$, there are only two incoming waves at every interaction point.

(4) We can repeat the above construction as long as the number of jump discontinuities does not diverge within a finite time.

(5) Our construction has no breakdown. The interaction between two wave fronts does not generate new waves, since the shock and rarefaction curves coincide, so we have no non-physical waves.

By previous observations, the approximate solution u^h is constructed for all $t \in [0, \infty)$. The total number of fronts is finite. The function $u^h(x,t)$ takes values in D_2.

Now, we need to show that

$$TV(u^h(.,t)) \leq TV(u_0).$$

For this end, we need to estimate the change in the values of $TV(u^h(.,t))$ across time τ where two fronts interact.

Let β denote a 1-shock wave, let o denote a 1-rarefaction wave and let γ denote the contact discontinuity.

The local interaction estimate are obtained in the following way.

Lemma 4. *Possible types of interaction between two waves and the solutions of the corresponding Riemann problems are*

(1) $\gamma + \beta \to \beta + \gamma$,
(2) $\gamma + o \to o + \gamma$,
(3) $\beta + o(o + \beta) \to \zeta = \beta + o$,

where ζ can be 1-shock wave or 1-rarefaction wave depending on the signal that it takes.

Let $t > 0$ be a time fixed, where no interaction occurs. We define the functional

$$V(t) \equiv \sum |\alpha|,$$

where the summation ranges over all fronts in $u^h(.,t)$.

Let $\tau > 0$ be a time where two fronts interact. By Lemma 4, we have

$$\Delta V(\tau) = V(\tau^+) - V(\tau^-) = 0.$$

Thus, $TV(u^h(.,t)) \leq TV(u_0)$, $\forall\ t > 0$.

Following [2], for each $h \in \mathbb{N}$, we can construct an approximate solution u^h by wave-front tracking method, satisfying

$$\begin{cases} u^h(-\infty, t) = p_0, & \forall\ t > 0, \\ u^h(x,t) \in D_2, & \forall\ (x,t) \in \mathbb{R} \times [0,\infty), \\ TV(u^h(.,t)) \leq TV(u_0), & \forall\ t > 0, \\ ||u^h(.,t) - u^h(.,s)||_{L^1} \leq L|t-s|, & \forall\ s,\, t > 0, \end{cases}$$

where L is independent of s and t. By Helly's Theorem, we have that there exists a subsequence $(u^{h_m}) \subset (u^h)$ of approximate solutions satisfying

$$(U_{h_m})_t + F(U_{h_m})_x \to 0,$$

when $h_m \to \infty$, in the weak sense. We conclude that u^{h_m} converge in the weak sense to a weak solution u.

7. Locally bounded variation

Now, we consider the initial data

$$u_0(x) = (\rho_0(x), v_0(x)), \quad x \in \mathbb{R},$$

where ρ_0, v_0 are bounded functions with locally bounded total variation and $\rho_0(x) \geq \underline{\rho} > 0$, $\forall\ x \in \mathbb{R}$.

Given $k \in \mathbb{N}$, we define u_0^k by

$$u_0^k(x) = u_0(j2^{-k}), \quad x \in [j2^{-k}, (j+1)2^{-k}),$$

where $j \in \mathbb{Z}$. We note that $u_0^k \to u_0$ in L^1_{loc}, when $k \to \infty$.

Let $N \in \mathbb{N}$ fixed. We consider the initial data

$$v_1(x) \equiv \begin{cases} u_0(-N), & x < -N, \\ u_0(x), & -N \leq x \leq N, \\ u_0(N), & x > N, \end{cases}$$

and the sequence of functions (v_0^k) defined by

$$v_0^k(x) \equiv \begin{cases} u_0^k(-N), & x < -N, \\ u_0^k(x), & -N \leq x \leq N, \\ u_0^k(N), & x > N. \end{cases}$$

We note that v_1 is a bounded function with bounded total variation satisfying

$$\begin{cases} ||v_1||_\infty \leq ||u_0||_\infty, \\ TV(v_1) = TV_{[-N,N]}(u_0). \end{cases}$$

Moreover, the sequence of functions (v_0^k) satisfies

$$\begin{cases} ||v_0^k||_\infty \leq ||v_1||_\infty, \\ TV(v_0^k) = TV(v_1), \\ ||v_0^k - v_1||_{L^1_{loc}} \to 0. \end{cases}$$

Each function v_0^k, $k \in \mathbb{N}$, has only a number finite of discontinuities.

Using the wave-front tracking method, we contruct a sequence of approximate solutions $(v^{k,1})$, which converges in L^1_{loc} to the a function v^1, which is a weak solution to the Cauchy problem with initial data v_1 in $[-N, N]$.

Now, we consider $[-2N, 2N]$, the initial data

$$v_2(x) \equiv \begin{cases} u_0(-2N), & x < -2N, \\ u_0(x), & -2N \le x \le 2N, \\ u_0(2N), & x > 2N, \end{cases}$$

and the sequence of functions $(v_0^{k,1})$ defined by

$$v_0^{k,1}(x) \equiv \begin{cases} u_0^{k,1}(-2N), & x < -2N, \\ u_0^{k,1}(x), & -2N \le x \le 2N, \\ u_0^{k,1}(2N), & x > 2N, \end{cases}$$

where $(u_0^{k,1}) \subset (u_0^k)$ is found in the first application of the wave-front tracking.

We apply the wave-front tracking method, using the initial data v_2 and the sequence of functions $(v_0^{k,1})$. Hence, we obtain a sequence of approximate solutions $(v^{k,2})$, which converge to a function v^2, which is a weak solution to the Cauchy problem with initial data v_2 in $[-2N, 2N]$.

We note that $(v^{k,2})$ is a subsequence of $(v^{k,1})$ in $[-N, N]$

Let $\bar{v}$ be such that $|u_0(x)| < \bar{v}$, $\forall\ x \in \mathbb{R}$. We note that $v^2(x,t)$,

$$(x,t) \in \left\{(x,t); 0 \le t < \frac{N}{\bar{v}},\quad -N + \bar{v}t < x < N - \bar{v}t\right\},$$

is solely determined by the restriction of $v_2(x)$ on the interval $[-N, N]$, moreover $v_2(x) = v_1(x)\ \forall\ x \in [-N, N]$ (initial data), (finite domain of dependence property).

Thus $v^2(x,t) = v^1(x,t)$ a.e.

$$(x,t) \in \left\{(x,t); 0 \le t < \frac{N}{\bar{v}},\quad -N + \bar{v}t < x < N - \bar{v}t\right\}.$$

We can repeat the above arguments to construct a sequence of functions (v^n), continuous on the right, such that v^n is a weak solution to the Cauchy problem in the subset

$$\left\{(x,t); 0 \le t < n\frac{N}{\bar{v}},\quad -nN + \bar{v}t < x < nN - \bar{v}t\right\},$$

and $v^n(x,t) = v^{n-1}(x,t)$ a.e. in the subset

$$\left\{(x,t); 0 \le t < (n-1)\frac{N}{\bar{v}},\quad -(n-1)N + \bar{v}t < x < (n-1)N - \bar{v}t\right\},$$

$\forall\ n \in \mathbb{N}$.

We define the function

$$u(x,t) \equiv v^n(x,t),\quad -nN + \bar{v}t < x < nN - \bar{v}t,\quad 0 \le t < n\tfrac{N}{\bar{v}},$$

where $n \in \mathbb{N}$. The function u is a weak solution to the Cauchy problem with initial data u_0.

In analogous way, we can construct a weak solution to the Cauchy problem when the total variation of the initial data is locally bounded using the Glimm method.

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E-mail address: [1]ricaedem@dex.ufla.br

[1]Federal University of Lavras P. O. Box 3037 37200-000, Lavras-MG, Brazil

E-mail address: [2]dcik@dm.ufscar.br

[2]Federal University of Sao Carlos P. O. Box 676 13565-905, Sao Carlos-SP, Brazil