A GENERALIZATION OF BOCHNER'S EXTENSION THEOREM TO ROUGH TUBES

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Abstract. This paper generalizes Bochner's extension theorem to tubes $X + i\mathbb{R}^m$ where the set $X \subset \mathbb{R}^m$ is not necessarily a manifold.

1. Introduction and preliminaries

A classical theorem of Bochner [Bo] states that a holomorphic function defined on a tube $\mathcal{M} = \Omega + i\mathbb{R}^m$ in $\mathbb{C}^m$ for some domain $\Omega \subset \mathbb{R}^m$ extends as a holomorphic function to the convex hull $\mathrm{ch}(\mathcal{M}) = \mathrm{ch}(\Omega) + i\mathbb{R}^m$. A CR version of this extension theorem was proved by Kazlow in [K] and Boivin and Dwilewicz in [BD] (see also [B1], [Ko]). In these papers, the tube above is replaced by a tube CR manifold $\mathcal{M} = X + i\mathbb{R}^m$ where X is a connected submanifold of $\mathbb{R}^m$ of class at least C^2. It was then proved that for any h a CR function, there exists a function H holomorphic on the interior of the hull of $\mathcal{M}$ and which extends h. The study of extension of CR functions on tube manifolds with tempered growth in the tube direction was first considered in [B3] and has been generalized quite recently to rough tubes [HHdS].

In this work we investigate the validity of such extension phenomenon for tubes $\mathcal{M}$ when X is a more general set without assuming any growth restrictions, in particular, X and hence $\mathcal{M}$ may not be a manifold. We will need to describe the analogues of CR functions in this more general setup which we will address next. When $\mathcal{M} = X + i\mathbb{R}^m$ is a tube CR manifold, by the Baouendi-Treves approximation theorem ([BT1], [T], [B2], [BCH]), any continuous CR function h on $\mathcal{M}$ is locally the uniform

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limit of a sequence $\{h_j\}$ of entire functions on $\mathbb{C}^m$. In fact, in this tube situation, the approximation is valid semiglobally, in the sense that the sequence h_j converges uniformly to h on every compact subset $K \Subset \mathcal{M}$. This motivates the following definition for more general tubes:

Definition 1.1. *Let $X \subset \mathbb{R}^m$ be a connected set and let $\mathcal{M} = X + i\mathbb{R}^m$. We say a continuous function f on $\mathcal{M}$ is a generalized CR function if there is a sequence $\{f_j\}$ of entire functions on $\mathbb{C}^m$ that converges uniformly to f on every compact subset $K \Subset X + i\mathbb{R}^m$.*

Using an appropriate version of the Baouendi-Treves approximation theorem, it is clear that when $\mathcal{M} = X + i\mathbb{R}^m$ is a C^1 tube CR manifold, a continuous function is CR if and only if it is a generalized CR function. Generalized CR functions also coincide with solutions of systems of vector fields in tube structures that are not necessarily CR structures. Indeed, suppose $\varphi(t) = (\varphi_1(t), \ldots, \varphi_m(t))$ is a C^1 function defined on a domain $U \subset \mathbb{R}^n$. Let $X = \varphi(U)$. Consider the n tube vector fields

$$L_j = \frac{\partial}{\partial t_j} - i \sum_{k=1}^{m} \frac{\partial \varphi_k}{\partial t_j}(t) \frac{\partial}{\partial x_k}, \quad 1 \leq j \leq n.$$

It is easy to see that any generalized CR function h on $\mathcal{M} = X + i\mathbb{R}^m$ is a weak solution of the system of equations $L_j h = 0$. Conversely, if h is a continuous function on $\mathcal{M}$ which is a weak solution of the system $L_j h = 0$, then the Baouendi-Treves approximation scheme can be used to show that h is a generalized CR function on $\mathcal{M}$. When the set X is more general, there may be no vector fields associated to the tube over X and so the generalized CR functions may not be solutions of equations. For tubes $\mathcal{M} = X + i\mathbb{R}^m$ over some special sets X which are not necessarily submanifolds of $\mathbb{R}^m$, other extensions of the notion of CR function can be found in the literature. For instance, for starshaped X a notion of CR functions was given in [K] when f is defined on $\mathcal{M} = X + i\mathbb{R}^m$ under the assumption that it is smooth in the sense of Whitney. Similarly, in [BD], the case in which X is a simplex and f is just continuous was considered. It is not hard to see that for these special choices of X, a continuous function is a generalized CR function in the sense of Definition 1.1 if and only if it is a CR function in the sense given in previous works.

We denote by $\mathrm{ch}(X)$ the convex hull of X, which is the intersection of all convex sets that contain X. Then $\mathrm{ch}(X)$ is contained in the intersection of all affine subspaces that contain X, which is itself an affine subspace Π. We will denote by $\mathrm{Int}\,\mathrm{ch}(X)$ the interior of $\mathrm{ch}(X)$ relative to Π.

In this paper we will extend the main result in [BD] to tube sets $\mathcal{M} = X + i\mathbb{R}^m$ such that $X \subset \mathbb{R}^m$ is rectifiably connected, that is, for every pair of points $x_0, x_1 \in X$, there is a map $\gamma : [0,1] \to \mathbb{R}^m$ such that $\gamma(0) = x_0$, $\gamma(1) = x_1$ and $\gamma([0,1]) \subset X$ has a finite length.

The tools used in [BD] when X is a manifold include Tumanov's theorem (on holomorphic extendability to wedges of CR functions at minimal points), and the edge-of-the-wedge theorem of Airapetjan and Henkin [AH]. These methods require that X be a manifold and that it be of regularity at least of class C^2. For example, it is known (see [R] and [S]) that even the classical edge-of-the-wedge theorem may fail when the edge is only Lipschitz. Even when X is a manifold, we believe that our approach is simpler than the methods employed in [BD].

Section 2 contains the proof of the main result. In section 3 we present a refinement of some of Kazlow's results in [K]. Section 4 contains an application of our main theorem. In Section 5 we state and prove a second extension result that implies the classical Bochner's extension theorem.

2. Statement and proof of the main result

The linear biholomorphism of $\mathbb{C}^m$ given by $z \mapsto iz$ maps a tube $\mathcal{M} = A + i\mathbb{R}^m$ onto a "horizontal" tube $\mathbb{R}^m + iA$ with real tube direction, which is better positioned for application of the FBI transform. In what follows, for any set $A \subset \mathbb{R}^m$, the notation $T(A)$ will denote the tube $\mathbb{R}^m + iA$ and refer to $T(A)$ as the tube over A. We will also consider truncated tubes $T_r(A) \doteq B_r(0) + iA$, $r > 0$. If $z_0 = x_0 + iy_0 \in T(X)$ we will be interested in approaching nontangentially the point z_0 by points in $T(\mathrm{Int}\,\mathrm{ch}(X))$. By a truncated cone $\Gamma_{y_0} \subset \mathrm{Int}\,\mathrm{ch}(X)$ we will mean a set of the form

$$\Gamma_{y_0} = \{y_0 + v \in \Pi : \quad v \in \Gamma,\ |v| < \delta\}$$

where $\Gamma \subset \mathbb{R}^m$ is a convex open cone with vertex at the origin, Π is the affine space spanned by X and $\delta > 0$. The main result of this paper is as follows:

Theorem 2.1. *Assume that $X \subset \mathbb{R}^m$ is arcwise connected by rectifiable arcs and let $f(z)$ be a continuous generalized CR function on $T(X)$. Then $f(z)$ can be extended to a function $F(z)$ defined on the tube over $X \cup \operatorname{Int}\operatorname{ch}(X)$ that is a continuous generalized CR function on $T\big(\operatorname{Int}\operatorname{ch}(X)\big)$ and nontangentially continuous at $T(X)$ in the following sense: for every point $y_0 \in X$ there is a truncated cone $\Gamma_{y_0} \subset \operatorname{Int}\operatorname{ch}(X)$ with vertex at y_0 such that the restriction of $F(z)$ to $\mathbb{R}^m + i\overline{\Gamma}_{y_0}$ is continuous at $x + iy_0$ for every $x \in \mathbb{R}^m$.*

Note that when $\operatorname{Int}\operatorname{ch}(X)$ is an open set of $\mathbb{R}^m$ the extension $F(z)$ will be holomorphic on $\mathrm{T}(\operatorname{Int}\operatorname{ch}(X))$. Let Π be the affine space spanned by X. The proof of Theorem 2.1 will be split into two cases depending on whether the dimension of Π is m (i.e., $\Pi = \mathbb{R}^m$) or the dimension of Π is $\ell < m$ (i.e., Π is a proper affine subspace of $\mathbb{R}^m$).

2.1. Case 1: $\mathbf{\Pi = \mathbb{R}^m}$.

In this case we assume that the convex hull of X has nonempty interior in $\mathbb{R}^m$ so a generalized CR function on $T\big(\operatorname{Int}\operatorname{ch}(X)\big)$ is just a holomorphic function on $T\big(\operatorname{Int}\operatorname{ch}(X)\big)$.

We recall that a rectifiable curve in $\mathbb{R}^m$ joining the points a and b is a continuous function of bounded variation $\gamma : [0,1] \to \mathbb{R}^m$ such that $\gamma(0) = a$ and $\gamma(1) = b$. The weak derivative of γ is a vector valued measure $\gamma' = \mu$ and the integration by parts formula

$$\int_{[0,1]} \psi(t) \cdot d\mu = \psi(1) \cdot \gamma(1) - \psi(0) \cdot \gamma(0) - \int_0^1 \psi'(t) \cdot \gamma(t)\, dt$$

holds if, say, $\psi \in C^1([0,1]; \mathbb{R}^m)$. For $\psi \equiv e_j = (0, 0, \ldots, 1, \ldots, 0)$ one gets

$$\int_{[0,1]} e_j \cdot d\mu = e_j \cdot (\gamma(1) - \gamma(0)),$$

that implies the fundamental theorem of calculus

$$\int_{[0,1]} d\mu = \gamma(1) - \gamma(0).$$

If $F(x)$ is a differentiable complex-valued function defined in a neighborhood of $\gamma([0,1])$, $\tilde{\gamma} = F \circ \gamma$ is a rectifiable curve in $\mathbb{C}$ with $\tilde{\gamma}' = \tilde{\mu} = (F' \circ \gamma) \cdot \mu$ that joins $F(\gamma(0))$ to $F(\gamma(1))$. Hence,

$$\int_{[0,1]} (F' \circ \gamma) \cdot d\mu = F(\gamma(1)) - F(\gamma(0)). \tag{2.1}$$

For $r > 0$ and $K \subseteq X$, let $A(T_r(K))$ denote the space of continuous functions h on $T_r(K) = B_r(0) + iK$ with the property that for each compact subset $S \Subset T_r(K)$, there is a sequence of holomorphic functions h_j defined on a neighborhood of S in $\mathbb{C}^m$ that converge uniformly to h on S.

Lemma 2.1. *Let $X \subset \mathbb{R}^m$ contain the origin. Assume $y^* \in X$ satisfies $\xi^0 \cdot y* < 0$ and there exists a rectifiable curve γ contained in X (i.e., $\gamma([0,1]) \subset X$) joining 0 to y^*. Then there exists $r_0 > 0$ and a compact set $K \Subset X$ such that if $r \geq r_0$ and $f \in A(T_{3r}(K))$, $(0, \xi^0)$ is not in the analytic wave-front set of $f_0(x) = f(x + i0)$.*

Proof: We will assume without loss of generality that $|\xi^0| = 1$. It will be enough to show that for some $\psi(x) \in C_c^\infty(B_{2r}(0))$ with $\psi(0) \neq 0$, the FBI transform of the compactly supported function $u = \psi(x) f_0(x)$ given by

$$F_u(x, \xi) = \int_{\mathbb{R}^m} e^{\,i(x-x')\cdot\xi - \kappa|\xi||x-x'|^2} u(x')\, dx'$$

satisfies the following decay condition: there is a neighborhood V of $0 \in \mathbb{R}^m$, an open cone $\Gamma \subset \mathbb{R}^m \backslash 0$, $\xi^0 \in \Gamma$ and constants $\kappa, c_1, c_2 > 0$ such that

$$|F_u(x, \xi)| \leq c_1 e^{-c_2|\xi|} \quad \forall (x, \xi) \in V \times \Gamma. \tag{2.2}$$

Let $K = \gamma([0,1])$ and fix $f \in A(T_{3r}(K))$. Let $f_j(z)$ be one of the holomorphic functions that approximates $f(z)$ on $T_{3r}(K)$ and, keeping

x and ξ fixed, consider the function of y

$$\begin{aligned} F_j(y) &= \int_{\mathbb{R}^m} e^{i(x-x'-iy)\cdot\xi-\kappa|\xi|[x-x'-iy]^2} f_j(x'+iy)\psi(x')\,dx' \\ &= \int_{\mathbb{R}^m} E(x,x'+iy,\xi)f_j(x'+iy)\psi(x')\,dx' \end{aligned} \tag{2.3}$$

where we have written $[\zeta]^2 \doteq \zeta_1^2 + \cdots + \zeta_m^2$ for $\zeta = (\zeta_1, \ldots, \zeta_m) \in \mathbb{C}^m$. Apply (2.1) to F_j where γ is the rectifiable curve that joins the origin to y^* to get

$$-F_j(0) = \int_{[0,1]} (F_j' \circ \gamma) \cdot d\mu - F_j(y^*). \tag{2.4}$$

Note that as $j \to \infty$, $F_j(0) \to F_u(x,\xi)$, so it is enough to estimate both terms on the right hand side of (2.4) uniformly in $j \in \mathbb{N}$. To estimate $|F_j(y^*)|$ for $|x| \le r$ and $\xi/|\xi|$ close to ξ^0 we observe that

$$\begin{aligned} |F_j(y^*)| \le & M_j \int_{\operatorname{supp}\psi} e^{|\xi|(y^*\cdot\xi/|\xi|)-\kappa|\xi|(|x-x'|^2-|y^*|^2)}\,dx', \\ M_j =& \sup\left\{|f_j(x'+iy^*)| :\ |x'| \le 2r,\ x' \in \operatorname{supp}\psi\right\}. \end{aligned}$$

By the choice of ξ^0, we may find a sufficiently thin cone Γ around ξ^0 such that $y^*\cdot\xi/|\xi| \le -2c_2 < 0$ if $\xi \in \Gamma$, for some $c_2 > 0$ that depends on ξ^0 alone. Next we choose $\kappa > 0$ sufficiently small so that $\kappa|y^*|^2 < c_2$. With these choices we get

$$|F_j(y^*)| \le C_\psi\, e^{-c_2|\xi|}, \quad |x| \le 1,\ \xi \in \Gamma, \tag{2.5}$$

since $\sup_j M_j < \infty$ because $f_j(z)$ converges uniformly on compact subsets of $T_{3r}(K)$. Let us now estimate

$$\left|\int_{[0,1]} (F_j' \circ \gamma) \cdot d\mu\right| \le |\mu|([0,1]) \sup_{0\le t\le 1} |F_j'(\gamma(t))|. \tag{2.6}$$

To compute $F_j'(y)$ we differentiate (2.3) under the integral sign with respect to y_k and note that $x'+iy \mapsto E(x,x'+iy,\xi)f_j(x'+iy)$ is holomorphic, which allows us to express its derivative with respect to y_k in terms of a derivative with respect to x_k'. Integrating by parts we get

$$\frac{\partial F_j}{\partial y_k}(y) = i\int_{\mathbb{R}^m} E(x,x'+iy,\xi)f_j(x'+iy)\frac{\partial\psi}{\partial x_k'}(x')\,dx'. \tag{2.7}$$

We now choose $\psi(x')$ supported in $|x'| < \frac{5r}{2}$ such that $\psi(x') = 1$ for $|x'| \le 2r$, with a large $r > 0$ to be determined. Assume that $|x| \le r$. Since the integrand on the right hand side of (2.7) vanishes for $|x'| \le 2r$, we may assume that $r \le |x' - x| \le 3r$ in (2.7) so

$$\left|\frac{\partial F_j}{\partial y_k}(\gamma(t))\right| \le C_\psi\, N_j\, e^{-|\xi|(\kappa r^2 - L - \kappa L^2))}$$

where $N_j = \sup\{|f_j(x' + i\gamma(t))| : |x'| \le 3r,\, t \in [0,1]\}$ and $L = \sup_{0\le t\le 1} |\gamma(t)|$. We now choose $r > 0$ large enough to guarantee that $\kappa r^2 - L - \kappa L^2 > c_2$ and as before, the uniform convergence of $f_j(z)$ on compact subsets of $T_{3r}(K)$ makes it clear that $\sup_j N_j < \infty$. Then (2.6) gives

$$\left|\int_{[0,1]} (F_j' \circ \gamma) \cdot d\mu\right| \le C_\psi\, e^{-c_2|\xi|}, \quad |x| \le \frac{r}{2},$$

which in view of (2.4) and (2.5) proves (2.2). □

Remark 2.1: The proof of Lemma 2.1 was adapted from the proof of [BT2, Thm. 1.1]

We now consider a set X, $0 \in X \subset \mathbb{R}^m$ with the following property: any two points in X can be joined by a rectifiable curve contained in X. We will also assume that $\operatorname{Int}\operatorname{ch}(X)$, the interior relative to $\mathbb{R}^m$ of the convex hull of X, is not empty.

Lemma 2.2. *Let X be as above and choose a point $0 \ne p \in \operatorname{Int}\operatorname{ch}(X)$. Then there exist $\delta, r > 0$, a neighborhood ω of the origin $0 \in \mathbb{R}^m$ and an open cone $\Gamma \subset \mathbb{R}^m$ with axis the ray through 0 and p, and a compact set $K \subseteq X$ such that every function h in $A(T_r(K))$ extends as a holomorphic function $H(z)$ to the wedge $\mathcal{W} = \omega + i\Gamma_\delta$, where $\Gamma_\delta = \{v \in \Gamma : |v| < \delta\}$. The extension is continuous up to the edge.*

Proof: Let $p \in \operatorname{Int}\operatorname{ch}(X)$, and assume that for some $\rho > 0$, the open ball $B_{2\rho}(p)$ is contained in $\operatorname{Int}\operatorname{ch}(X)$. Let $\Gamma^{2\rho}$ be the truncated conical region that is the convex hull of the ball $B_{2\rho}(p)$ and the point $0 \in X$, that is,

$$\Gamma^{2\rho} = \{\lambda v :\, v \in B_{2\rho}(p),\, 0 < \lambda < 1\}.$$

The set $\Gamma^{2\rho}$ is contained in $\operatorname{Int}\operatorname{ch}(X)$. Let

$$\mathcal{C}^{2\rho} = \{\xi \in \mathbb{R}^m \setminus 0 : \ \xi \cdot \eta < 0 \text{ for some } \ \eta \in \Gamma^{2\rho}\}.$$

Note that $\mathcal{C}^{2\rho}$ is the complement of the polar $(\Gamma^{2\rho})^0$. We choose ρ small enough and pick a finite number of points $q_j \in X$, $j = 1, \dots, N$ with the property that every point v in the ball $B_{2\rho}(p)$ is a convex combination of the q_j's (*cf.* Lemma 5.1 in the appendix). It follows that for each $\xi \in \mathcal{C}^{2\rho}$, there is $j \in \{1, \dots, N\}$ such that

$$q_j \cdot \xi < 0. \tag{2.8}$$

Define

$$\Gamma^\rho = \{\lambda v : \ v \in B_\rho(p),\ 0 < \lambda < 1\},$$

and

$$\mathcal{C}^{\rho} = \{\xi \in \mathbb{R}^m \setminus 0 : \ \xi \cdot \eta < 0 \text{ for some } \ \eta \in \Gamma^{\rho}\}.$$

Observe that if S^{m-1} denotes the unit sphere in $\mathbb{R}^m$, we have

$$\overline{\mathcal{C}^\rho} \cap S^{m-1} \subset \mathcal{C}^{2\rho} \cap S^{m-1}.$$

By (2.8) and the compactness of $\overline{\mathcal{C}^\rho} \cap S^{m-1}$, we can find a finite number of open sets $U_1, \dots, U_k$ such that

$$\overline{\mathcal{C}^\rho} \cap S^{m-1} \subset \bigcup_{j=1}^{k} U_j,$$

and for each $j \in \{1, \dots, k\}$, there exists $j' \in \{1, \dots, N\}$ satisfying

$$q_{j'} \cdot \eta < 0 \quad \text{for all } \eta \in U_j.$$

According to our hypothesis, every such point $q_{j'}$ can be joined to the origin by a rectifiable curve $\gamma_{j'}$ contained in X. Set

$$K = \bigcup_{j=1}^{N} \gamma_j.$$

If $h \in A(T_r(K))$, by Lemma 2.1, $(0, \eta)$ cannot belong to the analytic wave front set of $h_0(x) = h(x + i0)$ for any $\eta \in \bigcup_{j=1}^{k} U_j$ provided $r > 0$ is large enough. Since $\mathcal{C}^\rho$ is the complement of the polar $(\Gamma^\rho)^0$, for such h, we have

$$WF_a h_0(x)|_0 \subset \{0\} \times (\Gamma^\rho)^0 .$$

Passing to a smaller $\rho > 0$, the same argument shows more: there is a neighborhood $\omega \subset \mathbb{R}^m$ of the origin such that for any generalized continuous CR function $h(z)$ defined on $T_r(X)$, the restriction $h_0(x) = h(x + i0)$ satisfies

$$WF_a h_0(x)|_\omega \subset \omega \times (\Gamma^\rho)^0 .$$

Since $(\Gamma^\rho)^0$ is a closed convex cone, by [BT2, Lemma 3.1], for every $0 \in \omega' \Subset \omega$, and every open, convex cone Γ containing $(\Gamma^\rho)^0$, there exists $\delta > 0$, and a holomorphic function $H(z)$ defined on the wedge

$$\mathcal{W} = \omega' + i\,\mathrm{Int}(\Gamma^0_\delta), \quad \Gamma^0_\delta = \{v \in \Gamma^0 :\ |v| < \delta\}$$

with tempered growth at the edge such that $bH(x) = h_0(x)$, $x \in \omega'$. By [BER, Theorem 7.2.6] $H(z)$ is continuous up to the edge $\omega' + i0$ of $\mathcal{W}$. Fix once for all a cone Γ containing $(\Gamma^\rho)^0$. Set $\xi^0 = p/|p|$. Since $p \in \Gamma^\rho$ it is clear that $\xi^0 \in (\Gamma^\rho)^0 \subset \Gamma$. Although the aperture of Γ^0_δ is independent of the CR function h, δ and ω' might depend on h. We now show that shrinking $\delta > 0$ and $0 \in \omega' \subset \mathbb{R}^m$ we may choose $\mathcal{W}$ independent of h. We denote by $\mathcal{A}(\mathcal{W})$ the space of holomorphic functions on $\mathcal{W}$ which possess a boundary value at the edge and consider the space Y of functions in $A(T_r(K))$ with the topology of uniform convergence on compact subsets of $T_r(K)$. For $k = 1, 2, \dots$, define

$$Y_k = \{h \in Y :\ h(x + i0) = bH(x),\ H \in \mathcal{A}(\mathcal{W}_k),\ \|H\|_{L^\infty(\mathcal{W}_k)} \le k\},$$

where $\mathcal{W}_k$ is the wedge $B_{1/k}(0) + i\Gamma^0_k$, with $\Gamma^0_k = \{v \in \Gamma^0 :\ |v| < \frac{1}{k}\}$. Observe that

$$Y = \bigcup_{k=1}^{\infty} Y_k.$$

Each Y_k is closed, convex and stable under multiplication by complex numbers c, $|c| \le 1$. To see this, suppose h_j is a sequence in Y_k that converges to some h in Y. Let H_j denote the extension of $h_j(x + i0)$ to $\mathcal{W}_k$. By Montel's theorem, there is a subsequence $\{H_{j_\nu}\}$ which converges uniformly on compact subsets of $\mathcal{W}_k$ to a holomorphic function H. Since H is a bounded holomorphic function on $\mathcal{W}_k$, it has a distribution boundary value bH on $B_{1/k}(0)$. We therefore only need to show that $bH = h$. Let $\psi \in C_0^\infty(B_{1/k})$ and $y \in \Gamma^0_k$. If $p(z)$ is holomorphic on

$\mathcal{W}_k$ and continuous up to the edge, then we have

$$\int p(x)\psi(x)\,dx = \int p(x+iy)\psi(x)\,dx + i\iint_{0<t<1} p(x+ity)\sum_{j=1}^{m} y_j \frac{\partial\psi}{\partial x_j}(x)\,dxdt.$$

This formula follows from applying Stokes theorem to the $m-$form

$$p(x+ity)\psi(x)\,dx_1\wedge\cdots\wedge dx_m$$

on the set

$$\{x+ity : x\in B_{1/k}(0),\ 0<t<1\}.$$

In particular, for each ν,

$$\int H_{j_\nu}(x)\psi(x)\,dx = \int H_{j_\nu}(x+iy)\psi(x)\,dx + i\iint_{0<t<1} H_{j_\nu}(x+ity)\sum_{j=1}^{m} y_j \frac{\partial\psi}{\partial x_j}(x)\,dxdt.$$

Since the sequence $\{H_{j_\nu}(x+iy)\}$ is bounded and converges to $H(x+iy)$ on $\mathcal{W}_k$, letting $\nu\to\infty$ in the preceding equation, we get:

$$\int h(x)\psi(x)\,dx = \int H(x+iy)\psi(x)\,dx + i\iint_{0<t<1} H(x+ity)\sum_{j=1}^{m} y_j \frac{\partial\psi}{\partial x_j}(x)\,dxdt.$$

Since H is bounded, we conclude that

$$\int h(x)\psi(x)\,dx = \lim_{\Gamma_k\ni y\to 0}\int H(x+iy)\psi(x)\,dx.$$

This shows that the boundary value of H is h and hence Y_k is closed. Since Y is a Fréchet space, by Baire's category theorem, one of the Y_k's must have a nonempty interior and a standard argument shows that the origin is an interior point of that Y_k. □

Let X, $\mathcal{W}=\omega+i\Gamma_\delta$, $K\subset X$, and $r>0$ be as in Lemma 2.2. If $f\in A(T_r(K))$, there exists $F(z)\in\mathcal{A}(\mathcal{W})$ such that $bF(x)=f(x+i0)$, $x\in\omega$. We claim that

$$\sup_{\mathcal{W}}|F(z)| \le \sup_{|x|<r,\, y\in K}|f(x+iy)|. \tag{2.9}$$

Indeed, if for some $z_0 \in \mathcal{W}$, $|F(z_0)| > \sup_{|x|<r,\, y\in K} |f(x+iy)|$, the function

$$\frac{1}{f(x+iy) - F(z_0)}$$

would be an element of $A(T_r(K))$ that could not have a holomorphic extension to $\mathcal{W}$. Now, let $f_j(z)$ be a sequence of holomorphic functions that converges to $f(z)$ on compact subsets of $T(X)$. It follows from (2.9) that

$$\sup_{\mathcal{W}} |f_j(z) - f_k(z)| \leq \sup_{|x|<r,\, y\in K} |f_j(x+iy) - f_k(x+iy)|.$$

Since the sequence f_j converges uniformly on $\{|x| \leq r\} + iK$, we conclude that $f_j(z)$ converges uniformly on $\mathcal{W}$ up to the edge. If follows that $f_j(z)$ converges in $\mathcal{W}$ to the extension $F(z) \in \mathcal{A}(\mathcal{W})$ of $f(z)$. Fix now $x_0 \in \mathbb{R}^m$ and consider the generalized CR function on $T(X)$ given by $f_{x_0}(z) = f(z - x_0)$. Applying the previous reasoning to f_{x_0} we conclude that, for every $x_0 \in \mathbb{R}^m$, $f_j(z)$ converges uniformly on $x_0 + \mathcal{W}$. Note that $\bigcup_{x_0\in\mathbb{R}^m} x_0 + \mathcal{W} = \mathbb{R}^m + i\Gamma_\delta$. We have proved

Lemma 2.3. *Let $X \subset \mathbb{R}^m$ and $\mathcal{W} = \omega + i\Gamma_\delta$ be as in Lemma 2.2 and assume $f(z)$ is a continuous generalized* CR *function on $T(X)$. If $f_j(z)$ is a sequence of entire functions converging to f on compact subsets of $T(X)$, the tube $T(\Gamma_\delta) = \mathbb{R}^m + i\Gamma_\delta$ is contained in the domain of convergence of the sequence $(f_j(z))$. Furthermore, $f_j(z)$ converges in $T(\Gamma_\delta)$ to the extension of $f(z)$ to $T(\Gamma_\delta)$.*

End of the proof of Theorem 2.1 under Case 1: Given a point p in $\operatorname{Int}\operatorname{ch}(X)$ and choosing $\rho > 0$ small we have determined a conical region contained in $\operatorname{Int}\operatorname{ch}(X)$ with vertex at the origin $0 \in X$ such that the tube over a truncation of that conical region is inside the domain of convergence of any approximating sequence $f_j(z)$ that converges to some continuous generalized CR function $f(z)$ defined on $T(X)$. Of course, the same procedure can be applied to any other point $q \in X$. Consider the union of all truncated cones with vertices at $q \in X$ constructed in this manner. We obtain an open set $U \subset \operatorname{Int}\operatorname{ch}(X) \subset \mathbb{R}^m$ with the tube $T(U)$ contained in the domain of convergence of $f_j(z)$ and the limit of the sequence in $T(U)$ defines a holomorphic function $\tilde{f}(z) \in \mathcal{A}(T(U))$ that extends $f(z)$ in the sense that every point $q \in T(X)$ is in the edge of a wedge $\mathcal{W}_q$ and the boundary value of the restriction of $\tilde{f}$ to $\mathcal{W}_q$ is equal to the restriction of $f(z)$ to the edge of $\mathcal{W}_q$. By the classical Bochner's theorem (or by Corollary 5.2 in the appendix), $\tilde{f}$ can be extended to a holomorphic function $F(z)$ defined on the tube over the convex hull of U which is precisely the tube over $\operatorname{Int}\operatorname{ch}(X)$. $\square$

The proof of the case $\Pi = \mathbb{R}^m$ has been completed; however before tackling the next case, it will be useful to point out a consequence of what we have already proved. Consider the space $\mathcal{B}$ of functions $F(z)$ defined on the tube over $X \cup \operatorname{Int}\operatorname{ch}(X)$ which are holomorphic on $T\big(\operatorname{Int}\operatorname{ch}(X)\big)$ and continuous up to $T(X)$ in the sense given in Theorem 2.1. Hence, for every point $y_0 \in X$ there is a cone $\Gamma_{y_0} \subset \operatorname{Int}\operatorname{ch}(X)$ (that we fix) such that the restriction of $F(z)$ to $\mathbb{R}^m + i\overline{\Gamma}_{y_0}$ is continuous. We endow $\mathcal{B}$ with the topology of uniform convergence on sets of the form $K_1 + iK_2$ where $K_1 \Subset \mathbb{R}^m$ and either $K_2 \Subset X$ or $K_2 \Subset \operatorname{Int}\operatorname{ch}(X)$ or $K_2 \Subset \overline{\Gamma}_{y_0}$, $y_0 \in X$. With this topology $\mathcal{B}$ becomes a Fréchet space.

By restricting $F(z) \in \mathcal{B}$ we obtain an element of the space $\mathcal{G}$ of generalized CR functions on $T(X)$. The fact that the restriction is a generalized CR function can be seen by using the Baouendi-Treves approximation scheme as in section 4. The space $\mathcal{G}$ is a Fréchet space with the topology of uniform convergence over compact subsets of X and the restriction map is a continuous linear operator from $\mathcal{B}$ to $\mathcal{G}$. Then, a standard application of the open mapping theorem to the restriction map, which is onto by Case 1 and is one to one by uniqueness of the trace, shows that it has a continuous inverse. This implies the following

Lemma 2.4. *Let $K_1 \Subset \mathbb{R}^m$, let $y_0 \in X$ and let K_2 be either a compact subset of* $\operatorname{Int}\operatorname{ch}(X)$ *or else $K_2 = \overline{\Gamma}_{y_0}$. There exist a compact subset $\widehat{K} \Subset T(X)$ and a constant $C > 0$ such that*

$$\sup_{K_1 + iK_2} |F(z)| \leq C \sup_{\widehat{K}} |F(z)|, \quad F \in \mathcal{B}. \tag{2.10}$$

REMARK 2.2: The constant C in (2.10) may be taken equal to 1, as follows from applying (2.10) to powers $F^k(z)$, $k \in \mathbb{N}$, and letting $k \to \infty$.

2.2. **Case 2: $\Pi \neq \mathbb{R}^m$.** To avoid trivial cases we suppose X is not a single point. We may assume without loss of generality that $\Pi = \mathbb{R}^\ell \times \{0\}$, $1 \leq \ell < m$. Let $f(z)$ be a generalized CR function on $T(X)$ and let $f_j(z)$ be a sequence of entire functions on $\mathbb{C}^m$ that converges to $f(z)$ on compact subsets of $T(X)$. Write $z = (z', z'')$, $z' \in \mathbb{C}^\ell$, $z'' \in \mathbb{C}^{m-\ell}$, $x'' = (x_{\ell+1}, \ldots, x_m) \in \mathbb{R}^{m-\ell}$, and consider the functions

$$\tilde{f}_j(z'; x'') = f_j(z', x'' + i0)$$

which are entire holomorphic functions of z' and depend continuously on the variable x'' that may be regarded as a parameter. For fixed x'', the function $z' \mapsto \tilde{f}(z'; x'') = f(z', x'' + i0)$ is a generalized CR function on the tube $\mathbb{R}^\ell + iX \subset \mathbb{C}^\ell$ and since the convex hull of X has interior points relative to $\mathbb{R}^\ell$ we may apply to $\tilde{f}(z'; x'')$ the results of Case 1 (with ℓ in place of m). In particular, for every point $(y_0', 0) \in X$ there is a cone $\Gamma_{y_0'} \subset \operatorname{Int}\operatorname{ch}(X)$ as described in the proof of Case 1 and Lemma 2.4 holds. Let $K_1' \Subset \mathbb{R}^\ell$, let $K_1'' \Subset \mathbb{R}^{m-\ell}$ and let K_2 be either a compact subset of $\operatorname{Int}\operatorname{ch}(X)$ or $K_2 = \overline{\Gamma}_{y_0'}$. For fixed $x'' \in K_1''$, apply (2.10) to the entire function $z' \mapsto f_j(z', x'') - f_k(z', x'')$ to get, for some $\widehat{K}_1 \Subset \mathbb{R}^\ell + iX$,

$$\begin{aligned}\sup_{z' \in K_1' + iK_2} |(f_j - f_k)(z', x'')| &\leq \sup_{z' \in \widehat{K}_1} |(f_j - f_k)(z', x'')| \\ &\leq \sup_{x'' \in K_1''} \sup_{z' \in \widehat{K}_1} |(f_j - f_k)(z', x'')| \doteq \epsilon_{jk}.\end{aligned}$$

Since $f_j(z)$ converges uniformly over compact subsets of $T(X)$ we see that $\lim_{j,k\to\infty} \epsilon_{jk} = 0$ and this implies that $f_j(z', x'')$ is a uniform Cauchy sequence on $K_1' + iK_2$. It follows that the sequence $(f_j(z))$ converges uniformly over compact subsets of the tube $T\big(\operatorname{Int}\operatorname{ch}(X)\big)$ to a generalized CR function $F(z)$ that extends $f(z)$. Furthermore, taking K_2 of the form $K_2 = \overline{\Gamma}_{y_0'}$ we see that $F(z)$ has the required nontangential continuity property at points of $T(X)$. □

3. Tubes over starshaped sets

Assume that $X \subset \mathbb{R}^m$ is starshaped from the origin (i.e. $x \in X \implies \epsilon x \in X$ for $0 \leq \epsilon \leq 1$) with nonempty interior. Given $r > 0$, we consider the truncated tube $T_r(X)$ over X. Using that X is starshaped we will be able to obtain in this case a sharper version of Lemma 2.2 for CR functions defined on arbitrarily short tubes $T_r(X)$. The following lemma generalizes Kazlow's [K] results. A similar result was proved in [K] when X is a a manifold satisfying some additional properties (see the note after the proof of Lemma 9.3 in [K]).

Lemma 3.1. *Let X be starshaped and assume there is a point $y^* \neq 0$ in $\operatorname{Int}\operatorname{ch}(X)$. For any $r > 0$ there exist $\delta > 0$, a neighborhood ω of the origin $0 \in \mathbb{R}^m$ and an open cone $\Gamma \subset \mathbb{R}^m$ with axis the ray through 0 and y^* such that every generalized CR function $f(z)$ defined on $T_r(X)$ extends as a holomorphic function $F(z)$ to the wedge $\mathcal{W} = \omega + i\Gamma_\delta$, where $\Gamma_\delta = \{v \in \Gamma : |v| < \delta\}$. The extension is continuous up to the edge.*

Proof: The proof will be obtained by combining the proof of Lemma 2.2 with a variation of the proof of Lemma 2.1. Choose $\rho > 0$ sufficiently small, and points $x_1, \ldots, x_N \in X$ such that every point in the ball $B_{2\rho}(y^*) \subset \operatorname{ch}(\mathrm{X})$ can be written as a convex combination of the points $x_1, \ldots, x_N$. For $0 < \epsilon < 1$ set

$$\Gamma(\epsilon) = \{\lambda v : v \in B_\rho(y^*),\, 0 < \lambda < \epsilon\},$$

Assume that for some fixed $0 < \epsilon < 1$ we can show that for any continuous generalized CR function $f(z)$ on the truncated tube $T_r(X)$, the analytic wave front set of the restriction $f_0(x) = f(x + i0)$ satisfies

$$WF_a f_0(x)|_0 \subset \{0\} \times (\Gamma(\epsilon))^0 = \{0\} \times (\Gamma(1))^0 . \tag{3.1}$$

Then, as in Lemma 2.2, we will be able to conclude that there exists a truncated open cone Γ_δ with axis the ray through 0 and y^* and a neighborhood $\omega \subset \mathbb{R}^m$ of the origin such that $f_0(x)$ has a holomorphic extension to the wedge $\mathcal{W} = \omega + i\Gamma_\delta$ and $\mathcal{W}$ can be taken not to depend on $f(z)$. In order to show (3.1), it will be enough to show that if $|\xi^0| = 1$ and $\xi^0 \cdot \eta < 0$ for some $\eta \in \Gamma(\epsilon)$, then $(0, \xi^0)$ is not in the analytic wave-front set of $f_0(x)$. We will show that by proving that the FBI transform of $u = \psi(x) f_0(x)$ given by

$$F_u(x, \xi) = \int_{\mathbb{R}^m} e^{\, i(x-x')\cdot\xi - |\xi||x-x'|^2} u(x')\, dx' \tag{3.2}$$

has exponential decay on $V \times \Gamma_1$, where $\psi(x)$ is a cut-off function supported in $|x| < r$ with $\psi(0) \neq 0$ and Γ_1 is a thin cone around the ray $\{t\xi^0,\, t > 0\}$.

For $0 < \epsilon < 1$ we have that any point in the ball $\epsilon B_{2\rho}(y^*)$ can be written as a convex combination of the points $\epsilon x_1, \ldots, \epsilon x_N$ and so, if $\xi \in \Gamma_1$ where Γ_1 is a sufficiently thin cone around the ray along ξ^0, there exists $k \in \{1, \ldots N\}$ such that

$$\epsilon x_k \cdot \xi \leq -\epsilon c|\xi|$$

for some $c > 0$ independent of ϵ. Furthermore, ϵx_k can be joined to the origin by the segment $[0, \epsilon x_j] \subset X$ which has length $\leq C\epsilon$. To estimate (3.2) we proceed almost exactly as in the proof of Lemma 2.2 so we will be sketchy and restrict ourselves to mentioning the few points where the proof must be modified. If $f_j(z)$ is a member of the sequence of entire functions that approximates $f(z)$ on $T_r(X)$, we deform the integral over $\mathbb{R}^m$

$$I_j = \int_{\mathbb{R}^m} e^{\, i(x-x')\cdot\xi - |\xi||x-x'|^2} f_j(x')\psi(x')\, dx'$$

into $\mathbb{R}^m + i\epsilon x_k$ and write $I_j = I_{1,j} + I_{2,j}$ where

$$I_{1,j} = \int_{\mathbb{R}^m} e^{i(x-x'-i\epsilon x_k)\cdot\xi - |\xi|[x-x'-i\epsilon x_k]^2} f_j(x' + i\epsilon x_k)\psi(x')\, dx'$$

and

$$I_{2,j} = i\int_0^{\epsilon} ds \int_{\mathbb{R}^m} e^{i(x-x'-i\epsilon s x_k)\cdot\xi - |\xi|[x-x'-i\epsilon s x_k]^2} f_j(x' + i\epsilon s x_k)\frac{\partial\psi}{\partial x'_k}(x')\, dx'.$$

The main point is the control of the exponential terms in each one of the integrals. For $I_{1,j}$ the exponential is bounded by

$$e^{\epsilon x_k\cdot\xi + \epsilon^2|x_k|^2|\xi|} \leq e^{-\epsilon|\xi|(c-\epsilon C^2)}$$

so, for $\epsilon > 0$ small, if we let $\Gamma_1 \ni |\xi| \to \infty$ we will have that $|I_{1,j}| = O(e^{-c_2|\xi|})$. At this point we choose $\psi \in C_c^\infty(B_r(0))$ such that $\psi(x') \equiv 1$

for $|x'| \le r/2$, so assuming that $|x| < r/4$ we will have that $|x - x'| \ge r/2$ on the support of the integrand of $I_{2,j}$. Hence, the exponential in $I_{2,j}$ is bounded by

$$e^{-|\xi|(r^2/4 - C\epsilon - C^2\epsilon^2)}$$

which for $\epsilon > 0$ sufficiently small gives $|I_{2,j}| = O(e^{-c_2|\xi|})$ for some $c_2 > 0$. The terms that multiply the exponential terms are majorized uniformly in j as in Lemma 2.1. Thus, letting $j \to \infty$ we obtain an estimate

$$|F_u(x,\xi)| \le c_1 e^{-c_2|\xi|}, \quad |x| < r/4,\, \xi \in \Gamma_1.$$

This allows us to prove the validity of (3.1) and determine $\mathcal{W}$ such that $f(z)$ has a holomorphic extension $F(z)$ to $\mathcal{W}$ which is continuous up to the edge. We leave the details to the reader. □

4. Applications

Let $X_1, \ldots, X_k$ be C^1 (or Lipschitz) connected submanifolds of $\mathbb{R}^m$ of possibly varying dimensions such that $X_1 \cap \cdots \cap X_k \neq \emptyset$. Set $X = \bigcup_{i=1}^k X_i$ and assume that the interior of the convex hull of X is nonempty. Observe that each tube $\mathcal{M}_j = X_j + i\mathbb{R}^m$ is a CR submanifold of $\mathbb{C}^m$.

Lemma 4.1. *Let G be a continuous function on the set $\mathcal{M} = \bigcup_{j=1}^k \mathcal{M}_j$ which is CR on each tube $\mathcal{M}_j$, $j = 1, \ldots, k$. Then G extends to a holomorphic function on the tube* $\operatorname{Int}\operatorname{ch}(X) + i\mathbb{R}^m$.

Proof: By Theorem 2.1, we only need to show that G is a generalized CR function on the tube $\mathcal{M}$. This is the same as showing that there is a sequence of entire functions $E_j(z)$ that converges to $G(z)$ uniformly over compact sets of $\mathcal{M}_j$, $1 \le j \le k$. The construction of the sequence will not depend on j and we may reason with a fixed value of j, e.g., $j = 1$. Without loss of generality, assume that the origin is contained in each X_i. Choose a function $h(y) \in C_c^\infty(\mathbb{R}^m)$ satisfying $h(y) = 0$ for $|y| \ge 2$, $h(y) = 1$ for $|y| \le 1$ and set $h_r(y) \doteq h(y/r)$. For $\tau, r > 0$, using the terminology of the Baouendi-Treves approximation scheme, the approximation operator is

$$E_{\tau,r}G(z) = (\tau/\pi)^{m/2} \int_{\mathbb{R}^m} e^{\tau[z-i\eta]^2} G(0+i\eta)\, h_r(\eta)\, d\eta, \quad z \in \mathbb{C}^m, \tag{4.1}$$

while the modified approximation operator is

$$F_{\tau,r}G(x,y) = (\tau/\pi)^{m/2} \int_{\mathbb{R}^m} e^{-\tau|y-\eta|^2} G(x+i\eta)\, h_r(\eta)\, d\eta, \qquad x+iy \in \mathcal{M}. \tag{4.2}$$

In (4.1) we used the notation $[\zeta]^2 \doteq \zeta_1^2 + \cdots + \zeta_m^2$ for $\zeta = (\zeta_1, \ldots, \zeta_m) \in \mathbb{C}^m$. Since (4.2) defines an approximation of the identity (convolution with a Gaussian) it follows that, for fixed $x \in X$, $F_{\tau,r}G(x,y)$ converges uniformly to $G(x+iy)h_r(y)$ as $\tau \to \infty$. We will now study the convergence on

$$\mathcal{M}_1 = \{x+iy : x \in X_1\}.$$

Fix $z = x+iy \in \mathcal{M}_1$. Let $\psi : [0,1] \to X$ be a C^1 curve such that $\psi(0) = 0$ and $\psi(1) = x$. Consider the vector field on $[0,1] \times \mathbb{R}^m$ given by

$$L = \frac{\partial}{\partial s} + i \sum_{j=1}^m \psi_j'(s) \frac{\partial}{\partial \eta_j}.$$

Observe that if $g(s,\eta)$ is a Lipschitz function, then

$$dg(s,\eta) = (Lg)\, ds - i \sum_{j=1}^m \frac{\partial g}{\partial \eta_j} d(\psi_j(s) + i\eta_j), \tag{4.3}$$

as can be seen by applying both sides to the vector fields $\frac{\partial}{\partial s}$ and $\frac{\partial}{\partial \eta_j}$, $1 \le j \le m$. If $H(x+iy)$ is an entire function on $\mathbb{C}^m$, then

$$L(H(\psi(s)+i\eta) = 0 \text{ on } [0,1] \times \mathbb{R}^m. \tag{4.4}$$

Since G is a CR function, it follows that

$$L\left(e^{\tau(z-\psi(s)-i\eta)^2} G(\psi(s)+i\eta)\right) = 0. \tag{4.5}$$

Let $\gamma = \psi([0,1])$. By Stokes theorem we have:

$$F_{\tau,r}G(x,y) - E_{\tau,r}G(z)$$
$$= (\tau/\pi)^{m/2} \int_{\mathbb{R}^m} \int_\gamma d\left(e^{\tau(z-\psi(s)-i\eta)^2} G(\psi(s)+i\eta)\right) h_r(\eta)\, d\eta.$$

Using the latter, (4.5), and integration by parts, we get:

$$F_{\tau,r}G(x,y) - E_{\tau,r}G(z)$$

$$= i\sum_{j=1}^{m}(\tau/\pi)^{m/2}\int_{\mathbb{R}^m}\int_{\gamma} e^{\tau(z-\psi(s)-i\eta)^2}G(\psi(s)+i\eta)\psi_j'(s)\,\frac{\partial h_r(\eta)}{\partial \eta_j}\,ds\wedge d\eta.$$

Assume now that $z = x + iy$ varies in a compact set K in $\mathcal{M}_1$. We choose $r > 0$ so that $|x|+|y| \le r/2$ whenever $x+iy \in K$. We may also assume that the curves ψ are chosen for such x so that their derivatives are uniformly bounded. Since $\partial_{\eta_j} h_r(\eta) = 0$ for $|\eta| \le r$ and for $|\eta| \ge 2r$, we get constants $C, c > 0$ such that

$$|F_{\tau,r}G(x,y) - E_{\tau,r}G(z)| \le Ce^{-c\tau},$$

showing that the entire functions $E_{\tau,r}G(z)$ converge uniformly, as $\tau \to \infty$, to $G(z)h_r(y) = G(z)$ for $z \in K$, since $h_r(y) \equiv 1$ for $|y| \le 1$. Using a diagonal procedure, we may find sequences $r_j, \tau_j \to \infty$ such that $E_j(z) \doteq E_{\tau_j,r_j}G(z) \to G(z)$ uniformly on compact subsets of $\mathcal{M}_1$ or, more generally, on compact subsets of each $\mathcal{M}_j$, $j = 1,\dots,k$. $\square$

A simple special case of the preceding lemma is the following

Corollary 4.1. *Let $G(z_1, z_2, \dots, z_m)$ be a function of m variables $z_j = x_j + iy_j$, initially defined and continuous on the set where $x_j = 0$, $j = 1,\dots,m$. Suppose that for each fixed j, and fixed*

$$iy_1,\dots,iy_{j-1},iy_{j+1},\dots,iy_m,$$

the function

$$G(iy_1,\dots,iy_{j-1},z_j,iy_{j+1},\dots,iy_m)$$

as a function of z_j is holomorphically extendable into the strip $|x_j| < 1$. Then G extends as a holomorphic function on the tube

$$\{x+iy \in \mathbb{C}^m : \, |x_1|+|x_2|+\cdots+|x_m| < 1\}.$$

Proof: Let X be the union of the one-dimensional segments in $\mathbb{R}^m$ defined by

$$X = \{(x_1,0\dots,0)\} \cup \{(0,x_2,0,\dots,0)\} \cup \cdots \cup \{(0,\dots,0,x_m)\},$$

where each $x_j \in \mathbb{R}$ and $|x_j| < 1$. By the previous lemma G is a generalized CR function on the tube $\mathcal{M}$. Every pair of points in the set

X can be joined by a Lipschitz curve. Therefore, by our main theorem, the function G extends to a holomorphic function on the convex hull of $X + i\,\mathbb{R}^m$ which is exactly the set

$$\{x + iy \in \mathbb{C}^m : \, |x_1| + |x_2| + \cdots + |x_m| < 1\}.$$

□

Remark 4.1: The preceding lemma was proved in [KS] (Lemma 21) under the additional assumption that the function G is bounded on the set $X + i\mathbb{R}^m$.

5. Appendix

The extension $F(z)$ to the interior of the convex hull of the tube over X of a generalized function $f(z)$ defined on the tube over X in Case 1 of Theorem 2.1 was accomplished in two stages: first $f(z)$ was extended to the tube over an open set $U \subset \operatorname{Int}\operatorname{ch}(X)$ with the property that $\operatorname{Int}\operatorname{ch}(X) = \operatorname{ch}(U)$ and from this intermediate extension, the final one was obtained by invoking the classical Bochner's extension theorem. In this section we describe another approach that under appropriate circumstances grants the extension in a single stroke. In particular, this shows that Theorem 2.1 can be proved without assuming Bochner's extension theorem.

Definition 5.1. *Fix $\alpha > 0$. We say that the set $X \subset \mathbb{R}^m$ is C^α-pathwise connected if every pair of points $x_0, x_1 \in X$ can be joined by a curve of class C^α contained in X, i.e., if there is a map $\gamma : [0, 1] \to \mathbb{R}^m$ of class C^α such that $\gamma(0) = x_0$, $\gamma(1) = x_1$ and $\gamma([0, 1]) \subset X$.*

Theorem 5.1. *Suppose $X \subset \mathbb{R}^m$ is C^α-pathwise connected, and let f be a generalized CR function on $X + i\mathbb{R}^m$. Assume that the interior of $\operatorname{ch}(X)$ in $\mathbb{R}^m$ is nonempty and let $(f_j(z))$ be a sequence of entire functions on $\mathbb{C}^m$ that converges uniformly to f on compact subsets of $X + i\mathbb{R}^m$. Then the sequence $(f_j(z))$ converges uniformly on compact subsets of $\operatorname{Int}\operatorname{ch}(X) + i\mathbb{R}^m$.*

We will begin by stating and proving the two lemmas which are instrumental in the proof of Theorem 5.1.

In the next lemma Π denotes as in previous sections the affine space spanned by X.

Lemma 5.1. *Assume that $X \subset \mathbb{R}^m$ and let $p \in \operatorname{Int}\operatorname{ch}(X)$. There exists $\epsilon > 0$ and a finite number of points $x_1, x_2, \ldots, x_N \in X$ such that for every $q \in B_\epsilon(p) \cap \Pi$ there exist numbers $t_j(q) \geq \epsilon$, $j = 1, \ldots, N$, such that*

$$\sum_{j=1}^{N} t_j(q) = 1 \quad \textit{and} \quad q = \sum_{j=1}^{N} t_j(q)\, x_j. \tag{5.1}$$

An important point in equation (5.1) is that all points in $q \in \operatorname{Int}\operatorname{ch}(X)$ close to p are obtained through convex combinations of a set of points $\{x_1, \ldots, x_N\}$ that does not change with q. Although this expression is not unique if $N > m+1$, it will be technically important that among all possible convex combinations we can single out those with coefficients bounded away from zero.

Proof: In what follows, for a set $S \subset \mathbb{R}^m$, $\Pi(S)$ will denote the affine space generated by S and we keep writing $\operatorname{ch}(S)$ to denote the convex hull of S and $\operatorname{Int}\operatorname{ch}(S)$ to denote the interior of $\operatorname{ch}(S)$ in $\Pi(S)$. By using an affine mapping, we may and will assume that $\Pi(X) = \mathbb{R}^n \simeq \mathbb{R}^n \times \{0\} \subset \mathbb{R}^m$. Let $p \in \operatorname{Int}\operatorname{ch}(X)$. Then for some $\epsilon > 0$, $p + B_\epsilon(0) \subset \operatorname{ch}(X)$, where $B_\epsilon(0)$ denotes the ball of radius ϵ in $\mathbb{R}^n$. Hence, if $\{e_j\}_{j=1}^n$ is the usual basis of $\mathbb{R}^n$, for $\delta > 0$ sufficiently small, the points

$$w_j = p + \delta e_j, \quad j = 1, \ldots, n, \quad w_0 = p - \delta \left(\sum_{i=1}^{n} e_i \right)$$

are all in $\operatorname{ch}(X)$. We can find a finite set $V = \{x_1, \ldots, x_N\}$, $V \subset X$ such that for each $k = 0, \ldots, n$, we have:

$$w_k = \sum_{j=1}^{N} b_{kj} x_j$$

where $b_{kj} \geq 0$, and $\sum_{j=1}^{N} b_{kj} = 1$ for each k. We may assume that V is minimal for this property, i.e., the convex hull of any set with less than N elements cannot contain $\{w_0, \ldots, w_n\}$. With this proviso we have $\sum_{k=0}^{n} b_{kj} > 0$ for every $1 \leq j \leq N$. Thus

$$p = \sum_{k=0}^{n} \frac{w_k}{n+1} = \sum_{j=1}^{N} \left(\sum_{k=0}^{n} \frac{b_{kj}}{n+1} \right) x_j$$
$$= \sum_{j=1}^{N} t_j x_j$$

where $t_j > 0$, $1 \leq j \leq N$. More generally, we have:

$$\sum_{k=0}^{n} \tau_k w_k = \sum_{k=0}^{n} \sum_{j=1}^{N} \tau_k b_{kj} x_j$$
$$= \sum_{j=1}^{N} \left(\sum_{k=0}^{n} \tau_k b_{kj} \right) x_j$$
$$= \sum_{j=1}^{N} \psi_j(\tau) x_j$$

where by definition, $\psi_j(\tau) = \sum_{k=0}^{n} \tau_k b_{kj}$. Observe that the points $w_0, \ldots, w_n$ are affinely independent and belong to the image of the affine map Ψ given by

$$\mathbf{T} = \{\tau \in \mathbb{R}^{n+1} : \sum_{k=0}^{n} \tau_k = 1\} \longmapsto \Psi(\tau) = \sum_{j=1}^{N} \psi_j(\tau) x_j \in \mathbb{R}^n$$

which is therefore an open map onto $\mathbb{R}^n$. In particular, if we let $\tau = (\tau_0, \ldots, \tau_n)$ vary in a sufficiently small neighborhood U of the point $(\frac{1}{n+1}, \ldots, \frac{1}{n+1})$ in $\mathbf{T}$, we obtain a neighborhood $\Psi(U)$ of the point p in $\mathrm{ch}(X)$ in which every point is described by $\sum_{j=1}^{N} \psi_j(\tau) x_j = \sum_{j=1}^{N} t_j x_j$ with $\sum_{j=1}^{N} t_j = 1$ and $t_j(\tau) \geq \epsilon$, $1 \leq j \leq N$, for some positive ϵ. $\square$

We denote the unit disc in $\mathbb{C}$ by $\Delta = \{z \in \mathbb{C} : \ |z| < 1\}$ and by $\bar{\Delta}$ its closure. A continuous function $A : \bar{\Delta} \to \mathbb{C}^m$ is called an analytic disc in $\mathbb{C}^m$ if it is holomorphic on Δ. We say that the point $A(0)$ is the center of the analytic disc and the subset $A(\partial\Delta)$ is the boundary of the analytic disc.

The next lemma shows that every point p in the set $\operatorname{Int}\operatorname{ch}(X)$ is the center of an analytic disc with boundary contained in $X + i\mathbb{R}^m$. Furthermore, points close enough to p can be realized as centers of analytic discs with boundaries contained in a fixed compact susbset of $X + i\mathbb{R}^m$

Lemma 5.2. *Assume that $X \subset \mathbb{R}^m$ is C^α-pathwise connected and let $p \in \operatorname{Int}\operatorname{ch}(X)$. There exists $\delta > 0$, $R > 0$ and $Y \Subset X$ such that every point $q \in B_\delta(p) \cap \Pi$ is the center of an analytic disc with boundary contained in $Y + iB_R(p)$.*

Proof: We fix $p \in \operatorname{Int}\operatorname{ch}(X)$ and choose $\epsilon > 0$ and $x_1, \dots, x_N$ as in Lemma 5.1. We choose paths $\gamma_j(s)$, $1 \le j \le N$, connecting each point x_j to the next one (the point x_1 should be consider next to the point x_N). We consider an arbitrary point $q \in B_\epsilon(p) \cap \Pi$ and determine the coefficients $\epsilon \le t_j(q) \le 1$ from the expression (5.1) (later we will be more precise about the choice of q).

We will now define a closed path $u(e^{i\theta})$, $0 \le \theta \le 2\pi$ that will start and end at x_1 passing through the points $x_2, \dots, x_N$. If we could allow discontinuous "paths" we would define $u(e^{i\theta})$ to be equal to x_1 for $0 \le \theta \le 2\pi t_1$, equal to x_2 for $2\pi t_1 \le \theta \le 2\pi(t_1 + t_2), \dots$, and equal to x_N for $2\pi(t_1 + \cdots + t_{N-1}) \le \theta \le 2\pi$. Here we have written t_j instead of $t_j(q)$ to simplify the notation. We may modify this map replacing the jumps by connecting the point x_j to the adjacent point x_{j+1} with the path γ_j traversed at a very high speed. For any number $0 < \rho < 2\pi\epsilon$ to be chosen later we set

$$u(e^{i\theta}) = \begin{cases} x_1 & \text{if } 0 \le \theta \le 2\pi t_1 - \rho, \\ \gamma_1(\frac{\theta - 2\pi t_1 + \rho}{\rho}) & \text{if } 2\pi t_1 - \rho \le \theta \le 2\pi t_1, \\ x_2 & \text{if } 2\pi t_1 \le \theta \le 2\pi(t_1 + t_2) - \rho, \\ \gamma_2(\frac{\theta - 2\pi(t_1+t_2) + \rho}{\rho}) & \text{if } 2\pi(t_1 + t_2) - \rho \le \theta \le 2\pi(t_1 + t_2), \\ \dots\dots\dots\dots\dots\dots & \\ x_N & \text{if } 2\pi(t_1 + \cdots + t_{N-1}) \le \theta \le 2\pi - \rho, \\ \gamma_N(\frac{\theta - 2\pi(t_1+t_2+\cdots+t_N) + \rho}{\rho}) & \text{if } 2\pi - \rho \le \theta \le 2\pi, \end{cases}$$

Note that $\theta \mapsto u(e^{i\theta})$ is piecewise of class C^α and continuous, hence of class C^α. Set

$$K(w) = \frac{1}{2\pi}\int_0^{2\pi} \frac{e^{i\theta}+w}{e^{i\theta}-w}\, u(e^{i\theta})\, d\theta, \quad |w|<1.$$

Note that, since $u(e^{i\theta})$ is of class C^α, $K(w)$ can be continuously extended up to $|w|=1$ and

$$K(e^{i\theta}) = u(e^{i\theta}) + iHu(e^{i\theta})$$

where H denotes the Hilbert transform. Set $Y = u([0,2\pi])$ so $Y \Subset X$. Recall that the Hilbert transform is continuous on $C^\alpha(S^1)$, in particular,

$$\sup|Hu| \le C\,|u|_{C^\alpha}.$$

Then K is an analytic disc with boundary contained in $Y + iB_R(0)$ provided that $\|Hu\|_{L^\infty} < R$. The center of K is

$$\begin{aligned} K(0) &= \frac{1}{2\pi}\int_0^{2\pi} u(e^{i\theta})\, d\theta \\ &= \sum_{j=1}^N (t_j - (\rho/2\pi))x_j + \frac{\rho}{2\pi}\sum_{j=1}^N \int_0^1 \gamma_j(s)\, ds \\ &= \sum_{j=1}^N t_j x_j + \rho\, I = q + \rho I. \end{aligned}$$

Since $u(e^{i\theta}) \in X$, $0 \le \theta \le 2\pi$, and $d\theta/2\pi$ is a probability measure, $K(0) \in \mathrm{ch}(X) \subset \Pi$. Furthermore, $q \in \mathrm{ch}(X) \subset \Pi$ so we conclude that $\rho I = K(0) - q \in \Pi_0$, where Π_0 is the translate of Π that passes through the origin. Note that the vector $I \in \mathbb{R}^m$ depends on $x_1, \ldots, x_N$ and $\gamma_1, \ldots, \gamma_N$ but not on the coefficients $t_j(q)$, so I is independent of $q \in B_\epsilon(p)$. At this point we choose ρ small enough to grant that $\rho|I| < \epsilon/2$. Thus, if we choose q by setting $q = p - \rho I \in B_\epsilon(p)$ we get $K(0) = p$. Similarly, if $|r| < \epsilon/2$ and $p + r \in \Pi$, we may set $q = p + r - \rho I$ and check that $q \in B_\epsilon(p) \cap \Pi$. Thus, $K(0) = p + r$, showing that the statement of the lemma holds true with $\delta = \epsilon/2$. □

We may now easily prove Theorem 5.1. Consider a sequence of entire functions $f_j(z)$ that converge to $f(z)$ uniformly on compact subsets of $X + i\mathbb{R}^m$ and fix a point $p \in \mathrm{Int}\,\mathrm{ch}(X)$. Choose, $\delta > 0$, $R > 0$

and $Y \Subset X$ as in Lemma 5.2. Then every point $q \in \operatorname{Int}\operatorname{ch}(X)$ such that $|q-p| < \delta$ is the center of an analytic disc A with boundary $A(\partial\Delta) \subset Y + iB_R(0)$. By the maximum principle $f_j(q) = f_j \circ A(0)$ is a Cauchy sequence because $f_j(z)$ converges uniformly on the compact subset $Y + i\overline{B}(0,R)$ and this holds uniformly for $|q-p| < \delta$. Similarly, if $q + iy \in \operatorname{Int}\operatorname{ch}(X) + i\mathbb{R}^m$ and $|y| < R'$, $q + iy$ is the center of an analytic disc with boundary contained in $Y + i\overline{B}(0, R + R')$. Any compact subset of $\operatorname{Int}\operatorname{ch}(X) + i\mathbb{R}^m$ is contained in a compact set of the form $F + i\overline{B}(0, R'')$ with $F \Subset \operatorname{ch}(X)$. Since F may be covered by a finite number of balls $B_{\delta_k}(p_k) \cap \Pi$, this shows that $f_j(z)$ converges uniformly on compact subsets of $\operatorname{Int}\operatorname{ch}(X) + i\mathbb{R}^m$. $\square$

Let us now assume that the hypotheses of Theorem 5.1 hold and consider the set $\widehat{X} \subset \operatorname{ch}(X)$ defined by

$$\widehat{X} = \bigcup \{\Re A(\overline{\Delta}) : \quad A(\partial\Delta) \subset X + i\mathbb{R}^m\}.$$

where the union is taken over all the analytic discs with boundary contained in X. In view of what we have seen, it is clear that

$$X \cup \operatorname{Int}\operatorname{ch}(X) + i\mathbb{R}^m \subset \widehat{X} + i\mathbb{R}^m \subset \operatorname{ch}(X) + i\mathbb{R}^m$$

and if $f(z)$ is a generalized CR function on $X + i\mathbb{R}^m$, $f(z)$ has a natural extension to $\widehat{X} + i\mathbb{R}^m$. Indeed, if $f_j(z)$, $j = 1, 2, \ldots$, are entire functions converging to $f(z)$ on compact subsets of $X + i\mathbb{R}^m$, the domain of convergence of the sequence $f_j(z)$ contains $\widehat{X} + i\mathbb{R}^m$. Thus, the pointwise limit of the functions $f_j(z)$ determines a function $F(z)$ defined on $\widehat{X} + i\mathbb{R}^m$ that coincides with $f(z)$ on $X + i\mathbb{R}^m$ and is holomorphic on $\operatorname{Int}\operatorname{ch}(X) + i\mathbb{R}^m$. Furthermore, for any analytic disc A with boundary contained in $X + i\mathbb{R}^m$, the restriction $F \circ A(\zeta)$ of $F(z)$ to A is continuous and it is apparent that $F(z)$ is the only holomorphic function defined on $\operatorname{Int}\operatorname{ch}(X) + i\mathbb{R}^m$ with this property. This may be regarded as a sort of very weak nontangential continuity property that $F(z)$ possesses at points of $X + i\mathbb{R}^m$. Under the hypotheses of Theorem 5.1 we have the following

Corollary 5.1. *Any generalized CR function f on $X + i\mathbb{R}^m$ can be extended to a function $F(z)$ defined on $X \cup \operatorname{Int}\operatorname{ch}(X) + i\mathbb{R}^m$ that is* *holomorphic on* $\operatorname{Int}\operatorname{ch}(X) + i\mathbb{R}^m$ *and the restriction of* $F(z)$ *to any analytic disc with boundary in* $X + i\mathbb{R}^m$ *is continuous.*

If we take X to be a connected open subset of $\mathbb{R}^m$ we get

Corollary 5.2 (Bochner's tube theorem)**.** *Let* $X \subset \mathbb{R}^m$ *be an open connected set. Every holomorphic function* $f(z)$ *defined on* $X + i\mathbb{R}^m$ *can be extended as a holomorphic function defined on* $\operatorname{ch}(X) + i\mathbb{R}^m$*.*

We have shown that extensions of generalized CR functions on tubes over a connected set X exist under strengthened connectedness conditions on X. These conditions are not necessary and in some situations extensions can be obtained without that requirement, as the example below shows.

EXAMPLE 5.1: Let $X = I \cup G \subset \mathbb{R}^2$ be the set of points

$$I = \{0\} \times [-1, 1], \quad G = \{(x, \cos(2\pi/x)) : \quad 0 < x \leq 1\}.$$

Then X is connected but neither satisfies the hypotheses of Theorem 2.1 nor those of of Theorem 5.1 because it is not arcwise connected. Now let $f(z)$ be a generalized CR function of $X + i\mathbb{R}^2 \subset \mathbb{C}^2$. Applying anyone of our extension theorems with G in the place of X we obtain a holomorphic function $F(z)$ defined on

$$\operatorname{Int}\operatorname{ch}(X) + i\mathbb{R}^2 = \operatorname{Int}\operatorname{ch}(G) + i\mathbb{R}^2 \supset (0, 1/2) \times (-1, 1) + i\mathbb{R}^2.$$

Note that $f(z)$ can be recovered from the values of $F(z)$ on $\operatorname{Int}\operatorname{ch}(X) + i\mathbb{R}^2$ because the closure of $X \cap \operatorname{Int}\operatorname{ch}(X)$ contains the set I, which is the only non smooth part of X.

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