

Existence results for the Klein-Gordon-Maxwell equations in higher dimensions with critical exponents

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Abstract

In this paper we study the existence of radially symmetric solitary waves in \mathbb{R}^N for the nonlinear Klein-Gordon equations coupled with the Maxwell's equations when the nonlinearity exhibits critical growth. The main feature of this kind of problem is the lack of compactness arising in connection with the use of variational methods.

Keywords: Klein-Gordon-Maxwell system; radially symmetric solution; critical growth

1 Introduction

This article concerns the existence of solutions for the Klein-Gordon-Maxwell (\mathcal{KGM}) system in \mathbb{R}^N with critical Sobolev exponents

$$-\Delta u + [m_0^2 - (\omega + \phi)^2]u = \mu|u|^{q-2}u + |u|^{2^*-2}u \quad \text{in } \mathbb{R}^N, \quad (1)$$

$$\Delta \phi = (\omega + \phi)u^2 \quad \text{in } \mathbb{R}^N, \quad (2)$$

where $2 < q < 2^* = 2N/(N-2)$, $\mu > 0$, $m_0 > 0$ and $\omega \neq 0$ are real constants and also $u, \phi : \mathbb{R}^N \rightarrow \mathbb{R}$.

Such system has been first introduced by Benci and Fortunato [3] as a model which describes nonlinear Klein-Gordon fields in three-dimensional space interacting with the electromagnetic field. Further, in the quoted paper [4] they proved existence of solitary waves of the complement Klein-Gordon-Maxwell equations when the nonlinearity has subcritical behavior.

Some recent works have treated this problem still in the subcritical case and we cite a couple of them.

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D'Aprile and Mugnai [7] established the existence of infinitely many radially symmetric solutions for the subcritical (\mathcal{KGM}) system in \mathbb{R}^3 . They extended the interval of definition of the power in the nonlinearity exhibited in [4]. For related works, see [11] and [12].

Non-existence results and a treatment of the (\mathcal{KGM}) system in bounded domains can be found in ([6], [8], [9], [10] and references therein).

With this *Ansatz* Cassani [6] proved the existence of nontrivial radially symmetric solutions in \mathbb{R}^3 for the critical case. He was able to show that

- if $|m_0| > |\omega|$ and $4 < q < 2^*$, then for each $\mu > 0$ there exists at least a radially symmetric solution for system (1)-(2).
- if $|m_0| > |\omega|$ and $q = 4$, then system (1)-(2) also has at least a radially symmetric solution by supposing μ sufficiently large.

The goal of this paper is to complement Theorem 1.2 from Cassani in [6] and also extend it in higher dimensions as follows

Theorem 1. *Assume either $|m_0| > |\omega|$ and $4 \leq q < 2^*$ or $|m_0|\sqrt{q-2} > |\omega|\sqrt{2}$ and $2 < q < 4$.*

Then system (1)-(2) has at least one radially symmetric (nontrivial) solution (u, ϕ) with $u \in H^1(\mathbb{R}^N)$ and $\phi \in \mathcal{D}^{1,2}(\mathbb{R}^N)$ provided that

- i) $N = 4$ and $N \geq 6$ for $2 < q < 2^*$ if $\mu > 0$;*
- ii) $N = 5$ and either $2 < q < \frac{8}{3}$ if $\mu > 0$ or $\frac{8}{3} \leq q < 2^*$ if μ is sufficiently large;*
- iii) $N = 3$ and either $4 < q < 2^*$ if $\mu > 0$ or $2 < q \leq 4$ if μ is sufficiently large.*

In order to get this result we will explore the Brézis and Nirenberg technique and some of its variants. See e.g. [13].

2 Preliminary Results

We want to find solutions of the system (1)-(2) where $u \in H^1(\mathbb{R}^N)$ and $\phi \in \mathcal{D}^{1,2}(\mathbb{R}^N)$.

Here $H^1 \equiv H^1(\mathbb{R}^N)$ denotes the usual Sobolev space endowed with the norm

$$\|u\|^2 = \int_{\mathbb{R}^N} (|\nabla u|^2 + u^2) dx \quad (3)$$

and $\mathcal{D}^{1,2} \equiv \mathcal{D}^{1,2}(\mathbb{R}^N)$ denotes the completion of $C_0^\infty(\mathbb{R}^N)$ with respect to the norm

$$\|u\|_{\mathcal{D}^{1,2}}^2 = \int_{\mathbb{R}^N} |\nabla u|^2 dx. \quad (4)$$

The (\mathcal{KGM}) system are the Euler-Lagrange equations related to the functional

$$F : H^1 \times \mathcal{D}^{1,2} \rightarrow \mathbb{R}$$

defined as

$$\begin{aligned} F(u, \phi) = & \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla u|^2 - |\nabla \phi|^2 + [m_0^2 - (\omega + \phi)^2] u^2) dx + \\ & - \frac{\mu}{q} \int_{\mathbb{R}^N} |u|^q dx - \frac{1}{2^*} \int_{\mathbb{R}^N} |u|^{2^*} dx, \end{aligned} \quad (5)$$

which by standard arguments is C^1 on $H^1 \times \mathcal{D}^{1,2}$.

The functional F is strongly indefinite. To avoid this difficulty, we reduce the study of (5) to the study of a functional in the only variable u , as it has been done by the aforementioned authors.

Now we need some technical results.

Proposition 2. *For every $u \in H^1$, there exists a unique $\phi = \Phi[u] \in \mathcal{D}^{1,2}$ which solves (2). Furthermore, in the set $\{x|u(x) \neq 0\}$ we have $-\omega \leq \Phi[u] \leq 0$ if $\omega > 0$ and $0 \leq \Phi[u] \leq -\omega$ if $\omega < 0$.*

Proof. The proof of the uniqueness of $\Phi[u] \in \mathcal{D}^{1,2}(\mathbb{R}^N)$ is very similar to the one proved in dimension three by [4].

Following the same idea of [7], fix $u \in H^1$ and consider $\omega > 0$. If we multiply (2) by $(\omega + \Phi[u])^- = \min\{\omega + \Phi[u], 0\}$, which is an admissible test function, we get

$$-\int_{\{x|\omega+\Phi[u]<0\}} |\nabla\Phi[u]|^2 - \int_{\{x|\omega+\Phi[u]<0\}} (\omega + \Phi[u])^2 u^2 = 0$$

so that $\Phi[u] \geq -\omega$ where $u \neq 0$. Otherwise, if $\omega < 0$ and multiplying (2) by $(\omega + \Phi[u])^+ = \max\{\omega + \Phi[u], 0\}$ and repeating the same argument, we obtain $\Phi[u] \leq -\omega$, for $u \neq 0$.

Finally observe that by Stampacchia's lemma, if $\omega > 0$ then $\phi \leq 0$, and if $\omega < 0$, $\phi \geq 0$ (for details see [6]). □

In view of Proposition 2, we can define

$$\Phi : H^1 \rightarrow \mathcal{D}^{1,2}$$

which is of class C^1 (see [8]) and maps each $u \in H^1$ in the unique solution of (2), then

$$-\Delta\Phi[u] + u^2\Phi[u] = -\omega u^2. \quad (6)$$

Taking the product of (6) with $\Phi[u]$ and integrating by parts, we obtain

$$\int_{\mathbb{R}^N} |\nabla\Phi[u]|^2 dx = -\int_{\mathbb{R}^N} \omega u^2 \Phi[u] dx - \int_{\mathbb{R}^N} u^2 \Phi[u]^2 dx. \quad (7)$$

Now consider the functional

$$J : H^1 \rightarrow \mathbb{R}, \quad J(u) := F(u, \Phi[u])$$

which is also of class C^1 .

By the definition of F and using (7), J can be written in the following form

$$\begin{aligned} J(u) = & \frac{1}{2} \int_{\mathbb{R}^N} \left(|\nabla u|^2 + (m_0^2 - \omega^2)u^2 + |\nabla\Phi[u]|^2 + \Phi[u]^2 u^2 \right) dx + \\ & -\frac{\mu}{q} \int_{\mathbb{R}^N} |u|^q dx - \frac{1}{2^*} \int_{\mathbb{R}^N} |u|^{2^*} dx, \end{aligned} \quad (8)$$

while for J' we have, $\forall v \in H^1$,

$$\begin{aligned} \langle J'(u), v \rangle = & \\ = & \int_{\mathbb{R}^N} \left(\nabla u \cdot \nabla v + [m_0^2 - (\omega + \Phi[u])^2]uv - \mu|u|^{q-2}uv - |u|^{2^*-2}uv \right) dx. \end{aligned} \quad (9)$$

Proposition 3. *Let $(u, \phi) \in H^1 \times \mathcal{D}^{1,2}$. Then the following statements are equivalent:*

- i) (u, ϕ) is a critical point for F ;*
- ii) u is a critical point for J and $\phi = \Phi[u]$.*

Proof. See [4]. □

Hence, we look for critical points of J .

3 Proof of the Main Result

In order to overcome the lack of compactness due to the invariance under group of translations of J , we restrict ourselves to radial functions. More precisely, we look at the functional J on the subspace

$$H_r^1 = \{u \in H^1 : u(x) = u(|x|)\}$$

compactly embedded into L_r^p , $2 < p < 2^*$, where $L_r^p = \{u \in L^p : u(x) = u(|x|)\}$.

We also point out that any critical point $u \in H_r^1$ of $J|_{H_r^1}$ is also a critical point of J by the Principle of symmetric criticality of Palais (see [15]).

Now we show that the functional J verifies the Mountain-Pass Geometry, more exactly J satisfies the following lemma

Lemma 4. *The functional J satisfies*

- (i) There exist positive constants α, ρ such that $J(u) \geq \alpha$ for $\|u\| = \rho$.*
- (ii) There exists $u_1 \in H_r^1(\mathbb{R}^N)$ with $\|u_1\| > \rho$ such that $J(u_1) < 0$.*

Proof. Using the Sobolev embeddings, we have

$$J(u) \geq C_1 \|u\|^2 - C_2 \|u\|^q - C_3 \|u\|^{2^*},$$

where C_1, C_2 and C_3 are positive constants. Since $q > 2$, there exists $\alpha, \rho > 0$ such that $\inf_{\|u\|=\rho} J(u) > \alpha$, showing (i).

Let $u \in H_r^1$, then for $t \geq 0$

$$\begin{aligned} J(tu) &= \frac{t^2}{2} \int_{\mathbb{R}^N} (|\nabla u|^2 + (m_0^2 - \omega^2)u^2) dx + \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla \Phi[tu]|^2 + \Phi[tu]^2 (tu)^2) dx + \\ &\quad - \frac{\mu}{q} t^q \int_{\mathbb{R}^N} |u|^q dx - \frac{t^{2^*}}{2^*} \int_{\mathbb{R}^N} |u|^{2^*} dx. \end{aligned} \tag{10}$$

By Proposition (2) we get the estimate ¹

$$- \int_{\mathbb{R}^N} \omega u^2 \Phi[u] dx \leq \int_{\mathbb{R}^N} \omega^2 u^2 dx,$$

then using equation (7) and the last inequality in (8), we obtain

$$J(tu) \leq C_4 t^2 \|u\|^2 + \frac{\omega^2}{2} t^2 \|u\|_2^2 - \frac{\mu}{q} t^q \|u\|_q^q - \frac{1}{2^*} t^{2^*} \|u\|_{2^*}^{2^*}.$$

Since $q > 2$, there exists $u_1 \in H_r^1$, $u_1 := tu$ with t sufficiently large such that $\|u_1\| > \rho$ and $J(u_1) < 0$, proving (ii). □

¹From now on we use $\Phi(\Phi_n)$ instead of $\Phi[u]$ ($\Phi[u_n]$).

Applying a variant of the Ambrosetti-Rabinowitz Mountain Pass Theorem [1] we obtain a $((PS)_c)$ sequence $\{u_n\} \subset H_r^1$ such that

$$J(u_n) \rightarrow c \text{ and } J'(u_n) \rightarrow 0,$$

where

$$c := \inf_{\gamma \in \Gamma} \max_{0 \leq t \leq 1} J(\gamma(t)), \quad c \geq \alpha \quad (11)$$

and

$$\Gamma = \{\gamma \in \mathcal{C}([0, 1], H_r^1(\mathbb{R}^N)) \mid \gamma(0) = 0, \gamma(1) = u_1\}. \quad (12)$$

An important tool in our analysis will be the next lemma:

Lemma 5. *The $(PS)_c$ sequence $\{u_n\}$ is bounded.*

Proof. By hypothesis, let $\{u_n\} \subset H_r^1$ be such that $-\langle J'(u), v \rangle \leq o(1)\|u_n\|$ and $|J(u_n)| \leq M$, for some positive constant M . Then from (8) and (9),

$$\begin{aligned} qM + o(1)\|u_n\| &\geq qJ(u_n) - \langle J'(u_n), u_n \rangle = \\ &= \left(\frac{q}{2} - 1\right) \int_{\mathbb{R}^N} (|\nabla u_n|^2 + [m_0^2 - \omega^2]u_n^2) dx + \left(2 - \frac{q}{2}\right) \int_{\mathbb{R}^N} \omega \Phi_n u_n^2 dx + \\ &\quad + \int_{\mathbb{R}^N} \Phi_n^2 u_n^2 dx + \left(1 - \frac{q}{2^*}\right) \int_{\mathbb{R}^N} |u_n|^{2^*} dx \\ &\geq \left(\frac{q-2}{2}\right) \int_{\mathbb{R}^N} (|\nabla u_n|^2 + [m_0^2 - \omega^2]u_n^2) dx - \omega \left(\frac{q-4}{2}\right) \int_{\mathbb{R}^N} \Phi_n u_n^2 dx. \end{aligned} \quad (13)$$

There are two cases to be considered: either $2 < q < 4$ or $4 \leq q < 2^*$.

If $4 \leq q < 2^*$, then by Proposition 2 and inequality (13)

$$\begin{aligned} qM + o(1)\|u_n\| &\geq C\|u_n\|^2 + \omega \left(\frac{q-4}{2}\right) \int_{\mathbb{R}^N} (-\Phi_n) u_n^2 dx \\ &\geq C\|u_n\|^2 \end{aligned}$$

and we deduce that $\{u_n\}$ is bounded in H_r^1 .

But if $2 < q < 4$ and using again (13) and Proposition 2 we get

$$\begin{aligned} qM + o(1)\|u_n\| &\geq \left(\frac{q-2}{2}\right) \int_{\mathbb{R}^N} |\nabla u_n|^2 dx + \left(\frac{(q-2)m_0^2 - 2\omega^2}{2}\right) \int_{\mathbb{R}^N} |u_n^2| dx \\ &\geq C\|u_n\|^2, \end{aligned}$$

where $(q-2)m_0^2 - 2\omega^2 > 0$ by hypothesis, which implies that $\{u_n\}$ is again bounded in H_r^1 . \square

In view of the previous lemma we have that $\{\Phi_n\}$ is bounded in $\mathcal{D}_r^{1,2}$ because

$$\begin{aligned} \|\Phi_n\|_{\mathcal{D}_r^{1,2}}^2 &\leq \int_{\mathbb{R}^N} |\nabla \Phi_n|^2 dx + \int_{\mathbb{R}^N} |\Phi_n^2 u_n^2| dx \\ &= -\omega \int_{\mathbb{R}^N} |\Phi_n u_n^2| dx \leq C\omega \|\Phi_n\|_{\mathcal{D}_r^{1,2}} \|u_n\|_{2.2^*/(2^*-1)}^2. \end{aligned}$$

So, passing to a subsequence if necessary, we may assume

$$\begin{aligned} u_n &\rightharpoonup u, \quad \text{weakly in } H_r^1, \quad n \rightarrow \infty, \\ \Phi_n &\rightharpoonup \phi, \quad \text{weakly in } \mathcal{D}_r^{1,2}, \quad n \rightarrow \infty. \end{aligned}$$

Lemma 6. $\phi = \Phi[u]$ and $\Phi_n \rightarrow \Phi$ strongly in $\mathcal{D}_r^{1,2}$.

Proof. The proof is essentially as in Lemma 3.2 of [6], which can be easily extended in dimension N . \square

Moreover, since the Sobolev embeddings $H_r^1 \hookrightarrow L_r^s$, $2 < s < 2^*$, are compact we conclude that

$$u_n \rightarrow u, \quad \text{strongly in } L_r^s, \quad \text{for } 2 < s < 2^*, \quad n \rightarrow \infty.$$

Now we show that the pair (u, Φ) satisfies the (\mathcal{KGM}) system in the weak sense. Indeed, since $J'(u_n) \rightarrow 0$ we have $\forall v \in H_r^1$,

$$\begin{aligned} \int_{\mathbb{R}^N} \left(\nabla u_n \nabla v + (m_0^2 - \omega^2) u_n v \right) dx &= \int_{\mathbb{R}^N} u_n \Phi_n^2 v dx + 2\omega \int_{\mathbb{R}^N} \Phi_n u_n v dx + \\ &+ \mu \int_{\mathbb{R}^N} |u_n|^{q-2} u_n v dx + \int_{\mathbb{R}^N} |u_n|^{2^*-2} u_n v dx + o(1) \end{aligned} \quad (14)$$

We will prove that

$$\int_{\mathbb{R}^N} u_n \Phi_n^2 v dx + 2\omega \int_{\mathbb{R}^N} \Phi_n u_n v dx \xrightarrow{n \rightarrow \infty} \int_{\mathbb{R}^N} u \Phi^2 v dx + 2\omega \int_{\mathbb{R}^N} \Phi u v dx, \quad (15)$$

$$\int_{\mathbb{R}^N} |u_n|^{q-2} u_n v dx \xrightarrow{n \rightarrow \infty} \int_{\mathbb{R}^N} |u|^{q-2} u v dx \quad (16)$$

and

$$\int_{\mathbb{R}^N} |u_n|^{2^*-2} u_n v dx \xrightarrow{n \rightarrow \infty} \int_{\mathbb{R}^N} |u|^{2^*-2} u v dx \quad (17)$$

Verification of (15).

Using the generalized Hölder inequality, note that

$$\begin{aligned} \int_{\mathbb{R}^N} |\Phi u - \Phi_n u_n| |v| dx &\leq \|\Phi - \Phi_n\|_{2^*} \|u\|_{2 \cdot 2^*/(2^*-2)} \|v\|_2 + \\ &+ \|\Phi_n\|_{2^*} \|u - u_n\|_{2 \cdot 2^*/(2^*-2)} \|v\|_2 \end{aligned}$$

and

$$\begin{aligned} \int_{\mathbb{R}^N} |u \Phi^2 - u_n \Phi_n^2| |v| dx &\leq \|\Phi - \Phi_n\|_{2^*} \|\Phi + \Phi_n\|_{2^*} \|u_n\|_{2^*} \|v\|_2 + \\ &+ \|u - u_n\|_{2 \cdot 2^*/(2^*-2)} \|\Phi^2\|_{2 \cdot 2^*/(2^*-2)} \|v\|_{2 \cdot 2^*/(2^*-2)}. \end{aligned}$$

Then, by Lemma 6 we get (15).

Verification of (16)-(17).

The convergence in (16) follows from the compactness of the embedding $H_r^1 \hookrightarrow L_r^q$ and the convergence in (17) holds since $\{u_n\}$ is bounded in $L_r^{2^*}$.

Hence by (15), (16) and (17) together with (14), we conclude that (u, Φ) is a weak solution for (\mathcal{KGM}) system.

Due to the lack of compactness, we must prove that actually u does not vanish.

Lemma 7. *The number c given in (11) satisfies*

$$0 < c < \frac{1}{N} S^{N/2}, \quad (18)$$

where S is the best Sobolev constant, namely

$$S := \inf_{\substack{u \in \mathcal{D}^{1,2}(\mathbb{R}^N) \\ u \neq 0}} \frac{\int |\nabla u|^2 dx}{\left(\int |u|^{2^*} dx \right)^{2/2^*}}.$$

For a moment, suppose Lemma 7 holds true, we will prove that $u \neq 0$. Consider $u \equiv 0$. Since $J'(u_n) \rightarrow 0$ and $u_n \rightarrow 0$ in L_r^q as $n \rightarrow \infty$, we may assume

$$\int_{\mathbb{R}^N} \left(|\nabla u_n|^2 + (m_0^2 - \omega^2) u_n^2 \right) dx \xrightarrow{n \rightarrow \infty} \ell$$

and

$$\int_{\mathbb{R}^N} |u_n|^{2^*} dx \xrightarrow{n \rightarrow \infty} \ell, \quad \ell \geq 0.$$

Consequently,

$$J(u_n) \xrightarrow{n \rightarrow \infty} \left(\frac{1}{2} - \frac{1}{2^*} \right) \ell$$

where now $\ell > 0$, since $c > 0$.

By the definition of S ,

$$S \leq \frac{\int_{\mathbb{R}^N} \left(|\nabla u_n|^2 + (m_0^2 - \omega^2) u_n^2 \right) dx}{\left(\int |u|^{2^*} dx \right)^{2/2^*}} \xrightarrow{n \rightarrow \infty} \ell^{2/N},$$

from what we conclude that

$$c = \left(\frac{1}{2} - \frac{1}{2^*} \right) \ell \geq \frac{1}{N} S^{N/2},$$

contradicting Lemma 7.

Proof of Lemma 7. This proof uses a technique by Brézis and Nirenberg [5] and some of its variants. However we follow more closely Miyagaki [13].

It suffices to show that

$$\sup_{t \geq 0} J(tv_0) < \frac{1}{N} S^{\frac{N}{2}} \quad (19)$$

for some $v_0 \in H_r^1, v_0 \neq 0$.

Indeed, observing that $J(tv_0) \rightarrow -\infty$ as $t \rightarrow \infty$ and letting $\gamma \in \Gamma$ we have

$$J(\gamma(t)) \leq \sup_{t \geq 0} J(tv_0), \quad 0 \leq t \leq 1 \quad (20)$$

so that

$$c \leq \sup_{t \geq 0} J(tv_0) < \frac{1}{N} S^{\frac{N}{2}}.$$

In order to prove (20) consider $R > 0$ fixed and a cut-off function $\varphi \in C_0^\infty$ such that

$$\varphi|_{B_R} = 1, \quad 0 \leq \varphi \leq 1 \text{ in } B_{2R} \quad \text{and} \quad \text{supp } \varphi \subset B_{2R}.$$

Let $\varepsilon > 0$ and define $w_\varepsilon := u_\varepsilon \varphi$ where $u_\varepsilon \in \mathcal{D}^{1,2}$ is the well known Talenti's function (see [14])

$$u_\varepsilon(x) = \frac{[N(N-2)\varepsilon]^{\frac{N-2}{4}}}{\left(\varepsilon + |x|^2\right)^{\frac{N-2}{2}}}, \quad x \in \mathbb{R}^N, \varepsilon > 0$$

and also consider $v_\varepsilon \in C_0^\infty$ given by

$$v_\varepsilon := \frac{w_\varepsilon}{\|w_\varepsilon\|_{L^{2^*}(B_{2R})}}. \quad (21)$$

From the estimates given in [5] we have, as $\varepsilon \rightarrow 0$,

$$X_\varepsilon := \|\nabla v_\varepsilon\|_2^2 \leq S + O(\varepsilon^\delta), \quad \text{where } \delta = \frac{N-2}{2}. \quad (22)$$

Since $\lim_{t \rightarrow \infty} J(tv_\varepsilon) = -\infty \forall \varepsilon$, there exists $t_\varepsilon \geq 0$ such that $\sup_{t \geq 0} J(tv_\varepsilon) = J(t_\varepsilon v_\varepsilon)$ and we may assume without loss of generality that $t_\varepsilon \geq C_0 > 0$.

Claim 1. The following estimate holds

$$t_\varepsilon \leq \left(\int_{B_{2R}} |\nabla v_\varepsilon|^2 dx + \int_{B_{2R}} m_0^2 v_\varepsilon^2 dx \right)^{1/(2^*-2)} := r_\varepsilon. \quad (23)$$

Proof of Claim 1: Letting $\gamma(t) := J(tv_\varepsilon)$ we have, for $t > r_\varepsilon$,

$$\begin{aligned} \gamma'(t) &= J'(tv_\varepsilon)(v_\varepsilon) \\ &= tr_\varepsilon^{2^*-2} - t^{2^*-1} - t \int_{B_{2R}} (\omega + \phi[tv_\varepsilon])^2 v_\varepsilon^2 dx - \mu t^{q-1} \int_{B_{2R}} |v_\varepsilon|^q dx \\ &< 0. \end{aligned}$$

□

Now, the function of t

$$\frac{t^2}{2} r_\varepsilon^{2^*-2} - \frac{t^{2^*}}{2^*}$$

is increasing on $[0, r_\varepsilon)$, hence using (22) we conclude that

$$\begin{aligned} J(t_\varepsilon v_\varepsilon) &\leq \frac{1}{N} \left(S + O(\varepsilon^\delta) + \int_{B_{2R}} m_0^2 v_\varepsilon^2 dx \right)^{N/2} - \frac{t_\varepsilon^2}{2} \int_{B_{2R}} \omega^2 v_\varepsilon^2 dx + \\ &\quad + Ct_\varepsilon^4 \|v_\varepsilon\|_{2 \cdot 2^*/(2^*-1)}^4 - \frac{\mu}{q} t_\varepsilon^q \int_{B_{2R}} |v_\varepsilon|^q dx. \end{aligned}$$

Recalling that

$$(a+b)^\alpha \leq a^\alpha + \alpha(a+b)^{\alpha-1}b$$

which is valid for $a, b \geq 0$, $\alpha \geq 1$ we obtain

$$J(t_\varepsilon v_\varepsilon) \leq \frac{1}{N} S^{N/2} + O(\varepsilon^\delta) + K_1 \int_{B_{2R}} m_0^2 v_\varepsilon^2 dx + \\ - K_2 \int_{B_{2R}} \omega^2 v_\varepsilon^2 dx - \mu K_3 \int_{B_{2R}} |v_\varepsilon|^q dx + K_4 \|v_\varepsilon\|_{2 \cdot 2^*/(2^*-1)}^4,$$

where $K_i(\varepsilon) \geq K_0 > 0$.

We contend that

Claim 2.

$$\lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon^\delta} \left(\int_{B_{2R}} (v_\varepsilon^2 - \mu v_\varepsilon^q) dx + \|v_\varepsilon\|_{2 \cdot 2^*/(2^*-1)}^4 \right) = -\infty. \quad (24)$$

Assuming (24) for a while we have

$$J(t_\varepsilon v_\varepsilon) < \frac{1}{N} S^{N/2}, \quad \varepsilon \text{ small}$$

showing (19) and thus Lemma 7.

Proof of Claim 2:

As in [5], we obtain

$$\int_{B_{2R}} |w_\varepsilon|^{2^*} dx = (N(N-2))^{N/2} \int_{\mathbb{R}^N} \frac{1}{(1+|x|^2)^N} dx + O(\varepsilon^{N/2}) \quad (25)$$

so, in view of (21), it suffices evaluate (24) with w_ε instead of v_ε . In order to prove (24) we must show

$$\lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon^\delta} \left[\int_{B_R} (w_\varepsilon^2 - \mu w_\varepsilon^q) dx + \left(\int_{B_R} |w_\varepsilon|^{\frac{4N}{N+2}} dx \right)^{\frac{N+2}{N}} \right] = -\infty \quad (26)$$

and also that

$$\frac{1}{\varepsilon^\delta} \left[\int_{B_{2R} \setminus B_R} (v_\varepsilon^2 - \mu v_\varepsilon^q) dx + \left(\int_{B_{2R} \setminus B_R} |v_\varepsilon|^{\frac{4N}{N+2}} dx \right)^{\frac{N+2}{N}} \right] \quad (27)$$

is bounded.

Verification of (26). Let

$$I_\varepsilon := \frac{1}{\varepsilon^\delta} \left[\int_{B_R} (w_\varepsilon^2 - \mu w_\varepsilon^q) dx + \left(\int_{B_R} |w_\varepsilon|^{\frac{4N}{N+2}} dx \right)^{\frac{N+2}{N}} \right]$$

On B_R , by changing variables we have

$$I_\varepsilon \leq \varepsilon^{1-\delta} \left[C_1 \int_0^{\frac{R}{\sqrt{\varepsilon}}} \frac{r^{N-1}}{(1+r^2)^{N-2}} dr - \mu C_2 \varepsilon^{-\frac{(N-2)q}{4} + \frac{N}{2} - 1} \int_0^{\frac{R}{\sqrt{\varepsilon}}} \frac{r^{N-1}}{(1+r^2)^{(N-2)q/2}} dr \right. \\ \left. + C_3 \varepsilon^{\frac{4-N}{2}} \left(\int_0^{\frac{R}{\sqrt{\varepsilon}}} \frac{r^{N-1}}{(1+r^2)^{\frac{2N(N-2)}{N+2}}} dr \right)^{\frac{N+2}{N}} \right] \quad (28)$$

where C_i depends only on N .

Now we distinguish the cases $N \geq 6$, $N = 5$, $N = 4$ and $N = 3$ as follows:

Case 1. $N \geq 6$

It is not difficult to see that for $N \geq 6$ and $q > 2$ all integrals in (28) are convergent as $\varepsilon \rightarrow 0$. Besides we also have $-\frac{(N-2)}{4}q + \frac{N}{2} - 1 > \frac{4-N}{2}$ for $2 < q < 2^*$, then $I_\varepsilon \rightarrow -\infty$ as $\varepsilon \rightarrow 0$, proving (26).

Case 2. $N = 5$

As in *Case 1* all integrals in (28) are convergent as $\varepsilon \rightarrow 0$ for $N = 5$ and $2 < q < 2^*$.

There are two cases to be considered: either $2 < q < \frac{8}{3}$ or $\frac{8}{3} \leq q < 2^*$. For $2 < q < \frac{8}{3}$ we immediately see that $-\frac{(N-2)}{4}q + \frac{N}{2} - 1 > \frac{4-N}{2}$ and for $\frac{8}{3} \leq q < 2^*$ we choose $\mu = e^{1/\varepsilon}$. So in both cases we get $I_\varepsilon \rightarrow -\infty$ as $\varepsilon \rightarrow 0$.

Case 2. $N = 4$

Using the fact that $q < 2^* = 4$ and by computing

$$\int_0^{\frac{R}{\sqrt{\varepsilon}}} \frac{r^3}{(1+r^2)^2} dr = \frac{1}{2} \left(\log\left(1 + \frac{R^2}{\varepsilon}\right) + \frac{\varepsilon}{\varepsilon + R^2} - 1 \right)$$

and

$$\int_0^{\frac{R}{\sqrt{\varepsilon}}} \frac{r^3}{(1+r^2)^4} dr = \frac{1}{12} - \frac{\varepsilon^2(\varepsilon + 3R^2)}{12(\varepsilon + R^2)^3}$$

we get

$$\begin{aligned} I_\varepsilon &\leq \frac{C_1}{2} \left(\log\left(1 + \frac{R^2}{\varepsilon}\right) + \frac{\varepsilon}{\varepsilon + R^2} - 1 \right) - \mu C_2 \varepsilon^{\frac{2-q}{2}} \left(\frac{1}{12} - \frac{\varepsilon^2(\varepsilon + 3R^2)}{12(\varepsilon + R^2)^3} \right) + \\ &+ C_3 \left(\int_0^{\frac{R}{\sqrt{\varepsilon}}} \frac{r^3}{(1+r^2)^{8/3}} \right)^{3/2} \end{aligned}$$

But since

$$\lim_{\varepsilon \rightarrow 0} \frac{\varepsilon^{\frac{2-q}{2}}}{\log\left(1 + \frac{R^2}{\varepsilon}\right)} = +\infty$$

we conclude that $I_\varepsilon \rightarrow -\infty$ as $\varepsilon \rightarrow 0$.

Case 3. $N = 3$

By simple computations, one gets

$$\int_0^{\frac{R}{\sqrt{\varepsilon}}} \frac{r^2}{1+r^2} dr = \frac{R}{\sqrt{\varepsilon}} - \arctan\left(\frac{R}{\sqrt{\varepsilon}}\right)$$

then, arguing as in the proof of the case $N = 4$,

$$\begin{aligned} I_\varepsilon &\leq C_1 R - C_1 \varepsilon^{1/2} \arctan\left(\frac{R}{\sqrt{\varepsilon}}\right) - \mu C_2 \varepsilon^{\frac{4-q}{4}} \int_0^{\frac{R}{\sqrt{\varepsilon}}} \frac{r^2}{(1+r^2)^{q/2}} dr + \\ &+ C_3 \varepsilon \left(\int_0^{\frac{R}{\sqrt{\varepsilon}}} \frac{r^2}{(1+r^2)^{6/5}} dr \right)^{5/3} \\ &\leq C_1 R - \mu C_2 \varepsilon^{\frac{4-q}{4}} \int_0^{\frac{R}{\sqrt{\varepsilon}}} \frac{r^2}{(1+r^2)^{q/2}} dr + C_3 R^{5/3} \varepsilon^{1/6} \end{aligned}$$

We have to distinguish two cases: either $2 < q \leq 4$ or $4 < q < 2^*$.

The case $4 < q < 2^*$ was proved by Cassani [6]. However, we can also show (26) using the last inequality, since the integral $\int_0^\infty \frac{r^2}{(1+r^2)^{q/2}} dr$ converges.

If $2 < q \leq 4$ and noting that $\int_0^\infty \frac{r^2}{(1+r^2)^{q/2}} dr \geq \frac{\pi}{4}$ we conclude

$$I_\varepsilon \leq C_4 - \frac{\pi}{4} \mu C_2 \varepsilon^{\frac{4-q}{4}}$$

Finally, choosing $\mu = \varepsilon^{-\frac{1}{2}}$, we infer that $I_\varepsilon \rightarrow -\infty$ as $\varepsilon \rightarrow 0$.

Hence this proves (26).

Verification of (27). We have

$$\begin{aligned} & \frac{1}{\varepsilon^\delta} \left[\int_{B_{2R} \setminus B_R} (v_\varepsilon^2 dx - \mu v_\varepsilon^q) dx + \left(\int_{B_{2R} \setminus B_R} |v_\varepsilon|^{2 \cdot 2^*/(2^*-1)} dx \right)^{2 \cdot (2^*-1)/2^*} \right] \leq \\ & \leq \frac{C_1}{\varepsilon^\delta} \int_{B_{2R} \setminus B_R} \varphi^2 u_\varepsilon^2 dx + \frac{C_3}{\varepsilon^\delta} \left(\int_{B_{2R} \setminus B_R} \varphi^{2 \cdot 2^*/(2^*-1)} |u_\varepsilon|^{2 \cdot 2^*/(2^*-1)} dx \right)^{2 \cdot (2^*-1)/2^*} \\ & \leq C_1 \varepsilon \|\varphi\|_{H^1(B_{2R} \setminus B_R)}^2 + C_2 \varepsilon^{2+\delta} \|\varphi^{2^*/(2^*-1)}\|_{H^1(B_{2R} \setminus B_R)}^{2 \cdot (2^*-1)/2^*} \end{aligned}$$

where we choose R large such that $u_\varepsilon^2 \leq \varepsilon^{1+\delta}$, $\forall |x| \geq \delta$. Then we conclude that equation (27) is bounded.

Consequently, the proof of Claim 2 is complete. \square

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