SOME GENERALIZATIONS OF THE BORSUK-ULAM THEOREM

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ABSTRACT. Let S^n be the n-dimensional sphere, $A : S^n \to S^n$ the antipodal involution and R^n the n-dimensional euclidean space. The famous Borsuk-Ulam Theorem states that, if $f : S^n \to R^n$ is any continuous map, then there exists a point $x \in S^n$ such that $f(x) = f(A(x))$. In this paper we discuss some generalizations and variants of this theorem concerning the replacement either of the domain (S^n, A) by other free involution pairs (X, T), or of the target space R^n by more general topological spaces. For example, we consider the cases where: i) (S^2, A) is replaced by a product involution $(X, T) \times (Y, S) = (X \times Y, T \times S)$, where X and Y are Hausdorff and pathwise connected topological spaces, the involution T is free and the fundamental group of X is a torsion group; ii) R^n is replaced by $M^r \times N^s$, where M^r and N^s are closed manifolds with dimensions r and s, respectively, and $r + s = n$; iii) (S^2, A) is replaced by a product involution as described in i), and R^2 is replaced by the 2-dimensional torus T^2. We remark that i) includes the case in which $(X, T) \times (Y, S) = (X, T)$, by taking $(Y, S) = (\{point\}, identity)$, and in particular the popular 2-dimensional Borsuk-Ulam Theorem.

1. **Introduction**

Generalizations of the Borsuk-Ulam Theorem as mentioned in the abstract can be placed in the following general setting: let X, Y be topological spaces, where X is equipped with a free involution $T : X \to X$, that is, with $T(x) \neq x$ for every $x \in X$. We say that $\{(X, T), Y\}$ satisfies the Borsuk-Ulam Theorem (in an abbreviated form, *satisfies BUT*) if, given any continuous map $f : X \to Y$, there exists at least one point $x \in X$ so that $f(x) = f(T(x))$. Results of this type obtained by replacing (S^n, A) by more general free involution pairs (X, T) can be found, for example, in [2], [10], [12] and [13]. In general lines, in these

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papers S^n is replaced by spaces X subject to certain homological conditions, and the free involutions $T: X \to X$ are arbitrary. Results referring to the replacement of R^n by other spaces can be found, for example, in [11] ($Y=$ a differentiable manifold), [8] ($Y=$ a compact topological manifold), [1] ($Y=$ a generalized manifold), [4], [5], [6] and [7] ($Y=$ a finite CW-complex). Let X be a Hausdorff and pathwise connected topological space, equipped with a free involution $T: X \to X$. Let X/T be the orbit space of X by T and $p: X \to X/T$ the quotient map. Take a point $a \in X$, and consider the homomorphism induced in the fundamental groups, $p_*: \pi_1(X, a) \to \pi_1(X/T, \overline{a})$, where $\overline{a} = p(a)$. Denote by $h_X: \pi_1(X, a) \to H_1(X)$ the Hurewicz homomorphism, where $H_1(X)$ is the one-dimensional Z-homology group of X. In Section 2, we will obtain the following algebraic criterion for $\{(X,T), R^2\}$ satisfy BUT:

Theorem 2.1. *Set $G = \pi_1(X/T, \overline{a}) - p_*(\pi_1(X, a))$. If there exists $\beta \in G$ such that $h_{X/T}(\beta)$ is a torsion element in $H_1(X/T)$, then $\{(X,T), R^2\}$ satisfies BUT.*

As a consequence, we get that involution pairs as those mentioned in the abstract (item i)), together with the target space R^2, satisfy BUT, and in particular the fact that if $\pi_1(X, a)$ is a torsion group, then for any free involution $T: X \to X$, $\{(X,T), R^2\}$ satisfies BUT, which includes the popular 2-dimensional Borsuk-Ulam Theorem. Another consequence is the fact that $\{(S,T), R^2\}$ satisfies BUT, where S is any closed orientable surface with Euler characteristic congruent to 2 *mod* 4 (which includes S^2) and T is any free involution on S.

In Section 3, we consider the following weak version of the Borsuk-Ulam Theorem (WBUT): if G is a topological group, we say that $\{(X,T), G\}$ satisfies WBUT if, for every map $f: X \to G$, there exists $x \in X$ such that $f(x) = f(T(x))$ *(mod 2-torsion)*. We will see that, for all involution pairs (X,T) considered in Section 2, $\{(X,T), T^2\}$ satisfies WBUT, where T^2 is the 2-dimensional torus, considered with its additive structure (*mod* 1).

Given a topological space X, we define $BUT(X)$ as the smallest natural number n so that $\{(S^n, A), X\}$ satisfies BUT. In Section 4 we make some considerations about this number, which is a topological invariant. We will see that, if X is a finite n-dimensional CW-complex, then $n \leq BUT(X) \leq 2n$, and if X is a closed n-dimensional manifold, then $BUT(X) = n$ or $n+1$. This raises the question of finding $BUT(X)$ for specific n-dimensional CW-complexes X (or specific closed n-dimensional manifolds X). For example, we will see that, if X is a closed n-dimensional manifold satisfying the fact that its top-dimensional nonzero Z_2-cohomology class $\alpha \in H^n(X, Z_2)$ is a cup product of lower dimensional classes, then $BUT(X) = n$.

2. A result related to the 2-dimensional Borsuk-Ulam Theorem in terms of the fundamental group

We will prove the algebraic criterion for $\{(X,T), R^2\}$ satisfy BUT given by Theorem 2.1, maintaining the notation used to state the result; we will use simple facts concerning covering spaces. If $\sigma : I = [0,1] \to X$ is a path with $\sigma(0) = a$, we denote by $[\sigma]$ the homotopy equivalence class of σ relative to the base point a; we set σ^{-1} for the inverse path $t \to \sigma(1-t)$. To prove the result, suppose by contradiction that there exists a continuous map $f : X \to R^2$ with $f(x) \neq f(T(x))$ for every $x \in X$. Then a standard and well known construction yields an equivariant map $F : X \to S^1$, that is, satisfying $F(T(x)) = -F(x)$ for every $x \in X$. Set $q : S^1 \to S^1/A$ for the quotient map. Because F is equivariant, it induces a continuous map $\overline{F} : X/T \to S^1/A$ in such a way that the diagram

$$\begin{array}{ccc} X & \xrightarrow{F} & S^1 \\ \downarrow p & & \downarrow q \\ X/T & \xrightarrow{\overline{F}} & S^1/A \end{array}$$

is commutative. Set $F(a) = z$, that is, $q(z) = \overline{z} = \overline{F}(\overline{a})$. Take $\beta \in G$ so that $h_{X/T}(\beta)$ is a torsion element in $H_1(X/T)$, and consider the commutative diagram

$$\begin{array}{ccc}
\pi_1(X/T,\overline{a}) & \xrightarrow{\overline{F}_*} & \pi_1(S^1/A,\overline{z}) \\
\Big\downarrow{\scriptstyle h_{X/T}} & & \Big\downarrow{\scriptstyle h_{S^1/A}} \\
H_1(X/T) & \xrightarrow{\overline{F}_*} & H_1(S^1/A).
\end{array}$$

Then $\overline{F}_* h_{X/T}(\beta)$ is a torsion element in $H_1(S^1/A) \cong Z$, which means that $\overline{F}_* h_{X/T}(\beta) = 0$. Now choose a loop α in X/T which represents β. Then there exists a lifting $\overline{\alpha} : I \to X$ for α with $\overline{\alpha}(0) = a$. Since β does not belong to the image of p_*, one necessarily has that $\overline{\alpha}(1) = T(a)$, and since F is equivariant, $F\overline{\alpha}$ is a path in S^1 with initial point z and final point $-z$. Choose generators $c \in \pi_1(S^1, z)$, $d \in \pi_1(\frac{S^1}{A}, \overline{z})$, and a path μ in S^1 with initial point z and final point $-z$, so that $q\mu$ is a loop in $\frac{S^1}{A}$ representing d. Then the usual product of paths $(F\overline{\alpha}).(\mu^{-1})$ is a loop in S^1 with base point z, which means that, in $\pi_1(S^1, z)$, $[(F\overline{\alpha}).(\mu^{-1})] = rc$ for some $r \in Z$. Since q has degree two, on then has $q_*([(F\overline{\alpha}).(\mu^{-1}])) = [qF\overline{\alpha}] - [q\mu] = [qF\overline{\alpha}] - d = rq_*(c) = \pm 2rd$. Thus $[qF\overline{\alpha}] \neq 0$, and since $h_{S^1/A}$ is an isomorphism, $h_{S^1/A}([qF\overline{\alpha}]) \neq 0$. This contradicts the fact that $h_{S^1/A}([qF\overline{\alpha}]) = \overline{F}_* h_{X/T}(\beta) = 0$.

Corollary 2.2. *Let X be a Hausdorff and pathwise connected space, and (X, T) a free involution pair. If the fundamental group of X is a torsion group (which includes the case in which $X = S^2$), then $\{(X, T), R^2\}$ satisfies BUT.*

Proof. Choose a base point $a \in X$ and write $p : X \to X/T$ for the quotient map. Set $p(a) = \overline{a}$. One has that $p : X \to X/T$ is a two-fold covering, with $\{Identity, T\}$ being the group of deck transformations of this covering. In this way, the quotient group

$$\frac{\pi_1(X/T, \overline{a})}{p_*(\pi_1(X, a))}$$

is isomorphic to Z_2, the cyclic group of two elements. This means that $p_*(\pi_1(X, a))$ is a subgroup of $\pi_1(X/T, \overline{a})$ of index two. The fact that $\pi_1(X, a)$ is a torsion group then implies that $\pi_1(X/T, \overline{a})$ is a torsion group. In fact, if $\beta \in \pi_1(X/T, \overline{a}) - p^*(\pi_1(X, a))$, then $\beta^2 \in p^*(\pi_1(X, a))$ and so β^2 is a torsion element. In this way, β is a torsion element, which ends the proof. $\square$

Corollary 2.3. *Let X, Y be Hausdorff and pathwise connected spaces, and (X, T) a free involution pair like those of Theorem 2.1 (which particularly includes the case in which the fundamental group of X is a torsion group). Let (Y, S) be a involution pair, which is not necessarily free. Then $\{(X, T) \times (Y, S), R^2\}$ satisfies BUT.*

Proof. Take points $a \in X$, $c \in Y$, and write $p : X \to X/T$ and $q : X \times Y \to X \times Y/T \times S$ for the quotient maps. Set $p(a) = \overline{a}$ and $q(a, c) = \overline{(a, c)}$. Consider the maps $\theta : X \to X \times Y$, $\Phi : X \times Y \to X$, $\theta(x) = (x, c)$, $\Phi(x, y) = x$. Then θ and Φ induce maps $\overline{\theta} : X/T \to X \times Y/T \times S$, $\overline{\Phi} : X \times Y/T \times S \to X/T$ so that $\overline{\Phi} \circ \overline{\theta}$ is the identity map. Take $\beta \in G = \pi_1(X/T, \overline{a}) - p_*(\pi_1(X, a))$ with $h_{X/T}(\beta)$ being a torsion element in $H_1(X/T)$. Then $\overline{\theta}_*(h_{X/T}(\beta))$ is a torsion element in $H^1(X \times Y/T \times S)$, and since $\overline{\theta}_* \circ h_{X/T} = h_{X \times Y/T \times S} \circ \overline{\theta}_*$, it suffices to show that $\overline{\theta}_*(\beta) \in \pi_1(X \times Y/T \times S, \overline{(a, c)})$ does not belong to $q_*(\pi_1(X \times Y, (a, c))$. Otherwise, suppose $\overline{\theta}_*(\beta) = q^*(\omega)$ for some $\omega \in \pi_1(X \times Y, (a, c))$. Then $\beta = \overline{\Phi}_*(\overline{\theta}_*(\beta)) = \overline{\Phi}_*(q^*(\omega)) = p_*(\Phi_*(\omega))$, which contradicts the fact that $\beta \notin p_*(\pi_1(X, a))$. □

Remark. Concerning product involution pairs $(X \times Y, T \times S)$ with T without fixed points, we note that if $\{(X, T), Z\}$ satisfies BUT and S has a fixed point, then it is easy to prove directly that $\{(X \times Y, T \times S), Z\}$ satisfies BUT. However, if S does not have fixed points, then we have no topological way to prove that $\{(X \times Y, T \times S), Z\}$ satisfies BUT, even if also $\{(Y, S), Z\}$ satisfies BUT.

Corollary 2.4. (D. L. Gonçalves, [3]) *Let S be a closed orientable surface with Euler characteristic congruent to 2 mod 4 and T a free involution on S. Then $\{(S, T), R^2\}$ satisfies BUT.*

Proof. S/T is a non-orientable closed surface with odd Euler characteristic, and in this case it is well known that there exists an element $\beta \in \pi_1(S/T)$ so that $h_S(\beta) \in H_1(S)$ is a torsion element, and that β belongs to $\pi_1(S/T) - p_*(\pi_1(S))$. □

3. A weak Borsuk-Ulam theorem for maps into the 2-dimensional torus

Let (X, T) be a free involution pair and G a topological group. Set $i : G \to G$ for the involution $i(g) = g^{-1}$ and $2G$ for the set $\{g \in G \ / \ i(g) = g\}$; evidently, the neutral element $e \in G$ belongs to $2G$. Note that the validity of BUT for $\{(X, T), G\}$ is equivalent to the fact that, for every $f : X \to G$, $F^{-1}(e)$ is nonempty, where $F : (X, T) \to (G, i)$ is the equivariant map $F(x) = f(x).(f(T(x)))^{-1}$. This motivates the following extension of the BUT property: we say that $\{(X, T), G\}$ satisfies the *weak Borsuk-Ulam Theorem* (in an abbreviated form, satisfies *WBUT*) if $F^{-1}(2G)$ is nonempty for every $f : X \to G$. If $2G = \{e\}$, BUT is equivalent to WBUT; for example, this happens with $G = R^n$, considered with its additive structure. We want to consider the case in which G is the 2-dimensional torus $T^2 = \frac{[0,1]\times[0,1]}{\sim}$, where $\sim$ identifies $(t, 0)$ to $(t, 1)$ and $(0, t)$ to $(1, t)$, considered with its additive structure $(mod\ 1)$.

Theorem 3.1. *Let (X, T) be an involution pair like those of Theorem 2.1. Then $\{(X, T), T^2\}$ satisfies WBUT.*

Proof. The argument follows the lines of the proof of Theorem 2.1, but with more technical sophistication. One has $2T^2 = \{r_1 = (0, 0), r_2 = (0, \frac{1}{2}), r_3 = (\frac{1}{2}, 0), r_4 = (\frac{1}{2}, \frac{1}{2})\}$. Consider $K \subset T^2$, $K = ([0, 1] \times \{\frac{1}{4}\}) \cup ([0, 1] \times \{\frac{3}{4}\}) \cup (\{\frac{1}{4}\} \times [0, 1]) \cup (\{\frac{3}{4}\}) \times [0, 1])/\sim$. K is invariant under the map $i : T^2 \to T^2$, and (K, i) is a free involution pair. Write $q : K \to K/i$ for the quotient map. We need to describe the homomorphism $q_* : \pi_1(K, p) \to \pi_1(K/i, q(p))$, where the base point p is $p = (\frac{1}{4}, \frac{1}{4})$. To do this, set $A = \{(t, \frac{1}{4}),\ t \in I\}$, $B = \{(t, \frac{3}{4}),\ t \in I\}$, $C = \{(\frac{1}{4}, t),\ t \in I\}$, $D = \{(\frac{3}{4}, t),\ t \in I\}$ and $E = (\{(t, \frac{1}{4}),\ \frac{1}{4} \leq t \leq \frac{3}{4}\}) \cup (\{(\frac{3}{4}, t),\ \frac{1}{4} \leq t \leq \frac{3}{4}\}) \cup (\{(t, \frac{3}{4}),\ \frac{1}{4} \leq t \leq \frac{3}{4}\}) \cup (\{(\frac{1}{4}, t),\ \frac{1}{4} \leq t \leq \frac{3}{4}\})$. Then $\pi_1(K, p)$ is a free group in the generators a, b, c, d and e, which can be represented by loops whose images are $\frac{A}{\sim}$, $\frac{B}{\sim}$, $\frac{C}{\sim}$, $\frac{D}{\sim}$ and $\frac{E}{\sim}$, respectively. Up to isomorphism, $i_* : \pi_1(K, p) \to \pi_1(K, i(p))$ is the degree 2 isomorphism given by $i_*(a) = b$, $i_*(c) = d$ and $i_*(e) = e$; further, $i : \frac{E}{\sim} \to \frac{E}{\sim}/i$ is a two-fold covering. Therefore

$\pi_1(K/i, q(p))$ is a free group in generators x, y and z, and up to isomorphism $q_* : \pi_1(K, p) \to \pi_1(K/i, q(p))$ can be described by $q_*(a) = x$, $q_*(b) = zxz^{-1}$, $q_*(c) = y$, $q_*(d) = zyz^{-1}$ and $q_*(e) = z^2$. If a word $\mathcal{J}$ in a free group has a letter u, denote by $s(u)$ the algebraic sum of the powers of u occuring in $\mathcal{J}$. Then, if $\mathcal{J} \in \pi_1(K)$ is a word in the letters a, b, c, d and e, $q_*(\mathcal{J}) \in \pi_1(K/i)$ is a word in the letters x, y and z with $s(x) = s(a) + s(b)$, $s(y) = s(c) + s(d)$ and $s(z) = 2s(e)$.

We are now ready to proceed with the proof. Suppose by contradiction one has a map $f : X \to T^2$ so that the corresponding equivariant map $F : X \to T^2$ maps X into $T^2 - 2T^2$. We assert that there is an equivariant homotopy equivalence $h : (T^2 - 2T^2, i) \to (K, i)$. In fact, note that $T^2 - K$ is the disjoint union of four open disks, each one with one of the r_i's in the center. Then h can be constructed in an equivariant way by using the radial projection around r_i, for each $i = 1, 2, 3, 4$. This gives the equivariant map $g = hF : X \to K$ and the commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{g} & K \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle q} \\ X/T & \xrightarrow{\overline{g}} & K/i, \end{array}$$

where $p : X \to X/T$ is the quotient map. Choose an initial base point $v \in X$. Since K is pathwise connected, up to isomorphism the corresponding base point $g(v) \in K$ can be replaced by $p = (\frac{1}{4}, \frac{1}{4})$. Thus, without loss, in what follows $\pi_1(K)$ will always be considered with base point p, and so we will omit mention to base points. One has the commutative diagram

$$\begin{array}{ccc} \pi_1(X) & \xrightarrow{g_*} & \pi_1(K) \\ {\scriptstyle p_*}\downarrow & & \downarrow{\scriptstyle q_*} \\ \pi_1(X/T) & \xrightarrow{\overline{g_*}} & \pi_1(K/i) \\ {\scriptstyle h_{X/T}}\downarrow & & \downarrow{\scriptstyle h_{K/i}} \\ H_1(X/T) & \xrightarrow{\overline{g_*}} & Z \times Z \times Z. \end{array}$$

Take $\beta \in G = \pi_1(X/T) - p_*(\pi_1(X))$ as described in the hypothesis. Then, using the fact that $Z \times Z \times Z$ has not torsion elements, we get that $\overline{g}_* h_{X/T}(\beta) =$

0. On the other hand, choosing a loop α in X/T which represents β, there exists a lifting $\overline{\alpha} : I \to X$ for α with endpoints forming an orbit of T, and because g is equivariant, $g\overline{\alpha}$ is a path in K joining the points $w, i(w) \in K$, where $w = g(v)$. Now we can choose a path μ in K joining w to $i(w)$ so that $q\mu$ is a loop in K/i representing z: if $w \in E$, μ can be taken as a direct path in E from w to $i(w)$, and if $w \notin E$, μ can be taken as a path in K which runs through two consecutive edges of the square E. Then the path $(g\overline{\alpha}).(\mu^{-1})$ is a loop in K, which means that $[(g\overline{\alpha}).(\mu^{-1})] \in \pi_1(K)$ is a word $\mathcal{J}$ in the letters a, b, c, d and e. It follows that $q_*([(g\overline{\alpha}).(\mu^{-1}])) = [qg\overline{\alpha}].[q\mu^{-1}] = [qg\overline{\alpha}].z^{-1} = q_*(\mathcal{J})$. By the description of q_*, $q_*(\mathcal{J})$ is a word in the letters x, y and z such that the sum of the powers of z is even, and thus $[qg\overline{\alpha}]$ is a word in x, y, z with the sum of powers of z being odd. Then the coordinate of $h_{K/i}([qg\overline{\alpha}])$ corresponding to the $h_{K/i}(z)$-factor is nonzero, which gives that $h_{K/i}([qg\overline{\alpha}]) = \overline{g}_* h_{X/T}(\beta)$ is nonzero. □

Next we will give a source of explicit examples of free involutions pairs (X, T) for which $\{(X, T), T^2\}$ satisfies WBUT. This will be strongly based on the paper [3] of D. Gonçalves. Suppose $T, P : X \to X$ are free involutions on X. We say that (X, T) and (X, P) are equivalent if there is an equivariant homeomorphism $f : (X, T) \to (X, P)$. This is an equivalence relation on the set of all free involutions on X; denote by $Inv(X)$ the set of equivalence classes. $Inv(X)$ may be empty; for example, if X has odd Euler characteristic, or if X is a smooth closed manifold that does not bound. The equivalence classes $(X, [T])$ are suitable objects to study the BUT (WBUT) property; in fact, if (X, T) and (X, P) are equivalent, then, for every space Y, $\{(X, T), Y\}$ satisfies BUT (WBUT) if and only if $\{(X, P), Y\}$ satisfies BUT (WBUT). In [3], D. Gonçalves studied the BUT property for $\{(S, T), R^2\}$, where S is a closed surface (that is, a 2-dimensional closed manifold) and S is any free involution on S. It was shown that, if S is orientable, or if S is nonorientable and the Euler characteristic of S is even, then $Inv(S)$ is nonempty (in the remaining cases, it is known that $Inv(S)$ is empty, because S does not bound). Further, it was shown that $Inv(S)$ has $2^r - 1$ elements, where r is the number of elements of a canonical system of generators of $\pi_1(S)$. For any S for which $Inv(S)$ is nonempty, the elements of $Inv(S)$ which satisfy the mentioned BUT property were explicitly determined; for example, if S is orientable and the Euler characteristic of S is congruent to $2 \ mod \ 4$ (which includes the 2-dimensional sphere), then $\{(S,T), R^2\}$ satisfies BUT for every class $[T]$ (see Corolary 2.4).

Corollary 3.2. *Let S be a closed surface for which $Inv(S)$ is nonempty. If (S,T) is a free involution pair, then $\{(S,T), T^2\}$ satisfies WBUT if and only if $\{(S,T), R^2\}$ satisfies BUT.*

Proof. Suppose that $\{(S,T), R^2\}$ satisfies BUT. By doing a case-by-case inspection on these (S,T), and taking into account the classification theorem of surfaces, we can see that, for each such a (S,T), there exists an element $\beta \in \pi_1(S/T)$ so that $h_S(\beta) \in H_1(S)$ is a torsion element, and with β belonging to $\pi_1(S/T) - p_*(\pi_1(S))$. Now suppose that $\{(S,T), R^2\}$ does not satisfy BUT. Then there exists a continuous map $f : S \to R^2$ such that $f(x) \neq f(T(x))$ for every $x \in S$. Take a homeomorphism $g : R^2 \to B$, where B is an open ball centered in $(0,0)$ with radius equal to $\frac{1}{16}$, and consider the usual universal covering $h : R^2 \to T^2$. Then the equivariant map $F : (S,T) \to (T^2, i)$ which corresponds to the map $hgf : S \to T^2$ clearly satisfies the fact that $F^{-1}(2T^2)$ is empty. □

To increase the source of free involutions (X,T) for which $\{(X,T), T^2\}$ satisfies WBUT, one still has

Corollary 3.3. *Consider product involutions $(X \times Y, T \times S)$ like those of Corollary 2.3. Then $\{(X \times Y, T \times S), T^2$ satisfies WBUT.*

4. A topological invariant coming from the Borsuk-Ulam theorem

Let X be a topological space. Taking into account the standard equivariant inclusion $S^{n-1} \to S^n$, it is easy to see that, if $\{(S^n, A), X\}$ satisfies BUT, then $\{(S^m, A), X\}$ satisfies BUT for every $m > n$, and if $\{(S^n, A), X\}$ does not satisfy BUT, then the same is true for $\{(S^m, A), X\}$ with $m < n$. Then either $\{(S^n, A), X\}$ does not satisfy BUT for every natural number n, or there exists the smallest natural number n for which $\{(S^n, A), X\}$ satisfies BUT. In the first case, we write $BUT(X) = \infty$, and in the second $BUT(X) = n$. If there exists a continuous injective map $X \to Y$ (and in particular if X is a subspace of Y), then $BUT(X) \leq BUT(Y)$, and in particular $BUT(X)$ is a topological invariant (but not a homotopic invariant). Evidently, $BUT(\{point\}) = 0$ and $BUT(X) > 0$ if X has at least two points; in this case, $BUT(X) = 1$ if X is a discrete space. If $X = \{a, b\}$ has two points and is equipped with the trivial topology, then $BUT(X) = \infty$; in fact, using induction on n we can construct, for every $n \geq 0$, a subset $P \subset S^n$ such that $P \cap A(P) = \emptyset$ and $P \cup A(P) = S^n$. Next, we consider the map $f : S^n \to X$ that sends P into a and $A(P)$ into b. We also have $BUT(S^\infty) = \infty$, where $S^\infty = lim_n(S^n)$ with the weak topology. The Borsuk-Ulam Theorem implies that $BUT(R^n) \leq n$, and in fact $BUT(R^n) = n$. This is a very special particular case of the following

Theorem 4.1. *Suppose X is a finite n-dimensional CW-complex. Then*

i) $n \leq BUT(X) \leq 2n$;

ii) if X is a not closed topological manifold (which includes R^n), then $BUT(X) = n$;

iii) if X is a closed topological manifold, then $BUT(X) = n$ or $n+1$. In this case, if X satisfies the fact that its top-dimensional nonzero Z_2-cohomology class $\alpha \in H^n(X, Z_2)$ is a cup product of lower dimensional classes, then $BUT(X) = n$.

Proof. In fact, the stated results follow immediately from known results of the literature. Inside the interior of an n-cell of X we can take a copy homeomorphic of S^{n-1}, which means that $BUT(X) \geq n$. On the other hand, in [5], D. Gonçalves, P. Pergher and J. Jaworowski proved that, if $m \geq 2n$, then $\{(S^m, A), X\}$ satisfies BUT. This gives i). In [11], P. E. Conner and E. E. Floyd proved that if X is a differentiable manifold and $m > n$, then $\{(S^m, A), X\}$ satisfies BUT; in this case, they also proved that, if $m = n$ and $f : S^m \to X$ is a continuous map satisfying the fact that its induced homomorphism in Z_2-cohomology $f^* : H^n(X) \to H^n(S^m)$ is trivial, then there exists $x \in S^m$ with

$f(x) = f(A(x))$. In [8], Munkholm showed the same result without the differentiability hypotheses, but with X compact. Recently, in [1], this Conner-Floyd result was proved for X a generalized manifold not necessarily compact (see the remark below). Since $H^n(X, Z_2) = 0$ if X is a not closed manifold, ii) and the first statement of iii) are established. The second statement of iii) follows from the fact that, for any map $f : S^n \to X$, $f^* : H^n(X) \to H^n(S^n)$ is a ring homomorphism and $H^j(S^n) = 0$ if $0 < j < n$. □

Remark. Using iii), we get that $BUT(X) = dim(X)$ if $X = M^r \times N^s$, where M^r and N^s are closed manifolds with dimensions $r, s > 0$. The same is valid for real, complex and quaternionic projective spaces, Dold manifolds and projective space bundles associated to real, complex or quaternionic vector bundles over closed manifolds. One has $BUT(S^n) = n + 1$, and in fact it is the only example we know of closed manifold with this property.

Remark. A *generalized manifold of dimension* n is a topological space X which is an ENR and, for every $x \in X$, $H_*(X, X - \{x\}; Z)$ is isomorphic to $H_*(R^n, R^n - \{0\}; Z)$. Recently, such manifolds have been extensively studied. In [1], C. Biasi, E. L. Santos and D. de Mattos showed that the Conner-Floyd theorem mentioned above remains valid if we replace manifolds by generalized manifolds. In this way, $BUT(X) \leq n + 1$.

Remark. If X is a compact metric space of topological dimension n, then X can be imbedded in R^{2n+1}, which gives that $BUT(X) \leq 2n + 1$ (for the definition of topological dimension, see for example [9]). The same argument shows that, if $BUT(X) = \infty$, then X cannot be imbedded in a euclidean space.

Remark. Take X any pathwise connected topological space, and consider $X^* = (X \times X) - \{(x, y) \in X \times X \ / \ x = y\}$; note that on X^* one has the free involution $T_X(x, y) = (y, x)$. If $H_n(X^*/T_X, Z_2) = 0$, then $BUT(X) \leq n$ (see [12; Theorem 3]).

Remark. The theorem established in this section raises the question of improving the estimative $n \leq BUT(X) \leq 2n$ for special families of n-dimensional

CW-complexes X; for example, see the improvement for manifolds. In this setting, the next natural case after manifolds is the one-point union of two closed n-dimensional manifolds, $M^n \vee V^n$. The first unsolved case is $BUT(M^2 \vee V^2)$, where at least one of these closed surfaces is non-orientable. In this case, $BUT(X)$ may be 3 or 4.

Remark. We point the following additional questions: i) to estimate $BUT(X \times Y)$ in terms of $BUT(X)$ and $BUT(Y)$ (certainly, $maximum\{BUT(X), BUT(Y)\} \leq BUT(X \times Y)$); ii) to find a space X with $BUT(X)$ finite and such that X cannot be imbedded in a euclidean space, or to show that, if $BUT(X)$ is finite, then X can be imbedded in some euclidean space.

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