ON CODIMENSIONS k IMMERSIONS OF m-MANIFOLDS FOR $k = m - 3$, $k = m - 5$ AND $k = m - 6$

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Abstract. Let us consider M a closed smooth connected m-manifold, N a smooth n- manifold and $f : M \to N$ a continuous map with codimension $k = n - m > 0$. In this paper, under certain conditions, we prove that f is homotopic to an immersion, in the following cases: $m \equiv 1(4)$ and codimension $k = m - 3$; $m \equiv 2(4)$, $m \equiv 3(8)$ and codimension $k = m - 5$; $m \equiv 2(4)$, $m \equiv 3(8)$ and codimension $k = m - 6$. This work complements some results of Biasi et al. (Manus. Math. **126**, 527-530, 2008 and Manus. Math. **104**, 97-110, 2001); Koschorke (Lecture Notes in Mathematics, vol. 1350. Springer, Heidelberg, 1988); and of Bang-He Li and Gui-Song Li (Math. Proc. Camb. Phil. Soc. **112**, 281-285, 1992).

1. Introduction

Let M be a smooth m-manifold, N be a smooth n-manifold, and let $f : M \to N$ be a continuous map with codimension $k = n - m > 0$. If $k = m - 1$ it was proved in [6] that f is always homotopic to an immersion. If $k = m - 2$, in general, f is not homotopic to an immersion (see [7]). For this case, recently C. Biasi and A. K. M. Libardi in [1] proved that, on certain conditions, f is homotopic to an immersion. More specifically, Theorem 2.1 of [1] states that: *"Let M be a closed smooth connected manifold of dimension $m \equiv 1(4)$, and let N be a smooth manifold of dimension $2m - 2$. Let $f : M \to N$ be a continuous map such that $f_* : H_1(M; Z_2) \to \check{H}_1(f(M); Z_2)$ is injective. Then f is homotopic to an immersion."*

In this paper, we prove a version of [1, Theorem 2.1] on existence of immersions for codimensions $k = m - 3$, $k = m - 5$ and $k = m - 6$. Specifically, we prove the following theorem:

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Theorem 1.1. *Let M be a closed smooth connected manifold of dimension m, N be a smooth manifold of dimension $2m-i$ and let $f : M \to N$ be a continuous map. Let us suppose that $f_* : H_{i-1}(M; Z_2) \to \check{H}_{i-1}(f(M); Z_2)$ is injective and that one of the following conditions is satisfied:*

(a) $i = 3$ *and* $m \equiv 1(4)$,

$H_1(M; Z_2) = 0$ *and* $H^{m-2}(M; Z)$ *does not have elements of order 4.*

(b) $i = 5$ *and* $m \equiv 2(4)$,

$H_1(M; Z_2) = H_2(M; Z_8) = H_3(M; Z_2) = 0.$

(c) $i = 5$ *and* $m \equiv 3(8)$,

$H_3(M; Z_4) = 0$, $H_1(M; Z_2) = H_1(M; Z_{12}) = 0$ *(or* $H_1(M; Z) = 0$*) and* $H^{m-4}(M; Z)$ *does not have elements of order 4.*

(d) $i = 6$ *and* $m \equiv 2(4)$,

$H_1(M; Z_2) = H_2(M; Z_{24}) = H_3(M; Z_2) = H_4(M; Z_2) = 0$ *and* $H^{m-5}(M; Z)$ *does not have elements of order 4.*

(e) $i = 6$ *and* $m \equiv 3(8)$,

$f^* w_1(N) = w_1(M)$, $H_2(M; Z_2) = H_3(M; Z_8) = H_4(M; Z_2) = 0.$

Then f is homotopic to an immersion.

We observe that this theorem complement some results of [1, 2], [4, 5] and [9], and that this result cover all possible cases which could be obtained by using the Paechter's table [11].

We remark that if $f : M \to f(M) \subset N$ is a homotopy equivalence, where M is a closed smooth connected manifold of dimension m and N is a smooth manifold of dimension $n = 2m - i$, then the induced map $f_* : H_*(M; Z_2) \to \check{H}_*(f(M); Z_2)$ is injective. Therefore, by Theorem 1.1, in the following examples f is homotopic to an immersion.

i) For $i = 3$, $m \equiv 1(4)$, we take $M = S^2 \times S^{m-2}$, $m \neq 3$. In this case, $H_1(M; Z_2) = 0$ and $H^{m-2}(M; Z) = Z$.

ii) $i = 5$ and $m = 8k + 3$, $k \geq 1$, we take $M = S^{8k-1} \times CP^2$. In this case, $H_3(M; Z_4) = 0$, $H_1(M; Z) = 0$ and $H^{m-4}(M; Z) = Z$.

2. Preliminaries and Notations

Let M be a closed connected smooth manifold of dimension m, N be a smooth manifold of dimension n, and $f : M \to N$ a continuous map with codimension $k = n - m > 0$. Let $[M] \in H_m(M; Z_2)$ denote the fundamental class of M and

$U_f \in H^k(N;Z_2)$ the Poincaré dual of $f_*[M] \in H_n^C(N;Z_2)$, i.e., $f_*[M] = D(U_f) = U_f \frown [N]$, where $[N] \in H_n^C(N;Z_2)$ is the fundamental class of N and D is the Poincaré dual isomorphism.

We define a homology class $\theta(f) \in H_{m-k}(M;Z_2)$ by

$$\theta(f) = D(f^*(U_f) - w_k(f)) \in H_{m-k}(M;Z_2),$$

where $w_k(f) = w_k(\nu_f)$ is the k^{th} Stiefel-Whitney class of the stable normal bundle of f (see [2] for more details). We recall that $\theta(f)$ denotes the primary obstruction for f be homotopic to an embedding, that is, if f is homotopic to an embedding then $\theta(f) = 0$.

Throughout the paper, $\check{H}$ denotes the Čech singular homology group, $\pi^p_{m,q}$ denotes the $(m+p)^{th}$ homotopy group of the Stiefel manifold $V_{m+q,q}$, i.e., $\pi^p_{m,q} = \pi_{m+p}(V_{m+q,q})$ and ρ denotes the reduction modulo two homomorphism.

3. Proof of the Theorem 1.1

In this section, let us observe that all obstruction classes lie in cohomological groups with local coefficients system, however, under our assumptions the local coefficients system is always trivial.

Proof of Theorem 1.1(a). Let us consider $f_1 : M \to N \times R$ defined by $f_1(x) = (f(x), 0)$, for each $x \in M$. Since $H_1(M;Z_2) = 0$ by our assumption, we have that $f_{1*} : H_1(M;Z_2) \to \check{H}_1(f_1(M);Z_2)$ is an injective map. Also $m \equiv 1(4)$ and $N \times R$ has dimension $2m-2$. Therefore, it follows from [1, Theorem 2.1] that f_1 is homotopic to an immersion g_1.

On the other hand, Lemmas 6.1 and 6.2 of [2] imply that ${f_1}^*(U_{f_1}) = 0 \in H^{m-2}(M;Z_2)$. If we consider $\tilde{f} : M \to f_1(M)$, defined by $\tilde{f}(x) = f_1(x)$, $x \in M$, by [3, Theorem 1.1], we have that $\tilde{f}_*(\theta(f_1)) = 0$, where $\tilde{f}_* : H_2(M;Z_2) \to \check{H}_2(f_1(M);Z_2)$. Since f_* is injective by assumption, we have that $\tilde{f}_*$ is injective. Therefore, $0 = \theta(f_1) = D({f_1}^*(U_{f_1}) - w_{m-2}(f_1))$ and $w_{m-2}(f_1) = w_{m-2}(g_1) = w_{m-2}(f) = 0$.

Let us consider the stable normal bundle $\nu^{m-2}_{g_1}$. Since the Stiefel manifold $V_{m-2,1}$ is $(m-4)$–connected, $\nu^{m-2}_{g_1}$ has a 1–frame field over the $(m-3)$–skeleton of M. The obstruction to the existence of a 1–frame field over the $(m-2)$–skeleton of M is the class $W_{m-2}(g_1) \in H^{m-2}(M;\pi_{m-3}(V_{m-2,1}))$. Since $m-3 = 4s+2$ it follows from [11] that $\pi_{m-3}(V_{m-2,1}) = \pi^0_{m-3,1} = \pi^0_{4s+2,1} = Z$. By our assumption, $H^{m-2}(M;Z)$

does not have elements of order 4 and since $\rho(W_{m-2}(g_1)) = w_{m-2}(g_1) = 0$, where

$$\rho : H^{m-2}(M; \pi_{m-3}(V_{m-2,1})) \to H^{m-2}(M; Z_2),$$

we have that $W_{m-2}(g_1)$ vanishes. Then there exists a 1–frame field over the $(m-2)$–skeleton of M. The obstruction to the existence of a 1–frame field over the $(m-1)$–skeleton of M is the class $W_{m-1}(g_1) \in H^{m-1}(M; \pi_{m-2}(V_{m-2,1})) = H^{m-1}(M; \pi^1_{m-3,1})$ and $\pi^1_{m-3,1} = Z_2$ (see [11]). By hypothesis, $H_1(M; Z_2) = 0$, then $W_{m-1}(g_1) \in H^{m-1}(M; \pi_{m-2}(V_{m-2,1}))$ vanishes and there exists a 1–frame field over the $(m-1)$–skeleton of M.

Now, let us consider $g_q : M \to (N \times R) \times R^{q-1}$ defined by $g_q(x) = (g_1(x), 0)$, $q \geq 4$. Since $\nu^{m-2}_{g_1}$ has a 1–frame field over the $(m-1)$–skeleton of M, we have that $\nu^{m-3+q}_{g_q}$ has a q–frame field over the $(m-1)$–skeleton of M and the obstruction to the existence of a q–frame field over the m–skeleton of M is the class $W_m(g_q) \in H^m(M; \pi_{m-1}(V_{m-3+q,q})) = H^m(M; \pi^2_{m-3,q})$. Since $\pi^2_{m-3,q} = 0$, for $q \geq 4$ (see [11]), we have that the obstruction class $W_m(g_q) \in H^m(M; \pi_{m-1}(V_{m-3+q,q}))$ vanishes. Hence by a result of Hirsch [8], we conclude that f is homotopic to an immersion.

$\square$

Proof of Theorem 1.1(b)-(c). In this proof we consider simultaneously the cases $m \equiv 2(4)$ and $m \equiv 3(8)$. Let us consider $f_1 : M \to N \times R$ defined by $f_1(x) = (f(x), 0)$, for each $x \in M$. By Lemmas 6.1 and 6.2 of [2] we see that ${f_1}^*(U_{f_1}) = 0 \in H^{m-4}(M; Z_2)$. If we consider $\tilde{f} : M \to f_1(M)$, defined by $\tilde{f}(x) = f_1(x)$, $x \in M$, by [3, Theorem 1.1], we have that $\tilde{f}_*(\theta(f_1)) = 0$, where $\tilde{f}_* : H_4(M; Z_2) \to \check{H}_4(f_1(M); Z_2)$. Since f_* is injective by our assumption, we have that $\tilde{f}_*$ is injective. Therefore $0 = \theta(f_1) = D({f_1}^*(U_{f_1}) - w_{m-4}(f_1))$ and $w_{m-4}(f_1) = w_{m-4}(f) = 0$.

Let us consider $f_q : M \to N \times R^q$, $q \geq 6$. Since $N \times R^q$ has dimension $2m - 5 + q \geq 2m + 1$ we have that f_q is homotopic to an immersion g_q. Therefore, $w_{m-4}(g_q) = w_{m-4}(f_q) = w_{m-4}(f_1) = w_{m-4}(f) = 0$.

Let us consider the stable normal bundle $\nu^{m-5+q}_{g_q}$. Since the Stiefel manifold $V_{m-5+q,q}$ is $(m-6)$–connected, $\nu^{m-5+q}_{g_q}$ has a q–frame field over the $(m-5)$–skeleton of M. The obstruction to the existence of a q–frame field over the $(m-4)$–skeleton of M is the class $W_{m-4}(g_q) \in H^{m-4}(M; \pi_{m-5}(V_{m-5+q,q}))$. Since $m - 5 = 4s + 1$ or $m - 5 = 4s + 2$ and $q \geq 6$ it follows from [11] that $\pi_{m-5}(V_{m-5+q,q}) = \pi^0_{m-5,q} = \pi^0_{4s+1,q} = Z_2$ or $\pi_{m-5}(V_{m-5+q,q}) = \pi^0_{m-5,q} = \pi^0_{4s+2,q} = Z$. In the first case, the homomorphism

$$\rho : H^{m-4}(M; \pi_{m-5}(V_{m-5+q,q})) \to H^{m-4}(M; Z_2)$$

is an isomorphism and $\rho(W_{m-4}(g_q)) = w_{m-4}(g_q) = 0$. In the second case, $H^{m-4}(M;Z)$ does not have elements of order 4 by our assumption and $\rho(W_{m-4}(g_q)) = w_{m-4}(g_q) = 0$. In both cases, the obstruction class $W_{m-4}(g_q)$ vanishes and then there exists a q–frame field over the $(m-4)$–skeleton of M.

The obstruction class to the existence of a q–frame field over the $(m-3)$–skeleton of M is the class $W_{m-3}(g_q) \in H^{m-3}(M;\pi_{m-4}(V_{m-5+q,q}))$, which vanishes, because $\pi_{m-4}(V_{m-5+q,q}) = \pi^1_{m-5,q} = \pi^1_{4s+1,q} = Z_2$ or $\pi_{m-4}(V_{m-5+q,q}) = \pi^1_{m-5,q} = \pi^1_{4s+2,q} = Z_4$, for $q \geq 6$ (see [11]) and by hypothesis, $H_3(M;Z_2) = 0$ or $H_3(M;Z_4) = 0$.

The next obstruction class is $W_{m-2}(g_q) \in H^{m-2}(M;\pi_{m-3}(V_{m-5+q,q}))$. Since the homotopy group $\pi_{m-3}(V_{m-5+q,q}) = \pi^2_{m-5,q} = \pi^2_{4s+1,q} = Z_8$ or $\pi_{m-3}(V_{m-5+q,q}) = \pi^2_{m-5,q} = \pi^2_{4s+2,q} = 0$, for $q \geq 6$ and by assumption $H_2(M;Z_8) = 0$, we have that the obstruction class $W_{m-2}(g_q)$ vanishes and there exists a q–frame field over the $(m-2)$–skeleton of M.

The next obstruction class is $W_{m-1}(g_q) \in H^{m-1}(M;\pi_{m-2}(V_{m-5+q,q}))$. Since the homotopy group $\pi_{m-2}(V_{m-5+q,q}) = \pi^3_{m-5,q} = \pi^3_{4s+1,q} = Z_2$ or $\pi_{m-2}(V_{m-5+q,q}) = \pi^3_{m-5,q} = \pi^3_{4s+2,q} = Z_{12}$, for $q \geq 6$ (see [11]) and by assumption $H_1(M;Z_2) = 0$ or $H_1(M;Z_{12}) = 0$, we have that the obstruction class $W_{m-1}(g_q)$ vanishes and there exists a q–frame field over the $(m-1)$–skeleton of M.

Finally, the obstruction to the existence of a q–frame field over the m–skeleton of M is the class $W_m(g_q) \in H^m(M;\pi_{m-1}(V_{m-5+q,q}))$. Since $m-5 = 4s+5$ or $m-5 = 8s-2$ and $q \geq 6$ it follows from [11] that $\pi_{m-1}(V_{m-5+q,q}) = \pi^4_{m-5,q} = \pi^4_{4s+5,q} = 0$ or $\pi_{m-1}(V_{m-5+q,q}) = \pi^4_{m-5,q} = \pi^4_{8s-2,q} = 0$, and consequently, $W_m(g_q)$ vanishes. Therefore, by Hirsch [8], we conclude that f is homotopic to an immersion. □

Proof of Theorem 1.1(d). Let us consider $f_1 : M \to N \times R$ defined by $f_1(x) = (f(x), 0)$, for each $x \in M$. Since $H_4(M;Z_2) = 0$, we have that $f_{1*} : H_4(M;Z_2) \to \check{H}_4(f_1(M);Z_2)$ is an injective map. Also $m \equiv 2(4)$, $H_1(M;Z_2) = H_2(M;Z) = H_3(M;Z_2) = 0$ and $N \times R$ has dimension $2m-5$. By Theorem 1.1(b), we see that f_1 is homotopic to an immersion g_1.

On the other hand, since $f_* : H_5(M;Z_2) \to \check{H}_5(f(M);Z_2)$ is injective, by similar argument to that in the proof of Theorem 1.1(b)-(c), we have that $0 = \theta(f_1) = D({f_1}^*(U_{f_1}) - w_{m-5}(f_1))$ and $w_{m-5}(f_1) = w_{m-5}(f) = 0$.

Let us consider the stable normal bundle $\nu^{m-5}_{g_1}$. Since $V_{m-5,1}$ is $(m-7)$–connected, $\nu^{m-5}_{g_1}$ has a 1–frame field over the $(m-6)$–skeleton of M. Therefore, the obstruction to the existence of a 1–frame field over the $(m-5)$–skeleton of

M is the class $W_{m-5}(g_1) \in H^{m-5}(M; \pi_{m-6}(V_{m-5,1}))$. Since $m-6 = 4s$ it follows from [11] that $\pi_{m-6}(V_{m-5,1}) = \pi^0_{m-6,1} = \pi^0_{4s,1} = Z$. Since $H^{m-5}(M; Z)$ does not have elements of order 4 and $\rho(W_{m-5}(g_1)) = w_{m-5}(g_1) = 0$, where $\rho : H^{m-5}(M; \pi_{m-6}(V_{m-5,1})) \to H^{m-5}(M; Z_2)$, we have that $W_{m-5}(g_1)$ vanishes. Then, there exists a 1–frame field over the $(m-5)$–skeleton of M.

The next obstruction class is $W_{m-4}(g_1) \in H^{m-4}(M; \pi_{m-5}(V_{m-5,1}))$. Since the homotopy group $\pi_{m-5}(V_{m-5,1}) = \pi^1_{m-6,1} = \pi^1_{4s,1} = Z_2$ (see [11]) and by hypothesis, $H_4(M; Z_2) = 0$, we see that the obstruction class $W_{m-4}(g_1)$ vanishes. Thus, there exists a 1–frame field over the $(m-4)$–skeleton of M.

The obstruction to the existence of a 1-frame field over the $(m-3)$-skeleton of M is the class $W_{m-3}(g_1) \in H^{m-3}(M; \pi_{m-4}(V_{m-5,1}))$, which vanishes, because $\pi_{m-4}(V_{m-5,1}) = \pi^2_{m-6,1} = \pi^2_{4s,1} = Z_2$ and by hypothesis $H_3(M; Z_2) = 0$.

The next obstruction class is $W_{m-2}(g_1) \in H^{m-2}(M; \pi_{m-3}(V_{m-5,1}))$. Since $m - 6 = 4s$, we have that $m-6 = 8s$ or $m-6 = 8s+4$. Therefore, $\pi_{m-3}(V_{m-5,1}) = \pi^3_{m-6,1} = Z_{24}$ (see [11]) and by assumption $H_2(M; Z_{24}) = 0$, we have that the obstruction class $W_{m-2}(g_1)$ vanishes. Thus, there exists a 1–frame field over the $(m-2)$–skeleton of M.

The obstruction to the existence of a 1–frame field over the $(m-1)$–skeleton of M is the class $W_{m-1}(g_1) \in H^{m-1}(M; \pi_{m-2}(V_{m-5,1}))$. Since $\pi_{m-2}(V_{m-5,1}) = \pi^4_{m-6,1} = \pi^4_{4s,1} = 0$, we see that $W_{m-1}(g_1)$ vanishes. Thus, there exists a 1–frame field over the $(m-1)$–skeleton of M.

Finally, the obstruction to the existence of a 1–frame field over the m–skeleton of M is the class $W_m(g_1) \in H^m(M; \pi_{m-1}(V_{m-5,1}))$, which vanishes, since $\pi_{m-1}(V_{m-5,1}) = {\pi^5}_{m-6,1} = {\pi^5}_{4s,1} = 0$. Then, by Hirsch [8], we conclude that f is homotopic to an immersion. □

Proof of Theorem 1.1(e). Let us consider $f_1 : M \to N \times R$ defined by $f_1(x) = (f(x), 0)$, for each $x \in M$. Since $f_* : H_5(M; Z_2) \to \check{H}_5(f(M); Z_2)$ is injective, by similar argument to that in the proof of Theorem 1.1(b)-(c), we have that $0 = \theta(f_1) = D({f_1}^*(U_{f_1}) - w_{m-5}(f_1))$ and $w_{m-5}(f_1) = w_{m-5}(f) = 0$.

Also, if we consider $f_q : M \to N \times R^q$, $q \geq 7$, $N \times R^q$ has dimension $2m-6+q \geq 2m+1$. Therefore f_q is homotopic to an immersion g_q and $w_{m-5}(g_q) = w_{m-5}(f_q) = w_{m-5}(f_1) = w_{m-5}(f) = 0$.

Let us consider the stable normal bundle $\nu^{m-6+q}_{g_q}$. Since the Stiefel manifold $V_{m-6+q,q}$ is $(m-7)$–connected, $\nu^{m-6+q}_{g_q}$ has a q-frame field over the $(m-6)$–skeleton of M. The obstruction to the existence of a q-frame field over the $(m-5)$–skeleton of M is the class $W_{m-5}(g_q) \in H^{m-5}(M; \pi_{m-6}(V_{m-6+q,q}))$. Since $m-6 = 4s+1$ it follows from [11] that $\pi_{m-6}(V_{m-6+q,q}) = \pi^0_{m-6,q} = \pi^0_{4s+1,q} = Z_2$. In this case, the homomorphism

$$\rho : H^{m-5}(M; \pi_{m-6}(V_{m-6+q,q})) \to H^{m-5}(M; Z_2)$$

is an isomorphism and $\rho(W_{m-5}(g_q)) = w_{m-5}(g_q) = 0$. Therefore, $W_{m-5}(g_q)$ vanishes and there exists a q-frame field over the $(m-5)$–skeleton of M.

The next obstruction class is $W_{m-4}(g_q) \in H^{m-4}(M; \pi_{m-5}(V_{m-6+q,q}))$. Since the homotopy group $\pi_{m-5}(V_{m-6+q,q}) = \pi^1_{m-6,q} = \pi^1_{4s+1,q} = Z_2$, for $q \geq 7$ (see [11]) and by hypothesis, $H_4(M; Z_2) = 0$, we have that the obstruction class $W_{m-4}(g_q)$ vanishes. Thus there exists a q–frame field over the $(m-4)$–skeleton of M.

The obstruction to the existence of a q–frame field over the $(m-3)$–skeleton of M is the class $W_{m-3}(g_q) \in H^{m-3}(M; \pi_{m-4}(V_{m-6+q,q}))$, which vanishes, because $\pi_{m-4}(V_{m-6+q,q}) = \pi^2_{m-6,q} = \pi^2_{4s+1,q} = Z_8$, for $q \geq 7$ and by hypothesis $H_3(M; Z_8) = 0$.

The next obstruction class is $W_{m-2}(g_q) \in H^{m-2}(M; \pi_{m-3}(V_{m-6+q,q}))$. Since the homotopy group $\pi_{m-3}(V_{m-6+q,q}) = \pi^3_{m-6,q} = \pi^3_{4s+1,q} = Z_2$, for $q \geq 7$ (see [11]) and by assumption $H_2(M; Z_2) = 0$, we have that the obstruction class $W_{m-2}(g_q)$ vanishes and there exists a q–frame field over the $(m-2)$–skeleton of M.

The next obstruction class is $W_{m-1}(g_q) \in H^{m-1}(M; \pi_{m-2}(V_{m-6+q,q}))$. Since $m-6 = 4s+5$ it follows from [11] that $\pi_{m-2}(V_{m-6+q,q}) = \pi^4_{m-6,q} = \pi^4_{4s+5,q} = 0$, and consequently, $W_{m-1}(g_q)$ vanishes. Thus, there exists a q–frame field over the $(m-1)$–skeleton of M.

Finally, the obstruction to the existence of a q–frame field over the m–skeleton of M is the class $W_m(g_q) \in H^m(M; \pi_{m-1}(V_{m-6+q,q}))$. Since $m-6 = 8s+5$ and $q \geq 7$ it follows from [11] that $\pi_{m-1}(V_{m-6+q,q}) = \pi^5_{m-6,q} = \pi^5_{8s+5,q} = 0$, and consequently, $W_m(g_q)$ vanishes. Then, by Hirsch [8], we have that f is homotopic to an immersion. □

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