Global Hypoellipticity, global solvability and normal form for a class of real vector fields on a torus and application

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Abstract

The main purpose of this paper is to present a class of real vector fields defined on a torus for which the concept of global hypoelliptic-ity and global smooth solvability are equivalents. Furthermore, a such vector field is globally hypoelliptic if and only if its adjoint is glob-ally hypoelliptic and therefore we can reduce it to its normal form. As application, we study global C^∞ solvability for certain classes of subLaplacians.

1 Introduction

In this paper we will analyze the following questions about a real vector field $L = \sum_{j=1}^{N} a_j(x)\partial_{x_j}$, where $a_j \in C^\infty(\mathbb{T}^N)$, $j = 1, \ldots, N$, which does not vanish on $\mathbb{T}^N$:

Q1: Is it true that L is globally hypoelliptic in $\mathbb{T}^N$ iff its adjoint is globally hypoelliptic in $\mathbb{T}^N$?

Concerning this question we recall that if its adjoint L^* is globally hy-poelliptic in $\mathbb{T}^N$ then it follows from Theorem 2.2 in Chen and Chi [CC] that

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L is globally hypoelliptic in $\mathbb{T}^N$. We do not know if the converse of Theorem 2.2 in [CC] is true. However, Himonas, Petronilho and dos Santos [HPS] have proved it when $N = 2$.

Q2: Is it true that L is globally C^∞ solvable in $\mathbb{T}^N$ iff either its adjoint, L^*, or L is globally hypoelliptic in $\mathbb{T}^N$?

Q3: Is it true that L is globally C^∞ solvable in $\mathbb{T}^N$ iff L^* is globally C^∞ solvable in $\mathbb{T}^N$?

Concerning these questions it is well known that a standard functional analysis argument applies: the global hypoellipticity of L^* implies the global solvability of L. Since global hypoellipticity of L^* implies global hypoellipticity of L, as we have commented above, it follows that L is globally C^∞ solvable in $\mathbb{T}^N$. Also, it is well known that global solvability, even global C^∞ solvability, of L in $\mathbb{T}^N$, in general, does not imply that either L^* or L is globally hypoelliptic in $\mathbb{T}^N$. For instance, consider

$$L = \partial_t + \lambda\partial_x$$

with $(t, x) \in \mathbb{T}^2$ and λ is a rational number. The vector field L is globally C^∞ solvable in $\mathbb{T}^2$ but $L^*(= -L)$ and L are not globally hypoelliptic in $\mathbb{T}^2$.

We also would like to point out that the question Q1 is related with the following question:

Q4: If L is globally hypoelliptic in $\mathbb{T}^N$ is there a diffeomorphism of $\mathbb{T}^N$ onto $\mathbb{T}^N$ that takes L into

$$\sum_{j=1}^{N} A_j \partial_{y_j} \tag{1.1}$$

where the real numbers A_j satisfying some Diophantine condition?

We now recall some well-known results about question **Q4**.

Theorem 1.1 *(See Hounie [Hou]) Let X be a real vector field on $\mathbb{T}^2$ and suppose that X does not vanish on $\mathbb{T}^2$. If X is a globally hypoelliptic vector field on $\mathbb{T}^2$ then there exists a diffeomorphism of $\mathbb{T}^2$ onto $\mathbb{T}^2$ that takes X into*

$$f(s,t)(\partial_s + A\partial_t) \tag{1.2}$$

where the constant A is an irrational non-Liouville number and $f \in C^\infty(\mathbb{T}^2)$ is a non-vanishing function.

In $\mathbb{T}^n$ there exists a new reduction theorem for real vector fields due to Chen and Chi [CC]:

Theorem 1.2 *Let X be a real vector field on $\mathbb{T}^N$. Then, the transposed of X is a globally hypoelliptic operator on $\mathbb{T}^N$ if and only if there exist coordinates y on $\mathbb{T}^N$ in which X admits the form*

$$X = \sum_{j=1}^{N} A_j \partial_{y_j} \tag{1.3}$$

with the real numbers $A_1, \cdots, A_N$ satisfying the following Diophantine condition: there exist positive constants C and K such that

$$\left| \sum_{j=1}^{N} \xi_j A_j \right| \geq \frac{C}{(1+|\xi|)^K}, \ \forall\, \xi \in \mathbb{Z}^N \backslash \{0\}. \tag{1.4}$$

Remark 1.3 Theorem 1.2 gives new results on normal forms of real vector fields on $\mathbb{T}^N$ even for $N = 2$ (cf. Theorem 1.4 in Greenfield and Wallach [GW2]).

We notice that if L is globally hypoelliptic in $\mathbb{T}^N$ and the question **Q1** has a positive answer then it follows from Theorem 1.2 that question **Q4** has a positive answer as well.

The main goal here is to present a special class of real vector fields defined in a torus such that the concepts of global hypoellipticity, global C^∞ solvability and reduction to a normal form are all equivalents (see Theorem 3.1).

As an application we consider the following class of subLaplacians defined on the torus $\mathbb{T}^{n+m}$ by $P = -\Delta_t - L_x^2$ where $t \in \mathbb{T}^n$, $x \in \mathbb{T}^m$, $L = \sum_{j=1}^{m} a_j(x)\partial_{x_j}$ and we prove, under some conditions, that P is globally C^∞ solvable in $\mathbb{T}^{n+m}$ if and only if L is globally C^∞ solvable in $\mathbb{T}^m$ (see Theorem 4.3) . As alluded above, in order to study global solvability of an operator is natural to study global hypoellipticity for its adjoint. Thus, we start by analyzing the global hypoellipticity for the operator P^*. When L is a real vector field that does not vanish on $\mathbb{T}^m$ we prove that P^* is globally hypoelliptic in $\mathbb{T}^{n+m}$ iff L^* is globally hypoelliptic in $\mathbb{T}^m$, but when the kernel of L^* is spanned by a single smooth and non-vanishing function then we show that P is globally hypoelliptic in $\mathbb{T}^{n+m}$ iff L is globally hypoelliptic in $\mathbb{T}^m$

(see Theorem 4.1). Theorem 4.1 generalizes part of Theorem 6.2 un [HPS]. Finally we present an example where the hypotheses of our Theorem 4.3 about global C^∞ solvability of operator P are fulfilled.

For more results on the problem of global hypoellipticity we refer the reader to the works by Amano [A], Fujiwara and Omori [FO], Greenfield and Wallach [GW1], Himonas [Hi], Himonas and Petronilho [HP1], [HP2] and Omori and Kobayashi [OK], and the references therein.

For results on interesting open problems of local and global solvability we refer the reader to the following papers as well as the references therein: Albanese and Zanghirati [AZ], Bergamasco, Cordaro and Petronilho [BCP], Gramchev, Popivanov and Yoshino [GPY1] and [GPY2], Petronilho [P1] and [P2].

For results on diffeomorphisms that are globally conjugated to a rotation we refer the reader to Brjuno [Br], Herman [He], Yoccoz [Y] and references therein.

This paper is structured as follows. In section 2 we provide some definitions in order to, in section 3, state our result about global hypoellipticity, global solvability and reduction to the normal formal for certain class of real vector fields. In section 4 we apply the results obtained in the previous section in order to study global solvability and global hypoellipticity for certain classes of subLaplacians.

2 Definitions

Let P be a linear partial differential operator with coefficients in $C^\infty(\mathbb{T}^N)$.

Definition 2.1 We say that

$$P : C^\infty(\mathbb{T}^N) \to C^\infty(\mathbb{T}^N)$$

is *globally C^∞ solvable in $\mathbb{T}^N$* if for every

$$f \in \{h \in C^\infty(\mathbb{T}^N) : \langle w, h\rangle = 0, \forall\ w \in \operatorname{Ker} P^*\}$$

there exists $u \in C^\infty(\mathbb{T}^N)$ such that $Pu = f$, where

$$P^* : D'(\mathbb{T}^N) \to D'(\mathbb{T}^N)$$

is the transpose of P.

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Definition 2.3 We say that

$$P : D'(\mathbb{T}^N) \to D'(\mathbb{T}^N)$$

is *globally hypoelliptic in* $\mathbb{T}^N$ if the conditions $u \in D'(\mathbb{T}^N)$ and $Pu \in C^\infty(\mathbb{T}^N)$ imply that $u \in C^\infty(\mathbb{T}^N)$.

Remark 2.4 It is easily seen that if P is globally solvable in $\mathbb{T}^N$ and globally hypoelliptic in $\mathbb{T}^N$ then, P is globally C^∞ solvable in $\mathbb{T}^N$.

3 Results

In the next result we present a class of vector fields for which the questions **Q1**, **Q2** and **Q3** holds true for any dimension and we can write these vector fields in its normal form.

Theorem 3.1 *Let $L = \sum_{j=1}^{N} a_j(x)\partial_{x_j}$, be a smooth real vector field on $\mathbb{T}^N$. Assume that Ker $L^* = [w]$ where $w \in C^\infty(\mathbb{T}^N)$ and $w(x) \neq 0,\ \forall\ x \in \mathbb{T}^N$. Then the following conditions are equivalents:*

1. *There exists a diffeomorphism $\tau : \mathbb{T}^N \to \mathbb{T}^N, y = \tau(x)$ such that*

$$L = \sum_{j=1}^{N} A_j \partial_{y_j}$$

with the numbers $A_1, \cdots, A_N$ satisfying the condition: there exist $K > 0, C > 0$ such that

$$\left|\sum_{j=1}^{N} \xi_j A_j\right| \geq \frac{C}{(1+|\xi|)^K}, \ \forall \ (\xi_1, \cdots, \xi_N) \in \mathbb{Z}^N \backslash \{0\}. \qquad (3.1)$$

2. *L^* is globally hypoelliptic in $\mathbb{T}^N$.*
3. *L is globally hypoelliptic in $\mathbb{T}^N$.*
4. *L is globally C^∞ solvable in $\mathbb{T}^N$.*
5. *L^* is globally C^∞ solvable in $\mathbb{T}^N$.*

Remark 3.2 Before start the proof we would like to point out that for the class of vector fields studied in Theorem 3.1 global C^∞ solvability implies global solvability.

Remark 3.3 We recall that if L^* is globally hypoelliptic in $\mathbb{T}^N$ then Ker $L^* = [w]$ where $w \in C^\infty(\mathbb{T}^N)$ and $w(x) \neq 0, \ \forall \ x \in \mathbb{T}^N$, see Theorem 2.1 in [CC]. On the other hand, there exist real vector fields such its adjoint are not globally hypoelliptic in $\mathbb{T}^N$ but its Kernel is spanned by just one function w as described above. For instance, consider in $\mathbb{T}^2$ the vector field

$$L = \partial_t + \lambda \partial_x$$

where λ is a Liouville number. It is well known that $L^* = -L$ is not globally hypoelliptic in $\mathbb{T}^2$ but

$$\text{Ker}\, L^* = \text{Ker}\, L = [1].$$

Proof: It follows from [CC] that *1.* $\Leftrightarrow$ *2.*, *2.* $\Rightarrow$ *3.* and *2.* $\Rightarrow$ *4.*
3. $\Rightarrow$ *2.* : Let $u \in D'(\mathbb{T}^N)$ be such that

$$L^* u = f \in C^\infty(\mathbb{T}^N). \qquad (3.2)$$

We now notice that for any $v \in D'(\mathbb{T}^N)$ we have

$$\begin{aligned} L^*(wv) &= -L(wv) - \left(\sum_{j=1}^{N} \frac{\partial a_j}{\partial x_j}\right)(wv) \\ &= -(Lw)v - wLv - \left(\sum_{j=1}^{N} \frac{\partial a_j}{\partial x_j}\right)(wv) \qquad (3.3) \\ &= (L^*w)v - wLv = -wLv \end{aligned}$$

since $L^*w = 0$.

It follows from (3.2) and (3.3) that

$$f = L^*u = L^*(w\frac{1}{w}u) = -wL(\frac{1}{w}u).$$

Therefore

$$L(\frac{1}{w}u) = -\frac{1}{w}f \in C^\infty(\mathbb{T}^N). \tag{3.4}$$

By using the hypotheses that L is globally hypoelliptic in $\mathbb{T}^N$ it follows from (3.4) that

$$\frac{1}{w}u = g \in C^\infty(\mathbb{T}^N)$$

in turns implies that $u \in C^\infty(\mathbb{T}^N)$. Hence L^* is globally hypoelliptic in $\mathbb{T}^N$.

$4 \Rightarrow 3$: Let $u \in D'(\mathbb{T}^N)$ be such that

$$Lu = f \in C^\infty(\mathbb{T}^N). \tag{3.5}$$

Let $h \in \text{Ker } L^*$. Then $h = cw$ with c been a constant since $\text{Ker } L^* = [w]$. Thus,

$$\int_{\mathbb{T}^N} fh = \langle f, h\rangle = c\langle f, w\rangle = c\langle Lu, w\rangle = c\langle u, L^*w\rangle = c\langle u, 0\rangle = 0.$$

Thanks to this fact and thanks to the hypotheses that L is globally C^∞ solvable in $\mathbb{T}^N$ there exists $v \in C^\infty(\mathbb{T}^N)$ such that $Lv = f$. Therefore we can conclude that $Lu = Lv$. Hence,

$$L(u - v) = 0.$$

It follows from this and from (3.3) that

$$L^*[w(u - v)] = -wL(u - v) = 0.$$

Again, thanks to the fact that $\text{Ker } L^* = [w]$ it follows from $L^*[w(u-v)] = 0$ that

$$u - v = c = \text{constant}$$

in turns implies that $u \in C^\infty(\mathbb{T}^N)$ and therefore L is globally hypoelliptic in $\mathbb{T}^N$. $\square$

3. $\Rightarrow$ *5.* : Since *3.* $\Rightarrow$ *2.* and also *3.* implies that L^* is globally solvable in $\mathbb{T}^N$ it follows from Remark 2.4 that L^* is globally C^∞ solvable in $\mathbb{T}^N$. $\square$

5. $\Rightarrow$ *2.* : Let $u \in D'(\mathbb{T}^N)$ be such that

$$L^*u = f \in C^\infty(\mathbb{T}^N). \tag{3.6}$$

For $T \in \operatorname{Ker} L$ we have $\langle T, f\rangle = \langle T, L^*u\rangle = \langle LT, u\rangle = 0$. Thus it follows from the hypothesis that there exists $v \in C^\infty(\mathbb{T}^N)$ such that

$$L^*v = f \tag{3.7}$$

since L^* is globally C^∞ solvable. It follows from (3.6) and (3.7) that $u - v \in \operatorname{Ker} L^* = [w]$ and therefore $u \in C^\infty(\mathbb{T}^N)$. $\square$

Remark 3.4 The hypothesis that Ker $L^* = [w]$ where $w \in C^\infty(\mathbb{T}^N)$ and $w(x) \neq 0,\ \forall\ x \in \mathbb{T}^N$ is superflous for the proofs: *1.* $\Leftrightarrow$ *2.*, *2.* $\Rightarrow$ *3.* and *2.* $\Rightarrow$ *4.*, while we need it for the proofs *4.* $\Rightarrow$ *3.* and *5.* $\Rightarrow$ *2.* Also, for the proofs *3.* $\Rightarrow$ *2.* and *3.* $\Rightarrow$ *5.* it is suffices the existence of the function $w \in C^\infty(\mathbb{T}^N)$ and $w(x) \neq 0,\ \forall\ x \in \mathbb{T}^N$ such that $L^*w = 0$.

Remark 3.5 It follows from (3.3) that Ker $L = \{\text{constants}\}$, provided Ker $L^* = [w]$ where $w \in C^\infty(\mathbb{T}^N)$ with $w(x) \neq 0,\ \forall\ x \in \mathbb{T}^N$.

We now present a sufficient condition in order to guarantee that the existence of a function $w \in C^\infty(\mathbb{T}^N)$ such that $L^*w = 0$ with $w(x) \neq 0,\ \forall\ x \in \mathbb{T}^N$ implies that Ker $L^* = [w]$, more precisely,

Lemma 3.6 *If L is globally hypoelliptic in $\mathbb{T}^N$ and there exists a function $w \in C^\infty(\mathbb{T}^N)$ such that $L^*w = 0$ with $w(x) \neq 0,\ \forall\ x \in \mathbb{T}^N$ then Ker $L^* = [w]$.*

Proof: Let $T \in D'(\mathbb{T}^N)$ be such that $L^*T = 0$. We set $g = \frac{1}{w}T \in D'(\mathbb{T}^N)$. Thus, we have $T = wg$ and it follows from (3.3) that

$$0 = L^*T = L^*(wg) = -w(Lg).$$

Hence, $Lg = 0$. Since L is globally hypoelliptic in $\mathbb{T}^N$ we obtain $g \in C^\infty(\mathbb{T}^N)$ and therefore $T = wg \in C^\infty(\mathbb{T}^N)$.

Since L is a vector field we have $L(g^k) = 0, \forall\ k \in \{0, 1, 2, \dots\}$. Hence Ker $L \supset \{1, g, g^2, \dots, g^j, \dots\}$.

Now an application of the Rellich lemma implies that dim Ker $L < \infty$, since L is globally hypoelliptic in $\mathbb{T}^N$. Thus $1, g, g^2, \dots, g^k$ are linearly dependent for k sufficiently large. This implies

$$\sum_{j=0}^{k} a_j g^j = 0$$

with $a_j \in \mathbb{C}$ and not all a_j zero. Hence g can take at most k different values. Since $\mathbb{T}^N$ is connected, g is constant. Thus we can conclude that $T = cw$ with c a constant and therefore $T \in [w]$.

On the other hand if $T = cw$ with c a constant it is easily seen that $L^*T = 0$. □

4 Application

We start this section by recalling that in [HPS] the authors consider the operator $P = -\partial_t^2 - (\partial_{x_1} + a(t, x_1, x_2)\partial_{x_2})^2$ and under the hypothesis that all points in $\mathbb{T}^3$ are of infinite type for the vector fields $X = -\partial_t$ and $L = \partial_{x_1} + a(t, x_1, x_2)\partial_{x_2}$ they have proved that P is globally hypoelliptic in $\mathbb{T}^3$ iff the vector field $L = \partial_{x_1} + a(x_1, x_2)\partial_{x_2}$ is globally hypoelliptic in $\mathbb{T}^2$. Note that the infinite type condition implies that the coefficient a depend only on the variables x_1 and x_2 and their operator can be written as $P = -\partial_t^2 - (\partial_{x_1} + a(x_1, x_2)\partial_{x_2})^2$. Thus the study of the global hypoellipcticity for P is equivalent to the study of the global hypoellipticity for the vector field L.

Motivated by this situation we will analyze global C^∞ solvability and global hypoellipticity for the following class of subLaplacian:

$$P = -\Delta_t - L_x^2 \tag{4.1}$$

where $t \in \mathbb{T}^n$ and $L_x = \sum_{j=1}^m a_j(x)\partial_{x_j}$, does not vanish on $\mathbb{T}^m$ and $a_j \in C^\infty(\mathbb{T}^m; \mathbb{R})$, $j = 1, \dots, m$. We will use the notation

$$P = -\Delta - L^2.$$

It is well known that a linear operator is globally solvable in $\mathbb{T}^N$ provided its adjoint is globally hypoelliptic in $\mathbb{T}^N$. Thus, in order to study global C^∞ solvability for our operator P, given by (4.1), we start by studying the global hypoellipticity for P^*. More precisely, we prove the following

Theorem 4.1 *Let P be given by (4.1). Then,*

1) P^* is globally hypoelliptic in $\mathbb{T}^{n+m}$ if and only if L^* is globally hypoelliptic in $\mathbb{T}^m$.

2) If we assume that Ker $L^* = [w]$ where w is as in Theorem 3.1, then P is globally hypoelliptic in $\mathbb{T}^{n+m}$ if and only if L is globally hypoelliptic in $\mathbb{T}^m$.

Proof of 1).

Necessity: Assume that L^* is not globally hypoelliptic in $\mathbb{T}^m$. Thus, there exists $u \in D'(\mathbb{T}^m)\backslash\{C^\infty(\mathbb{T}^m)\}$ such that $L^*u = f \in C^\infty(\mathbb{T}^m)$ and therefore we have

$$(L^*)^2u = L^*(L^*u) = L^*f = g \in C^\infty(\mathbb{T}^m).$$

Since we have $(L^2)^* = (L^*)^2$ it follows from that last equality that

$$P^*u = -\Delta u - (L^*)^2u = -(L^*)^2u = -g \in C^\infty(\mathbb{T}^{n+m}),$$

which implies that P^* is not globally hypoelliptic in $\mathbb{T}^{n+m}$.

Sufficiency: Since L^* is globally hypoelliptic in $\mathbb{T}^m$ it follows from [CC] that there exists a diffeomorphism in $\mathbb{T}^m$, $y = \tau(x)$, such that $L = \sum_{j=1}^m a_j(x)\partial_{x_j}$ can be written as

$$L_y = \sum_{j=1}^m A_j \partial_{y_j}$$

where the real constants A_j satisfy the following Diophantine condition: there exist $K > 0$, $C > 0$ such that

$$\left| \sum_{j=1}^m k_j A_j \right| \geq \frac{C}{(1 + |k|)^K}, \quad \forall \; k = (k_1, \ldots, k_m) \in \mathbb{Z}^m \backslash \{0\}. \tag{4.2}$$

In this new variables we can write

$$P^* = -\Delta - (L_y + b(y))^2 = -\Delta - Q^2$$

where we are setting

$$Q = L_y + b(y)$$

with $b \in C^\infty(\mathbb{T}^m; \mathbb{R})$.

We shall show that P^* is globally hypoelliptic im $\mathbb{T}^{n+m}$. For this, let $u \in D'(\mathbb{T}^{n+m})$ be such that $P^*u = f \in C^\infty(\mathbb{T}^{n+m})$.

Since L_y is globally C^∞ solvable in $\mathbb{T}^m$ there exists $w(y) \in C^\infty(\mathbb{T}^m)$ such that

$$L_y w = -b(y) + b_{00}$$

where $b_{00} = \frac{1}{(2\pi)^m} \int_{\mathbb{T}^m} b(y)dy$.

We now set $v = e^{-w}u$. Thus,

$$L_y u = (L_y w)e^w v + e^w L_y v = (-b(y) + b_{00})e^w v + e^w L_y v$$

and therefore we have

$$\begin{aligned} Qu &= L_y u + b(y)u = (-b(y) + b_{00})e^w v + e^w L_y v + b(y)e^w v \\ &= b_{00}e^w v + e^w L_y v = e^w(L_y v + b_{00}v) \end{aligned}$$

and

$$\begin{aligned} Q^2 u &= (L_y + b(y))e^w(L_y v + b_{00}v) \\ &= (L_y w)e^w(L_y v + b_{00}v) + e^w L_y(L_y v + b_{00}v) + b(y)e^w(L_y v + b_{00}v) \\ &= b_{00}e^w(L_y v + b_{00}v) + e^w L_y(L_y v + b_{00}v) \\ &= e^w\left[(L_y + b_{00})(L_y v + b_{00}v)\right] = e^w(L_y + b_{00})^2 v, \end{aligned}$$

i.e., we have

$$Q^2 u = e^w(L_y + b_{00})^2 v.$$

Thus, we can write

$$\begin{aligned} f &= P^* u = -\Delta_t u - e^w(L_y + b_{00})^2 v \\ &= e^w\left[-\Delta_t v - (L_y + b_{00})^2 v\right], \end{aligned}$$

and therefore we have

$$-\Delta_t v - (L_y + b_{00})^2 v = e^{-w} f \doteq g \in C^\infty(\mathbb{T}^{n+m}).$$

We now will analyze the operator with constant coefficients:

$$R = -\Delta_t - (L_y + b_{00})^2 = -\sum_{j=1}^{n} \partial_{t_j}^2 - (\sum_{k=1}^{m} A_j \partial_{y_j} + b_{00})^2.$$

By taking the Fourier transform in the equation $Rv = g$ we obtain

$$[|\tau|^2 + (\sum_{k=1}^{m} A_j \xi_j - ib_{00})^2]\hat{v}(\tau, \xi) = \hat{g}(\tau, \xi). \tag{4.3}$$

For $\tau = 0$ we have

$$\hat{v}(0,\xi) = \frac{\hat{g}(0,\xi)}{(\sum_{k=1}^{m} A_j\xi_j - ib_{00})^2}. \tag{4.4}$$

From now on we will use C_N to represent a constant that depend on N, which may change a finite number of times.

The fact that $g \in C^\infty(\mathbb{T}^3)$ implies that given $N \in \mathbb{N}$ there exists $C_N > 0$ such that

$$|\hat{g}(\tau,\xi)| \leq C_N(1 + |(\tau,\xi)|)^{-N}, \ \forall \ (\tau,\xi) \in \mathbb{Z}^{n+m}.$$

It follows from this, from (4.2) and from (4.4) that given $N \in \mathbb{N}$ there exists $C_N > 0$ such that

$$|\hat{v}(0,\xi)| \leq C_N(1 + |\xi|)^{-N}, \ \forall \ \xi \in \mathbb{Z}^m.$$

We now assume that $\tau \neq 0$ and we split this case in two subcases:

i) $b_{00} = 0$**:** given $N \in \mathbb{N}$ there exists $C_N > 0$ such that

$$|\hat{v}(\tau,\xi)| = \left| \frac{\hat{g}(\tau,\xi)}{|\tau|^2 + (\sum_{k=1}^{m} A_j\xi_j)^2} \right| \leq |\hat{g}(\tau,\xi)| \leq (1 + |(\tau,\xi)|)^{-N}, \tag{4.5}$$

for $\tau \in \mathbb{Z}^n \backslash \{0\}$ and $\xi \in \mathbb{Z}^m$.

ii-1) $b_{00} \neq 0$ **and** $|\tau|^2 \geq b_{00}^2 + 1$**:** since

$$\left| |\tau|^2 + (\sum_{k=1}^{m} A_j\xi_j - ib_{00})^2 \right| \geq |\tau|^2 + (\sum_{k=1}^{m} A_j\xi_j)^2 - b_{00}^2 \geq |\tau|^2 - b_{00}^2 \geq 1$$

for $(\tau,\xi) \in \mathbb{Z}^{n+m}$ and $|\tau|^2 \geq b_{00}^2 + 1$, the estimate for $|\hat{v}(\tau,\xi)|$ follows as in the case i).

ii-2) $b_{00} \neq 0$ **and** $\tau \neq 0$**,** $|\tau|^2 < b_{00}^2 + 1$**:** in this case we note that

$$\left| |\tau|^2 + (\sum_{k=1}^{m} A_j\xi_j - ib_{00})^2 \right| \geq 2|b_{00}| |\sum_{k=1}^{m} A_j\xi_j|$$

and therefore it follows from (4.2) that there exist positive constants C and K such that

$$\left| |\tau|^2 + (\sum_{k=1}^{m} A_j\xi_j - ib_{00})^2 \right| \geq 2C|b_{00}|(1 + |\xi|)^{-K}, \ \forall \ \xi \neq 0.$$

Summing up, we have proved that given $N \in \mathbb{N}$ there exists $C_N > 0$ such that

$$|\hat{v}(\tau, \xi)| \leq C_N(1 + |(\tau, \xi)|)^{-N}, \tag{4.6}$$

for $(\tau, \xi) \in \mathbb{Z}^{n+m} \backslash \{F\}$ where F is a finite subset, in turns implies that $v \in C^\infty(\mathbb{T}^{n+m})$ and therefore P^* is globally hypoelliptic in $\mathbb{T}^{n+m}$. □

Proof of 2).

Necessity: We assume that P is globally hypoelliptic in $\mathbb{T}^{n+m}$ and let $u \in \mathcal{D}'(\mathbb{T}^m)$ be such that $Lu = f \in C^\infty(\mathbb{T}^m)$. Then $Pu = -L^2 u = -Lf \in C^\infty(\mathbb{T}^m)$. Hence, $u \in C^\infty(\mathbb{T}^m)$. The proof of the necessity is complete.

Sufficiency: Since L is globally hypoelliptic in $\mathbb{T}^m$ and we are assuming that Ker $L^* = [w]$ with w as in Theorem 3.1 it follows from Theorem 3.1 that there exists new variables, y, such that the vector field L can be written as

$$L = \sum_{j=1}^{m} A_j \partial_{y_j}$$

where the real numbers A_j satisfy the Diophantine condition (3.1). Thus, we can write P as

$$P = -\Delta - (\sum_{j=1}^{m} A_j \partial_{y_j})^2$$

and therefore, P is globally hypoelliptic in $\mathbb{T}^{n+m}$. This completes the proof of item 2) of Theorem 4.1. □

We shall need the following

Lemma 4.2 *Let P be given by* (4.1). *If P is globally C^∞ solvable in $\mathbb{T}^{n+m}$ then L^2 is globally C^∞ solvable $\mathbb{T}^m$.*

Proof: Suppose that L^2 is not globally C^∞ solvable. Then there exists $f \in \mathcal{E}(L^2) = \{h \in C^\infty(\mathbb{T}^m) : \langle h, w \rangle = 0, \ \forall\, w \in \text{Ker}(L^2)^*\}$ such that $L^2 u \neq f, \ \forall\, u \in C^\infty(\mathbb{T}^m)$.

We now define

$$g(t, x) = \sum_{(\tau,\xi) \in \mathbb{Z}^{n+m}} \hat{g}(\tau, \xi) e^{i(t\cdot\tau + x\cdot\xi)}$$

where

$$\hat{g}(\tau, \xi) = \begin{cases} \hat{f}(\xi) & \text{if } \tau = 0 \\ 0, & \text{otherwise} \end{cases}$$

We now are going to prove that $g \in \mathcal{E}(P)$.

Since $f \in C^\infty(\mathbb{T}^m)$ we have $g \in C^\infty(\mathbb{T}^{n+m})$. Let $w \in \text{Ker } P^*$. Then, $P^*w = 0$ and it is equivalent to $-\Delta w - (L^*)^2 w = 0$. By using a t-partial Fourier series the last equation is equivalent to

$$|\tau|^2 \hat{w}(\tau, x) - (L^*)^2 \hat{w}(\tau, x) = 0,\ \tau \in \mathbb{Z}^n,\ x \in \mathbb{T}^m.$$

In particular, for $\tau = 0$ we obtain from the last equality that

$$(L^*)^2 \hat{w}(0, x) = 0,\ x \in \mathbb{T}^m,$$

i.e., $\hat{w}(0,x) \in \text{Ker } (L^*)^2$. Therefore, $\hat{w}(0,x) \in \text{Ker } (L^2)^*$ since $(L^2)^* = (L^*)^2$.

Since $f \in \mathcal{E}(L^2)$ and $\hat{w}(0,x) \in \text{Ker } (L^2)^*$ we have

$$\langle f, \hat{w}(0,x) \rangle = 0. \tag{4.7}$$

We now recall that for $T \in D'(\mathbb{T}^{n+m})$ and $\varphi \in C^\infty(\mathbb{T}^{n+m})$ the following formula holds:

$$\langle T, \varphi \rangle = (2\pi)^n \sum_{\tau \in \mathbb{Z}^n} \langle \hat{T}(-\tau, x), \hat{\varphi}(\tau, x) \rangle.$$

Applying this formula with $T = w$ and $\varphi = g$ and taking the definition of g into account we obtain

$$\langle w, g \rangle = (2\pi)^n \langle \hat{w}(0,x), f(x) \rangle. \tag{4.8}$$

It follows from (4.7) and (4.8) that $\langle w, g \rangle = 0$ and therefore we have proved that $g \in \mathcal{E}(P)$.

Since by hypotheses P is globally C^∞ solvable there exists $u \in C^\infty(\mathbb{T}^{n+m})$ such that $Pu = g$. The equation $Pu = g$ is equivalent to

$$(|\tau|^2 + L^2)\hat{u}(\tau, x) = \hat{g}(\tau, x),\ \tau \in \mathbb{Z}^n,\ x \in \mathbb{T}^m.$$

In particular it follows from the last equality that

$$L^2 \hat{u}(0, x) = \hat{g}(0, x) = f(x),$$

what is a contradiction with the fact that we are assuming that L^2 is not globally C^∞ solvable. $\square$

We are now ready to prove the following result:

Theorem 4.3 *We assume that* $Ker\, L^* = Ker\,(L^*)^2 = [w]$*, where* $w \in C^\infty(\mathbb{T}^m)$ *and* $w(x) \neq 0,\ \forall\ x \in \mathbb{T}^m$*. Then* P *is globally* C^∞ *solvable in* $\mathbb{T}^{n+m}$ *if and only if* L *is globally* C^∞ *solvable in* $\mathbb{T}^m$*.*

Proof:
Necessity: Assuming that P is globally C^∞ solvable in $\mathbb{T}^{n+m}$ it follows from Lemma 4.2 that L^2 is globally C^∞ solvable in $\mathbb{T}^m$. We will show that L is globally C^∞ solvable in $\mathbb{T}^m$. For this, let $f \in \mathcal{E}(L)$. Thanks to the fact that $\mathrm{Ker}\, L^* = \mathrm{Ker}\,(L^*)^2$ we can conclude that $f \in \mathcal{E}(L^2)$. Since L^2 is globally C^∞ solvable in $\mathbb{T}^2$ and $f \in \mathcal{E}(L^2)$ there exists $u \in C^\infty(\mathbb{T}^m)$ such that $L^2 u = f$ and therefore $v = Lu \in C^\infty(\mathbb{T}^m)$ satisfies the equation

$$Lv = f,$$

i.e., L is globally C^∞ solvable in $\mathbb{T}^m$.
Sufficiency: Since, by hypotheses, L is globally C^∞ solvable in $\mathbb{T}^m$ and $\mathrm{Ker}\ L^* = [w]$ where $w \in C^\infty(\mathbb{T}^m)$ and $w(x) \neq 0,\ \forall\ x \in \mathbb{T}^m$ it follows from Theorem 3.1 that L^* is globally hypoelliptic in $\mathbb{T}^m$. It now follows from Theorem 4.1 that P^* is globally hypoellipitic in $\mathbb{T}^{n+m}$ and therefore P is globally solvable in $\mathbb{T}^{n+m}$. But, it also follows from Theorem 3.1 and Theorem 4.1 that P is globally hypoelliptic in $\mathbb{T}^{n+m}$ and therefore we now can conclude that P is globally C^∞ solvable in $\mathbb{T}^{n+m}$. $\square$

We now will present an example of vector field, X, for which we have $\mathrm{Ker}\, X^* = \mathrm{Ker}\,(X^*)^2 = [w]$, where $w \in C^\infty(\mathbb{T}^m)$ and $w(x) \neq 0,\ \forall\ x \in \mathbb{T}^m$.

Example 4.4

We consider in $\mathbb{T}^3$ the vector field

$$X = \partial_t + a(t)\partial_x + b(t)\partial_y$$

where $a, b \in C^\infty(\mathbb{T})$ are real-valued. We are assuming that the vector (a_0, b_0) is a non-Liouville vector, where

$$a_0 = \frac{1}{2\pi}\int_0^{2\pi} a(t)dt,\ b_0 = \frac{1}{2\pi}\int_0^{2\pi} b(t)dt.$$

We recall that a vector $v = (v_1, v_2) \in \mathbb{R}^2$ is said to be non-Liouville if there exist $C > 0$ and $K > 0$ such that for any $\eta \in \mathbb{Z}$ and $\xi \in \mathbb{Z}^2\backslash\{0\}$ we have

$$|\eta - v \cdot \xi| \geq \frac{C}{|\xi|^K}.$$

We shall need the following

Lemma 4.5 *Under the above conditions the vector field X is globally hypoelliptic in $\mathbb{T}^3$.*

Proof. Let $u \in D'(\mathbb{T}^3)$ be such that

$$Xu = f \in C^\infty(\mathbb{T}^3). \tag{4.9}$$

By taking the Fourier transform with respect to variables x and y in (4.9) we obtain

$$\partial_t \hat{u}(t,\xi,\eta) + i\xi a(t)\hat{u}(t,\xi,\eta) + i\eta b(t)\hat{u}(t,\xi,\eta) = \hat{f}(t,\xi,\eta). \tag{4.10}$$

For each $(\xi,\eta) \in \mathbb{Z}^2$ the operator in equation (4.10) is elliptic in t and therefore by elliptic theory we have $\hat{u}(\cdot,\xi,\eta) \in C^\infty(\mathbb{T})$. We set

$$A(t) = \int_0^t a(s)ds - a_0 t, \;\; B(t) = \int_0^t b(s)ds - b_0 t$$

and

$$v(t,\xi,\eta) = e^{i\xi A(t) + i\eta B(t)}\hat{u}(t,\xi,\eta).$$

Thus, $v(t,\xi,\eta)$ satisfies the equation

$$\partial_t v(t,\xi,\eta) + i(a_0\xi + b_0\eta)v(t,\xi,\eta) = \hat{f}(t,\xi,\eta)e^{i\xi A(t)+i\eta B(t)} \doteq g(t,\xi,\eta). \tag{4.11}$$

For $(\xi,\eta) \in \mathbb{Z}^2\backslash\{0\}$ such that $\frac{1}{e^{2\pi i(a_0\xi+b_0\eta)}-1} \neq 0$ we have

$$v(t,\xi,\eta) = \frac{1}{e^{2\pi i(a_0\xi+b_0\eta)} - 1}\int_0^{2\pi} e^{i(a_0\xi+b_0\eta)s} g(t+s,\xi,\eta)ds. \tag{4.12}$$

Since the vector (a_0,b_0) is non-Liouville it is easily seen that there exist $C > 0$ and $K > 0$ such that

$$|e^{2\pi i(a_0\xi+b_0\eta)} - 1| \geq \frac{C}{|(\xi,\eta)|^K} \tag{4.13}$$

for all $(\xi,\eta) \in \mathbb{Z}^2\backslash\{0\}$.

For $\xi = \eta = 0$ we have

$$v(t,0,0) = \int_0^t g(s,0,0)ds + C \tag{4.14}$$

with $\int_0^{2\pi} g(s,0,0)ds = 0$ and C is a constant.

It is standard to prove that (4.12), (4.13) and (4.14) imply that X is globally hypoelliptic in $\mathbb{T}^3$. $\square$

Remark 4.6 In particular, (4.12), (4.13) and (4.14) imply that

$$\text{Ker}\, X = [1]. \tag{4.15}$$

We now notice that

$$X^* = -X \text{ and } (X^*)^2 = X^2.$$

Suppose that $X^2u = 0$. Then, $X(Xu) = 0$ and therefore it follows from (4.15) that there exists a constant C such that $Xu = C$. Thus, C must satisfies the compatibility condition

$$(2\pi)^3 C = \langle C, 1\rangle = \langle Xu, 1\rangle = \langle u, X^*1\rangle = \langle u, -X1\rangle = \langle u, 0\rangle = 0.$$

Hence, $C = 0$ and we have proved that

$$\text{Ker}X^2 \subset \text{Ker}X.$$

Summing up we have proved that

$$\text{Ker}\, X^* = \text{Ker}\,(X^*)^2.$$

It follows from Theorem 3.1 that the vector field X is globally C^∞ solvable in $\mathbb{T}^3$ and it follows from Theorem 4.3 that the operator $P = -\Delta - X^2$ is globally C^∞ solvable in $\mathbb{T}^{n+3}$.

Remark 4.7 Since X is globally C^∞ solvable in $\mathbb{T}^3$ then let $\varphi_1, \varphi_2, \varphi_3 \in C^\infty(\mathbb{T}^3)$ be such that

$$X\varphi_1 = 0,\ X\varphi_2 = a_0 - a(t),\ X\varphi_3 = b_0 - b(t).$$

For this example it is easy to see that the diffeomorphism $\tau = \tau(t,x,y)$ which is guaranteed by Theorem 3.1 is given by

$$\begin{aligned}\tau_1(t,x,y) &= t + \varphi_1(t,x,y),\\ \tau_2(t,x,y) &= x + \varphi_2(t,x,y),\\ \tau_3(t,x,y) &= y + \varphi_3(t,x,y)\end{aligned}$$

and the constants A_1, A_2, A_3 are given by

$$A_1 = 1,\ A_2 = a_0,\ A_3 = b_0.$$

It is also easily seen that the condition (a_0, b_0) is a non-Liouville vector implies the Diophantine condition (3.1).

References

[AZ] A. A. Albanese and L. Zanghirati, *Global hypoellipticity and global solvability in Gevrey classes on the n-dimensional torus*, J. Differential Equations **199**, (2004), 256-268.

[A] K. Amano, *The Global Hypoellipticity of a class of degenerate elliptic-parabolic operators*, Proc. Japan Acad. **60, Ser. 4**, (1984), 312-314.

[BCP] A. P. Bergamasco, P. D. Cordaro and G. Petronilho, *Global solvability for a class of complex vector fields on the two-torus*, Comm. in PDE **29**, no. 5 and 6, (2004), 785-819.

[Br] A. D. Brjuno, *Analytic form of differential equations*, Trans. Moscow Math. Soc. **25**, (1971), 131-288 and **26**, (1972), 199-239.

[CC] W. Chen and M.Y. Chi, *Hypoelliptic vector fields and almost periodic motions on the torus T^n*, Comm. in PDE **25(1-2)**, (2000), 337–354.

[FO] D. Fujiwara and H. Omori, *An example of a globally hypoelliptic operator*, Hokkaido Math. J. **12**, (1983), 293–297.

[GPY1] T. Gramchev, P. Popivanov and M. Yoshino, *Global solvability and hypoellipticity on the torus for a class of differential operators with variable coefficients*, Proc. Japan Acad., **68**, (1992), 53-57.

[GPY2] T. Gramchev, P. Popivanov and M. Yoshino, *Global properties in spaces of generalized functions on the torus for second order differential operators with variable coefficients*, Rend. Sem. Mat. Pol. Torino **51**, no.2, (1993), 145-172.

[GW1] S. J. Greenfield and N. R. Wallach, *Global hypoellipticity and Liouville numbers*, Proc. Amer. Math. Soc., **31** (1972), 112–114.

[GW2] S. J. Greenfield and N. R. Wallach, *Globally hypoelliptic vector field*, Topology **12**, (1973), 247-253.

[He] M. R. Herman, *Sur la conjugaison différentiable des difféomorphismes du cercle à des rotations*, Publ. I.H.E.S. **49**, (1979), 5-233.

[Hi] A. A. Himonas, *On degenerate elliptic operators of infinite type*, Math. Z. **220**, (1995), 449-460.

[HP1] A. A. Himonas and G. Petronilho, *Global hypoellipticity and simultaneous approximability*, J. Funct. Anal., **170**, (2000), 356–365.

[HP2] A. A. Himonas and G. Petronilho, *On C^∞ and Gevrey Regularity of Sublaplacians*, Trans. Amer. Math. Soc. **358**, (2006), no. 11, 4809–4820.

[HPS] A. A. Himonas, G. Petronilho and L. A. C. dos Santos, *Analytic, Gevrey and C^∞ global regularity for a class of subLaplacinas*, preprinted.

[Hou] J. Hounie, *Globally hypoelliptic vector field on compact surfaces*, Comm. in PDE **7(4)**, (1982), 343-370.

[OK] H. Omori and T. Kobayashi, *Global hypoellipticity of subelliptic operators on closed manifolds*, Hokkaido Math. J., **28**, (1999), 613-633.

[P1] G. Petronilho, *Global solvability and simultaneously approximable vectors*, J. Differential Equations, **184**, (2002), 48-61.

[P2] G. Petronilho, *On Gevrey solvability and regularity*, Math. Nachr., **282**, no. 3, (2009), 470-481.

[Y] J.C. Yoccoz, *Conjugaison différentiable des difféomorphismes du cercle dont le nombre de rotation vérifie une condition Diophantienne*, Ann. Sci. École Norm. Sup. **IV** Sér. 17, (1984), 333-359.

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