THE ANALYTIC TORSION OF THE CONE OVER AN ODD DIMENSIONAL MANIFOLD

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Abstract. We study the analytic torsion of the cone over an orientable odd dimensional compact connected Riemannian manifold W. We prove that the logarithm of the analytic torsion of the cone decomposes as the sum of the logarithm of the root of the analytic torsion of the boundary of the cone, plus a topological term, plus a further term that is a rational linear combination of local Riemannian invariants of the boundary. We show that this last term coincides with the anomaly boundary term appearing in the Cheeger Müller theorem [4] [19] for a manifold with boundary, according to Brüning and Ma [1]. We also prove Poincaré duality for the analytic torsion of a cone.

1. Introduction and statement of the results

Analytic torsion was originally introduced by Ray and Singer in [22], as an analytic counter part of the Reidemeister torsion of Reidemeister, Franz and de Rham. The first important result in this context, nowadays known as the Cheeger-Müller Theorem, was achieved by W. Müller [19] and J. Cheeger [4], who proved that for a compact connected Riemannian manifold without boundary, the analytic torsion and the Reidemeister torsion coincide, as conjectured by Ray and Singer in [22]. The next natural question along this line of investigation was to answer the same problem for manifolds with boundary. It was soon realized that the answer to such a question was an highly non trivial one. W. Lück proved in [16] that in the case of a product metric near the boundary the boundary term is topological, and depends only upon the Euler characteristic of the boundary. The answer to the general case required 20 more years of work, and is contained in a recent paper of Brüning and Ma [1] (see also [6]). The new contribution of the boundary, beside the topological one given by Lück, is called anomaly boundary term, has a quite complicate expression, but only depends on some local quantities constructed from the metric tensor near the boundary (see Section 2.3 for details). The next natural step is to study the analytic torsion for spaces with singularities, and the simplest singular space is the cone over a manifold, CW. Cones and spaces with conical singularities have been deeply investigated by J. Cheeger in a series of works [4] [5] (see also [20]). Due to this investigation, all information on L^2-forms, Hodge theory, and Laplace operator on forms on CW are available. Further information on the class of regular singular operators, that contains the Laplace operator on CW, are given in works of Brüning and Seely (see in particular [3]). As a result it is not difficult to obtain a complete description of the eigenvalues of the Laplace operator on CW in terms of the eigenvalues of the Laplace operator on W. With all these tools available, namely on one side the formula for the boundary term, and on the other some representation of the eigenvalues of the Laplace operator on the cone, it is natural to tackle the problem of investigating the analytic torsion of CW. A possible extension of the Cheeger Müller theorem could follow, or

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not. Indeed, in case of conical singularity such an extension would require intersection R torsion more than classical R torsion (see [7]). However, if the section is a rational homology manifold, then the two torsions coincide (see [4], end of Section 2), and the classical Cheeger Müller theorem is expected to extend. If $\mathcal{C}(W)$ is the chain complex associated to some cell decomposition of W, then the algebraic mapping cone $Cone(\mathcal{C}(W))$ gives the chain complex for a cell decomposition of CW. It is then easy to see that the R torsion of CW only depends on the choice of a base for the zero dimensional homology. Even if Poincaré duality does not hold, it does hold between top and bottom dimension, and therefore we can fix the base for the zero homology using the Riemannian structure and harmonic forms (see [22] Section 3, see also [13]). The result for the R torsion is $\tau(CW) = \sqrt{\mathrm{Vol(CW)}}$. On the other side, one wants the analytic torsion. The analytic tools necessary to deal with the zeta functions appearing in the definition of the analytic torsion, constructed with the eigenvalues of the Laplace operator on CW, are available by works of M. Spreafico [27] [28] [30]. In these works, the zeta function associated to a general class of double sequences is investigated. In particular, a decomposition result is presented and formulas for the zeta invariants of a decomposable sequence are given (see Section 2.4). This technique applies to the case of the zeta functions appearing in the definition of the analytic torsion on CW. This approach was used in [11] to study the cone over an odd low dimensional sphere, and is applied here to the general case, proving a conjecture stated in [11] (see Corollary 1.1). This method was originally used to deal with the zeta determinant of the Laplace operator on a cone in [26] (see also [31]), and consequently in [29] [32] [13] and [11] to study the zeta determinants of the Laplacian on forms and analytic torsion type invariants. In particular, in [32] a general formula for the analytic torsion of a cone is given. The formula is obtained using a method introduced by one of the authors of this paper in some older works [26] [28] and some results of M. Lesch [14] [15], and it is not particularly illuminating as it is stated, since essentially it is just an application of the formulas given in those works. In the abstract of [32], it is stated that the result is obtained 'by generalizing some computational methods of M. Spreafico', however such generalization is already contained in [28] and [29], and an even further generalization is contained in the preprint [31], of which the author of [32] seems not to be aware.

We are now ready to state the main results of this paper (we referee to the on line version of this work [12] for further developments and results), for we fix some notation. Let (W, g) be an orientable compact connected Riemannian manifold of finite dimension m without boundary and with Riemannian structure g. We denote by $C_l W$ the cone over W with the Riemannian structure

$$dx \otimes dx + x^2 g,$$

on $CW - \{pt\}$, where pt denotes the tip of the cone and $0 < x \leq l$ (see Section 3.1 for details). The formal Laplace operator on forms on $CW - \{pt\}$ has a suitable L^2-self adjoint extension $\Delta_{\mathrm{abs/rel}}$ on $C_l W$ with absolute or relative boundary conditions on the boundary $\partial C_l W$ (see Section 3.3 for details), with pure discrete spectrum $\mathrm{Sp}\Delta_{\mathrm{abs/rel}}$. This permits to define the associated zeta function

$$\zeta(s, \Delta_{\mathrm{abs/rel}}) = \sum_{\lambda \in \mathrm{Sp}_+ \Delta_{\mathrm{abs/rel}}} \lambda^{-s},$$

for $\mathrm{Re}(s) > \frac{m+1}{2}$. This zeta function has a meromorphic analytic continuation to the whole complex s-plane with at most isolated poles (see Section 4 for details). It is then possible to define the analytic torsion of the cone (the trivial representation of the fundamental group is assumed)

$$\log T_{\rm abs/rel}(C_lW) = \frac{1}{2}\sum_{q=0}^{m+1}(-1)^q q\zeta'(0,\Delta^{(q)}_{\rm abs/rel}).$$

In this setting, we have the following results (analogous results with relative boundary conditions also follow by Poincaré duality on the cone, proved in Theorem 4.1 below).

Theorem 1.1. *The analytic torsion on the cone C_lW on an orientable compact connected Riemannian manifold (W,g) of odd dimension $2p-1$ is*

$$\log T_{\rm abs}(C_lW) = \frac{1}{2}\sum_{q=0}^{p-1}(-1)^{q+1}{\rm rk}H_q(W;\mathbb{Q})\log\frac{2(p-q)}{l} + \frac{1}{2}\log T(W,l^2g) + {\rm S}(\partial C_lW),$$

where the singular term ${\rm S}(\partial C_lW)$ only depends on the boundary of the cone:

$${\rm S}(\partial C_lW) = \frac{1}{2}\sum_{q=0}^{p-1}\sum_{j=0}^{p-1}\sum_{k=0}^{j}\operatorname*{Res_0}_{s=0}\Phi_{2k+1,q}(s)\binom{-\frac{1}{2}-k}{j-k}\sum_{h=0}^{q}(-1)^h\operatorname*{Res_1}_{s=j+\frac{1}{2}}\zeta\left(s,\tilde\Delta^{(h)}\right)(q-p+1)^{2(j-k)},$$

where the functions $\Phi_{2k+1,q}(s)$ are some universal functions, explicitly known by some recursive relations, and $\tilde\Delta$ is the Laplace operator on forms on the section of the cone.

It is important to observe that the singular term ${\rm S}(\partial C_lW)$ is a universal linear combination of local Riemannian invariants of the boundary, for the residues of the zeta function of the section are such linear combination (see Section 7 for details).

Theorem 1.2. *With the notation of Theorem 1.1, the singular term of the analytic torsion of the cone C_lW coincides with the anomaly boundary term of Brüning and Ma, namely ${\rm S}(\partial C_lW) = A_{\rm BM,abs}(\partial C_lW)$.*

See Section 2.3 for the definition of $A_{\rm BM,abs}(\partial C_lW)$. If W is an odd sphere, we have:

Corollary 1.1. *The natural extension of Cheeger Müller theorem for manifold with boundary is valid for the cone over an odd dimensional sphere, namely*

$$\log T_{\rm abs}(C_lS^{2p-1}) = \log\tau(C_lS^{2p-1}) + A_{\rm BM,abs}(\partial C_lS^{2p-1}).$$

In particular, if a denotes the radius of the sphere, then

$$A_{\rm BM,abs}(\partial C_lS^{2p-1}_a) = \frac{(2p-1)!}{4^p(p-1)!}\sum_{k=0}^{p-1}\frac{1}{(p-1-k)!(2k+1)}\sum_{j=0}^{k}\frac{(-1)^{k-j}2^{j+1}}{(k-j)!(2j+1)!!}a^{2k+1}.$$

The result in the corollary should be understood as a particular case of the still unproved general result that the analytic torsion and the intersection R-torsion of a cone coincide up to the boundary term, for the intersection R-torsion is the classical R-torsion for the cone over a sphere. We point out that we also have a purely combinatoric proof of the result stated in Corollary 1.1, independent from Theorem 1.2. To contain space, we omit the proof here; it will appear somewhere else (see also [12]).

We conclude with a remark on the even dimensional case, namely when the dimension of the section W is even. It is clear enough that all the arguments used in the odd dimensional case go through also in the even dimensional case, and that the anomaly boundary term is the one of Brüning and Ma. So we obtain formulas for the analytic torsion as in the theorems above.

However, in the even dimensional case some further term appears: this was described in some details for $W = S^2$ in [13]. Since we do not have a clear understanding of this new term yet, we prefer to omit the non particularly illuminating formulas for the even dimensional case here.

2. Preliminaries and notation

We introduce in this section some notation necessary in the following. As usual (W, g) is a compact connected oriented Riemannian manifold.

2.1. **Manifolds with boundary.** If W has a boundary ∂W, then there is a natural splitting near the boundary of ΛW as direct sum of vector bundles $\Lambda T^*\partial W \oplus N^*W$, where N^*W is the dual to the normal bundle to the boundary. Locally, let ∂_x denote the outward pointing unit normal vector to the boundary, and dx the corresponding one form, then near the boundary we have the collar decomposition $Coll(\partial W) = (-\epsilon, 0] \times \partial W$, and if y is a system of local coordinates on the boundary, then (x, y) is a local system of coordinates in $Coll(\partial W)$. The metric tensor decomposes near the boundary in this local system as $g = dx \otimes dx + g_\partial(x)$, where $g_\partial(x)$ is a family of metric structure on ∂W such that $g_\partial(0) = i^*g$, where $i : \partial W \to W$ denotes the inclusion. The smooth forms on W near the boundary decompose as $\omega = \omega_{\rm tan} + \omega_{\rm norm}$, where $\omega_{\rm norm}$ is the orthogonal projection on the subspace generated by dx, and $\omega_{\rm tan}$ is in $C^\infty(W) \otimes \Lambda(\partial W)$. We write $\omega = \omega_1 + dx \wedge \omega_2$, where $\omega_j \in C^\infty(W) \otimes \Lambda(\partial W)$, and

$$\star\omega_2 = -dx \wedge \star\omega. \tag{2.1}$$

Define absolute and relative boundary conditions by

$$B_{\rm abs}(\omega) = \omega_{\rm norm}|_{\partial W} = \omega_2|_{\partial W} = 0, \qquad B_{\rm rel}(\omega) = \omega_{\rm tan}|_{\partial W} = \omega_1|_{\partial W} = 0.$$

Note that, if $\omega \in \Omega^q(W)$, then $B_{\rm abs}(\omega) = 0$ if and only if $B_{\rm rel}(\star\omega) = 0$, $B_{\rm rel}(\omega) = 0$ implies $B_{\rm rel}(d\omega) = 0$, and $B_{\rm abs}(\omega) = 0$ implies $B_{\rm abs}(d^\dagger\omega) = 0$. Let $\mathcal{B}(\omega) = B(\omega) \oplus B((d + d^\dagger)(\omega))$. Then the operator $\Delta = (d + d^\dagger)^2$ with boundary conditions $\mathcal{B}(\omega) = 0$ is self adjoint, and if $\mathcal{B}(\omega) = 0$, then $\Delta\omega = 0$ if and only if $(d + d^\dagger)\omega = 0$. Note that $\mathcal{B}$ correspond to

$$\mathcal{B}_{\rm abs}(\omega) = 0 \qquad \text{if and only if} \qquad \begin{cases} \omega_{\rm norm}|_{\partial W} = 0, \\ (d\omega)_{\rm norm}|_{\partial W} = 0, \end{cases} \tag{2.2}$$

$$\mathcal{B}_{\rm rel}(\omega) = 0 \qquad \text{if and only if} \qquad \begin{cases} \omega_{\rm tan}|_{\partial W} = 0, \\ (d^\dagger\omega)_{\rm tan}|_{\partial W} = 0, \end{cases} \tag{2.3}$$

2.2. **The form valued zeta functions and the analytic torsion.** The Laplace operator $\Delta^{(q)}$ with boundary conditions $\mathcal{B}_{\rm abs/rel}$ has a pure point spectrum ${\rm Sp}\Delta^{(q)}_{\rm abs/rel}$ consisting of real non negative eigenvalues. The sequence ${\rm Sp}_+\Delta^{(q)}_{\rm abs/rel}$ is a totally regular sequence of spectral type accordingly to Section 2.4, and the *forms valued zeta function* is the associated zeta function, defined by

$$\zeta(s, \Delta^{(q)}_{\rm abs/rel}) = \zeta(s, {\rm Sp}_+\Delta^{(q)}_{\rm abs/rel}) = \sum_{\lambda \in {\rm Sp}_+\Delta^{(q)}_{\rm abs/rel}} \lambda^{-s},$$

when ${\rm Re}(s) > \frac{m}{2}$. The *analytic torsion* $T_{\rm abs/rel}((W, g); \rho)$ of (W, g) with respect to the representation $\rho : \pi_1(W) \to O(k, \mathbb{R})$ is defined by

$$\log T_{\rm abs/rel}((W, g); \rho) = \frac{1}{2}\sum_{q=1}^m (-1)^q q \zeta'(0, \Delta^{(q)}_{\rm abs/rel}).$$

The following duality holds for analytic torsion [16]

$$\log T_{\rm abs}((W,g);\rho) = (-1)^{m+1}\log T_{\rm rel}((W,g);\rho). \tag{2.4}$$

We will omit the representation in the notation whenever we mean the trivial representation. Next, recall some classical results of Hodge theory in order to define *closed, coclosed, exact* and *coexact zeta functions*. We restrict ourselves to the case of a manifold without boundary (see [22] for the case of manifold with boundary). Setting $\mathcal{H}^q(W,E_\rho) = \{\omega\in\Omega^{(q)}(W,E_\rho)\mid \Delta\omega = 0\}$, the space of the q-harmonic forms, we have the Hodge decomposition

$$\Omega^q(W,E_\rho) = \mathcal{H}^q(W,E_\rho)\oplus d\Omega^{q-1}(W,E_\rho)\oplus d^\dagger\Omega^{q+1}(W,E_\rho). \tag{2.5}$$

This induces a decomposition of the eigenspace of a given eigenvalue $\lambda\neq 0$ of $\Delta^{(q)}$ into the spaces of *closed forms* and *coclosed forms*: $\mathcal{E}^{(q)}_\lambda = \mathcal{E}^{(q)}_{\lambda,{\rm cl}}\oplus\mathcal{E}^{(q)}_{\lambda,{\rm ccl}}$, where

$$\mathcal{E}^{(q)}_{\lambda,{\rm cl}} = \{\omega\in\Omega^q(W,E_\rho)\mid \Delta\omega=\lambda\omega,\ d\omega=0\},\quad \mathcal{E}^{(q)}_{\lambda,{\rm ccl}} = \{\omega\in\Omega^q(W,E_\rho)\mid \Delta\omega=\lambda\omega,\ d^\dagger\omega=0\}.$$

Define *exact forms* and *coexact forms* by

$$\mathcal{E}^{(q)}_{\lambda,{\rm ex}} = \{\omega\in\Omega^q(W,E_\rho)\mid \Delta\omega=\lambda\omega,\ \omega=d\alpha\},\quad \mathcal{E}^{(q)}_{\lambda,{\rm cex}} = \{\omega\in\Omega^q(W,E_\rho)\mid \Delta\omega=\lambda\omega,\ \omega=d^\dagger\alpha\}.$$

Note that, if $\lambda\neq 0$, then $\mathcal{E}^{(q)}_{\lambda,{\rm cl}} = \mathcal{E}^{(q)}_{\lambda,{\rm ex}}$, and $\mathcal{E}^{(q)}_{\lambda,{\rm ccl}} = \mathcal{E}^{(q)}_{\lambda,{\rm cex}}$, and we have an isometry

$$\phi:\mathcal{E}^{(q)}_{\lambda,{\rm cl}}\to\mathcal{E}^{(q-1)}_{\lambda,{\rm cex}},\quad \phi:\omega\mapsto\frac{1}{\sqrt{\lambda}}d^\dagger\omega, \tag{2.6}$$

whose inverse is $\frac{1}{\sqrt{\lambda}}d$. Also, the restriction of the Hodge star defines an isometry $\star: d^\dagger\Omega^{(q+1)}(W)\to d\Omega^{(m-q-1)}(W)$, and that composed with the previous one gives the isometries:

$$\frac{1}{\sqrt{\lambda}}d\star:\mathcal{E}^{(q)}_{\lambda,{\rm cl}}\to\mathcal{E}^{(m-q+1)}_{\lambda,{\rm cex}},\quad \frac{1}{\sqrt{\lambda}}d^\dagger\star:\mathcal{E}^{(q)}_{\lambda,{\rm ccl}}\to\mathcal{E}^{(m-q-1)}_{\lambda,{\rm ex}}. \tag{2.7}$$

By the very definition, we have

$$\zeta(s,\Delta^{(q)}) = \sum_{\lambda\in{\rm Sp}_+\Delta^{(q)}}\dim\mathcal{E}^{(q)}_\lambda\lambda^{-s} = \zeta_{\rm cl}(s,\Delta^{(q)}) + \zeta_{\rm ccl}(s,\Delta^{(q)}),$$

where

$$\zeta_{\rm cl}(s,\Delta^{(q)}) = \sum_{\lambda\in{\rm Sp}_+\Delta^{(q)}}\dim\mathcal{E}^{(q)}_{\lambda,{\rm cl}}\lambda^{-s},\qquad \zeta_{\rm ccl}(s,\Delta^{(q)}) = \sum_{\lambda\in{\rm Sp}_+\Delta^{(q)}}\dim\mathcal{E}^{(q)}_{\lambda,{\rm ccl}}\lambda^{-s}.$$

Since, by (2.6), $\zeta_{\rm cl}(s,\Delta^{(q)}) = \zeta_{\rm ccl}(s,\Delta^{(q-1)})$, we obtain from the above relations the following formulas for the torsion of a closed m dimensional manifold W:

$$\log T((W,g);\rho) = \frac{1}{2}\sum_{q=1}^m(-1)^q q\zeta'(0,\Delta^{(q)}) = \frac{1}{2}\sum_{q=1}^m(-1)^q\zeta'_{\rm cl}(0,\Delta^{(q)}) = -\frac{1}{2}\sum_{q=0}^{m-1}(-1)^q\zeta'_{\rm ccl}(0,\Delta^{(q)}).$$

In particular, using again duality, for an odd dimensional manifold W of dimension $m=2p-1$,

$$
\begin{aligned}
\log T((W,g);\rho) &= \sum_{q=1}^{p-1}(-1)^q\zeta_{\rm cl}'(0,\Delta^{(q)})+\frac{(-1)^p}{2}\zeta_{\rm cl}'(0,\Delta^{(p)})\\
&= -\sum_{q=0}^{p-2}(-1)^q\zeta_{\rm ccl}'(0,\Delta^{(q)})+\frac{(-1)^p}{2}\zeta_{\rm ccl}'(0,\Delta^{(p-1)}).
\end{aligned}
\tag{2.8}
$$

2.3. **The Cheeger Müller theorem for manifolds with boundary, and the anomaly boundary term of Brüning and Ma.** In case of a smooth orientable compact connect Riemannian manifold (W,g) with boundary ∂W, for any representation ρ of the fundamental group (for simplicity assume $\mathrm{rk}(\rho)=1$), the analytic torsion is given by the Reidemeister torsion plus some further contributions. It was shown by J. Cheeger in [4] that this further contribution only depends on the boundary, and W. Lück proved the following formula in the case of a product metric near the boundary, where $\chi(X)$ denotes the Euler characteristic of X [16]

$$\log T_{\rm abs}((W,g);\rho)=\log\tau(W;\rho)+\frac{1}{4}\chi(\partial W)\log 2.$$

In the general case a further contribution appears, that measures how the metric is far from a product metric. A formula for this new anomaly boundary contribution is contained in some recent result of Brüning and Ma [1]. More precisely, in [1] (equation (0.6)) is given a formula for the ratio of the analytic torsion of two metrics, g_1 and g_0. Using their notation for $\mathbb{Z}/2$ graded algebras, we identify an antisymmetric endomorphism ϕ of finite dimensional vector space V (over a field of characteristic zero) with the element $\hat\phi=\frac{1}{2}\sum_{j,k=1}^m\langle\phi(v_j),v_k\rangle\hat v_j\wedge\hat v_k$, of $\widehat{\Lambda^2 V}$. For the elements $\langle\phi(v_j),v_k\rangle$ are the entries of the tensor representing ϕ in the base $\{v_k\}$, and this is an antisymmetric matrix. Now assume that r is an antisymmetric endomorphism of Λ^2V. Then, $(R_{jk}=\langle r(v_j),v_k\rangle)$ is a tensor of two forms in $\Lambda^2 V$. We extend the above construction identifying R with the element

$$\hat R=\frac{1}{2}\sum_{j,k=1}^m\langle r(v_j),v_k\rangle\wedge\hat v_j\wedge\hat v_k, \tag{2.9}$$

of $\Lambda^2V\wedge\widehat{\Lambda^2V}$. This can be generalized to higher dimensions. In particular, all the construction can be done taking the dual V^* instead of V. Accordingly to [1], we define the following forms

$$
\begin{aligned}
\mathcal{S}_j &= \frac{1}{2}\sum_{k=1}^{m-1}(i^*\omega_j-i^*\omega_0)_{0k}\wedge\hat e_k^*\\
\widehat{i^*\Omega}_j &= \frac{1}{2}\sum_{k,l=1}^{m-1}i^*\Omega_{j,kl}\wedge\hat e_k^*\wedge\hat e_l^*,\qquad \hat\Theta=\frac{1}{2}\sum_{k,l=1}^{m-1}\Theta_{kl}\wedge\hat e_k^*\wedge\hat e_l^*.
\end{aligned}
\tag{2.10}
$$

Here, ω_j are the connection one forms, and Ω_j, $j=0,1$, the curvature two forms associated to the metrics g_0 and g_1, respectively, while Θ is the curvature two form of the boundary (with the metric induced by the inclusion), and $\{e_k\}_{k=0}^{m-1}$ is an orthonormal base of TW (with respect to the metric g). Then, set

$$B(\nabla_j)=\frac{1}{2}\int_0^1\int^B \mathrm{e}^{-\frac{1}{2}\hat\Theta-u^2\mathcal{S}_j^2}\sum_{k=1}^\infty\frac{1}{\Gamma\left(\frac{k}{2}+1\right)}u^{k-1}\mathcal{S}_j^k du. \tag{2.11}$$

Taking $g_1 = g$, and g_0 an opportune deformation of g, that is a product metric near the boundary, and a flat vector bundle F, the formula of [1] reads

$$\log \frac{T_{\rm abs}((W,g_1);\rho)}{T_{\rm abs}((W,g_0);\rho)} = \frac{1}{2}\int_{\partial W} B(\nabla_1).$$

Note that the right side of this equation is (as expected) a local quantity, and is well defined if there exists a regular collar neighborhood of the boundary. If this is the case, we define the Brüning and Ma *anomaly boundary term* (with absolute BC) by

$$A_{\rm BM,abs}(\partial W) = \frac{1}{2}\int_{\partial W} B(\nabla_1), \tag{2.12}$$

and we have

$$\log T_{\rm abs}((W,g);\rho) = \log\tau(W;\rho) + \frac{1}{4}\chi(\partial W)\log 2 + A_{\rm BM,abs}(\partial W). \tag{2.13}$$

2.4. **Zeta determinants.** This section is essentially contained in Section 4 of [11], to which we refer for details. Given a sequence $S = \{a_n\}_{n=1}^\infty$ of spectral type, we define the *zeta function* by

$$\zeta(s,S) = \sum_{n=1}^\infty a_n^{-s},$$

when $\mathrm{Re}(s) > \mathsf{e}(S)$, and by analytic continuation otherwise, and for all $\lambda \in \rho(S) = \mathbb{C} - S$, we define the *Gamma function* by the canonical product,

$$\frac{1}{\Gamma(-\lambda,S)} = \prod_{n=1}^\infty \left(1 + \frac{-\lambda}{a_n}\right) \mathrm{e}^{\sum_{j=1}^{\mathsf{g}(S)} \frac{(-1)^j}{j}\frac{(-\lambda)^j}{a_n^j}}. \tag{2.14}$$

Given a double sequence $S = \{\lambda_{n,k}\}_{n,k=1}^\infty$ of non vanishing complex numbers with unique accumulation point at the infinity, finite exponent $s_0 = \mathsf{e}(S)$ and genus $p = \mathsf{g}(S)$, we use the notation S_n (S_k) to denote the simple sequence with fixed n (k), we call the exponents of S_n and S_k the *relative exponents* of S, and we use the notation $(s_0 = \mathsf{e}(S), s_1 = \mathsf{e}(S_k), s_2 = \mathsf{e}(S_n))$; we define *relative genus* accordingly.

Definition 2.1. *Let $S = \{\lambda_{n,k}\}_{n,k=1}^\infty$ be a double sequence with finite exponents (s_0, s_1, s_2), genus (p_0, p_1, p_2), and positive spectral sector Σ_{θ_0,c_0}. Let $U = \{u_n\}_{n=1}^\infty$ be a totally regular sequence of spectral type of infinite order with exponent r_0, genus q, domain $D_{\phi,d}$. We say that S is spectrally decomposable over U with power κ, length ℓ and asymptotic domain $D_{\theta,c}$, with $c = \min(c_0, d, c')$, $\theta = \max(\theta_0, \phi, \theta')$, if there exist positive real numbers κ, ℓ (integer), c', and θ', with $0 < \theta' < \pi$, such that:*

1. *the sequence $u_n^{-\kappa} S_n = \left\{\frac{\lambda_{n,k}}{u_n^\kappa}\right\}_{k=1}^\infty$ has spectral sector $\Sigma_{\theta',c'}$, and is a totally regular sequence of spectral type of infinite order for each n;*
2. *the logarithmic Γ-function associated to S_n/u_n^κ has an asymptotic expansion for large n uniformly in λ for λ in $D_{\theta,c}$, of the following form*

$$\log\Gamma(-\lambda, u_n^{-\kappa}S_n) = \sum_{h=0}^{\ell} \phi_{\sigma_h}(\lambda) u_n^{-\sigma_h} + \sum_{l=0}^{L} P_{\rho_l}(\lambda) u_n^{-\rho_l}\log u_n + o(u_n^{-r_0}), \tag{2.15}$$

where σ_h and ρ_l are real numbers with $\sigma_0 < \cdots < \sigma_\ell$, $\rho_0 < \cdots < \rho_L$, the $P_{\rho_l}(\lambda)$ are polynomials in λ satisfying the condition $P_{\rho_l}(0) = 0$, ℓ and L are the larger integers such that $\sigma_\ell \le r_0$ and $\rho_L \le r_0$.

Define the following functions, ($\Lambda_{\theta,c} = \left\{ z \in \mathbb{C} \mid |\arg(z-c)| = \frac{\theta}{2} \right\}$, oriented counter clockwise):

$$\Phi_{\sigma_h}(s) = \int_0^\infty t^{s-1} \frac{1}{2\pi i} \int_{\Lambda_{\theta,c}} \frac{\mathrm{e}^{-\lambda t}}{-\lambda} \phi_{\sigma_h}(\lambda) d\lambda dt. \tag{2.16}$$

By Lemma 3.3 of [30], for all n, we have the expansions:

$$\begin{aligned} \log\Gamma(-\lambda, S_n/u_n^\kappa) &\sim \sum_{j=0}^\infty a_{\alpha_j,0,n}(-\lambda)^{\alpha_j} + \sum_{k=0}^{p_2} a_{k,1,n}(-\lambda)^k \log(-\lambda), \\ \phi_{\sigma_h}(\lambda) &\sim \sum_{j=0}^\infty b_{\sigma_h,\alpha_j,0}(-\lambda)^{\alpha_j} + \sum_{k=0}^{p_2} b_{\sigma_h,k,1}(-\lambda)^k \log(-\lambda), \end{aligned} \tag{2.17}$$

for large λ in $D_{\theta,c}$. We set (see Lemma 3.5 of [30])

$$\begin{aligned} A_{0,0}(s) &= \sum_{n=1}^\infty \left(a_{0,0,n} - \sum_{h=0}^{\ell} b_{\sigma_h,0,0} u_n^{-\sigma_h} \right) u_n^{-\kappa s}, \\ A_{j,1}(s) &= \sum_{n=1}^\infty \left(a_{j,1,n} - \sum_{h=0}^{\ell} b_{\sigma_h,j,1} u_n^{-\sigma_h} \right) u_n^{-\kappa s},\ 0 \le j \le p_2. \end{aligned} \tag{2.18}$$

Theorem 2.1. *Let S be spectrally decomposable over U as in Definition 2.1. Assume that the functions $\Phi_{\sigma_h}(s)$ have at most simple poles for $s = 0$. Then, $\zeta(s, S)$ is regular at $s = 0$, and*

$$\zeta(0, S) = -A_{0,1}(0) + \frac{1}{\kappa} \sum_{h=0}^{\ell} \operatorname*{Res_1}_{s=0} \Phi_{\sigma_h}(s) \operatorname*{Res_1}_{s=\sigma_h} \zeta(s, U),$$

$$\begin{aligned} \zeta'(0, S) = &- A_{0,0}(0) - A'_{0,1}(0) + \frac{\gamma}{\kappa} \sum_{h=0}^{\ell} \operatorname*{Res_1}_{s=0} \Phi_{\sigma_h}(s) \operatorname*{Res_1}_{s=\sigma_h} \zeta(s, U) \\ &+ \frac{1}{\kappa} \sum_{h=0}^{\ell} \operatorname*{Res_0}_{s=0} \Phi_{\sigma_h}(s) \operatorname*{Res_1}_{s=\sigma_h} \zeta(s, U) + {\sum_{h=0}^{\ell}}' \operatorname*{Res_1}_{s=0} \Phi_{\sigma_h}(s) \operatorname*{Res_0}_{s=\sigma_h} \zeta(s, U), \end{aligned}$$

where the notation $\sum'$ means that only the terms such that $\zeta(s, U)$ has a pole at $s = \sigma_h$ appear in the sum.

Remark 2.1. *We call regular part of $\zeta(0, S)$ the first term appearing in the formula given in the theorem, and regular part of $\zeta'(0, S)$ the first two terms. The other terms we call singular part.*

Corollary 2.1. *Let $S_{(j)} = \{\lambda_{(j),n,k}\}_{n,k=1}^\infty$, $j = 1, ..., J$, be a finite set of double sequences that satisfy all the requirements of Definition 2.1 of spectral decomposability over a common sequence U, with the same parameters κ, ℓ, etc., except that the polynomials $P_{(j),\rho}(\lambda)$ appearing in condition (2) do not vanish for $\lambda = 0$. Assume that some linear combination $\sum_{j=1}^J c_j P_{(j),\rho}(\lambda)$, with complex coefficients, of such polynomials does satisfy this condition, namely that $\sum_{j=1}^J c_j P_{(j),\rho}(\lambda) = 0$. Then, the linear combination of the zeta function $\sum_{j=1}^J c_j \zeta(s, S_{(j)})$ is regular at $s = 0$ and satisfies the linear combination of the formulas given in Theorem 2.1.*

We conclude recalling some formulas for the zeta determinants of some simple sequences. The results are known to specialists, and can be found in different places. We will use the formulation of [25]. For positive real l and q, define the *non homogeneous quadratic Bessel zeta function* by

$$z(s,\nu,q,l)=\sum_{k=1}^\infty\left(\frac{j_{\nu,k}^2}{l^2}+q^2\right)^{-s},$$

for $\mathrm{Re}(s)>\frac{1}{2}$. Then, $z(s,\nu,q,l)$ extends analytically to a meromorphic function in the complex plane with simple poles at $s=\frac{1}{2},-\frac{1}{2},-\frac{3}{2},\ldots$. The point $s=0$ is a regular point and

$$z(0,\nu,q,l)=-\frac{1}{2}\left(\nu+\frac{1}{2}\right),\quad z'(0,\nu,q,l)=-\log\sqrt{2\pi l}\frac{I_\nu(lq)}{q^\nu}. \tag{2.19}$$

In particular, taking the limit for $q\to 0$,

$$z'(0,\nu,0,l)=-\log\frac{\sqrt{\pi}l^{\nu+\frac{1}{2}}}{2^{\nu-\frac{1}{2}}\Gamma(\nu+1)}.$$

3. Geometric setting and Laplace operator

3.1. **The finite metric cone.** Let (W,g) be an orientable compact connected Riemannian manifold of finite dimension m without boundary and with Riemannian structure g. Embedding W in the opportune Euclidean space $\mathbb{R}^k$, and $\mathbb{R}^k$ in some hyperplane of $\mathbb{R}^{k+h}$, with opportune h, disconnected from the origin, a geometric realization of the cone CW is the given by the set of the finite length l line segments joining the origin to the embedded copy of W. Let x the euclidean geodesic distance from the origin. We equip $CW-\{p\}$ with the Riemannian structure $dx\otimes dx+x^2g$, and we denote by $C_{(0,l]}W$ the space $(0,l]\times W$ with this metric. We denote by C_lW the compact space $\overline{C_{(0,l]}W}=C_{(0,l]}W\cup\{p\}$, and we call it the *(completed finite metric) cone* over W. We call the subspace $\{l\}\times W$ of C_lW, the *boundary of the cone*, and we denote it by ∂C_lW. This is of course diffeomorphic to W, and isometric to (W,l^2g). Following common notation, we will call (W,g) the *section* of the cone. Also following usual notation, a tilde will denotes operations on the section (of course $\tilde g=g$), and not on the boundary. All the results of Section 2.1 are valid. In particular, given a local coordinate system y on W, then (x,y) is a local coordinate system on the cone.

We now give the explicit form of $\star$, $d^\dagger$ and Δ. See [5] and [20] Section 5 for details. If $\omega\in\Omega^q(C_{(0,l]}W)$, set

$$\omega(x,y)=f_1(x)\omega_1(y)+f_2(x)dx\wedge\omega_2(y),$$

with smooth functions f_1 and f_2, and $\omega_j\in\Omega(W)$. Then a straightforward calculation gives

$$\star\omega(x,y)=x^{m-2q+2}f_2(x)\tilde\star\omega_2(y)+(-1)^q x^{m-2q}f_1(x)dx\wedge\tilde\star\omega_1(y), \tag{3.1}$$

$$\begin{aligned} d\omega(x,y)&=f_1(x)\tilde d\omega_1(y)+\partial_x f_1(x)dx\wedge\omega_1(y)-f_2(x)dx\wedge\tilde d\omega_2(y),\\ d^\dagger\omega(x,y)&=x^{-2}f_1(x)\tilde d^\dagger\omega_1(y)-\left((m-2q+2)x^{-1}f_2(x)+\partial_x f_2(x)\right)\omega_2(y)\\ &\quad-x^{-2}f_2(x)dx\wedge\tilde d^\dagger\omega_2(y), \end{aligned} \tag{3.2}$$

$$\begin{aligned} \Delta\omega(x,y)=&\left(-\partial_x^2 f_1(x)-(m-2q)x^{-1}\partial_x f_1(x)\right)\omega_1(y)+x^{-2}f_1(x)\tilde\Delta\omega_1(y)-2x^{-1}f_2(x)\tilde d\omega_2(y)\\ &+dx\wedge\Big(x^{-2}f_2(x)\tilde\Delta\omega_2(y)+\omega_2(y)\left(-\partial_x^2 f_2(x)-(m-2q+2)x^{-1}\partial_x f_2(x)\right.\\ &\left.+(m-2q+2)x^{-2}f_2(x)\right)-2x^{-3}f_1(x)\tilde d^\dagger\omega_1(y)\Big). \end{aligned} \tag{3.3}$$

3.2. **Riemannian tensors on the cone.** We give here the explicit form of the main Riemannian quantities on the cone. Recall that a tilde denotes quantities relative to the section, that we have local coordinate $(x, y_1, \ldots, y_m)$ on C_lW, and that the metric is $g_1 = dx \otimes dx + x^2 g$. Let $\{b_k\}_{k=1}^m$ be a local orthonormal base of TW, and $\{b_k^*\}_{k=1}^m$ the associated dual base. Then, $e_0 = \partial_x, e_0^* = dx, e_k = \frac{1}{x} b_k, e_k^* = x b_k^*, 1 \leq k \leq m$. Direct calculations give Cartan structure constants $c_{jk0} = 0, 1 \leq j,k \leq m, c_{0kl} = -c_{k0l} = -\frac{\delta_{kl}}{x}, 1 \leq k,l \leq m, c_{jkl} = \frac{1}{x}\tilde{c}_{jkl}, 1 \leq j,k,l \leq m$, and the Christoffel symbols are $\Gamma_{0kl} = 0, 1 \leq k,l \leq m, \Gamma_{j0k} = -\Gamma_{jk0} = \frac{\delta_{jk}}{x}, 1 \leq j,k \leq m, \Gamma_{jkl} = \frac{1}{x}\tilde{\Gamma}_{jkl}, 1 \leq j,k,l \leq m$. The connection one form matrix relatively to the metric g_1 has components

$$\begin{aligned} &\omega_{1,00} = 0, \\ &\omega_{1,0j} = -\omega_{1,j0} = -\frac{1}{x} e_j^* = -b_j^*, && 1 \leq j \leq m, \\ &\omega_{1,jk} = \sum_{h=1}^m \Gamma_{hkj} e_h^* = \frac{1}{x}\sum_{h=1}^m \tilde{\Gamma}_{hkj} e_h^* = \sum_{h=1}^m \tilde{\Gamma}_{hkj} b_h^* = \tilde{\omega}_{jk}, && 1 \leq j,k \leq m. \end{aligned} \tag{3.4}$$

To compute the curvature we calculate

$$d\omega_{1,0j} = -\sum_{l=1}^m (\partial_l b_j^*) \wedge dy_l = -\sum_{l,k=1}^m (\partial_l b_{kj}) dy_k \wedge dy_l,$$

where $b_j^* = \sum_{k=1}^m b_{kj} dy_k$, and, for $1 \leq j,k \leq m$, $d\omega_{1,jk} = \tilde{d}\tilde{\omega}_{jk}$; while

$$\begin{aligned} -(\omega_1 \wedge \omega_1)_{k0} = (\omega_1 \wedge \omega_1)_{0k} &= \sum_{l=0}^m \omega_{1,0l} \wedge \omega_{1,lk} = \sum_{l=1}^m \omega_{1,0l} \wedge \omega_{1,lk} = -\sum_{l=1}^m b_l^* \wedge \tilde{\omega}_{lk}, \\ (\omega_1 \wedge \omega_1)_{jk} &= \sum_{l=0}^m \omega_{1,jl} \wedge \omega_{1,lk} = \omega_{1,j0} \wedge \omega_{1,0k} + \sum_{l=1}^m \omega_{1,jl} \wedge \omega_{1,lk} = -b_j^* \wedge b_k^* + (\tilde{\omega} \wedge \tilde{\omega})_{jk}, \end{aligned}$$

for $1 \leq j,k \leq m$. The curvature two form has components

$$\begin{aligned} &\Omega_{1,00} = 0, \\ &\Omega_{1,0j} = -\sum_{l,k=1}^m (\partial_l b_{kj}) dy_k \wedge dy_l - \sum_{l=1}^m b_l^* \wedge \tilde{\omega}_{lk}, && 1 \leq j \leq m, \\ &\Omega_{1,jk} = \tilde{d}\tilde{\omega}_{jk} - b_j^* \wedge b_k^* + (\tilde{\omega} \wedge \tilde{\omega})_{jk} = \tilde{\Omega}_{jk} - b_j^* \wedge b_k^*, && 1 \leq j,k \leq m. \end{aligned}$$

Next, considering the metric $g_0 = dx \otimes dx + g$, similar calculations gives:

$$\omega_{0,0j} = 0, \quad 0 \leq j \leq m, \quad \omega_{0,jk} = \tilde{\omega}_{jk}, \quad 1 \leq j,k \leq m. \tag{3.5}$$

By equations (3.4) and (3.5),

$$\mathcal{S}_1 = -\frac{1}{2l}\sum_{k=1}^m e_k^* \wedge e_k^* = -\frac{l}{2}\sum_{k=1}^m b_k^* \wedge b_k^* = -\frac{1}{2}\sum_{k=1}^m b_k^* \wedge e_k^*, \tag{3.6}$$

$$\mathcal{S}_0 = 0. \tag{3.7}$$

We also need the curvature two form Θ on the boundary ∂C_lW. A similar calculation gives $\Theta_{jk} = \tilde{\Omega}_{jk}$. Note in particular that it is easy to verify the equation (1.16) of [1]: $\hat{\Theta} = \widehat{i^*\Omega_1} - 2\mathcal{S}_1^2$.

For

$$2\mathcal{S}_1^2 = -\frac{l^2}{2}\sum_{j,k=1}^m b_j^* \wedge b_k^* \wedge \hat b_j^* \wedge \hat b_k^*, \qquad \hat\Theta = \frac{l^2}{2}\sum_{j,k=1}^m \tilde\Omega_{jk} \wedge \hat b_j^* \wedge \hat b_k^*,$$

while $(i^*\Omega)_{jk} = \tilde\Omega_{jk} - b_j^* \wedge b_k^*$, gives

$$\widehat{i^*\Omega_{1,jk}} = \frac{l^2}{2}\sum_{j,k=1}^m \left(\tilde\Omega_{jk} - b_j^* \wedge b_k^*\right) \wedge \hat b_j^* \wedge \hat b_k^*.$$

3.3. **The Laplace operator on the cone and its spectrum.** We study the Laplace operator on forms on the space C_lW. This is essentially based on [5] and [3]. Let denote by $\mathcal{L}$ the formal differential operator defined by equation (3.3) acting on smooth forms on $C_{(0,l]}W$, $\Gamma(C_{(0,l]}W, \Lambda T^*C_{(0,l]}W)$. We define in Lemma 3.1 a self adjont operator Δ acting on $L^2(C_lW, \Lambda^{(q)}C_lW)$, and such that $\Delta\omega = \mathcal{L}\omega$, if $\omega \in \mathrm{dom}\Delta$. Then, in Lemma 3.2, we list all the solutions of the eigenvalues equation for $\mathcal{L}$. Eventually, in Lemma 3.3, we give the spectrum of Δ.

Lemma 3.1. *The formal operator $\mathcal{L}$ in equation (3.3) with the absolute/relative boundary conditions given in equations (2.2)/(2.3) on the boundary ∂C_lW defines a unique self adjoint semi bounded operator on $L^2(C_lW, \Lambda^{(q)}T^*C_lW)$, that we denote by the symbol $\Delta_{\rm abs}/\Delta_{\rm rel}$, respectively, with pure point spectrum.*

Proof. Let $L^{(q)}$ denote the minimal operator defined by the formal operator $\mathcal{L}^{(q)}$, with domain the q-forms with compact support in $C_{(0,l]}W$, namely $\mathrm{dom}L^{(q)} = \Gamma_0(C_{(0,l]}W, \Lambda T^*C_{(0,l]}W)$. The boundary values problem at the boundary $x = l$, i.e. ∂C_lW, is trivial, and gives the self adjoint extensions stated. The point $x = 0$ requires more work. First, note that $L^{(q)}$ reduces by unitary transformation to an operator of the type

$$D^2 + \frac{A(x)}{x^2}, \qquad D = -i\frac{d}{dx}, \tag{3.8}$$

where $A(x)$ is smooth family of symmetric second order elliptic operators [3] pg. 370. More precisely, there exists a unitary transformation ψ_q between the relevant spaces with the suitable L^2 structures, see [3] for details. Under the transformation ψ_q, $L^{(q)}$ has the form in equation (3.8), with $A(x)$ the constant smooth family of symmetric second order elliptic operators in $\Gamma(W, \Lambda^{(q)}T^*W \times \Lambda^{(q-1)}T^*W)$:

$$A(x) = A(0) = \begin{pmatrix} \tilde\Delta^{(q)} + \left(\frac{m}{2} - q\right)\left(\frac{m}{2} - q - 1\right) & 2(-1)^q \tilde d \\ 2(-1)^q \tilde d^\dagger & \tilde\Delta^{(q-1)} + \left(\frac{m}{2} + 2 - q\right)\left(\frac{m}{2} + 1 - q\right) \end{pmatrix}$$

Next, by its definition, $A(x)$ satisfies all the requirements at pg. 373 of [3], with $p = 1$ (in particular this follows from the fact that $A(x)$ is defined by the Laplacian on forms on a compact space). We can apply the results of Brüning and Seeley [3], observing that in the present case we are in what they call "constant coefficient case" (Section 3 of [3]). By Theorem 5.1 of [3], the operator L extends to a unique self adjoint bounded operator $\Delta^{(q)}$. Note that this extension is the Friedrich extension by Theorem 6.1 of [3]. Note also that bundary condition at $x = 0$ are necessary in general in the definition of the domain of $\Delta^{(q)}$, see (L2) (c), pg. 410 of [3] for these conditions.

Eventually, by Theorem 5.2 of [3], the square (here $p = 1$, so $m = 2$) of the resolvent of $\Delta^{(q)}$ is of trace class. This means that the resolvent is Hilbert Schmidt, and consequently the spectrum of $\Delta^{(q)}$ is pure point, by the spectral theorm for compact operators. Note that we do not need the cut off function γ appearing in Theorem 5.2 of [3], since here $0 < x \leq l$.

□

Lemma 3.2 ([5]). *Let $\{\varphi^{(q)}_{\rm har}, \varphi^{(q)}_{{\rm cex},n}, \varphi^{(q)}_{{\rm ex},n}\}$ be an orthonormal base of $\Gamma(W, \Lambda^{(q)}T^*W)$ consisting of harmonic, coexact and exact eigenforms of $\tilde\Delta^{(q)}$ on W. Let $\lambda_{q,n}$ denotes the eigenvalue of $\varphi^{(q)}_{{\rm cex},n}$ and $m_{{\rm cex},q,n}$ its multiplicity (so that $m_{{\rm cex},q,n} = \dim \mathcal{E}^{(q)}_{{\rm cex},n} = \dim \mathcal{E}^{(q)}_{{\rm ccl},n}$). Let J_ν be the Bessel function of index ν. Define $\alpha_q = \frac{1}{2}(1 + 2q - m)$ and $\mu_{q,n} = \sqrt{\lambda_{q,n} + \alpha_q^2}$. Then, assuming that $\mu_{q,n}$ is not an integer, all the solutions of the equation $\Delta u = \lambda^2 u$, with $\lambda \neq 0$, are convergent sums of forms of the following six types:*

$$
\begin{aligned}
\psi^{(q)}_{\pm,1,n,\lambda} =& x^{\alpha_q} J_{\pm\mu_{q,n}}(\lambda x)\varphi^{(q)}_{{\rm cex},n},\\
\psi^{(q)}_{\pm,2,n,\lambda} =& x^{\alpha_{q-1}} J_{\pm\mu_{q-1,n}}(\lambda x)\tilde d\varphi^{(q-1)}_{{\rm cex},n} + \partial_x(x^{\alpha_{q-1}} J_{\pm\mu_{q-1,n}}(\lambda x))dx \wedge \varphi^{(q-1)}_{{\rm cex},n}\\
\psi^{(q)}_{\pm,3,n,\lambda} =& x^{2\alpha_{q-1}+1}\partial_x(x^{-\alpha_{q-1}} J_{\pm\mu_{q-1,n}}(\lambda x))\tilde d\varphi^{(q-1)}_{{\rm cex},n}\\
&+ x^{\alpha_{q-1}-1} J_{\pm\mu_{q-1,n}}(\lambda x)dx \wedge \tilde d^\dagger \tilde d\varphi^{(q-1)}_{{\rm cex},n}\\
\psi^{(q)}_{\pm,4,n,\lambda} =& x^{\alpha_{q-2}+1} J_{\pm\mu_{q-2,n}}(\lambda x)dx \wedge \tilde d\varphi^{(q-2)}_{{\rm cex},n}\\
\psi^{(q)}_{\pm,E,\lambda} =& x^{\alpha_q} J_{\pm|\alpha_q|}(\lambda x)\varphi^{(q)}_{\rm har}\\
\psi^{(q)}_{\pm,O,\lambda} =& \partial_x(x^{\alpha_{q-1}} J_{\pm|\alpha_{q-1}|}(\lambda x))dx \wedge \varphi^{(q-1)}_{\rm har}.
\end{aligned}
$$

When $\mu_{q,n}$ is an integer the $-$ solutions must be modified including some logarithmic term (see for example [34] *for a set of linear independent solutions of the Bessel equation).*

Proof. The proof is a direct verification of the assertion, using the definitions in equations (3.1), (3.2), and (3.3). First, by Hodge theorem, there exist an orthonormal base of $\Lambda^{(q)}T^*W$ as stated. Thus, we decompose any form ω in this base. Second, we compute $\Delta\omega$, using this decomposition and the formula in equation (3.3). This gives some differential equations in the functions appearing as coefficients of the forms. All these differential equations reduce to equations of Bessel type. Third, we write all the solutions using Bessel functions. A complete proof for the case of the harmonic forms can be found in [20] Section 5. □

Note that the forms of types 1 and 3 are coexact, those of types 2 and 4 exacts. The operator d sends forms of types 1 and 3 in forms of types 2 and 4, while $d^\dagger$ sends forms of types 2 and 4 in forms of types 1 and 3, respectively. The Hodge operator sends forms of type 1 in forms of type 4, 2 in 3, and E in 0.

Corollary 3.1. *The functions $+$ in Lemma 3.1 are square integrable and satisfy the boundary conditions at $x = 0$ defining the domain of $\Delta_{\rm rel/abs}$. The functions $-$ either are not square integrable or do not satisfy these conditions.*

Remark 3.1. *All the $-$ solutions are either not square or their exterior derivative are not square integrable. Requiring the last condition in the definition of the domain of $\Delta_{\rm rel/abs}$, it follows that there are not boundary conditions at zero. This was observed by Cheeger for harmonic forms when the dimension is odd in* [5] *Section 3.*

Lemma 3.3. *The positive part of the spectrum of the Laplace operator on forms on C_lW, with absolute boundary conditions on ∂C_lW is:*

$$\begin{aligned}
\mathrm{Sp}_+\Delta^{(q)}_{\mathrm{abs}} &= \left\{m_{\mathrm{cex},q,n} : \hat{j}^2_{\mu_{q,n},\alpha_q,k}/l^2\right\}_{n,k=1}^{\infty} \cup \left\{m_{\mathrm{cex},q-1,n} : \hat{j}^2_{\mu_{q-1,n},\alpha_{q-1},k}/l^2\right\}_{n,k=1}^{\infty}\\
&\cup \left\{m_{\mathrm{cex},q-1,n} : j^2_{\mu_{q-1,n},k}/l^2\right\}_{n,k=1}^{\infty} \cup \left\{m_{\mathrm{cex},q-2,n} : j^2_{\mu_{q-2,n},k}/l^2\right\}_{n,k=1}^{\infty}\\
&\cup \left\{m_{\mathrm{har},q} : \hat{j}^2_{|\alpha_q|,\alpha_q,k}/l^2\right\}_{k=1}^{\infty} \cup \left\{m_{\mathrm{har},q-1} : \hat{j}^2_{|\alpha_{q-1}|,\alpha_q,k}/l^2\right\}_{k=1}^{\infty}.
\end{aligned}$$

With relative boundary conditions:

$$\begin{aligned}
\mathrm{Sp}_+\Delta^{(q)}_{\mathrm{rel}} &= \left\{m_{\mathrm{cex},q,n} : j^2_{\mu_{q,n},k}/l^2\right\}_{n,k=1}^{\infty} \cup \left\{m_{\mathrm{cex},q-1,n} : j^2_{\mu_{q-1,n},k}/l^2\right\}_{n,k=1}^{\infty}\\
&\cup \left\{m_{\mathrm{cex},q-1,n} : \hat{j}^2_{\mu_{q-1,n},-\alpha_{q-1},k}/l^2\right\}_{n,k=1}^{\infty} \cup \left\{m_{\mathrm{cex},q-2,n} : \hat{j}^2_{\mu_{q-1,n},-\alpha_{q-2},k}/l^2\right\}_{n,k=1}^{\infty}\\
&\cup \left\{m_{\mathrm{har},q} : j_{|\alpha_q|,k}/l^2\right\}_{k=1}^{\infty} \cup \left\{m_{\mathrm{har},q-1} : j^2_{|\alpha_{q-1}|,k}/l^2\right\}_{k=1}^{\infty},
\end{aligned}$$

where the $j_{\mu,k}$ are the zeros of the Bessel function $J_\mu(x)$, the $\hat{j}_{\mu,c,k}$ are the zeros of the function $\hat{J}_{\mu,c}(x) = cJ_\mu(x) + xJ'_\mu(x)$, $c \in \mathbb{R}$, α_q and $\mu_{q,n}$ are defined in Lemma 3.2.

Proof. By the Lemma 3.1, Lemma 3.2 and its corollary, we know that the + solutions of Lemma 3.2 determine a complete system of square integrable solutions of the eigenvalues equation $\Delta^{(q)}u = \lambda u$, with $\lambda \neq 0$, satisfying the boundary condition at $x = 0$. Since $\Delta^{(q)}_{\mathrm{abs/rel}}$ has pure point spectrum, in order to obtain a discrete resolution (more precisely the positive part of it) of $\Delta^{(q)}_{\mathrm{abs/rel}}$, we have to determine among these solutions those that belong to the domain of $\Delta^{(q)}_{\mathrm{abs/rel}}$, namely those that satisfy the boundary condition at $x = l$. By direct application of the BC we obtain the result. For example, for forms of type 3, we obtain the system

$$\begin{cases} x^{\alpha_{q-1}-1}J_{\mu_{q-1,n}}(\lambda x)\big|_{x=l} = 0,\\ \partial_x\left(x^{2\alpha_{q-1}+1}\partial_x(x^{-\alpha_{q-1}}J_{\mu_{q-1,n}}(\lambda x))\right) - \lambda x^{\alpha_{q-1}-1}J_{\mu_{q-2,n}}(\lambda x)\big|_{x=l} = 0,\end{cases}$$

that using classical properties of Bessel functions and their derivative, gives $\lambda = j_{\mu_{q-1},n,k}/l$. □

Lemma 3.4 ([5], [20])**.** *With the notation of Lemma 3.2, and $a_{\pm,q,n} = \alpha_q \pm \mu_{q,n}$, then all the solutions of the harmonic equation $\Delta u = 0$, are convergent sums of forms of the following four types:*

$$\begin{aligned}
\psi^{(q)}_{\pm,1,n} &= x^{a_{\pm,q,n}}\varphi^{(q)}_{\mathrm{ccl},n},\\
\psi^{(q)}_{\pm,2,n} &= x^{a_{\pm,q-1,n}}\tilde{d}\varphi^{(q-1)}_{\mathrm{ccl},n} + a_{\pm,q-1,n}x^{a_{\pm,q-1,n}-1}dx \wedge \varphi^{(q-1)}_{\mathrm{ccl},n},\\
\psi^{(q)}_{\pm,3,n} &= x^{a_{\pm,q-1,n}+2}\tilde{d}\varphi^{(q-1)}_{\mathrm{ccl},n} + a_{\mp,q-1,n}x^{a_{\pm,q-1,n}+1}dx \wedge \varphi^{(q-1)}_{\mathrm{ccl},n},\\
\psi^{(q)}_{\pm,4,n} &= x^{a_{\pm,q-2,n}+1}dx \wedge \tilde{d}\varphi^{(q-2)}_{\mathrm{ccl},n}.
\end{aligned}$$

Lemma 3.5. *Assume* $\dim W = 2p-1$ *is odd. Then*

$$\mathcal{H}^q_{\rm abs}(C_l W) = \begin{cases} \mathcal{H}^q(W), & 0 \leq q \leq p-1, \\ \{0\}, & p \leq q \leq 2p-1. \end{cases}$$

$$\mathcal{H}^q_{\rm rel}(C_l W) = \begin{cases} \{0\}, & 0 \leq q \leq p, \\ \left\{x^{2\alpha_q-1} dx \wedge \varphi^{(q-1)}, \varphi^{(q-1)} \in \mathcal{H}^{q-1}(W)\right\}, & p+1 \leq q \leq 2p. \end{cases}$$

Proof. First, by Remark 3.1, we need only to consider the + solutions in Lemma 3.4. The proof then follows by argument similar to the one used in the proof of Lemma 3.3. Let see one case in details. Consider $\psi^{(q)}_{+,1,n} = x^{a_{+,q,n}} \varphi^{(q)}_{{\rm ccl},n}$, where $a_{+,q,n} = \alpha_q + \mu_{q,n}$. In order that $\psi^{(q)}_{+,1,n}$ satisfies the absolute boundary condition (2.2), we need that

$$(d\psi^{(q)}_{+,1,n})_{\rm norm}\Big|_{\partial C_l W} = a_{+,q,n} l^{a_{+,q,n}-1} dx \wedge d\varphi^{(q)}_{{\rm ccl},n} = 0$$

and this is true if and only if $a_{+,q,n} = 0$. The condition $a_{+,q,n} = 0$ is equivalent to the conditions $\lambda_{q,n} = 0$, and $\alpha_q = -|\alpha_q|$. Therefore, $\varphi^{(q)}_{{\rm ccl},n}$ is harmonic, $0 \leq q \leq p-1$, and $\psi^{(q)}_{+,1,n} = \varphi^{(q)}_{{\rm ccl},n}$. □

4. Torsion zeta function and Poincaré duality for a cone

Using the description of the spectrum of the Laplace operator on forms $\Delta^{(q)}_{\rm abs/rel}$ given in Lemma 3.3, we define the zeta function on q-forms as in Section 2.2, by

$$\zeta(s, \Delta^{(q)}_{\rm abs/rel}) = \sum_{\lambda \in {\rm Sp}_+ \Delta^{(q)}_{\rm abs/rel}} \lambda^{-s},$$

for ${\rm Re}(s) > \frac{m+1}{2}$. The explicit knowledge of the behaviour of the large eigenvalues allows to completely determine the analytic continuation of the zeta function, by using the tools os Section 2.4. In particular , it is possible to prove that there can be at most a simple pole at $s = 0$. We will not do this here (but the interested reader can compare with [30]), because for our purpose it is more convenient to investigate the analytic properties of other zeta functions, resulting by a suitable different decomposition of the analytic torsion, as described here below. For we define the *torsion zeta function* by

$$t_{\rm abs/rel}(s) = \frac{1}{2} \sum_{q=1}^{m+1} (-1)^q q \zeta(s, \Delta^{(q)}_{\rm abs/rel}).$$

It is clear that the analytic torsion of $C_l W$ is (in the following we will use the simplified notation $T(C_l W)$ for $T((C_l W, g); \rho)$)

$$\log T_{\rm abs/rel}(C_l W) = t'_{\rm abs/rel}(0).$$

Our first result is a Poincaré duality (compare with Proposition 2.4, [16] and the result of [7]).

Theorem 4.1. Poincaré duality for the analytic torsion of a cone. *Let* (W, g) *be an orientable compact connected Riemannian manifold of dimension* m*, without boundary, then*

$$\log T_{\rm abs}(C_l W) = (-1)^m \log T_{\rm rel}(C_l W).$$

Proof. By Hodge duality in equation (2.7), the Hodge operator $\star$ sends forms of type $1, 2, 3, 4, E$, and O into forms of type $4, 3, 2, 1, O$, and E, respectively. Moreover, $\star$ sends q-forms satisfying absolute boundary conditions, as in equation (2.2), into $m+1-q$-forms satisfying relative boundary conditions, as in equation (2.3). Therefore, using the explicit description of the eigenvalues given in Lemma 3.3, it follows that $\mathrm{Sp}\Delta_{\rm abs}^{(q)} = \mathrm{Sp}\Delta_{\rm rel}^{(m+1-q)}$. Using the formulas in equations (3.1), (3.2), and (3.3), and the eigenforms in Lemma 3.2, a straightforward calculation shows that the forms of type 1, 3, and E are coexact, and those of type 2, 4, and O are exact, and that the operator d sends forms of type 1, 3, and E in forms of type 2, 4, and O, respectively, with inverse $d^\dagger$. Then, set

$$\begin{aligned}
F^{(q)}_{\rm ccl,abs} &= \left\{ m_{{\rm ccl},q,n} : \hat{j}^2_{\mu_{q,n},\alpha_q,k}/l^2 \right\}_{n,k=1}^{\infty} \cup \left\{ m_{{\rm ccl},q-1,n} : j^2_{\mu_{q-1,n},k}/l^2 \right\}_{n,k=1}^{\infty}\\
&\quad \cup \left\{ m_{{\rm ccl},q,0} : \hat{j}^2_{|\alpha_q|,\alpha_q,k}/l^2 \right\}_{k=1}^{\infty},\\
F^{(q)}_{\rm cl,abs} &= \left\{ m_{{\rm cl},q-1,n} : \hat{j}^2_{\mu_{q-1,n},\alpha_{q-1},k}/l^2 \right\}_{n,k=1}^{\infty} \cup \left\{ m_{{\rm cl},q-2,n} : j^2_{\mu_{q-2,n},k}/l^2 \right\}_{n,k=1}^{\infty}\\
&\quad \cup \left\{ m_{{\rm cl},q-1,0} : \hat{j}^2_{|\alpha_{q-1}|,\alpha_{q-1},k}/l^2 \right\}_{k=1}^{\infty}.
\end{aligned}$$

$F^{(q)}_{\rm ccl,abs}$ is the set of the eigenvalues of the coclosed q-forms with absolute boundary conditions, and $F^{(q)}_{\rm cl,abs}$ is the set of the eigenvalues of the closed q-forms with absolute boundary conditions. Since obviously $\mathrm{Sp}\Delta^{(q)}_{\rm abs} = F^{(q)}_{\rm ccl,abs} \cup F^{(q)}_{\rm cl,abs}$, and $F^{(q)}_{\rm ccl,abs} = F^{(q+1)}_{\rm cl,abs}$, we have that

$$\begin{aligned}
t_{\rm abs}(s) &= \frac{1}{2}\sum_{q=0}^{m+1} (-1)^q q \zeta(s, \Delta^{(q)}_{\rm abs}) = \frac{1}{2}\sum_{q=0}^{m+1} (-1)^q q \zeta(s, \Delta^{(m+1-q)}_{\rm rel})\\
&= (-1)^m t_{\rm rel}(s) + \frac{1}{2}(m+1)\sum_{q=0}^{m+1} (-1)^{m+1-q} \zeta(s, \Delta^{(q)}_{\rm rel})\\
&= (-1)^m t_{\rm rel}(s) + \frac{1}{2}(m+1)\sum_{q=0}^{m+1} (-1)^{q} \left(\zeta(s, F^{(q+1)}_{\rm ccl,abs}) + \zeta(s, F^{(q)}_{\rm cl,abs}) \right) = (-1)^m t_{\rm rel}(s).
\end{aligned}$$

Since by definition $\log T_{\rm abs}(W) = t'_{\rm abs}(0)$, the thesis follows. □

5. The torsion zeta function of the cone over an odd dimensional manifold

In this section we develop the main steps in order to obtain the proof of our theorems. This accounts essentially in the application of the tools described in Section 2.4 to some suitable sequences appearing in the definition of the torsion. So our first step is precisely to obtain this suitable description. This we do in this section. In the next two subsections, we will make the calculations necessary for the proof of our main theorems. We proceed assuming $\dim W = 2p-1$ odd, and assuming absolute boundary condition; for notational convenience, we will omit the *abs* subscript.

Lemma 5.1. *Here* $j'_{\nu,k} = \hat{j}_{\nu,0,k}$.

$$t(s) = \frac{l^{2s}}{2}\sum_{q=0}^{p-2}(-1)^q\left(\sum_{n,k=1}^{\infty} m_{\mathrm{cex},q,n}\left(2j_{\mu_{q,n},k}^{-2s} - \hat{j}_{\mu_{q,n},\alpha_q,k}^{-2s} - \hat{j}_{\mu_{q,n},-\alpha_q,k}^{-2s}\right)\right)$$
$$+ (-1)^{p-1}\frac{l^{2s}}{2}\left(\sum_{n,k=1}^{\infty} m_{\mathrm{cex},p-1,n}\left(j_{\mu_{p-1,n},k}^{-2s} - (j'_{\mu_{p-1,n},k})^{-2s}\right)\right)$$
$$- \frac{l^{2s}}{2}\sum_{q=0}^{p-1}(-1)^q \mathrm{rk}\mathcal{H}_q(\partial C_l W;\mathbb{Q})\sum_{k=1}^{\infty}\left(j_{-\alpha_{q-1},k}^{-2s} - j_{-\alpha_q,k}^{-2s}\right).$$

Proof. Using the eigenvalues in Lemma 3.3

$$l^{2s}\zeta(s,\Delta^{(q)}) = \sum_{n,k=1}^{\infty} m_{\mathrm{cex},q,n}\hat{j}_{\mu_{q,n},\alpha_q,k}^{-2s} + \sum_{n,k=1}^{\infty} m_{\mathrm{cex},q-1,n}\hat{j}_{\mu_{q-1,n},\alpha_{q-1},k}^{-2s} + \sum_{n,k=1}^{\infty} m_{\mathrm{cex},q-1,n}j_{\mu_{q-1,n},k}^{-2s}$$
$$+ \sum_{n,k=1}^{\infty} m_{\mathrm{cex},q-2,n}j_{\mu_{q-2,n},k}^{-2s} + \sum_{k=1}^{\infty} m_{\mathrm{har},q,0}\hat{j}_{|\alpha_q|,\alpha_q,k}^{-2s} + \sum_{k=1}^{\infty} m_{\mathrm{har},q-1,0}\hat{j}_{|\alpha_{q-1}|,\alpha_{q-1},k}^{-2s}.$$

Since for each fixed q, with $0 \leq q \leq 2p-2$,

$$(-1)^q q\sum_{n,k=1}^{\infty} m_{\mathrm{cex},q,n}\hat{j}_{\mu_{q,n},\alpha_q,k}^{-2s} + (-1)^{q+1}(q+1)\sum_{n,k=1}^{\infty} m_{\mathrm{cex},q,n}\hat{j}_{\mu_{q,n},\alpha_q,k}^{-2s}$$
$$+ (-1)^{q+1}(q+1)\sum_{n,k=1}^{\infty} m_{\mathrm{cex},q,n}j_{\mu_{q,n},k}^{-2s} + (-1)^{q+2}(q+2)\sum_{n,k=1}^{\infty} m_{\mathrm{cex},q,n}j_{\mu_{q,n},k}^{-2s}$$
$$+ q(-1)^q\sum_{k=1}^{\infty} m_{\mathrm{har},q,0}\hat{j}_{|\alpha_q|,\alpha_q,k}^{-2s} + (q+1)(-1)^{q+1}\sum_{k=1}^{\infty} m_{\mathrm{har},q,0}\hat{j}_{|\alpha_q|,\alpha_q,k}^{-2s}$$
$$=(-1)^q\left(\sum_{n,k=1}^{\infty} m_{\mathrm{cex},q,n}j_{\mu_{q,n},k}^{-2s} - \sum_{n,k=1}^{\infty} m_{\mathrm{cex},q,n}\hat{j}_{\mu_{q,n},\alpha_q,k}^{-2s}\right) + (-1)^{q+1}\sum_{k=1}^{\infty} m_{\mathrm{har},q,0}\hat{j}_{|\alpha_q|,\alpha_q,k}^{-2s}.$$

It follows that

$$t(s) = \frac{l^{2s}}{2}\sum_{q=0}^{2p-2}(-1)^q\sum_{n,k=1}^{\infty} m_{\mathrm{cex},q,n}\left(j_{\mu_{q,n},k}^{-2s} - \hat{j}_{\mu_{q,n},\alpha_q,k}^{-2s}\right) + \frac{l^{2s}}{2}\sum_{q=0}^{2p-1}(-1)^{q+1}\sum_{k=1}^{\infty} m_{\mathrm{har},q,0}\hat{j}_{|\alpha_q|,\alpha_q,k}^{-2s}.$$

Next, by Hodge duality on coexact q-forms on the section (see equation (2.7)) $\lambda_{q,n} = \lambda_{2p-2-q,n}$, and recalling the definition of the constants α_q and $\mu_{q,n}$ in Lemma 3.2, we have that $\alpha_q = \frac{1}{2}(1 + 2q - 2p + 1) = q - p + 1 = -\alpha_{2p-2-q}$, and $\mu_{q,n} = \mu_{2p-2-q,n}$. Thus, fixing q with $0 \leq q \leq p-2$,

$$(-1)^q\sum_{n,k=1}^{\infty} m_{\mathrm{cex},q,n}\left(j_{\mu_{q,n},k}^{-2s} - \hat{j}_{\mu_{q,n},\alpha_q,k}^{-2s}\right) + (-1)^{(2p-2-q)}\sum_{n,k=1}^{\infty} m_{\mathrm{cex},q,n}\left(j_{\mu_{q,n},k}^{-2s} - \hat{j}_{\mu_{q,n},-\alpha_q,k}^{-2s}\right)$$
$$=(-1)^q\sum_{n,k=1}^{\infty} m_{\mathrm{cex},q,n}\left(2j_{\mu_{q,n},k}^{-2s} - \hat{j}_{\mu_{q,n},\alpha_q,k}^{-2s} - \hat{j}_{\mu_{q,n},-\alpha_q,k}^{-2s}\right),$$

while when $q = p-1$, $\alpha_q = 0$. Therefore,

$$\begin{aligned} t(s) =& \frac{l^{2s}}{2}\sum_{q=0}^{p-2}(-1)^q \sum_{n,k=1}^{\infty} m_{\mathrm{cex},q,n}\left(2j_{\mu_{q,n},k}^{-2s} - \hat{j}_{\mu_{q,n},\alpha_q,k}^{-2s} - \hat{j}_{\mu_{q,n},-\alpha_q,k}^{-2s}\right)\\ &+ (-1)^{p-1}\frac{l^{2s}}{2}\sum_{n,k=1}^{\infty} m_{\mathrm{cex},p-1,n}\left(j_{\mu_{p-1,n},k}^{-2s} - (j'_{\mu_{p-1,n},k})^{-2s}\right)\\ &+ \frac{l^{2s}}{2}\sum_{q=0}^{2p-1}(-1)^{q+1}\sum_{k=1}^{\infty} m_{\mathrm{har},q,0}\hat{j}_{|\alpha_q|,\alpha_q,k}^{-2s}, \end{aligned}$$

where $j'_{\nu,k} = \hat{j}_{\nu,0,k}$ are the zeros of J'_ν. Eventually, consider the last sum in the previous equation. We will use some classical properties of Bessel function, see for example [34]. Recall $m = \dim W = 2p-1$, and therefore $\alpha_q = q-p+1$ is an integer. Moreover, α_q is negative for $0 \leq q < p-1$. Fixed such a q, we study the function $\hat{J}_{-\alpha_q,\alpha_q}(z) = \alpha_q J_{-\alpha_q}(z) + zJ'_{-\alpha_q}(z)$. Since

$$zJ'_\mu(z) = -zJ_{\mu+1}(z) + \mu J_\mu(z)$$

it follows that $\hat{J}_{-\alpha_q,\alpha_q}(z) = -zJ_{-\alpha_q+1}(z) = -zJ_{-\alpha_{q-1}}(z)$, and hence $\hat{j}_{|\alpha_q|,\alpha_q,k} = j_{-\alpha_{q-1},k}$. Next, fix q with $p-1 < q \leq 2p-1$, such that α_q is a positive integer. Then, since

$$zJ'_\mu(z) = zJ_{\mu-1}(z) - \mu J_\mu(z),$$

the function $\hat{J}_{\alpha_q,\alpha_q}(z) = \alpha_q J_{\alpha_q}(z) + zJ'_{\alpha_q}(z)$ coincides with $zJ_{\alpha_q-1}(z)$, and hence $\hat{j}_{|\alpha_q|,\alpha_q,k} = j_{\alpha_{q-1},k}$. Note that when $q = p-1$, $\alpha_{p-1} = 0$ and hence $j_{\alpha_{p-1},\alpha_{p-1},k} = j'_{0,k} = j_{1,k}$. Summing up,

$$\begin{aligned} \sum_{q=0}^{2p-1}(-1)^{q+1}\sum_{k=1}^{\infty} m_{\mathrm{har},q,0}\hat{j}_{|\alpha_q|,\alpha_q,k}^{-2s} =& \sum_{q=0}^{p-2}(-1)^{q+1}\sum_{k=1}^{\infty}\frac{m_{\mathrm{har},q,0}}{j_{-\alpha_{q-1},k}^{2s}} + (-1)^p\sum_{k=1}^{\infty}\frac{m_{\mathrm{har},p-1,0}}{j_{1,k}^{2s}}\\ &+ \sum_{q=p}^{2p-1}(-1)^{q+1}\sum_{k=1}^{\infty}\frac{m_{\mathrm{har},q,0}}{j_{\alpha_{q-1},k}^{2s}}, \end{aligned}$$

and since by Hodge duality $m_{q,0} = m_{2p-1-q,0}$,

$$\begin{aligned} =& \sum_{q=0}^{p-2}(-1)^{q+1}\sum_{k=1}^{\infty} m_{\mathrm{har},q,0}j_{-\alpha_{q-1},k}^{-2s} + (-1)^p\sum_{k=1}^{\infty} m_{\mathrm{har},p-1,0}j_{1,k}^{-2s} + \sum_{q=0}^{p-1}(-1)^q\sum_{k=1}^{\infty} m_{\mathrm{har},q,0}j_{-\alpha_q,k}^{2s}\\ =& \sum_{q=0}^{p-1}(-1)^{q+1}m_{\mathrm{har},q,0}\sum_{k=1}^{\infty}\left(j_{-\alpha_{q-1},k}^{-2s} - j_{-\alpha_q,k}^{-2s}\right). \end{aligned}$$

Since $m_{\mathrm{har},q,0} = \mathrm{rk}\mathcal{H}_q(\partial C_l W;\mathbb{Q})$, this completes the proof. □

It is convenient to introduce the following functions. We set

$$\begin{aligned} Z_q(s) &= \sum_{n,k=1}^{\infty} m_{\mathrm{cex},q,n}j_{\mu_{q,n},k}^{-2s}, & \dot{Z}_q(s) &= \sum_{n,k=1}^{\infty} m_{\mathrm{cex},q,n}(j'_{\mu_{q,n},k})^{-2s},\\ Z_{q,\pm}(s) &= \sum_{n,k=1}^{\infty} m_{\mathrm{cex},q,n}\hat{j}_{\mu_{q,n},\pm\alpha_q,k}^{-2s}, & z_q(s) &= \sum_{k=1}^{\infty}\left(j_{-\alpha_{q-1},k}^{-2s} - j_{-\alpha_q,k}^{-2s}\right), \end{aligned} \tag{5.1}$$

for $0\leq q\leq p-1$, and

$$
\begin{aligned}
t_{p-1}(s)&=Z_{p-1}(s)-\dot{Z}_{p-1}(s),\\
t_q(s)&=2Z_q(s)-Z_{q,+}(s)-Z_{q,-}(s),\quad 0\leq q\leq p-2.
\end{aligned}
\tag{5.2}
$$

Then,

$$
\begin{aligned}
t(s)=&\frac{l^{2s}}{2}\sum_{q=0}^{p-2}(-1)^q\left(2Z_q(s)-Z_{q,+}(s)-Z_{q,-}(s)\right)+(-1)^{p-1}\frac{l^{2s}}{2}\left(Z_{p-1}(s)-\dot{Z}_{p-1}(s)\right)\\
&-\frac{l^{2s}}{2}\sum_{q=0}^{p-1}(-1)^q\mathrm{rk}\mathcal{H}_q(\partial C_lW;\mathbb{Q})z_q(s)\\
=&\frac{l^{2s}}{2}\sum_{q=0}^{p-1}(-1)^q t_q(s)-\frac{l^{2s}}{2}\sum_{q=0}^{p-1}(-1)^q\mathrm{rk}\mathcal{H}_q(\partial C_lW;\mathbb{Q})z_q(s),
\end{aligned}
$$

and

$$
\begin{aligned}
\log T(C_lW)=t'(0)=&\frac{\log l^2}{2}\left(\sum_{q=0}^{p-1}(-1)^{q+1}r_qz_q(0)+\sum_{q=0}^{p-1}(-1)^q t_q(0)\right)\\
&+\frac{1}{2}\left(\sum_{q=0}^{p-1}(-1)^{q+1}r_qz_q'(0)+\sum_{q=0}^{p-1}(-1)^q t_q'(0)\right),
\end{aligned}
\tag{5.3}
$$

where $r_q=\mathrm{rk}\mathcal{H}_q(\partial C_lW;\mathbb{Q})$. In order to obtain the value of $\log T(C_lW)$ we use Theorem 2.1 and its corollary applied to the functions $z_q(s)$, $Z_q(s)$, $\dot{Z}_q(s)$, $Z_{q,\pm}(s)$. More precisely, the functions z_q were studied at the end of Section 2.4, and we will study the functions t_q in Sections 5.2 and 5.1, and eventually we sum up on the forms degree q in Section 6.

5.1. **The functions $t_q(s)$, $0\leq q\leq p-2$.** In this section we study the functions $t_q(s)$. For we apply Theorem 2.1 to the double sequences $S_q=\{m_{q,n}:j^2_{\mu_{q,n},k}\}_{n=1}^\infty$ and $S_{q,\pm}=\{m_{q,n}:j^2_{\mu_{q,n}\pm\alpha_q,k}\}_{n=1}^\infty$, since we have that $Z_q(s)=\zeta(s,S_q)$, $Z_{q,\pm}(s)=\zeta(s,S_{q,\pm})$, where $q=0,1,\ldots,p-2$, $\alpha_q=p-q-1$. Note that the sequence S_q coincides with the sequence S_{p-1} analysed in Section 5.2, with $q=p-1$. So we just need to study the other two sequences. First, we verify Definition 2.1. For we introduce the simple sequence $U_q=\{m_{q,n}:\mu_{q,n}\}_{n=1}^\infty$.

Lemma 5.2. *For all $0\leq q\leq p-1$, the sequence U_q is a totally regular sequence of spectral type with infinite order, $\mathsf{e}(U_q)=\mathsf{g}(U_q)=2p-1$, and*

$$
\zeta(s,U_q)=\zeta_{cex}\left(\frac{s}{2},\tilde{\Delta}^{(q)}+\alpha_q^2\right).
$$

The possible poles of $\zeta(s,U_q)$ are at $s=2p-1-h$, $h=0,2,4,\ldots$, and the residues are completely determined by the residues of the function $\zeta_{cex}(s,\tilde{\Delta}^{(q)})$, namely:

$$
\operatorname*{Res_1}_{s=2k+1}\zeta(s,U_q)=\sum_{j=0}^{p-1-k}\binom{-\frac{2k+1}{2}}{j}\operatorname*{Res_1}_{s=2(k+j)+1}\zeta_{cex}\left(\frac{s}{2},\tilde{\Delta}^{(q)}\right)\alpha_q^{2j}.
$$

Proof. By definition $U_q=\{m_{\mathrm{cex},q,n}:\mu_{q,n}\}_{n=1}^\infty$, where by Lemmas 3.2 and 3.3 $\mu_{q,n}=\sqrt{\lambda_{q,n}+\alpha_q^2}$, and the $\lambda_{q,n}$ are the eigenvalues of the operator $\tilde{\Delta}^{(q)}$ on the compact manifold W. Counting such eigenvalues according to multiplicity of the associated coexact eigenform, since the dimension of the eigenspace of $\lambda_{q,n}$ are finite, and $\lambda_{q,n} \sim n^{\frac{2}{m}}$ for large n. This gives order and genus. The last formula follows expanding the powers of the binomial in the definition of the zeta function. $\square$

Next, for $c \in \mathbb{C}$, define the functions

$$\hat{J}_{\nu,c}(z) = cJ_\nu(z) + zJ'_\nu(z).$$

Recalling the series definition of the Bessel function [9] 8.402, we obtain that near $z = 0$

$$\hat{J}_{\nu,c}(z) = \left(1 + \frac{c}{\nu}\right) \frac{z^\nu}{2^\nu \Gamma(\nu)}.$$

This means that the function $z^{-\nu}\hat{J}_{\nu,c}(z)$ is an even function of z. Let $\hat{j}_{\nu,c,k}$ be the positive zeros of $\hat{J}_{\nu,c}(z)$ arranged in increasing order. By the Hadamard factorization theorem, we have the product expansion

$$z^{-\nu}\hat{J}_{\nu,c}(z) = z^{-\nu}\hat{J}_{\nu,c}(z) \prod_{k=-\infty}^{+\infty} \left(1 - \frac{z}{\hat{j}_{\nu,c,k}}\right),$$

and therefore

$$\hat{J}_{\nu,c}(z) = \left(1 + \frac{c}{\nu}\right) \frac{z^\nu}{2^\nu \Gamma(\nu)} \prod_{k=1}^{\infty} \left(1 - \frac{z^2}{\hat{j}^2_{\nu,c,k}}\right).$$

Next, recalling that (when $-\pi < \arg(z) < \frac{\pi}{2}$)

$$J_\nu(iz) = \mathrm{e}^{\frac{\pi}{2} i\nu} I_\nu(z), \qquad J'_\nu(iz) = \mathrm{e}^{\frac{\pi}{2} i\nu} \mathrm{e}^{-\frac{\pi}{2} i} I'_\nu(z),$$

we obtain $\hat{J}_{\nu,c}(iz) = \mathrm{e}^{\frac{\pi}{2} i\nu} \left(cI_\nu(z) + zI'_\nu(z)\right)$. Thus, we define (for $-\pi < \arg(z) < \frac{\pi}{2}$)

$$\hat{I}_{\nu,c}(z) = \mathrm{e}^{-\frac{\pi}{2} i\nu} \hat{J}_{\nu,c}(iz), \tag{5.4}$$

and hence

$$\hat{I}_{\nu,\pm\alpha_q}(z) = \pm\alpha_q I_\nu(z) + zI'_\nu(z) = \left(1 \pm \frac{\alpha_q}{\nu}\right) \frac{z^\nu}{2^\nu \Gamma(\nu)} \prod_{k=1}^{\infty} \left(1 + \frac{z^2}{\hat{j}^2_{\nu,\pm\alpha_q,k}}\right). \tag{5.5}$$

Recalling the definition in equation (2.14) we have proved the following fact.

Lemma 5.3. *The logarithmic Gamma functions associated to the sequences $S_{q,\pm,n}$ have the following representations, when $\lambda \in D_{\theta,c'}$, with $c' = \frac{1}{2}\min(j^2_{\mu_{q,0}}, j^2_{\mu_{q,0},\pm\alpha_q})$,*

$$\begin{aligned}\log\Gamma(-\lambda, S_{q,\pm,n}) &= -\log \prod_{k=1}^{\infty} \left(1 + \frac{(-\lambda)}{\hat{j}^2_{\mu_{q,n},\pm\alpha_q,k}}\right) \\ &= -\log \hat{I}_{\mu_{q,n},\pm\alpha_q}(\sqrt{-\lambda}) + \mu_{q,n}\log\sqrt{-\lambda} - \mu_{q,n}\log 2 \\ &\quad - \log\Gamma(\mu_{q,n}) + \log\left(1 \pm \frac{\alpha_q}{\mu_{q,n}}\right).\end{aligned}$$

Proposition 5.1. *The double sequences $S_{q,\pm}$ have relative exponents $\left(p, \frac{2p-1}{2}, \frac{1}{2}\right)$, relative genus $(p, p-1, 0)$, and are spectrally decomposable over U_q with power $\kappa = 2$, length $\ell = 2p$ and domain $D_{\theta,c'}$. The coefficients σ_h appearing in equation (2.15) are $\sigma_h = h - 1$, with $h = 1, 2, \ldots, \ell = 2p$.*

Proof. The values of the exponents and genus follow by classical estimates of the zeros of the Bessel functions [34], and zeta function theory. In particular, to determinate $s_0 = p$, we use the Young inequality and the Plana theorem as in [24]. Note that $\alpha > \frac{1}{2}$, since $s_2 = \frac{1}{2}$. As observed, the existence of a complete asymptotic expansion of the Gamma function follows by Lemma 5.3. This implies that $S_{q,\pm,n}$ are sequences of spectral type. A direct inspection of the expansions shows that $S_{q,\pm,n}$ are totally regular sequences of infinite order. The existence of the uniform expansion follows using the uniform expansions for the Bessel functions and their derivative given for example in [21] (7.18) and Ex. 7.2, and classical expansion of the Euler Gamma function [9] 8.344. We refer to [11] Section 5 or to [13] Section 4 for details. This proves that $S_{q,\pm}$ are spectrally decomposable over U_q, with power $\kappa = 2$. The length ℓ of the decomposition is precisely $2p$. For $\mathsf{e}(U_q) = 2p - 1$, and therefore the larger integer such that $\sigma_h = h - 1 \leq 2p - 1$ is $2p$. □

Remark 5.1. *Only the term with $\sigma_h = 1$, $\sigma_h = 3$, ..., $\sigma_h = 2p-1$ namely $h = 2, 4, \dots, 2p$, appear in the formula of Theorem 2.1, since the unique poles of $\zeta(s, U_q)$ are at $s = 1$, $s = 3$, ... $s = 2p-1$.*

Since we aim to apply the version of Theorem 2.1 given in Corollary 2.1, for linear combination of two spectrally decomposable sequences, we inspect directly the uniform asymptotic expansion of $2S_q - S_{q,-} - S_{q,+}$. This give the functions ϕ_{σ_h}.

Lemma 5.4. *We have the the following asymptotic expansions for large n, uniform in λ, for $\lambda \in D_{\theta,c'}$,*

$$\begin{aligned}
2 \log\Gamma(-\lambda, S_{q,n}/\mu_{q,n}^2) &- \log\Gamma(-\lambda, S_{q,+,n}/\mu_{q,n}^2) - \log\Gamma(-\lambda, S_{q,-,n}/\mu_{q,n}^2) \\
&= -2\log I_{\mu_{q,n}}(\mu_{q,n}\sqrt{-\lambda}) + \log \hat{I}_{\mu_{q,n},\alpha_q}(\mu_{q,n}\sqrt{-\lambda}) + \log \hat{I}_{\mu_{q,n},-\alpha_q}(\mu_{q,n}\sqrt{-\lambda}) \\
&\quad - 2\log\mu_{q,n} - \log\left(1 - \frac{\alpha_q^2}{\mu_{q,n}^2}\right) \\
&= \log(1-\lambda) + \sum_{j=1}^{2p-1} \phi_{j,q}(\lambda)\frac{1}{\mu_{q,n}^j} + O\left(\frac{1}{(\mu_{q,n})^{2p}}\right).
\end{aligned}$$

Proof. Using the representations given in Lemmas 5.9 and 5.3, we obtain

$$\begin{aligned}
2\log\Gamma(-\lambda, S_{q,n}/\mu_{q,n}^2) &- \log\Gamma(-\lambda, S_{q,+,n}/\mu_{q,n}^2) - \log\Gamma(-\lambda, S_{q,-,n}/\mu_{q,n}^2) \\
&= -2\log I_{\mu_{q,n}}(\mu_{q,n}\sqrt{-\lambda}) + \log \hat{I}_{\mu_{q,n},\alpha_q}(\mu_{q,n}\sqrt{-\lambda}) + \log \hat{I}_{\mu_{q,n},-\alpha_q}(\mu_{q,n}\sqrt{-\lambda}) \\
&\quad - 2\log\mu_{q,n} - \log\left(1 - \frac{\alpha_q^2}{\mu_{q,n}^2}\right).
\end{aligned}$$

Recall the uniform expansions for the Bessel functions given for example in [21] (7.18) pg. 376, and Ex. 7.2,

$$I_\nu(\nu z) = \frac{\mathrm{e}^{\nu\sqrt{1+z^2}}\mathrm{e}^{\nu\log\frac{z}{1+\sqrt{1+z^2}}}}{\sqrt{2\pi\nu}(1+z^2)^{\frac{1}{4}}}\left(1 + \sum_{j=1}^{2p-1}\frac{U_j(z)}{\nu^j} + O\left(\frac{1}{\nu^{2p}}\right)\right),$$

where

$$U_0(w) = 1, \qquad U_j(w) = \frac{1}{2}w^2(1-w^2)\frac{d}{dw}U_{j-1}(w) + \frac{1}{8}\int_0^w (1-5t^2)U_{j-1}(t)dt,$$

with $w=\frac{1}{\sqrt{1+z^2}}$, and

$$I'_\nu(\nu z)=\frac{(1+z^2)^{\frac{1}{4}}\mathrm{e}^{\nu\sqrt{1+z^2}}\mathrm{e}^{\nu\log\frac{z}{1+\sqrt{1+z^2}}}}{\sqrt{2\pi\nu}z}\left(1+\sum_{j=1}^{2p-1}\frac{V_j(z)}{\nu^j}+O\left(\frac{1}{\nu^{2p}}\right)\right),$$

where

$$V_0(w)=1,\qquad V_j(w)=U_j(w)-\frac{w}{2}(1-w^2)U_{j-1}(w)-w^2(1-w^2)\frac{d}{dw}U_{j-1}(w).$$

Using these expansions, we obtain the following expansion for $\hat I_{\nu,\pm\alpha_q}(\nu z)$,

$$\begin{aligned}\hat I_{\nu,\pm\alpha_q}(\nu z)&=\pm\alpha_q I_\nu(\nu z)+\nu z I'_\nu(\nu z)\\&=\sqrt{\nu}(1+z^2)^{\frac{1}{4}}\frac{\mathrm{e}^{\nu\sqrt{1+z^2}}\mathrm{e}^{\nu\log\frac{z}{1+\sqrt{1+z^2}}}}{\sqrt{2\pi}}\left(1+\sum_{j=1}^{2p-1}W_{\pm\alpha_q,j}(z)\frac{1}{\nu^j}+O\left(\frac{1}{\nu^{2p}}\right)\right),\end{aligned}$$

where $W_{\pm\alpha_q,j}(z)=V_j(z)\pm\frac{\alpha_q}{\sqrt{1+z^2}}U_{j-1}(z)$. Thus,

$$\begin{aligned}\log\hat I_{\nu,\pm\alpha_q}(\nu z)=&\nu\sqrt{1+z^2}+\nu\log z-\nu\log(1+\sqrt{1+z^2})+\log\nu+\frac{1}{4}\log(1+z^2)\\&-\frac{1}{2}\log 2\pi\nu+\log\left(1+\sum_{j=1}^{2p-1}W_{\pm\alpha_q,j}(z)\frac{1}{\nu^j}+O\left(\frac{1}{\nu^{2p}}\right)\right).\end{aligned}$$

This gives,

$$\begin{aligned}&2\log\Gamma(-\lambda,S_{q,n}/\mu_{q,n}^2)-\log\Gamma(-\lambda,S_{q,+,n}/\mu_{q,n}^2)-\log\Gamma(-\lambda,S_{q,-,n}/\mu_{q,n}^2)\\&=\log(1-\lambda)-2\log\left(1+\sum_{j=1}^{2p-1}\frac{U_j(\sqrt{-\lambda})}{\mu_{q,n}^j}+O\left(\frac{1}{\mu_{q,n}^{2p}}\right)\right)\\&+\log\left(1+\sum_{j=1}^{2p-1}\frac{W_{+\alpha_q,j}(\sqrt{-\lambda})}{\mu_{q,n}^j}+O\left(\frac{1}{\mu_{q,n}^{2p}}\right)\right)+\log\left(1+\sum_{j=1}^{2p-1}\frac{W_{-\alpha_q,j}(\sqrt{-\lambda})}{\mu_{q,n}^j}+O\left(\frac{1}{\mu_{q,n}^{2p}}\right)\right).\end{aligned}$$

Expanding the logarithm as

$$\log\left(1+\sum_{j=1}^{\infty}\frac{a_j}{z^j}\right)=\sum_{j=1}^{\infty}\frac{l_j}{z^j},$$

where $a_0=1$, $a_1=l_1$ and $l_j=a_j-\sum_{k=1}^{j-1}\frac{j-k}{j}a_k l_{j-k}$, we have that

$$\begin{aligned}&2\log\Gamma(-\lambda,S_{q,n}/\mu_{q,n}^2)-\log\Gamma(-\lambda,S_{q,+,n}/\mu_{q,n}^2)-\log\Gamma(-\lambda,S_{q,-,n}/\mu_{q,n}^2)\\&\qquad=\log(1-\lambda)+\sum_{j=1}^{p}\left(-2l_{2j-1}(\lambda)+l^+_{2j-1}(\lambda)+l^-_{2j-1}(\lambda)\right)\frac{1}{\mu_{q,n}^{2j-1}}\\&\qquad+\sum_{j=1}^{p-1}\left(-2l_{2j}(\lambda)+l^+_{2j}(\lambda)+l^-_{2j}(\lambda)+\frac{\alpha_q^{2j}}{j}\right)\frac{1}{\mu_{q,n}^{2j}}+O\left(\frac{1}{\mu_{q,n}^{2p}}\right),\end{aligned}$$

where we denote by $l_j(\lambda)$ the term in the expansion relative to the sequence S (thus the one containing the $U_j(z)$) and by $l_j^{\pm}(\lambda)$ the terms relative to $S_\pm$ (thus the ones containing the $W_{\pm\alpha_q,j}(z)$). Setting

$$\begin{aligned}\phi_{q,2j-1}(\lambda) &= -2l_{2j-1}(\lambda) + l^+_{2j-1}(\lambda) + l^-_{2j-1}(\lambda)\\ \phi_{q,2j}(\lambda) &= -2l_{2j}(\lambda) + l^+_{2j}(\lambda) + l^-_{2j}(\lambda) + \frac{\alpha_q^{2j}}{j},\end{aligned} \tag{5.6}$$

the result follows. □

Remark 5.2. *Note that there are no logarithmic terms* $\log \mu_{q,n}$ *in the asymptotic expansion of the difference of the logarithmic Gamma function given in Lemma 5.4. So we can apply Corollary 2.1.*

Next, we give some results on the functions $\phi_{j,q}(\lambda)$, and $\Phi_{j,q}(s)$ defined in equation (2.16).

Lemma 5.5. *For all* j *and all* $0 \leq q \leq p-2$*, the functions* $\phi_{j,q}(\lambda)$ *are odd polynomial in* $w = \frac{1}{\sqrt{1-\lambda}}$

$$\phi_{2j-1,q}(\lambda) = \sum_{k=0}^{2j-1} a_{2j-1,q,k} w^{2k+2j-1}, \qquad \phi_{2j,q}(\lambda) = \sum_{k=0}^{2j} a_{2j,q,k} w^{2k+2j} + \frac{\alpha_q^{2j}}{j}.$$

The coefficients $a_{j,q,k}$ *are completely determined by the coefficients of the expansion given in Lemma 5.4.*

Proof. This follows by direct inspection of the last equality in the statement of Lemma 5.4. □

Lemma 5.6. *For all* j *and all* $0 \leq q \leq p-2$*,* $\phi_{j,q}(0) = 0$*.*

Proof. The proof is by induction on j. We will consider all the functions as functions of $w = \frac{1}{\sqrt{1-\lambda}}$. We use the following hypothesis for the induction, for $1 \leq k \leq j-1$:

$$\phi_{2k-1,q}(1) = 0, \qquad \phi_{2k,q}(1) = 0, \tag{5.7}$$

$$l^-_{2k-1}(1) - l^+_{2k-1}(1) = \frac{-2\alpha_q^{2k-1}}{2k-1}, \qquad l^-_{2k}(1) - l^+_{2k}(1) = 0, \tag{5.8}$$

where the functions $\phi_{j,q}(\lambda)$ are defined in equation (5.6), and the function $l(\lambda)$ in the course of the proof of Lemma 5.4. First, we verify the hypothesis for $j = 1$. The formulas in equation (5.8) follow by the definition when $k = 1$. For those in equation (5.7), we have by definition when $k = 1$ that

$$\begin{aligned}\phi_{1,q}(\lambda) &= -2l_1(\lambda) + l_1^+(\lambda) + l_1^-(\lambda) = -2U_1(\sqrt{-\lambda}) + V_1(\sqrt{-\lambda}) + V_1(\sqrt{-\lambda}) + (\alpha_q - \alpha_q)U_0(\sqrt{-\lambda})\\ &= -\frac{1}{(1-\lambda)^{\frac{1}{2}}} + \frac{1}{(1-\lambda)^{\frac{3}{2}}},\end{aligned}$$

and

$$\begin{aligned}\phi_{2,q}(\lambda) &= -2l_2(\lambda) + l_2^+(\lambda) + l_2^-(\lambda) + \alpha_q^2 = -2U_2(\sqrt{-\lambda}) + 2V_2(\sqrt{-\lambda}) + U_1(\sqrt{-\lambda})^2 - V_1(\sqrt{-\lambda})^2\\ &= -\frac{3}{2}\frac{1}{(1-\lambda)} + 2\frac{1}{(1-\lambda)^2} - \frac{3}{2}\frac{1}{(1-\lambda)^3} + 1,\end{aligned}$$

and hence formulas in (5.7) are also verified when $k = 1$. Second we prove that all formulas hold for $k = j$. Recalling that $U_k(1) = V_k(1)$ for all k, we have from the definition that

$$\begin{aligned}l^-_{2j-1}(1) - l^+_{2j-1}(1) &= U_{2j-1}(1) - \alpha_q U_{2j-2}(1) - U_{2j-1}(1) - \alpha_q U_{2j-2}(1)\\ &- \sum_{k=1}^{2j-2} \frac{2j-1-k}{2j-1}\left(U_k(1)(l^-_{2j-1-k}(1) - l^+_{2j-1-k}(1)) - \alpha_q U_{k-1}(1)(l^-_{2j-1-k}(1) + l^+_{2j-1-k}(1))\right),\end{aligned}$$

and hence, using the hypothesis we obtain

$$\begin{aligned}
l^-_{2j-1}(1) - l^+_{2j-1}(1) &= -2\alpha_q U_{2j-2}(1) + \sum_{k=1}^{j-1} \frac{2(j-k)}{2j-1}\alpha_q U_{2k-2}(1)\left(2l_{2(j-k)} - \frac{\alpha_q^{2(j-k)}}{j-k}\right)\\
&\quad - \sum_{k=1}^{j-1} U_{2k}(1)\frac{-2\alpha_q^{2(j-k)-1}}{2j-1} + \sum_{k=1}^{j-1}\frac{2(j-k)-1}{2j-1}2\alpha_q U_{2k-1}(1) l_{2(j-k)-1}(1)\\
&= -\frac{2\alpha_q^{2j-1}}{2j-1} - 2\alpha_q U_{2j-2}(1) + \frac{2\alpha_q}{2j-1}U_{2j-2}(1)\\
&\quad + \frac{2\alpha_q}{2j-1}\left(2(j-1)l_{2j-2} + \sum_{k=1}^{2j-3}(2j-2-k)\alpha_q U_k(1) l_{2j-2-k}(1)\right)\\
&= -\frac{2\alpha_q^{2j-1}}{2j-1} - 2\alpha_q U_{2j-2}(1) + \frac{2\alpha_q}{2j-1}U_{2j-2}(1) + \frac{2\alpha_q(2j-2)U_{2j-2}}{2j-1} = -\frac{2\alpha_q^{2j-1}}{2j-1},
\end{aligned}$$

thus proving the first formula in (5.8) for $k = j$. For the first formula in (5.7), $\phi_{2j-1,q}(1) = -2l_{2j-1}(1) + l^+_{2j-1}(1) + l^-_{2j-1}(1)$, and hence

$$\begin{aligned}
\phi_{2j-1,q}(1) &= \sum_{k=1}^{2j-2}\frac{2j-1-k}{2j-1}\left(U_k(1)(2l_{2j-1-k}(1) - l^+_{2j-1-k}(1) - l^-_{2j-1-k}(1))\right)\\
&\quad - \sum_{k=1}^{2j-2}\frac{2j-1-k}{2j-1}\alpha_q U_{k-1}(1)\left(l^-_{2j-1-k}(1) - l^+_{2j-1-k}(1)\right),
\end{aligned}$$

and using the induction hypothesis, and the previous formula with $k = j$ just proved, this means that

$$\begin{aligned}
\phi_{2j-1,q}(1) &= \sum_{k=1}^{j}\frac{2j-1-(2k-1)}{2j-1}U_{2k-1}(1)\frac{\alpha(i)^{2(j-k)}}{j-k}\\
&\quad - 2\sum_{k=1}^{j}\frac{2j-1-2k}{2j-1}\alpha(i)U_{2k-1}(1)\left(\frac{\alpha(i)^{2(j-k)-1}}{2(j-k)-1}\right) = 0.
\end{aligned}$$

and the first formula in (5.7) with $k = j$ follows. For the second formula in (5.8), using the hypothesis, we have

$$\begin{aligned}
l^-_{2j}(1) - l^+_{2j}(1) &= U_{2j}(1) - \alpha_q U_{2j-1}(1) - U_{2j}(1) - \alpha_q U_{2j-1}(1)\\
&\quad - \sum_{k=1}^{2j-1} \frac{2j-k}{2j}\left(U_k(1)(l^-_{2j-k}(1) - l^+_{2j-k}(1))\right) + \sum_{k=1}^{2j-1}\frac{2j-k}{2j}\left(\alpha_q U_{k-1}(1)(l^-_{2j-k}(1)+l^+_{2j-k}(1))\right)\\
&= -2\alpha_q U_{2j-1}(1) + \sum_{k=1}^{j-1}\frac{2j-2k}{2j}\alpha_q U_{2k-1}(1)\left(2l_{2j-2k}(1) - \frac{\alpha_q^{2j-2k}}{j-k}\right)\\
&\quad + \sum_{k=1}^{j}\frac{2(j-k)+1}{2j}\left(U_{2k-1}(1)\frac{2\alpha_q^{2(j-k)+1}}{2(j-k)+1} + \alpha_q U_{2k-2}(1) 2 l_{2(j-k)+1}(1)\right)\\
&= -2\alpha_q U_{2j-1}(1) + \frac{2\alpha_q U_{2j-1}(1)}{2j} + \frac{2\alpha_q}{2j}(2j-1)l_{2j-1}(1) + 2\alpha_q\sum_{k=2}^{2j-1}\frac{2j-k}{2j}U_{k-1}(1)l_{2j-k}(1)\\
&= -2\alpha_q U_{2j-1}(1) + \frac{\alpha_q}{j}\left(U_{2j-1}(1) + (2j-1)l_{2j-1}(1) + \sum_{k=1}^{2j-2}(2j-1-k)U_k(1)l_{2j-1-k}(1)\right)\\
&= -2\alpha_q U_{2j-1}(1) + \frac{(\alpha_q(2j-1)+\alpha_q)U_{2j-1}(1)}{j} = 0.
\end{aligned}$$

Eventually, for the second formula in (5.7)

$$\begin{aligned}
\phi_{2j,q}(1) &= -2l_{2j}(1) + l^+_{2j}(1) + l^-_{2j}(1) + \frac{\alpha_q^{2j}}{j}\\
&= \frac{\alpha_q^{2j}}{j} + \sum_{k=1}^{2j-1}\frac{2j-k}{2j}\left(U_k(1)(2l_{2j-k}(1) - l^+_{2j-k}(1) - l^-_{2j-k}(1))\right)\\
&\quad - \sum_{k=1}^{2j-1}\frac{2j-k}{2j}\alpha_q U_{k-1}(1)\left(l^-_{2j-k}(1) - l^+_{2j-k}(1)\right)\\
&= \sum_{k=1}^{j-1}\frac{2j-2k}{2j}U_{2k}(1)\frac{\alpha_q^{2(j-k)}}{j-k} - 2\sum_{k=2}^{j}\alpha_q U_{2k-2}(1)\frac{\alpha_q^{2(j-k)+1}}{2j} = 0,
\end{aligned}$$

□

Corollary 5.1. *For all j and all $0 \le q \le p-2$, $0 \le j \le p-1$, $\operatorname{Res}_{1\,s=0}\Phi_{2j+1,q}(s) = 0$.*

Next, we determine the terms $A_{0,0}(0)$ and $A'_{0,1}(0)$, defined in equation (2.18).

Lemma 5.7. *For all $0 \le q \le p-2$,*

$$\begin{aligned}
\mathcal{A}_{0,0,q}(s) &= 2A_{0,0,q}(s) - A_{0,0,q,+}(s) - A_{0,0,q,-}(s) = -\sum_{n=1}^{\infty}\log\left(1 - \frac{\alpha_q^2}{\mu_{q,n}^2}\right)\frac{m_{q,n}}{\mu_{q,n}^{2s}},\\
\mathcal{A}_{0,1,q}(s) &= 2A_{0,1,q}(s) - A_{0,1,q,+}(s) - A_{0,1,q,-}(s) = \zeta(2s, U_q).
\end{aligned}$$

Proof. For S_q equation (2.18) reads

$$A_{0,0,q}(s) = \sum_{n=1}^{\infty} m_{\mathrm{cex},q,n} \left(a_{0,0,n,q} - \sum_{j=1}^{p} b_{2j-1,0,0,q} \mu_{q,n}^{-2j+1} \right) \mu_{q,n}^{-2s},$$

$$A_{0,1,q}(s) = \sum_{n=1}^{\infty} m_{\mathrm{cex},q,n} \left(a_{0,1,n,q} - \sum_{j=1}^{p} b_{2j-1,0,1,q} \mu_{q,n}^{-2j+1} \right) \mu_{q,n}^{-2s};$$

for $S_{q,\pm}$:

$$A_{0,0,q,\pm}(s) = \sum_{n=1}^{\infty} m_{\mathrm{cex},q,n} \left(a_{0,0,n,q,\pm} - \sum_{j=1}^{p} b_{2j-1,0,0,q,\pm} \mu_{q,n}^{-2j+1} \right) \mu_{q,n}^{-2s},$$

$$A_{0,1,q,\pm}(s) = \sum_{n=1}^{\infty} m_{\mathrm{cex},q,n} \left(a_{0,1,n,q,\pm} - \sum_{j=1}^{p} b_{2j-1,0,1,q,\pm} \mu_{q,n}^{-2j+1} \right) \mu_{q,n}^{-2s}.$$

We need the expansions for large λ of $l_{2j-1}(\lambda)$, $l^{\pm}_{2j-1}(\lambda)$, for $j = 1, 2, \ldots, p$, $\log \Gamma(-\lambda, S_{q,n}/\mu_{q,n}^2)$ and $\log \Gamma(-\lambda, S_{q,\pm,n}/\mu_{q,n}^2)$. Using classical expansion for Bessel functions and their derivative (see [13] or [10] for details), we obtain

$$\begin{aligned}\log \Gamma(-\lambda, S_{q,n}/\mu_{q,n}^2) =& \frac{1}{2}\log 2\pi + \left(\mu_{q,n} + \frac{1}{2}\right) \log \mu_{q,n} - \mu_{q,n} \log 2 \\ &- \log \Gamma(\mu_{q,n} + 1) + \frac{1}{2}\left(\mu_{q,n} + \frac{1}{2}\right) \log(-\lambda) + O(\mathrm{e}^{-\mu_{q,n}\sqrt{-\lambda}}).\end{aligned}$$

For $S_{q,\pm}$, by the same expansions in the definition of the function $\hat{I}$, equation (5.5), we obtain

$$\hat{I}_{\nu,\pm\alpha_q}(z) \sim \frac{\sqrt{z}\mathrm{e}^z}{\sqrt{2\pi}} \left(1 + \sum_{k=1}^{\infty} b_k z^{-k}\right) + O(\mathrm{e}^{-z}),$$

and hence

$$\begin{aligned}\log \Gamma(-\lambda, S_{q,\pm,n}/\mu_{q,n}^2) =& \mu_{q,n}\sqrt{-\lambda} + \frac{1}{2}\log 2\pi + \left(\mu_{q,n} - \frac{1}{2}\right)\log \mu_{q,n} - \mu_{q,n} \log 2 \\ &- \log \Gamma(\mu_{q,n}) + \frac{1}{2}\left(\mu_{q,n} - \frac{1}{2}\right)\log(-\lambda) + \log\left(1 \pm \frac{\alpha_q}{\mu_{q,n}}\right) + O(\mathrm{e}^{-\mu_{q,n}\sqrt{-\lambda}}).\end{aligned}$$

This gives

$$\begin{aligned}a_{0,0,n,q} &= \frac{1}{2}\log 2\pi + \left(\mu_{q,n} + \frac{1}{2}\right)\log \mu_{q,n} - \mu_{q,n}\log 2 - \log \Gamma(\mu_{q,n} + 1),\\ a_{0,1,n,q} &= \frac{1}{2}\left(\mu_{q,n} + \frac{1}{2}\right),\\ a_{0,0,n,q,\pm} &= \frac{1}{2}\log 2\pi + \left(\mu_{q,n} - \frac{1}{2}\right)\log \mu_{q,n} - \log 2^{\mu_{q,n}}\Gamma(\mu_{q,n}) + \log\left(1 \pm \frac{\alpha_q}{\mu_{q,n}}\right),\\ a_{0,1,n,q,\pm} &= \frac{1}{2}\left(\mu_{q,n} - \frac{1}{2}\right),\end{aligned}$$

while the $b_{2j-1,0,0,q}$, $b_{2j-1,0,0,q,\pm}$ all vanish since the functions $l_{2j-1}(\lambda)$, $l^{\pm}_{2j-1}(\lambda)$ do not have constant terms. Therefore,

$$\begin{aligned}
&2a_{0,0,n,q} - a_{0,0,n,q,+} - a_{0,0,n,q,-} = -\log\left(1 - \frac{\alpha_q^2}{\mu_{q,n}^2}\right),\\
&2a_{0,1,n,q} - a_{0,1,n,q,+} - a_{0,1,n,q,-} = 1,
\end{aligned}$$

and the thesis follows. □

Applying Theorem 2.1 and its corollary, we obtain the values of $t_q(0)$ and $t'_q(0)$.

Proposition 5.2. *For $0 \leq q \leq p-2$,*

$$t_q(0) = t_{q,\rm reg}(0) + t_{q,\rm sing}(0), \qquad t'_q(0) = t'_{q,\rm reg}(0) + t'_{q,\rm sing}(0),$$

where

$$\begin{aligned}
&t_{q,\rm reg}(0) = -\zeta(0, U_q) = -\zeta_{\rm cex}\left(0, \tilde\Delta^{(q)} + \alpha_q^2\right), \qquad t_{q,\rm sing}(0) = 0,\\
&t'_{q,\rm reg}(0) = -\mathcal{A}_{q,0,0}(0) - \mathcal{A}'_{q,0,1}(0),\\
&t'_{q,\rm sing}(0) = \frac{1}{2}\sum_{j=0}^{p-1} \mathop{\rm Res_0}_{s=0} \Phi_{2j+1,q}(s) \mathop{\rm Res_1}_{s=2j+1} \zeta(s, U_q) = \frac{1}{2}\sum_{j=0}^{p-1} \mathop{\rm Res_0}_{s=0} \Phi_{2j+1,q}(s) \mathop{\rm Res_1}_{s=2j+1} \zeta_{\rm cex}\left(\frac{s}{2}, \tilde\Delta^{(q)} + \alpha_q^2\right).
\end{aligned}$$

Proof. By definition in equations (5.1) and (5.2),

$$t_q(0) = 2Z_q(0) - Z_{q,+}(0) - Z_{q,-}(0), \qquad t'_q(0) = 2Z'_q(0) - Z'_{q,+}(0) - Z'_{q,-}(0).$$

where $Z_q(s) = \zeta(s, S_q)$, and $Z_{q,\pm}(s) = \zeta(s, S_{q,\pm})$. By Proposition 5.1 and Lemma 5.4, we can apply Theorem 2.1 and its Corollary to the linear combination above of these double zeta functions. The regular part of $2Z_q(0) - Z_{q,+}(0) - Z_{q,-}(0)$ is then given in Lemma 5.7, while the singular part vanishes, since, by Corollary 5.1, the residues of the functions $\Phi_{k,q}(s)$ at $s = 0$ vanish. The regular part of $2Z'_q(0) - Z'_{q,+}(0) - Z'_{q,-}(0)$ again follows by Lemma 5.7. For the singular part, since by Proposition 5.1, $\kappa = 2$, $\ell = 2p$, and $\sigma_h = h - 1$, with $0 \leq h \leq 2p$, by Remark 5.1 we need only the odd values of $h - 1 = 2j + 1$, $0 \leq j \leq p-1$, and this gives the formula stated for $t'_{p-1,\rm sing}(0)$. □

5.2. **The function $t_{p-1}(s)$.** In this section we study the function $t_{p-1}(s)$. For we apply Theorem 2.1 to the double sequences $S_{p-1} = \{m_{p-1,n} : j^2_{\mu_{p-1,n},k}\}_{n=1}^\infty$ and $\dot S_{p-1} = \{m_{p-1,n} : (j'_{\mu_{p-1,n},k})^2\}_{n=1}^\infty$, since $Z_{p-1}(s) = \zeta(s, S_{p-1})$, $\dot Z_{p-1}(s) = \zeta(s, \dot S_{p-1})$. Spectral decomposition is with respect to the simple sequence $U_{p-1} = \{m_{p-1,n} : \mu_{p-1,n}\}_{n=1}^\infty$. Since the method is essentially the same as in the previous subsection, we just state the results here.

Lemma 5.8. *The sequence U_{p-1} is a totally regular sequence of spectral type with infinite order, $\mathsf{e}(U_{p-1}) = \mathsf{g}(U_{p-1}) = 2p-1$, and $\zeta(s, U_{p-1}) = \zeta_{cex}\left(\frac{s}{2}, \tilde\Delta^{(p-1)}\right)$, with possible simple poles at $s = 2p - 1 - h$, $h = 0, 2, 4, \ldots$.*

Lemma 5.9. *The logarithimic Gamma functions associated to the sequences $S_{p-1,n}/\mu_{p-1,n}^2$ and $\dot{S}_{p-1,n}/\mu_{p-1,n}^2$ have the following representations, with $\lambda\in D_{\theta,c}$, $0\leq\theta\leq\pi$, $c=\frac{\min(j^2_{\mu_{p-1,1}},(j'_{\mu_{p-1,1}})^2)}{2\mu^2_{p-1,1}}$,*

$$\begin{aligned}
\log\Gamma(-\lambda,S_{p-1,n}/\mu_{p-1,n}^2)&=-\log\prod_{k=1}^{\infty}\left(1+\frac{(-\lambda)\mu_{p-1,n}^2}{j^2_{\mu_{p-1,n},k}}\right)\\
&=-\log I_{\mu_{p-1,n}}(\mu_{p-1,n}\sqrt{-\lambda})+(\mu_{p-1,n})\log\sqrt{-\lambda}\\
&\quad+\mu_{p-1,n}\log(\mu_{p-1,n})-\mu_{p-1,n}\log 2-\log\Gamma(\mu_{p-1,n}+1),\\
\log\Gamma(-\lambda,\hat{S}_{p-1,n}/\mu_{p-1,n}^2)&=-\log\prod_{k=1}^{\infty}\left(1+\frac{(-\lambda)(\mu_{p-1,n})^2}{(j'_{\mu_{p-1,n},k})^2}\right)\\
&=-\log I'_{\mu_{p-1,n}}(\mu_{p-1,n}\sqrt{-\lambda})+(\mu_{p-1,n}-1)\log\sqrt{-\lambda}\\
&\quad+\mu_{p-1,n}\log(\mu_{p-1,n})-\mu_{p-1,n}\log 2-\log\Gamma(\mu_{p-1,n}+1).
\end{aligned}$$

Proposition 5.3. *The double sequences S_{p-1} and $\dot{S}_{p-1}$ have relative exponents $\left(p,\frac{2p-1}{2},\frac{1}{2}\right)$, relative genus $(p,p-1,0)$, and are spectrally decomposable over U_{p-1} with power $\kappa=2$, length $\ell=2p$ and domain $D_{\theta,c}$. The coefficients σ_h appearing in equation (2.15) are $\sigma_h=h-1$, with $h=0,1,\ldots,\ell=2p$.*

Remark 5.3. *Only the terms with $\sigma_h=1$, $\sigma_h=3$, …, $\sigma_h=2p-1$ namely $h=2,4,\ldots,2p$, appear in the formula of Theorem 2.1, since the unique non negative poles of $\zeta(s,U_{p-1})$ are at $s=1$, $s=3$, … $s=2p-1$, by Lemma 5.8.*

Lemma 5.10. *The difference of the logarithimic Gamma functions associated to the sequences $S_{p-1,n}/\mu_{p-1}^2$ and $\dot{S}_{p-1,n}/\mu_{p-1,n}^2$ have the following uniform asymptotic expansions for large n, $\lambda\in D_{\theta,c}$,*

$$\begin{aligned}
\log\Gamma(-\lambda,S_{p-1,n}/(\mu^p_{p-1,n})^2)&-\log\Gamma(-\lambda,\dot{S}_{p-1,n}/(\mu^p_{p-1,n})^2)=\\
&=-\log I(\mu_{p-1,n}\sqrt{-\lambda})+\log I'(\mu_{p-1,n}\sqrt{-\lambda})+\log\sqrt{-\lambda}\\
&=\frac{1}{2}\log(1-\lambda)+\sum_{j=1}^{2p-1}\phi_{j,p-1}(\lambda)\frac{1}{(\mu^p_{p-1,n})^j}+O\left(\frac{1}{\mu^{2p}_{p-1,n}}\right).
\end{aligned}$$

Proof. Proceeding as in the proof of Proposition 5.4

$$\begin{aligned}
\log\Gamma(-\lambda,S_{p-1,n}/(\mu_{p-1,n})^2)&-\log\Gamma(-\lambda,\dot{S}_{p-1,n}/(\mu_{p-1,n})^2)=\frac{1}{2}\log(1-\lambda)\\
&+\sum_{j=1}^{2p-1}\frac{1}{\mu^j_{p-1,n}}\left(V_j(\sqrt{-\lambda})-U_j(\sqrt{-\lambda})+\sum_{k=1}^{j-1}\frac{j-k}{j}\left(V_k(\sqrt{-\lambda})\dot{l}_{j-k}(\lambda)-U_k(\sqrt{-\lambda})l_{j-k}(\lambda)\right)\right)\\
&+O\left(\frac{1}{\mu^{2p}_{p-1,n}}\right),
\end{aligned}$$

where we denote by $\dot{l}_j(\lambda)$ the term in the expansion relative to the sequence $\dot{S}$ (thus the one containing the $V_j(z)$) and by $l_j(\lambda)$ the term relative to S (thus the one containing the $U_j(z)$).

Setting

$$\begin{aligned}\phi_{p-1,j}(\lambda) &= \dot{l}_j(\lambda) - l_j(\lambda)\\ &= V_j(\sqrt{-\lambda}) - U_j(\sqrt{-\lambda}) + \sum_{k=1}^{j-1} \frac{j-k}{j}\left(V_k(\sqrt{-\lambda})\dot{l}_{j-k}(\lambda) - U_k(\sqrt{-\lambda}) l_{j-k}(\lambda)\right),\end{aligned} \tag{5.9}$$

we have the formula stated in the thesis. □

Lemma 5.11. *For all j, the functions $\phi_{j,p-1}(\lambda)$ are odd polynomial in $w = \frac{1}{\sqrt{1-\lambda}}$*

$$\phi_{j,p-1}(\lambda) = \sum_{k=j}^{3j+1} a_{j,p-1,k} w^{2k+1}.$$

Lemma 5.12. *For all j, $\phi_{j,p-1}(0) = 0$.*

Corollary 5.2. *For all j, and $0 \leq j \leq p-1$, $\mathrm{Res}_{1\, s=0}\, \Phi_{2j+1,p-1}(s) = 0$.*

Lemma 5.13.

$$\begin{aligned}\mathcal{A}_{0,0,p-1}(s) &= A_{0,0,p-1}(s) - \dot{A}_{0,0,p-1}(s) = 0,\\ \mathcal{A}_{0,1,p-1}(s) &= A_{0,1,p-1}(s) - \dot{A}_{0,1,p-1}(s) = \frac{1}{2}\zeta(2s, U_{p-1}).\end{aligned}$$

Proposition 5.4.

$$t_{p-1}(0) = t_{p-1,\mathrm{reg}}(0) + t_{p-1,\mathrm{sing}}(0), \qquad t'_{p-1}(0) = t'_{p-1,\mathrm{reg}}(0) + t'_{p-1,\mathrm{sing}}(0),$$

where

$$\begin{aligned}
t_{p-1,\mathrm{reg}}(0) &= -\frac{1}{2}\zeta(0, U_{p-1}) = -\frac{1}{2}\zeta_{\mathrm{cex}}\left(0, \tilde{\Delta}^{(p-1)}\right), \qquad t_{p-1,\mathrm{sing}}(0) = 0,\\
t'_{p-1,\mathrm{reg}}(0) &= -\zeta'(0, U_{p-1}) = -\frac{1}{2}\zeta'_{\mathrm{cex}}\left(0, \tilde{\Delta}^{(p-1)}\right),\\
t'_{p-1,\mathrm{sing}}(0) &= \frac{1}{2}\sum_{j=0}^{p-1} \mathop{\mathrm{Res}_0}_{s=0} \Phi_{2j+1,q}(s) \mathop{\mathrm{Res}_1}_{s=2j+1} \zeta(s, U_{p-1}) = \frac{1}{2}\sum_{j=0}^{p-1} \mathop{\mathrm{Res}_0}_{s=0} \Phi_{2j+1,q}(s) \mathop{\mathrm{Res}_1}_{s=2j+1} \zeta_{\mathrm{cex}}\left(\frac{s}{2}, \tilde{\Delta}^{(p-1)}\right).
\end{aligned}$$

6. The analytic torsion, and the proof of Theorem 1.1

In this section we collect all the results obtained in the previous one in order to produce our formulas for the analytic torsion, thus proving Theorem 1.1, that follows from Propositions 6.1 and 6.2 below. By equation (5.3), the torsion is

$$\begin{aligned}\log T(C_l W) = t'(0) =& \frac{\log l^2}{2}\left(\sum_{q=0}^{p-1}(-1)^{q+1} r_q z_q(0) + \sum_{q=0}^{p-1}(-1)^q t_q(0)\right)\\ &+ \frac{1}{2}\left(\sum_{q=0}^{p-1}(-1)^{q+1} r_q z'_q(0) + \sum_{q=0}^{p-1}(-1)^q t'_q(0)\right).\end{aligned}$$

However, it is convenient to split the torsion in *regular* and *singular* part, accordingly to remark 2.1 and the results in Propositions 5.4 and 5.2. First, observe that the functions $z_q(s)$ where studied at the end of Section 2.4, where it is showed that there is no singular contribution to $z_q(0)$ and $z_q'(0)$. So $z_q(0)=z_{q,\rm reg}(0)$, and $z_q'(0)=z_{q,\rm reg}'(0)$. Therefore, we set

$$\log T(C_lW)=\log T_{\rm reg}(C_lW)+\log T_{\rm sing}(C_lW),$$

with

$$\begin{aligned}\log T_{\rm reg}(C_lW)=t_{\rm reg}'(0)=&\frac{\log l^2}{2}\left(\sum_{q=0}^{p-1}(-1)^{q+1}r_qz_q(0)+\sum_{q=0}^{p-1}(-1)^q t_{q,\rm reg}(0)\right)\\&+\frac{1}{2}\left(\sum_{q=0}^{p-1}(-1)^{q+1}r_qz_q'(0)+\sum_{q=0}^{p-1}(-1)^q t_{q,\rm reg}'(0)\right),\end{aligned}\tag{6.1}$$

$$\log T_{\rm sing}(C_lW)=t_{\rm sing}'(0)=\frac{\log l^2}{2}\sum_{q=0}^{p-1}(-1)^q t_{q,\rm sing}(0)+\frac{1}{2}\sum_{q=0}^{p-1}(-1)^q t_{q,\rm sing}'(0).\tag{6.2}$$

Lemma 6.1. *For all $0\leq q\leq p-1$,*

$$z_q(0)=-\frac{1}{2},\qquad\qquad z_q'(0)=\log 2+\log(p-q).$$

Proof. This follows by equation (2.19). □

Lemma 6.2.

$$\begin{aligned}t_{q,\rm reg}(0)&=-\zeta_{\rm cex}(0,\tilde\Delta^{(q)}), & 0\leq&q\leq p-2,\\ t_{q,\rm reg}'(0)&=-\zeta_{\rm cex}'(0,\tilde\Delta^{(q)}), & 0\leq&q\leq p-2,\\ t_{p-1,\rm reg}(0)&=-\frac{1}{2}\zeta_{\rm cex}(0,\tilde\Delta^{(p-1)}), & t_{p-1,\rm reg}'(0)&=-\frac{1}{2}\zeta_{\rm cex}'(0,\tilde\Delta^{(p-1)}).\end{aligned}$$

Proof. The first and the third formulas follows by Propositions 5.4 and 5.2, and the fact that for the zeta function associated to any sequence S, and any number b, $\zeta(0,S+b)=\zeta(0,S)$. For the derivatives, when $0\leq q\leq p-2$, by Proposition 5.2,

$$t_{q,\rm reg}'(0)=-\mathcal{A}_{0,0,q}(0)-\mathcal{A}_{0,1,q}'(0).$$

By Lemma 5.7

$$\mathcal{A}_{0,0,q}(s)=-\sum_{n=1}^\infty\log\left(1-\frac{\alpha_q^2}{\mu_{q,n}^2}\right)\frac{m_{{\rm cex},q,n}}{\mu_{q,n}^{2s}},\qquad \mathcal{A}_{0,1,q}(s)=\zeta(2s,U_q)=\sum_{n=1}^\infty\frac{m_{{\rm cex},q,n}}{\mu_{q,n}^{2s}}.$$

Recalling that $\mu_{q,n}=\sqrt{\lambda_{q,n}+\alpha_q}$, and expanding the binomial, we obtain

$$\begin{aligned}-\mathcal{A}_{0,0,q}(s)-\mathcal{A}_{0,1,q}'(s)&=\sum_{n=1}^\infty\log\left(1-\frac{\alpha_q^2}{\mu_{q,n}^2}\right)\frac{m_{{\rm cex},q,n}}{\mu_{q,n}^{2s}}-\sum_{n=1}^\infty\frac{m_{{\rm cex},q,n}}{\mu_{q,n}^{2s}}\log\mu_{q,n}^2\\&=\sum_{n=1}^\infty\log\lambda_{q,n}\frac{m_{{\rm cex},q,n}}{\mu_{q,n}^{2s}}=\sum_{n=1}^\infty\log\lambda_{q,n}\sum_{j=0}^\infty\binom{-s}{j}\frac{m_{{\rm cex},q,n}}{\lambda_{q,n}^{s+j}}\alpha_q^{2j}\\&=-\sum_{j=0}^\infty\binom{-s}{j}\zeta_{\rm ccl}'(s+j,\tilde\Delta^{(q)})\alpha_q^{2j},\end{aligned}$$

that gives the second formula. Eventually, the result for $t'_{p-1,\rm reg}(0)$ follows by Proposition 5.4 and the fact that $\alpha_{p-1}=0$ since the dimension is $m=2p-1$. □

Proposition 6.1.

$$\begin{aligned}\log T_{\rm reg}(C_lW) =&\frac{1}{2}\sum_{q=0}^{p-1}(-1)^q r_q\log\frac{l}{2}-\frac{1}{2}\sum_{q=0}^{p-1}(-1)^q r_q\log(p-q)+\frac{1}{2}\log T(W,g)\\ &-\left(\sum_{q=0}^{p-2}(-1)^q\zeta_{\rm ccl}(0,\tilde\Delta^{(q)})+\frac{1}{2}(-1)^{p-1}\zeta_{\rm ccl}(0,\tilde\Delta^{(p-1)})\right)\log l\\ =&\frac{1}{2}\sum_{q=0}^{p-1}(-1)^q r_q\log\frac{l}{2}-\frac{1}{2}\sum_{q=0}^{p-1}(-1)^q r_q\log(p-q)+\frac{1}{2}\log T(W,l^2g),\end{aligned}$$

where $r_q={\rm rk}\mathcal{H}_q(\partial C_lW;\mathbb{Q})$

Proof. Substitution in the formula in equation (6.1) of the values given in Lemmas 6.1 and 6.2 gives

$$\begin{aligned}\log T_{\rm reg}(C_lW) =&\frac{1}{2}\sum_{q=0}^{p-1}(-1)^q r_q\log\frac{l}{2}-\frac{1}{2}\sum_{q=0}^{p-1}(-1)^q r_q\log(p-q)\\ &-\left(\sum_{q=0}^{p-2}(-1)^q\zeta_{\rm ccl}(0,\tilde\Delta^{(q)})+\frac{1}{2}(-1)^{p-1}\zeta_{\rm ccl}(0,\tilde\Delta^{(p-1)})\right)\log l\\ &+\frac{1}{4}\left(2\sum_{q=0}^{p-2}(-1)^{q+1}\zeta'_{\rm ccl}(0,\tilde\Delta^{(q)})+(-1)^p\zeta'_{\rm ccl}(0,\tilde\Delta^{(p-1)})\right).\end{aligned}$$

By the second formula in equation (2.8)

$$\frac{1}{4}\left(2\sum_{q=0}^{p-2}(-1)^{q+1}\zeta'_{\rm ccl}(0,\tilde\Delta^{(q)})+(-1)^p\zeta'_{\rm ccl}(0,\tilde\Delta^{(p-1)})\right)=\frac{1}{2}\log T(W,g),$$

and this gives the first formula stated. For the second formula, note that the boundary of the cone ∂C_lW is the manifold W with metric l^2g. The restriction of the Laplace operator on the boundary is then $\Delta_{\partial C_lW}=\frac{\tilde\Delta}{l^2}$. Since for the zeta function associated to any sequence S, and any number a,

$$\zeta'(0,aS)=-\zeta(0,S)\log a+\zeta'(0,S),$$

a simple calculation shows that

$$\begin{aligned}&-\left(\sum_{q=0}^{p-2}(-1)^q\zeta_{\rm ccl}(0,\tilde\Delta^{(q)})+\frac{1}{2}(-1)^{p-1}\zeta_{\rm ccl}(0,\tilde\Delta^{(p-1)})\right)\log l^2\\ &+\frac{1}{2}\left(2\sum_{q=0}^{p-2}(-1)^{q+1}\zeta'_{\rm ccl}(0,\tilde\Delta^{(q)})+(-1)^p\zeta'_{\rm ccl}(0,\tilde\Delta^{(p-1)})\right)\\ =&t(0,W)\log l^2+t'(0,W)=\log T(\partial C_lW).\end{aligned}$$

□

Proposition 6.2.

$$\begin{aligned}
\log T_{\rm sing}(C_lW) =&\frac{1}{2}\sum_{q=0}^{p-1}(-1)^q\sum_{j=0}^{p-1}\operatorname*{Res_0}_{s=0}\Phi_{2j+1}(s)\operatorname*{Res_1}_{s=j+\frac{1}{2}}\zeta_{\rm cex}\left(s,\tilde\Delta^{(q)}+\alpha_q^2\right)\\
=&\frac{1}{2}\sum_{q=0}^{p-1}(-1)^q\sum_{j=0}^{p-1}\operatorname*{Res_0}_{s=0}\Phi_{2j+1}(s)\sum_{l=0}^{q}(-1)^l\operatorname*{Res_1}_{s=j+\frac{1}{2}}\zeta\left(s,\tilde\Delta^{(l)}+\alpha_q^2\right)\\
=&\frac{1}{2}\sum_{q=0}^{p-1}(-1)^q\sum_{j=0}^{p-1}\sum_{k=0}^{j}\operatorname*{Res_0}_{s=0}\Phi_{2k+1,q}(s)\binom{-\frac{1}{2}-k}{j-k}\operatorname*{Res_1}_{s=j+\frac{1}{2}}\zeta_{cex}\left(s,\tilde\Delta^{(q)}\right)\alpha_q^{2(j-k)}\\
=&\frac{1}{2}\sum_{q=0}^{p-1}\sum_{j=0}^{p-1}\sum_{k=0}^{j}\operatorname*{Res_0}_{s=0}\Phi_{2k+1,q}(s)\binom{-\frac{1}{2}-k}{j-k}\sum_{l=0}^{q}(-1)^l\operatorname*{Res_1}_{s=j+\frac{1}{2}}\zeta\left(s,\tilde\Delta^{(l)}\right)\alpha_q^{2(j-k)}.
\end{aligned}$$

Proof. The first formula follows by substitution in equation (6.2) of the values given in Propositions 5.4 and 5.2, and observing that, for the zeta function associated to any sequence S: $a\operatorname{Res_1}_{s=s_0}\zeta(as,S)=\operatorname{Res_1}_{s=as_0}\zeta(s,S)$. The second by duality, see Section 2.2,

$$\zeta_{ccl}\left(s,\tilde\Delta^{(q)}\right)=\zeta\left(s,\tilde\Delta^{(q)}\right)-\zeta_{cl}\left(s,\tilde\Delta^{(q)}\right)=\zeta\left(s,\tilde\Delta^{(q)}\right)-\zeta_{ccl}\left(s,\tilde\Delta^{(q-1)}\right)=\sum_{k=0}^{q}(-1)^{q+k}\zeta\left(s,\tilde\Delta^{(k)}\right).$$

The third formula follows by Lemmas 5.8 and 5.2, and some combinatorics, and the last by the previous ones. □

7. The proof of Theorem 1.2: low dimensional cases

We present a proof for the case $2p-1=3$. We also have a similar proof for the case $2p-1=5$, that we omit to spare space. The proof is in two parts: in the first we compute the anomaly boundary term, as defined in Section 3.2, in the second we compute the singular term in the analytic torsion, using Proposition 6.2. A proof of the general case by this method is unlikely, since we do not have general formulas for the higher coefficients $e_{q,j}$ appearing in the asymptotic expansion of the heat kernel of the Laplacian on forms. However, we decided to present the proof for $p=2$ here, since this together with the direct combinatoric proof of the the same result when the section of the cone is a sphere, mentioned in the introduction, makes the result in the general case a strong conjecture.

7.1. **Part 1.** Since $m=3$, the unique terms that give non trivial contribution in the Berezin integral appearing in equation (2.11) are those homogeneous of degree 3. By definition of exponential (recall that $\Theta=\tilde\Omega$, see Section 3.2), the terms of degree 3 in the integrand in equation (2.11) are

$$-\frac{2}{3\sqrt\pi}u^2\mathcal{S}_1^3-\frac{1}{\sqrt\pi}\hat{\tilde\Omega}S_1;$$

thus

$$\begin{aligned}
B(\nabla_1)&=\frac{1}{2}\int_0^1\int^B {\rm e}^{-\frac{1}{2}\hat{\tilde\Omega}-u^2\mathcal{S}_j^2}\sum_{k=1}^\infty\frac{1}{\Gamma\left(\frac{k}{2}+1\right)}u^{k-1}\mathcal{S}_j^k du\\
&=\frac{1}{2}\int_0^1\int^B\left(-\frac{2}{3\sqrt\pi}u^2\mathcal{S}_1^3-\frac{1}{\sqrt\pi}\hat{\tilde\Omega}S_1\right)du \qquad (7.1)\\
&=-\frac{1}{2\sqrt\pi}\int^B\hat{\tilde\Omega}S_1-\frac{1}{9\sqrt\pi}\int^B\mathcal{S}_1^3.
\end{aligned}$$

Equation (3.6) and direct calculations give

$$\mathcal{S}_1^3 = -\frac{1}{8}\left(\sum_{k=1}^m b_k^* \wedge \hat{e}_k^*\right)^3 = \frac{3}{4} dvol_g \wedge \hat{e}_1^* \wedge \hat{e}_2^* \wedge \hat{e}_3^*,$$

and

$$(b_1^* \wedge b_2^* \wedge \hat{e}_1^* \wedge \hat{e}_2^*) \wedge (b_3^* \wedge \hat{e}_3^*) = b_1^* \wedge b_2^* \wedge b_3^* \wedge \hat{e}_1^* \wedge \hat{e}_2^* \wedge \hat{e}_3^*.$$

Thus,

$$\int^B \mathcal{S}_1^3 = \frac{3}{4\pi^{\frac{3}{2}}} dvol_g.$$

By equations (3.6) and (2.10),

$$\hat{\tilde{\Omega}} \mathcal{S}_1 = -\frac{1}{4}\left(\sum_{k,l=1}^3 \tilde{\Omega}_{kl} \wedge \hat{e}_k^* \wedge \hat{e}_l^*\right) \wedge \left(\sum_{k=1}^3 b_k^* \wedge \hat{e}_k^*\right).$$

Direct calculations give

$$\begin{aligned}
\hat{\tilde{\Omega}} \mathcal{S}_1 &= -\frac{1}{2}(\Omega_{23} \wedge b_1^* - \Omega_{13} \wedge b_2^* + \Omega_{12} \wedge b_3^*) \wedge \hat{e}_1^* \wedge \hat{e}_2^* \wedge \hat{e}_3^* \\
&= -\frac{1}{2}(R_{2332} + R_{1331} + R_{1221}) \hat{e}_1^* \wedge \hat{e}_2^* \wedge \hat{e}_3^* \\
&= -\frac{1}{4}\tilde{\tau} \hat{e}_1^* \wedge \hat{e}_2^* \wedge \hat{e}_3^*,
\end{aligned}$$

and hence

$$\int^B \hat{\tilde{\Omega}} \mathcal{S}_1 = \frac{1}{4\pi^{\frac{3}{2}}} \sum_{k,l=1}^3 \tilde{R}_{kllk} dvol_g.$$

Substitution in equation (7.1) gives

$$B(\nabla_1) = \frac{1}{4\sqrt{\pi}} \int^B \hat{\tilde{\Omega}} S_1 - \frac{1}{9\sqrt{\pi}} \int^B \mathcal{S}_1^3 = \frac{1}{8\pi^2} \tilde{\tau} dvol_g - \frac{1}{12\pi^2} dvol_g.$$

By the formula in equation (2.12), the anomaly boundary term is

$$A_{\mathrm{BM,abs}}(\partial C_l W) = \frac{1}{16\pi^2} \int_{\partial C_l W} \tilde{\tau} dvol_g - \frac{1}{24\pi^2} \int_{\partial C_l W} dvol_g.$$

7.2. **Part 2.** By Proposition 6.2, with $p = 2$,

$$\log T_{\mathrm{sing}}(C_l W) = \frac{1}{2} \sum_{q=0}^1 (-1)^q \sum_{j=0}^1 \operatorname*{Res_0}_{s=0} \Phi_{2j+1,q}(s) \operatorname*{Res_1}_{s=j+\frac{1}{2}} \zeta_{\mathrm{cex}}\left(s, \tilde{\Delta}^{(q)} + \alpha_q^2\right).$$

Since $p = 2$, $\alpha_0 = -1$ and $\alpha_1 = 0$. Since there are no exact 0-forms

$$\zeta_{\mathrm{cex}}\left(s, \tilde{\Delta}^{(0)} + \alpha_0^2\right) = \zeta\left(s, \tilde{\Delta}^{(0)} + \alpha_0^2\right).$$

By Lemma 5.2,

$$\operatorname*{Res_1}_{s=\frac{3}{2}} \zeta\left(s, \tilde\Delta^{(0)} + \alpha_0^2\right) = \operatorname*{Res_1}_{s=\frac{3}{2}} \zeta\left(s, \tilde\Delta^{(0)}\right),$$

$$\operatorname*{Res_1}_{s=\frac{1}{2}} \zeta\left(s, \tilde\Delta^{(0)} + \alpha_0^2\right) = \operatorname*{Res_1}_{s=\frac{1}{2}} \zeta\left(s, \tilde\Delta^{(0)}\right) - \frac{1}{2}\operatorname*{Res_1}_{s=\frac{3}{2}} \zeta\left(s, \tilde\Delta^{(0)}\right).$$

By duality (see Section 2.2)

$$\zeta_{\rm cex}(s, \tilde\Delta^{(1)}) = \zeta(s, \tilde\Delta^{(1)}) - \zeta_{\rm ex}(s, \tilde\Delta^{(1)}) = \zeta(s, \tilde\Delta^{(1)}) - \zeta_{\rm cex}(s, \tilde\Delta^{(0)}),$$

and also

$$\operatorname*{Res_1}_{s=\frac{1}{2}} \zeta(s, \tilde\Delta^{(1)}) = -3\operatorname*{Res_1}_{s=\frac{1}{2}} \zeta(s, \tilde\Delta^{(0)}), \qquad \operatorname*{Res_1}_{s=\frac{3}{2}} \zeta(s, \tilde\Delta^{(1)}) = 3\operatorname*{Res_1}_{s=\frac{3}{2}} \zeta(s, \tilde\Delta^{(0)}).$$

Putting all together, we obtain

$$\begin{aligned}\log T_{\rm sing}(C_l W) = \frac{1}{2}\left(\operatorname*{Res_0}_{s=0} \Phi_{1,0}(s) + \operatorname*{Res_0}_{s=0} \Phi_{1,1}(s) + 3\operatorname*{Res_0}_{s=0} \Phi_{1,1}(s)\right)\operatorname*{Res_1}_{s=\frac{1}{2}} \zeta(s, \tilde\Delta^{(0)})\\ + \frac{1}{2}\left(\operatorname*{Res_0}_{s=0} \Phi_{3,1}(s) + \operatorname*{Res_0}_{s=0} \Phi_{3,0}(s) - \frac{1}{2}\operatorname*{Res_0}_{s=0} \Phi_{1,0}(s) - 3\operatorname*{Res_0}_{s=0} \Phi_{3,1}(s)\right)\operatorname*{Res_1}_{s=\frac{3}{2}} \zeta(s, \tilde\Delta^{(0)}).\end{aligned}$$

By Corollaries 5.2 (when $q = 1$), and 5.1 (when $q = 0$)

$$\operatorname*{Res_0}_{s=0} \Phi_{1,1}(s) = 1, \qquad \operatorname*{Res_0}_{s=0} \Phi_{3,1}(s) = \frac{2}{315},$$

$$\operatorname*{Res_0}_{s=0} \Phi_{1,0}(s) = 2, \qquad \operatorname*{Res_0}_{s=0} \Phi_{3,0}(s) = \frac{214}{315}.$$

This gives

$$\log T_{\rm sing}(C_l W) = 3\operatorname*{Res_1}_{s=\frac{1}{2}} \zeta(s, \tilde\Delta^{(0)}) - \frac{1}{6}\operatorname*{Res_1}_{s=\frac{3}{2}} \zeta(s, \tilde\Delta^{(0)}).$$

To complete the proof, recall from one side that for a compact connected Riemannian manifold (W, g) of dimension m there exists a full asymptotic expansion for the trace of the heat kernel of the Laplacian on forms for small t [8], ${\rm Tr}_{L^2} {\rm e}^{-t\Delta^{(q)}} = t^{-\frac{m}{2}}\sum_{j=0}^\infty e_{q,j} t^{\frac{j}{2}}$. The coefficients depend only on local invariants constructed from the metric tensor, are in principle calculable from it, and we have the following explicit formulas for the first ones are:

$$e_{q,0} = \frac{1}{(4\pi)^{\frac{m}{2}}}\binom{m}{q}\int_W dvol_g, \qquad e_{q,2} = \frac{1}{6(4\pi)^{\frac{m}{2}}}\left(\binom{m}{q} - 6\binom{m-2}{q-1}\right)\int_W \tau dvol_g.$$

From the other side, the sequence ${\rm Sp}_+\Delta^{(q)}$ of the positive eigenvalues of the metric Laplacian on forms is a totally regular sequence of spectral type, with finite exponent $\mathsf{e} = \frac{m}{2}$, genus $\mathsf{g} = [\mathsf{e}]$, spectral sector $\Sigma_{\theta,c}$ with some $0 < c < \lambda_1$, $\epsilon < \theta < \frac{\pi}{2}$, asymptotic domain $D_{\theta,c} = \mathbb{C} - \Sigma_{\theta,c}$, and infinite order [31]. Therefore, the zeta function $\zeta(s, {\rm Sp}_+\Delta^{(q)})$ has a meromorphic continuation to the whole complex plane up to simple poles at the values of $s = \frac{m-h}{2}$, $h = 0, 1, 2, \dots$, that are not negative integers nor zero, with residues

$${\rm Res_1}_{s=\frac{m-h}{2}}\, \zeta(s, {\rm Sp}_+\Delta^{(q)}) = \frac{e_{q,h}}{\Gamma(\frac{m-h}{2})}.$$

This facts imply that

$$\log T_{\rm sing}(C_l W) = \frac{1}{16\pi^2}\int_{\partial C_l W} \tilde\tau dvol_g - \frac{1}{24\pi^2}\int_{\partial C_l W} dvol_g.$$

8. The proof of Theorem 1.2: the general case

Since the argument is very closed to the one described in details in the previous sections, we will just sketch it here. We consider the *conical frustum* (or more precisely its external surface) that is the compact connected oriented Riemannian manifold

$$C_{[l_1,l_1]}W = [l_1, l_2]\times W,$$

with $0 < l_1 < l_2$, and with metric $dx\otimes dx + x^2 g$. We study the analytic torsion of $C_{[l_1,l_2]}$ with relative boundary conditions at $x = l_1$ and absolute boundary condition at $x = l_2$, and we respect to the trivial representation for the fundamental group. This idea was originally suggested to M.S. by W. Müller; see also the preprint [33], for a similar approach. We denote by $\partial_{1/2}C_{[l_1,l_2]}W$, or simply $\partial_{1/2}$, the two boundaries, and by $\log T_{{\rm rel}\ \partial_1,{\rm abs}\ \partial_2}(C_{[l_1,l_2]}W)$ the torsion.

8.1. **Spectrum.** First, we describe the spectrum of the Laplace operator on forms. The proofs of the next lemmas are analogous to the proofs of Lemmas 3.2 and 3.3 and will be omitted.

Lemma 8.1. *With the notation of Lemma 3.2, assuming that $\mu_{q,n}$ is not an integer, all the solutions of the equation $\Delta u = \lambda^2 u$, with $\lambda \neq 0$, are convergent sums of forms of the following twelve types:*

$$\begin{aligned}
\psi^{(q)}_{+,1,n,\lambda} =& x^{\alpha_q} J_{\mu_{q,n}}(\lambda x)\varphi^{(q)}_{{\rm cex},n},\\
\psi^{(q)}_{-,1,n,\lambda} =& x^{\alpha_q} Y_{\mu_{q,n}}(\lambda x)\varphi^{(q)}_{{\rm cex},n},\\
\psi^{(q)}_{+,2,n,\lambda} =& x^{\alpha_{q-1}} J_{\mu_{q-1,n}}(\lambda x)\tilde d\varphi^{(q-1)}_{{\rm cex},n} + \partial_x(x^{\alpha_{q-1}} J_{\mu_{q-1,n}}(\lambda x))dx\wedge\varphi^{(q-1)}_{{\rm cex},n},\\
\psi^{(q)}_{-,2,n,\lambda} =& x^{\alpha_{q-1}} Y_{\mu_{q-1,n}}(\lambda x)\tilde d\varphi^{(q-1)}_{{\rm cex},n} + \partial_x(x^{\alpha_{q-1}} Y_{\mu_{q-1,n}}(\lambda x))dx\wedge\varphi^{(q-1)}_{{\rm cex},n},\\
\psi^{(q)}_{+,3,n,\lambda} =& x^{2\alpha_{q-1}+1}\partial_x(x^{-\alpha_{q-1}} J_{\mu_{q-1,n}}(\lambda x))\tilde d\varphi^{(q-1)}_{{\rm cex},n}\\
&+ x^{\alpha_{q-1}-1} J_{\mu_{q-1,n}}(\lambda x)dx\wedge\tilde d^\dagger\tilde d\varphi^{(q-1)}_{{\rm cex},n},\\
\psi^{(q)}_{-,3,n,\lambda} =& x^{2\alpha_{q-1}+1}\partial_x(x^{-\alpha_{q-1}} Y_{\mu_{q-1,n}}(\lambda x))\tilde d\varphi^{(q-1)}_{{\rm cex},n}\\
&+ x^{\alpha_{q-1}-1} Y_{\mu_{q-1,n}}(\lambda x)dx\wedge\tilde d^\dagger\tilde d\varphi^{(q-1)}_{{\rm cex},n},\\
\psi^{(q)}_{+,4,n,\lambda} =& x^{\alpha_{q-2}+1} J_{\mu_{q-2,n}}(\lambda x)dx\wedge\tilde d\varphi^{(q-2)}_{{\rm cex},n},\\
\psi^{(q)}_{-,4,n,\lambda} =& x^{\alpha_{q-2}+1} Y_{\mu_{q-2,n}}(\lambda x)dx\wedge\tilde d\varphi^{(q-2)}_{{\rm cex},n},\\
\psi^{(q)}_{+,E,\lambda} =& x^{\alpha_q} J_{|\alpha_q|}(\lambda x)\varphi^{(q)}_{\rm har},\\
\psi^{(q)}_{-,E,\lambda} =& x^{\alpha_q} Y_{|\alpha_q|}(\lambda x)\varphi^{(q)}_{\rm har},\\
\psi^{(q)}_{+,O,\lambda} =& \partial_x(x^{\alpha_{q-1}} J_{|\alpha_{q-1}|}(\lambda x))dx\wedge\varphi^{(q-1)}_{{\rm har},n},\\
\psi^{(q)}_{-,O,\lambda} =& \partial_x(x^{\alpha_{q-1}} Y_{|\alpha_{q-1}|}(\lambda x))dx\wedge\varphi^{(q-1)}_{{\rm har},n}.
\end{aligned}$$

When $\mu_{q,n}$ is an integer the $-$ solutions must be modified including some logarithmic term (see for example [34] *for a set of linear independent solutions of the Bessel equation).*

Note that the forms of types 1, 3 and E are coexact, those of types 2, 4 and O exacts. The operator d sends forms of types 1, 3 and E in forms of types 2, 4 and O, while $d^\dagger$ sends forms of types 2, 4 and O in forms of types 1, 3 and E, respectively. The Hodge operator sends forms of type 1 in forms of type 4, 2 in 3, and E in O. Define the functions, for $c \neq 0$,

$$\begin{aligned} F_{\mu,c}(x) &= J_\mu(l_2x)(cY_\mu(l_1x) + l_1xY'_\mu(l_1x)) - Y_\mu(l_2x)(cJ_\mu(l_1x) + l_1xJ'_\mu(l_1x)),\\ \hat F_{\mu,c}(x) &= J_\mu(l_1x)(cY_\mu(l_2x) + l_2xY'_\mu(l_2x)) - Y_\mu(l_lx)(cJ_\mu(l_2x) + l_2xJ'_\mu(l_2x)),\end{aligned}$$

and when $c = 0$,

$$\begin{aligned} F_{\mu,0}(x) &= J_\mu(l_2x)Y'_\mu(l_1x) - Y_\mu(l_2x)J'_\mu(l_1x),\\ \hat F_{\mu,0}(x) &= J_\mu(l_1x)Y'_\mu(l_2x) - Y_\mu(l_lx)J'_\mu(l_2x).\end{aligned}$$

Lemma 8.2. *The positive part of the spectrum of the Laplace operator on forms on $C_{[l_1,l_2]}W$, with relative boundary conditions on $\partial_1 C_{[l_1,l_2]}W$ and absolute boundary conditions on $\partial_2 C_{[l_1,l_2]}W$ is:*

$$\begin{aligned} {\rm Sp}_+\Delta^{(q)}_{{\rm rel}\ \partial_1,{\rm abs\ b}_2} = &\left\{m_{{\rm cex},q,n} : \hat f^2_{\mu_{q,n},\alpha_q,k}\right\}_{n,k=1}^\infty \cup \left\{m_{{\rm cex},q-1,n} : \hat f^2_{\mu_{q-1,n},\alpha_{q-1},k}\right\}_{n,k=1}^\infty\\ &\cup \left\{m_{{\rm cex},q-1,n} : f^2_{\mu_{q-1,n},-\alpha_{q-1},k}\right\}_{n,k=1}^\infty \cup \left\{m_{{\rm cex},q-2,n} : f^2_{\mu_{q-2,n},-\alpha_{q-2},k}\right\}_{n,k=1}^\infty\\ &\cup \left\{m_{{\rm har},q,0} : \hat f^2_{|\alpha_q|,\alpha_q,k}\right\}_{k=1}^\infty \cup \left\{m_{{\rm har},q-1,0} : \hat f^2_{|\alpha_{q-1}|,\alpha_{q-1},k}\right\}_{k=1}^\infty.\end{aligned}$$

With absolute boundary conditions on $\partial_1 C_{[l_1,l_2]}W$ and relative boundary conditions on $\partial_2 C_{[l_1,l_2]}W$ is:

$$\begin{aligned} {\rm Sp}_+\Delta^{(q)}_{{\rm abs}\ \partial_1,{\rm rel}\ \partial_2} = &\left\{m_{{\rm cex},q,n} : f^{-2s}_{\mu_{q,n},\alpha_q,k}\right\}_{n,k=1}^\infty \cup \left\{m_{{\rm cex},q-1,n} : f^{-2s}_{\mu_{q-1,n},\alpha_{q-1},k}\right\}_{n,k=1}^\infty\\ &\cup \left\{m_{{\rm cex},q-1,n} : \hat f^{-2s}_{\mu_{q-1,n},-\alpha_{q-1},k}\right\}_{n,k=1}^\infty \cup \left\{m_{{\rm cex},q-2,n} : \hat f^{-2s}_{\mu_{q-1,n},-\alpha_{q-2},k}\right\}_{n,k=1}^\infty\\ &\cup \left\{m_{{\rm har},q} : f_{|\alpha_q|,\alpha_q,k}\right\}_{k=1}^\infty \cup \left\{m_{{\rm har},q-1} : f_{|\alpha_{q-1}|,\alpha_{q-1},k}\right\}_{k=1}^\infty,\end{aligned}$$

where the $f_{\mu,c,k}$ are the zeros of the function $F_{\mu,c}(x)$, the $\hat f_{\mu,c,k}$ are the zeros of the function $\hat F_{\mu,c}(x)$, $c \in \mathbb{R}$, α_q and $\mu_{q,n}$ are defined in Lemma 3.2.

8.2. **Torsion zeta function.** We define the *torsion zeta function* as in Section 2.2 by

$$t_{{\rm rel}\ \partial_1,{\rm abs}\ \partial_2}(s) = \frac{1}{2}\sum_{q=1}^{m+1}(-1)^q q\zeta(s,\Delta^{(q)}_{{\rm rel}\ \partial_1,{\rm abs}\ \partial_2}).$$

By a proof similar to the one of Theorem 4.1 we have the expected duality ($\dim(W) = m$):

$$\log T_{{\rm abs}\ \partial_1,{\rm rel}\ \partial_2}(C_{[l_1,l_2]}W) = (-1)^m \log T_{{\rm rel}\ \partial_1,{\rm abs}\ \partial_2}(C_{[l_1,l_2]}W).$$

We proceed assuming $\dim W = 2p-1$ odd, and assuming relative boundary condition on $\partial_1 C_{[l_1,l_2]}W$ and absolute boundary condition on ∂_2; for notational convenience, we will omit the *abs*, *rel* subscript. We define the functions

$$\begin{aligned} \hat F_c(x) &= J_c(l_2x)Y_{c-1}(l_1x) - Y_c(l_2x)J_{c-1}(l_1x),\\ F_c(x) &= J_c(l_1x)Y_{c-1}(l_2x) - Y_c(l_lx)J_{c-1}(l_2x).\end{aligned}$$

Note that, with these definitions $\hat F_0(x) = F_1(x)$ and $F_0(x) = \hat F_1(x)$ (remember that $Y_{-n}(x) = (-1)^n Y_n(x)$ and $J_{-n}(x) = (-1)^n J_n(x)$). The proof of the following lemma is analogous to the proof of Lemma 5.1. The main step is to prove that $\hat f_{|\alpha_q|,\alpha_q,k} = f_{-\alpha_{q-1},k}$, that $\hat f_{|\alpha_q|,\alpha_q,k} = \hat f_{\alpha_q,k}$, when

$p-1<q\leq 2p-1$, and that $\hat{f}_{0,0,k}=f_{1,k}$, where the $f_{c,k}$, $\hat{f}_{c,k}$ are the zeros of the functions F_c, $\hat{F}_c$, respectively.

Lemma 8.3.

$$\begin{aligned} t(s) =&\frac{1}{2}\sum_{q=0}^{p-2}(-1)^q\sum_{n,k=1}^{\infty} m_{\mathrm{cex},q,n}\left(f^{-2s}_{\mu_{q,n},\alpha_q,k}+f^{-2s}_{\mu_{q,n},-\alpha_q,k}-\hat{f}^{-2s}_{\mu_{q,n},\alpha_q,k}-\hat{f}^{-2s}_{\mu_{q,n},-\alpha_q,k}\right)\\ &+(-1)^{p-1}\frac{1}{2}\sum_{n,k=1}^{\infty} m_{\mathrm{cex},p-1,n}\left(f^{-2s}_{\mu_{p-1,n},0,k}-\hat{f}^{-2s}_{\mu_{p-1,n},0,k}\right)\\ &-\frac{1}{2}\sum_{q=0}^{p-1}(-1)^q\mathrm{rk}\mathcal{H}_q(W;\mathbb{Q})\sum_{k=1}^{\infty}\left(f^{-2s}_{-\alpha_{q-1},k}-\hat{f}^{-2s}_{-\alpha_{q-1},k}\right). \end{aligned}$$

We set

$$\begin{aligned} Z_{q,\pm}(s) &= \sum_{n,k=1}^{\infty} m_{\mathrm{cex},q,n} f^{-2s}_{\mu_{q,n},\pm\alpha_q,k}, & \hat{Z}_{q,\pm}(s) &= \sum_{n,k=1}^{\infty} m_{\mathrm{cex},q,n}\hat{f}^{-2s}_{\mu_{q,n},\pm\alpha_q,k},\\ Z_{p-1}(s) &= \sum_{n,k=1}^{\infty} m_{\mathrm{cex},p-1,n} f^{-2s}_{\mu_{p-1,n},0,k}, & \hat{Z}_{p-1,\pm}(s) &= \sum_{n,k=1}^{\infty} m_{\mathrm{cex},p-1,n}\hat{f}^{-2s}_{\mu_{p-1,n},0,k},\\ z_q(s) &= \sum_{k=1}^{\infty}\left(f^{-2s}_{-\alpha_{q-1},k}-\hat{f}^{-2s}_{-\alpha_{q-1},k}\right), \end{aligned} \tag{8.1}$$

for $0\leq q\leq p-1$, and

$$\begin{aligned} t_{p-1}(s) &= Z_{p-1}(s)-\hat{Z}_{p-1}(s),\\ t_q(s) &= Z_{q,+}(s)+Z_{q,-}(s)-\hat{Z}_{q,+}(s)-\hat{Z}_{q,-}(s), \quad 0\leq q\leq p-2. \end{aligned} \tag{8.2}$$

Then,

$$\begin{aligned} t(s) =&\frac{1}{2}\sum_{q=0}^{p-2}(-1)^q\left(Z_{q,+}(s)+Z_{q,-}(s)-\hat{Z}_{q,+}(s)-\hat{Z}_{q,-}(s)\right)+(-1)^{p-1}\frac{1}{2}\left(Z_{p-1}(s)-\hat{Z}_{p-1}(s)\right)\\ &-\frac{1}{2}\sum_{q=0}^{p-1}(-1)^q\mathrm{rk}\mathcal{H}_q(W;\mathbb{Q})z_q(s)\\ =&\frac{1}{2}\sum_{q=0}^{p-1}(-1)^q t_q(s)-\frac{1}{2}\sum_{q=0}^{p-1}(-1)^q\mathrm{rk}\mathcal{H}_q(\partial C_l W;\mathbb{Q})z_q(s), \end{aligned}$$

and

$$\log T_{\mathrm{rel}\ \partial_1,\mathrm{abs}\ \partial_2}(C_{[l_1,l_2]}W)=t'(0)=\frac{1}{2}\sum_{q=0}^{p-1}(-1)^q t'_q(0)-\frac{1}{2}\sum_{q=0}^{p-1}(-1)^q\mathrm{rk}\mathcal{H}_q(\partial C_l W;\mathbb{Q})z'_q(0). \tag{8.3}$$

8.3. **Expansions of the logarithmic Gamma functions.** We study the zeta functions $Z_{q,\pm}$, $\hat{Z}_{q,\pm}$, by the method of Section 2.4. The double series associated to these zeta functions, as defined in equation (8.1), are denoted by $S_{\pm\alpha_q}$, $\hat{S}_{\pm\alpha_q}$. We show that all these double sequences are spectrally decomposable on the sequence U_q, defined at the beginning of Section 5.1. We verify all requirements precisely as in Sections 5.2 and 5.1. First, we need suitable representation for the associated logarithmic Gamma functions. Proceeding as in Section 5.1, consider for example the function

$$F_{\mu,c}(z) = J_\mu(l_2 z)(cY_\mu(l_1 z) + l_1 z Y'_\mu(l_1 z)) - Y_\mu(l_2 z)(cJ_\mu(l_1 z) + l_1 z J'_\mu(l_1 z)).$$

Recalling the series definition of the Bessel function [9] pg. 910, near $z = 0$,

$$F_{\mu,c}(z) = \frac{1}{\pi}\left(\left(\frac{l_2^\mu}{l_1^\mu} + \frac{l_1^\mu}{l_2^\mu}\right) - \frac{c}{\mu}\left(\frac{l_2^\mu}{l_1^\mu} - \frac{l_1^\mu}{l_2^\mu}\right)\right)$$

Thus $F_{\mu,c}(z)$ is an even function of z, and we obtain the product representation

$$F_{\mu,c}(z) = \frac{1}{\pi}\left(\left(\frac{l_2^\mu}{l_1^\mu} + \frac{l_1^\mu}{l_2^\mu}\right) - \frac{c}{\mu}\left(\frac{l_2^\mu}{l_1^\mu} - \frac{l_1^\mu}{l_2^\mu}\right)\right)\prod_{k=1}^{+\infty}\left(1 - \frac{z^2}{f_{\mu,c,k}^2}\right).$$

Recalling that

$$Y_\mu(z) = \frac{\cos\mu\pi}{\sin\mu\pi}J_\mu(z) - \frac{1}{\sin\mu\pi}J_{-\mu}(z), \qquad I_{-\mu}(z) = \frac{2}{\pi}\sin\mu\pi K_\mu(z) + I_\mu(z),$$

and that (when $-\pi < \arg(z) \leq \frac{\pi}{2}$) $J_\mu(iz) = \mathrm{e}^{\frac{\pi}{2}i\mu}I_\mu(z)$, and $J'_\mu(iz) = \mathrm{e}^{\frac{\pi}{2}i\mu}\mathrm{e}^{-\frac{\pi}{2}i}I'_\mu(z)$, we obtain

$$\begin{aligned}
Y_\mu(iz) &= \left(\frac{\cos\mu\pi}{\sin\mu\pi}\mathrm{e}^{\frac{\pi}{2}i\mu} + \frac{\mathrm{e}^{-\frac{\pi}{2}i\mu}}{\sin\mu\pi}\right)I_\mu(z) - \frac{2}{\pi}\mathrm{e}^{-\frac{\pi}{2}i\mu}K_\mu(z),\\
Y'_\mu(iz) &= e^{-\frac{\pi}{2}i}\left(\frac{\cos\mu\pi}{\sin\mu\pi}\mathrm{e}^{\frac{\pi}{2}i\mu} + \frac{\mathrm{e}^{-\frac{\pi}{2}i\mu}}{\sin\mu\pi}\right)I'_\mu(z) - \frac{2}{\pi}e^{-\frac{\pi}{2}i}\mathrm{e}^{-\frac{\pi}{2}i\mu}K'_\mu(z).
\end{aligned}$$

So

$$F_{\mu,c}(iz) = \frac{2}{\pi}\left(-K_\mu(l_2 z)(cI_\mu(l_1 z) + l_1 z I'_\mu(l_1 z)) + I_\mu(l_2 z)(cK_\mu(l_1 z) + l_1 z K'_\mu(l_1 z))\right),$$

and if we define (for $-\pi < \arg(z) \leq \frac{\pi}{2}$) $G_{\mu,c}(z) = i^2 F_{\mu,c}(iz)$,

$$G_{\mu,c}(z) = \frac{1}{\pi}\left(\left(\frac{l_2^\mu}{l_1^\mu} + \frac{l_1^\mu}{l_2^\mu}\right) - \frac{c}{\mu}\left(\frac{l_2^\mu}{l_1^\mu} - \frac{l_1^\mu}{l_2^\mu}\right)\right)\prod_{k=1}^{+\infty}\left(1 + \frac{z^2}{f_{\mu,c,k}^2}\right).$$

Proceeding in the similar way

$$\begin{aligned}
\hat F_{\mu,c}(iz) &= \frac{2}{\pi}\left(K_\mu(l_1 z)(cI_\mu(l_2 z) + l_2 z I'_\mu(l_2 z)) - I_\mu(l_1 z)(cK_\mu(l_2 z) + l_2 z K'_\mu(l_2 z))\right),\\
\hat G_{\mu,c}(z) = \hat F_{\mu,c}(iz) &= \frac{1}{\pi}\left(\left(\frac{l_2^\mu}{l_1^\mu} + \frac{l_1^\mu}{l_2^\mu}\right) + \frac{c}{\mu}\left(\frac{l_2^\mu}{l_1^\mu} - \frac{l_1^\mu}{l_2^\mu}\right)\right)\prod_{k=1}^{+\infty}\left(1 + \frac{z^2}{\hat f_{\mu,c,k}^2}\right);
\end{aligned}$$

$$\begin{aligned}
F_{\mu,0}(iz) &= \frac{2}{\pi}\left(-K_\mu(l_2 z)I'_\mu(l_1 z) + I_\mu(l_2 z)K'_\mu(l_1 z)\right),\\
G_{\mu,0}(z) = i^2 F_{\mu,0}(iz) &= \frac{1}{l_1 z\pi}\left(\frac{l_2^\mu}{l_1^\mu} + \frac{l_1^\mu}{l_2^\mu}\right)\prod_{k=1}^{+\infty}\left(1 + \frac{z^2}{f_{\mu,c,k}^2}\right);\\
\hat F_{\mu,0}(iz) &= \frac{2}{\pi}\left(K_\mu(l_1 z)I'_\mu(l_2 z) - I_\mu(l_1 z)K'_\mu(l_2 z))\right),\\
\hat G_{\mu,0}(z) = \hat F_{\mu,0}(iz) &= \frac{1}{l_2 z\pi}\left(\frac{l_2^\mu}{l_1^\mu} + \frac{l_1^\mu}{l_2^\mu}\right)\prod_{k=1}^{+\infty}\left(1 + \frac{z^2}{\hat f_{\mu,0,k}^2}\right).
\end{aligned}$$

These give the following representations for the logarithmic Gamma functions with $z=\sqrt{-\lambda}$,

$$\begin{aligned}\log\Gamma(-\lambda,S_{\pm\alpha_q})&=-\log\prod_{k=1}^{\infty}\left(1+\frac{(-\lambda)}{f^2_{\mu_{q,n},\pm\alpha_q,k}}\right)\\&=-\log G_{\mu_{q,n},\pm\alpha_q}(\sqrt{-\lambda})+\log\frac{1}{\pi}+\log\left(\left(\frac{l_2^{\mu_{q,n}}}{l_1^{\mu_{q,n}}}+\frac{l_1^{\mu_{q,n}}}{l_2^{\mu_{q,n}}}\right)\mp\frac{\alpha_q}{\mu_{q,n}}\left(\frac{l_2^{\mu_{q,n}}}{l_1^{\mu_{q,n}}}-\frac{l_1^{\mu_{q,n}}}{l_2^{\mu_{q,n}}}\right)\right),\\ \log\Gamma(-\lambda,\hat S_{\pm\alpha_q})&=-\log\prod_{k=1}^{\infty}\left(1+\frac{(-\lambda)}{\hat f^2_{\mu_{q,n},\pm\alpha_q,k}}\right)\\&=-\log \hat G_{\mu_{n,q},\pm\alpha_q}(\sqrt{-\lambda})+\log\frac{1}{\pi}+\log\left(\left(\frac{l_2^{\mu_{q,n}}}{l_1^{\mu_{q,n}}}+\frac{l_1^{\mu_{q,n}}}{l_2^{\mu_{q,n}}}\right)\pm\frac{\alpha_q}{\mu_{q,n}}\left(\frac{l_2^{\mu_{q,n}}}{l_1^{\mu_{q,n}}}-\frac{l_1^{\mu_{q,n}}}{l_2^{\mu_{q,n}}}\right)\right);\\ \log\Gamma(-\lambda,S_0)&=-\log\prod_{k=1}^{\infty}\left(1+\frac{(-\lambda)}{f^2_{\mu_{p-1,n},0,k}}\right)\\&=-\log G_{\mu_{n,p-1},0}(\sqrt{-\lambda})-\frac12\log-\lambda-\log l_1+\log\frac{1}{\pi}+\log\left(\frac{l_2^{\mu_{q,n}}}{l_1^{\mu_{q,n}}}+\frac{l_1^{\mu_{q,n}}}{l_2^{\mu_{q,n}}}\right),\\ \log\Gamma(-\lambda,\hat S_0)&=-\log\prod_{k=1}^{\infty}\left(1+\frac{(-\lambda)}{\hat f^2_{\mu_{p-1,n},0,k}}\right)\\&=-\log \hat G_{\mu_{n,p-1},0}(\sqrt{-\lambda})-\frac12\log-\lambda-\log l_2+\log\frac{1}{\pi}+\log\left(\frac{l_2^{\mu_{p-1,n}}}{l_1^{\mu_{p-1,n}}}+\frac{l_1^{\mu_{p-1,n}}}{l_2^{\mu_{p-1,n}}}\right).\end{aligned}$$

These representations and uniform asymptotic expansions of Bessel functions and their derivative (see the proof of Lemma 5.10 for the functions I_ν and [21] pg. 376 for the functions K_ν) will give the expansion required in equation (2.15) of Definition 2.1. Let see one case in some details. We have

$$\begin{aligned}&\log\Gamma(-\lambda,S_{n,\pm\alpha_q}/\mu^2_{q,n})=\\&-\log G_{\mu_{n,q},\pm\alpha_q}(\mu_{q,n}\sqrt{-\lambda})+\log\frac{1}{\pi}+\log\left(\left(\frac{l_2^{\mu_{q,n}}}{l_1^{\mu_{q,n}}}+\frac{l_1^{\mu_{q,n}}}{l_2^{\mu_{q,n}}}\right)\mp\frac{\alpha_q}{\mu_{q,n}}\left(\frac{l_2^{\mu_{q,n}}}{l_1^{\mu_{q,n}}}-\frac{l_1^{\mu_{q,n}}}{l_2^{\mu_{q,n}}}\right)\right).\end{aligned}$$

Using the cited expansions we obtain

$$\begin{aligned}\log G_{\mu,c}(\mu z)&=\log\frac{1}{\pi}+\mu\left(\sqrt{1+l_2^2z^2}-\sqrt{1+l_1^2z^2}\right)+\mu\log\frac{l_2(1+\sqrt{1+l_1^2z^2})}{l_1(1+\sqrt{1+l_2^2z^2})}+\frac14\log\frac{(1+l_1^2z^2)}{(1+l_2^2z^2)}\\&+\log\left(1+\sum_{j=1}^{2p-1}\frac{1}{\mu^j}\left(U_j(l_2z)+(-1)^jW_{-c,j}(l_1z)+\sum_{k=1}^{j-1}(-1)^{j-k}U_k(l_2z)W_{-c,j-k}(l_1z)\right)+O(\mu^{-2p})\right).\end{aligned}$$

Thus

$$\begin{aligned}\log G_{\mu_{q,n},\pm\alpha_q}(\mu_{q,n}\sqrt{-\lambda})&=\mu_{q,n}\left(\sqrt{1-l_2^2\lambda}-\sqrt{1-l_1^2\lambda}\right)+\mu_{q,n}\log\frac{l_2(1+\sqrt{1-l_1^2\lambda})}{l_1(1+\sqrt{1-l_2^2\lambda})}\\&\quad+\log\frac{1}{\pi}+\frac14\log\frac{(1-l_1^2\lambda)}{(1-l_2^2\lambda)}+\sum_{j=1}^{2p-1}\frac{l_{j,\mp\alpha_q}(\lambda)}{\mu^j_{q,n}}+O(\mu_{q,n}^{-2p}),\end{aligned}$$

with

$$\begin{aligned}
&a_{0,\pm\alpha_q}(\lambda)=1, \qquad\qquad l_{1,\pm\alpha_q}(\lambda)=a_{1,\pm\alpha_q}(\lambda),\\
&a_{j,\pm\alpha_q}(\lambda)=U_j(l_2\sqrt{-\lambda})+(-1)^jW_{\pm\alpha_q,j}(l_1\sqrt{-\lambda})+\sum_{k=1}^{j-1}U_k(l_2\sqrt{-\lambda})(-1)^{j-k}W_{\pm\alpha_q,j-k}(l_1\sqrt{-\lambda}),\\
&l_{j,\pm\alpha_q}(\lambda)=a_{j,\pm\alpha_q}(\lambda)-\sum_{k=1}^{j-1}\frac{j-k}{j}a_{k,\pm\alpha_q}(\lambda)l_{j-k,\pm\alpha_q}(\lambda).
\end{aligned}$$

Substituting in the $\log\Gamma(-\lambda,S_{n,\pm\alpha_q}/\mu_{q,n}^2)$, we have

$$\begin{aligned}
&\log\Gamma(-\lambda,S_{n,\pm\alpha_q}/\mu_{q,n}^2)=-\mu_{q,n}\left(\sqrt{1-l_2^2\lambda}-\sqrt{1-l_1^2\lambda}\right)-\mu_{q,n}\log\frac{l_2(1+\sqrt{1-l_1^2\lambda})}{l_1(1+\sqrt{1-l_2^2\lambda})}\\
&-\frac{1}{4}\log\frac{(1-l_1^2\lambda)}{(1-l_2^2\lambda)}-\sum_{j=1}^{2p-1}\frac{l_{j,\mp\alpha_q}(\lambda)}{\mu_{q,n}^j}+\log\left(\left(\frac{l_2^{\mu_{q,n}}}{l_1^{\mu_{q,n}}}+\frac{l_1^{\mu_{q,n}}}{l_2^{\mu_{q,n}}}\right)\mp\frac{\alpha_q}{\mu_{q,n}}\left(\frac{l_2^{\mu_{q,n}}}{l_1^{\mu_{q,n}}}-\frac{l_1^{\mu_{q,n}}}{l_2^{\mu_{q,n}}}\right)\right)+O(\mu_{q,n}^{-2p}).
\end{aligned}$$

Proceeding in a similar way we obtain

$$\begin{aligned}
&\log\Gamma(-\lambda,\hat S_{n,\pm\alpha_q}/\mu_{q,n}^2)=-\mu_{q,n}\left(\sqrt{1-l_2^2\lambda}-\sqrt{1-l_1^2\lambda}\right)-\mu_{q,n}\log\frac{l_2(1+\sqrt{1-l_1^2\lambda})}{l_1(1+\sqrt{1-l_2^2\lambda})}\\
&-\frac{1}{4}\log\frac{(1-l_1^2\lambda)}{(1-l_2^2\lambda)}-\sum_{j=1}^{2p-1}\frac{\hat l_{j,\pm\alpha_q}(\lambda)}{\mu_{q,n}^j}+\log\left(\left(\frac{l_2^{\mu_{q,n}}}{l_1^{\mu_{q,n}}}+\frac{l_1^{\mu_{q,n}}}{l_2^{\mu_{q,n}}}\right)\pm\frac{\alpha_q}{\mu_{q,n}}\left(\frac{l_2^{\mu_{q,n}}}{l_1^{\mu_{q,n}}}-\frac{l_1^{\mu_{q,n}}}{l_2^{\mu_{q,n}}}\right)\right)+O(\mu_{q,n}^{-2p}),
\end{aligned}$$

with

$$\begin{aligned}
&\hat a_{0,\pm\alpha_q}(\lambda)=1, \qquad\qquad \hat l_{1,\pm\alpha_q}(\lambda)=\hat a_{1,\pm\alpha_q}(\lambda),\\
&\hat a_{j,\pm\alpha_q}(\lambda)=\hat W_{\pm\alpha_q,j}(l_2\sqrt{-\lambda})+(-1)^jU_j(l_1\sqrt{-\lambda})+\sum_{k=1}^{j-1}(-1)^kU_k(l_1\sqrt{-\lambda})\hat W_{\pm\alpha_q,j-k}(l_2\sqrt{-\lambda}),\\
&\hat l_{j,\pm\alpha_q}(\lambda)=\hat a_{j,\pm\alpha_q}(\lambda)-\sum_{k=1}^{j-1}\frac{j-k}{j}\hat a_{k,\pm\alpha_q}(\lambda)\hat l_{j-k,\pm\alpha_q}(\lambda);
\end{aligned}$$

$$\begin{aligned}
&\log\Gamma(-\lambda,\hat S_{n,0}/\mu_{p-1,n}^2)=-\mu_{p-1,n}\left(\sqrt{1-l_2^2\lambda}-\sqrt{1-l_1^2\lambda}\right)-\mu_{p-1,n}\log\frac{l_2(1+\sqrt{1-l_1^2\lambda})}{l_1(1+\sqrt{1-l_2^2\lambda})}\\
&-\frac{1}{4}\log\frac{(1-l_1^2\lambda)}{(1-l_2^2\lambda)}-\sum_{j=1}^{2p-1}\frac{\hat l_{j,0}(\lambda)}{\mu_{p-1,n}^j}+\log\left(\frac{l_2^{\mu_{p-1,n}}}{l_1^{\mu_{p-1,n}}}+\frac{l_1^{\mu_{p-1,n}}}{l_2^{\mu_{p-1,n}}}\right)+O(\mu_{p-1,n}^{-2p}),
\end{aligned}$$

with

$$\hat{a}_{0,0}(\lambda)=1, \qquad \hat{l}_{1,0}(\lambda)=\hat{a}_{1,0}(\lambda),$$
$$\hat{a}_{j,0}(\lambda)=V_j(l_2\sqrt{-\lambda})+(-1)^jU_j(l_1\sqrt{-\lambda})+\sum_{k=1}^{j-1}(-1)^kU_k(l_1\sqrt{-\lambda})V_{j-k}(l_2\sqrt{-\lambda}),$$
$$\hat{l}_{j,0}(\lambda)=\hat{a}_{j,0}(\lambda)-\sum_{k=1}^{j-1}\frac{j-k}{j}\hat{a}_{k,0}(\lambda)\hat{l}_{j-k,0}(\lambda);$$

$$\log\Gamma(-\lambda,S_{n,0}/\mu_{q,n}^2)=-\mu_{p-1,n}\left(\sqrt{1-l_2^2\lambda}-\sqrt{1-l_1^2\lambda}\right)-\mu_{p-1,n}\log\frac{l_2(1+\sqrt{1-l_1^2\lambda})}{l_1(1+\sqrt{1-l_2^2\lambda})}$$
$$-\frac{1}{4}\log\frac{(1-l_1^2\lambda)}{(1-l_2^2\lambda)}-\sum_{j=1}^{2p-1}\frac{l_{j,0}(\lambda)}{\mu_{p-1,n}^j}+\log\left(\frac{l_2^{\mu_{p-1,n}}}{l_1^{\mu_{p-1,n}}}+\frac{l_1^{\mu_{p-1,n}}}{l_2^{\mu_{p-1,n}}}\right)+O(\mu_{p-1,n}^{-2p}),$$

with

$$a_{0,0}(\lambda)=1, \qquad l_{1,0}(\lambda)=a_{1,0}(\lambda),$$
$$a_{j,0}(\lambda)=U_j(l_2\sqrt{-\lambda})+(-1)^jV_j(l_1\sqrt{-\lambda})+\sum_{k=1}^{j-1}U_k(l_2\sqrt{-\lambda})(-1)^{j-k}V_{j-k}(l_1\sqrt{-\lambda}),$$
$$l_{j,0}(\lambda)=a_{j,0}(\lambda)-\sum_{k=1}^{j-1}\frac{j-k}{j}a_{k,0}(\lambda)l_{j-k}(\lambda).$$

We conclude this section with the expansions for large λ, accordingly to equation (2.17). Using classical expansions of Bessel functions I_ν and K_ν and their derivative for large argument, we obtain the expansions of the functions G and $\hat{G}$, and then those for the Gamma functions:

$$\log\Gamma(-\lambda,S_{n,\pm\alpha_q}/\mu_{q,n}^2)$$
$$\sim-\mu_{q,n}(l_2-l_1)\sqrt{-\lambda}-\frac{1}{2}\log\frac{l_1}{l_2}+\log\left(\left(\frac{l_2^{\mu_{q,n}}}{l_1^{\mu_{q,n}}}+\frac{l_1^{\mu_{q,n}}}{l_2^{\mu_{q,n}}}\right)\mp\frac{\alpha_q}{\mu_{q,n}}\left(\frac{l_2^{\mu_{q,n}}}{l_1^{\mu_{q,n}}}-\frac{l_1^{\mu_{q,n}}}{l_2^{\mu_{q,n}}}\right)\right)+O\left(\frac{1}{\sqrt{-\lambda}}\right),$$
$$\log\Gamma(-\lambda,\hat{S}_{n,\pm\alpha_q}/\mu_{q,n}^2)$$
$$\sim-\mu_{q,n}(l_2-l_1)\sqrt{-\lambda}-\frac{1}{2}\log\frac{l_2}{l_1}+\log\left(\left(\frac{l_2^{\mu_{q,n}}}{l_1^{\mu_{q,n}}}+\frac{l_1^{\mu_{q,n}}}{l_2^{\mu_{q,n}}}\right)\pm\frac{\alpha_q}{\mu_{q,n}}\left(\frac{l_2^{\mu_{q,n}}}{l_1^{\mu_{q,n}}}-\frac{l_1^{\mu_{q,n}}}{l_2^{\mu_{q,n}}}\right)\right)+O\left(\frac{1}{\sqrt{-\lambda}}\right),$$
$$\log\Gamma(-\lambda,S_{n,0}/\mu_{p-1,n}^2)\sim-\mu_{p-1,n}(l_2-l_1)\sqrt{-\lambda}+\frac{1}{2}\log\frac{l_2}{l_1}+\log\left(\frac{l_2^{\mu_{p-1,n}}}{l_1^{\mu_{p-1,n}}}+\frac{l_1^{\mu_{p-1,n}}}{l_2^{\mu_{p-1,n}}}\right)+O\left(\frac{1}{\sqrt{-\lambda}}\right),$$
$$\log\Gamma(-\lambda,\hat{S}_{n,0}/\mu_{p-1,n}^2)\sim-\mu_{p-1,n}(l_2-l_1)\sqrt{-\lambda}+\frac{1}{2}\log\frac{l_1}{l_2}+\log\left(\frac{l_2^{\mu_{p-1,n}}}{l_1^{\mu_{p-1,n}}}+\frac{l_1^{\mu_{p-1,n}}}{l_2^{\mu_{p-1,n}}}\right)+O\left(\frac{1}{\sqrt{-\lambda}}\right).$$

8.4. **The function $t_q(s)$.** By definition in equation (8.2), we need to consider the difference between $\log\Gamma(-\lambda,S_{n,\pm\alpha_q}/\mu_{q,n})$ and $\log\Gamma(-\lambda,\hat{S}_{n,\pm\alpha_q}/\mu_{q,n})$. The expansions given in the previous subsection give expansion for large μ

$$\log\Gamma(-\lambda, S_{n,-\alpha_q}/\mu_{q,n}) + \log\Gamma(-\lambda, S_{n,\alpha_q}/\mu_{q,n}) - \log\Gamma(-\lambda, \hat{S}_{n,\alpha_q}/\mu_{q,n}) - \log\Gamma(-\lambda, \hat{S}_{n,-\alpha_q}/\mu_{q,n}) =$$

$$= \log\frac{(1-\lambda l_2^2)}{(1-\lambda l_1^2)} + \sum_{j=1}^{2p-1}\frac{1}{\mu_{q.n}^j}(\hat{l}_{j,\alpha_q}(\lambda) + \hat{l}_{j,-\alpha_q}(\lambda) - l_{j,\alpha_q}(\lambda) - l_{j,-\alpha_q}(\lambda)) + O(\mu_{q,n}^{-2p}),$$

and for large λ

$$\log\Gamma(-\lambda, S_{n,\alpha_q}/\mu_{q,n}) + \log\Gamma(-\lambda, S_{n,\alpha_q}/\mu_{q,n}) - \log\Gamma(-\lambda, \hat{S}_{n,\alpha_q}/\mu_{q,n}) - \log\Gamma(-\lambda, \hat{S}_{n,-\alpha_q}/\mu_{q,n}) =$$

$$= 2\log\frac{l_2}{l_1} + O\left(\frac{1}{\sqrt{-\lambda}}\right).$$

Proceeding as in the proof of Lemma 5.7, we obtain $a_{0,0,q,n} = 2\log\frac{l_2}{l_1}$, $a_{0,1,q,n} = 0$, $b_{2j-1,0,0,q} = 0$, $b_{2j-1,0,1,q} = 0$, and hence

$$A_{0,0,q}(s) = 2\log\frac{l_2}{l_1}\sum_{n=1}^{\infty}\frac{m_{q,n}}{\mu_{q,n}^{2s}} = 2\log\frac{l_2}{l_1}\sum_{j=0}^{\infty}\binom{-s}{j}\alpha_q^2 j\zeta_{\rm ccl}(s+j, \tilde{\Delta}^{(q)}), \qquad A_{0,1,q}(s) = 0.$$

This gives

$$A_{0,0,q}(0) = 2\log\frac{l_2}{l_1}\zeta_{{\rm ccl},q}(0, \tilde{\Delta}^{(q)}) = 2(-1)^q\log\frac{l_2}{l_1}\sum_{k=0}^{q}(-1)^k{\rm rk}\mathcal{H}^k(W,\mathbb{Q}),$$

and

$$t'_{q,{\rm reg}}(0) = 2(-1)^{q+1}\log\frac{l_2}{l_1}\sum_{k=0}^{q}(-1)^k{\rm rk}\mathcal{H}^k(W,\mathbb{Q}).$$

Similarly, we consider the difference of $\log\Gamma(-\lambda, S_{n,0}/\mu_{p-1,n})$ and $\log\Gamma(-\lambda, \hat{S}_{n,0}/\mu_{p-1,n})$ for the function t_{p-1}, and we obtain $a_{0,0,n,p-1} = \log\frac{l_2}{l_1}$, $a_{0,1,n,p-1} = 0$, $b_{2j-1,0,0,p-1} = 0$, $b_{2j-1,0,1,p-1} = 0$, and hence

$$A_{0,0,p-1}(s) = \log\frac{l_2}{l_1}\sum_{n=1}^{\infty}\frac{m_{p-1,n}}{\mu_{p-1,n}^{2s}} = \log\frac{l_2}{l_1}\zeta_{{\rm ccl},p-1}(s, \tilde{\Delta}^{(q)}), \qquad A_{0,1}(s) = 0,$$

$$A_{0,0,p-1}(0) = \log\frac{l_2}{l_1}\zeta_{{\rm ccl},q}(0, \tilde{\Delta}^{(p-1)}) = (-1)^{p-1}\log\frac{l_2}{l_1}\sum_{k=0}^{p-1}(-1)^k{\rm rk}\mathcal{H}^k(W,\mathbb{Q}),$$

and

$$t'_{p-1,{\rm reg}}(0) = (-1)^p\log\frac{l_2}{l_1}\sum_{k=0}^{p-1}(-1)^k{\rm rk}\mathcal{H}^k(W,\mathbb{Q}).$$

8.5. **The regular term of the torsion.** We use equation (8.3). First, note that as in Section 6 there is no singular contribution by the functions $z_q(s)$. Using equation (2.19), and recalling that $-\alpha_{q-1} = -(q-1-p+1) = p-q$, we compute as in Lemma 6.1

$$z'_q(0) = \log\frac{l_2}{l_1} - 2(p-q)\log\frac{l_2}{l_1}.$$

Therefore, substitution in equation (8.3) gives

$$\log T_{{\rm rel}\ \partial_1,{\rm abs}\ \partial_2,{\rm reg}}(C_{[l_1,l_2]}W) = t'_{reg}(0) = 0.$$

8.6. **The singular term of the torsion.** We show that the singular part of the torsion is twice the singular part of the torsion on the cone, namely that

$$\log T_{\text{rel } \partial_1,\text{abs } \partial_2,\text{sing}}(C_{[l_1,l_2]}W) = 2\log T_{\text{abs,sing}}(C_l W). \tag{8.4}$$

Lemma 8.4. *We have the equations:*

$$l_{j,\pm\alpha_q}(\lambda) = l_j(l_2^2\lambda) + (-1)^j l_j^{\mp}(l_1^2\lambda), \qquad \hat{l}_{j,\pm\alpha_q}(\lambda) = l_j^{\pm}(l_2^2\lambda) + (-1)^j l_j(l_1^2\lambda),$$
$$l_{j,0}(\lambda) = l_j(l_2^2\lambda) + (-1)^j \dot{l}_j(l_1^2\lambda), \qquad \hat{l}_{j,0}(\lambda) = \dot{l}_j(l_2^2\lambda) + (-1)^j l_j(l_1^2\lambda),$$

where the functions l_j, $\dot{l}_j$ are defined in the proof of Lemma 5.10, the functions $l_j^{\pm}$ in the proof of Lemma 5.4, and the other function in Subsection 8.3.

Proof. The proof is by induction. We give details for the first equation. For $j = 1$, we have

$$l_{1,\pm\alpha_q}(\lambda) = U_1(l_2\sqrt{-\lambda}) - W_{\mp,1}(l_1\sqrt{-\lambda}) = l_1(l_2^2\lambda) + (-1)^1 l_1(l_1^2\sqrt{-\lambda}).$$

Assume the equation is valid for all $n < j$. By definition

$$\begin{aligned} l_{j,\pm\alpha_q}(\lambda) - &\sum_{k=1}^{j-1} U_k(l_2\sqrt{-\lambda})(-1)^{j-k} W_{\mp\alpha_q,j-k}(l_1\sqrt{-\lambda}) \\ &= U_j(l_2\sqrt{-\lambda}) + (-1)^j W_{\mp\alpha_q,j}(l_1\sqrt{-\lambda}) - \sum_{k=1}^{j-1}\frac{j-k}{j} a_{k,\mp\alpha_q}(\lambda) l_{j-k,\mp\alpha_q}(\lambda), \end{aligned}$$

and using the inductive hypothesis for $l_{j-k,\mp\alpha_q}(\lambda)$, and collecting similar terms, this gives

$$\begin{aligned} l_{j,\pm\alpha_q}(\lambda) - &\sum_{k=1}^{j-1} U_k(l_2\sqrt{-\lambda})(-1)^{j-k} W_{\mp\alpha_q,j-k}(l_1\sqrt{-\lambda}) \\ =& l_j(l_2^2\lambda) + (-1)^j l_j^{\mp}(l_1^2\lambda) - \sum_{k=1}^{j-1}\frac{j-k}{j}(-1)^k W_{\mp\alpha_q,k}(l_1\sqrt{-\lambda}) l_{j-k}(l_2^2\lambda) \\ &- \sum_{k=1}^{j-1}\frac{j-k}{j}(U_k(l_2\sqrt{-\lambda}))(-1)^{j-k} l_{j-k}^{\mp}(l_1^2\lambda) \\ &- \sum_{k=1}^{j-1}\frac{j-k}{j}\sum_{h=1}^{k-1} U_h(l_2\sqrt{-\lambda})(-1)^{k-h} W_{\mp\alpha_q,k-h}(l_1\sqrt{-\lambda}) l_{j-k}(l_2^2\lambda) \\ &- \sum_{k=1} 1^{j-1}\frac{j-k}{j}\sum_{h=1}^{k-1} U_h(l_2\sqrt{-\lambda})(-1)^{k-h} W_{\mp\alpha_q,k-h}(l_1\sqrt{-\lambda}))(-1)^{j-k} l_{j-k}^{\mp}(l_1^2\lambda). \end{aligned}$$

Rearranging the summation's indices, this reads

$$
\begin{aligned}
&= l_j(l_2^2\lambda) + (-1)^j l_j^{\mp}(l_1^2\lambda) - \sum_{k=1}^{j-1}(-1)^k W_{\mp\alpha_q,k}(l_1\sqrt{-\lambda})U_{j-k}(l_2\sqrt{-\lambda})\\
&+ \sum_{k=1}^{j-1}(-1)^{j-k} W_{\mp\alpha_q,j-k}(l_1\sqrt{-\lambda})\sum_{h=1}^{k-1}\frac{k-h}{j}U_h(l_2\sqrt{-\lambda})l_{k-h}(l_2^2\lambda)\\
&+ \sum_{k=1}^{j-1}(-1)^k U_{j-k}(l_2\sqrt{-\lambda})\sum_{h=1}^{k-1}\frac{h}{j}W_{\mp\alpha_q,k-h}(l_1\sqrt{-\lambda})l_h^{\mp}(l_1^2\lambda)\\
&- \sum_{k=1}^{j-1}\frac{j-k}{j}l_{j-k}(l_2^2\lambda)\sum_{h=1}^{k-1}U_h(l_2\sqrt{-\lambda})(-1)^{k-h}W_{\mp\alpha_q,k-h}(l_1\sqrt{-\lambda})\\
&- \sum_{k=1}^{j-1}\frac{j-k}{j}(-1)^{j-k}l_{j-k}^{\mp}(l_1^2\lambda)\sum_{h=1}^{k-1}U_h(l_2\sqrt{-\lambda})(-1)^{k-h}W_{\mp\alpha_q,k-h}(l_1\sqrt{-\lambda}).
\end{aligned}
$$

Reordering the first two double sums as

$$
\begin{aligned}
&\sum_{k=1}^{j-1}(-1)^{j-k}W_{\mp\alpha_q,j-k}(l_1\sqrt{-\lambda})\sum_{h=1}^{k-1}\frac{k-h}{j}U_h(l_2\sqrt{-\lambda})l_{k-h}(l_2^2\lambda)\\
&\qquad = \sum_{k=1}^{j-1}\frac{j-k}{j}l_{j-k}(l_2^2\lambda)\sum_{h=1}^{k-1}U_h(l_2\sqrt{-\lambda})(-1)^{k-h}W_{\mp\alpha_q,k-h}(l_1\sqrt{-\lambda}),\\
&\sum_{k=1}^{j-1}(-1)^k U_{j-k}(l_2\sqrt{-\lambda})\sum_{h=1}^{k-1}\frac{k-h}{j}W_{\mp\alpha_q,h}(l_1\sqrt{-\lambda})l_{k-h}^{\mp}(l_1^2\lambda)\\
&\qquad = \sum_{k=1}^{j-1}\frac{j-k}{j}(-1)^{j-k}l_{j-k}^{\mp}(l_1^2\lambda)\sum_{h=1}^{k-1}U_h(l_2\sqrt{-\lambda})(-1)^{k-h}W_{\mp\alpha_q,k-h}(l_1\sqrt{-\lambda}),
\end{aligned}
$$

the result follows. □

We are now in the position of proving equation (8.4). Proceeding as in the proof of Propositions 5.4 and 5.2, the singular part of the torsion is given by some residua of the zeta function associated to the sequence U and some residua of the functions Φ. Since the sequence U is the same for the conical frustum and for the cone, and the range of the indices are the same, we only need to compare the functions Φ in the two cases. The functions Φ are defined in equation (2.16), we introduce the linear operator

$$
\Phi_{\sigma_h}(s) = \mathcal{T}(\phi_{\sigma_h}(\_))(s) = \int_0^\infty t^{s-1}\frac{1}{2\pi i}\int_{\Lambda_{\theta,c}}\frac{\mathrm{e}^{-\lambda t}}{-\lambda}\phi_{\sigma_h}(\lambda)d\lambda dt. \tag{8.5}
$$

Let use the notation $\phi^{\rm cone}$ and $\phi^{\rm frust}$. We have

$$
\begin{aligned}
\phi^{\rm cone}_{q,2j-1}(\lambda) &= -2l_{2j-1}(\lambda) + l^{+}_{2j-1}(\lambda) + l^{-}_{2j-1}(\lambda),\\
\phi^{\rm frust}_{q,j}(\lambda) &= -l_{j,\alpha_q}(\lambda) - l_{j,-\alpha_q}(\lambda) + \hat{l}_{j,\alpha_q}(\lambda) + \hat{l}_{j,-\alpha_q}(\lambda).
\end{aligned}
$$

Note that all the functions appearing in the definition of the functions $\phi(\lambda)$ are polynomial in $w=\frac{1}{\sqrt{1-\lambda}}$. Applying the formula in equation (8.5), we have that

$$\mathcal{T}(l_{j+}(l_2^2 \_))(s)=l_2^{2s}\mathcal{T}(l_{j+}(\_))(s),$$

and similarly for the other. Using Lemma 8.4, and odd indices, we obtain for example

$$\Phi_{2j-1}^{\rm frust}(s)=(l_2^{2s}+l_1^{2s})\Phi_{2j-1}^{\rm cone}(s).$$

Since by Corollaries 5.2 and 5.1 all the residua Res_1 of the function $\Phi_{2j-1}^{\rm cone}(s)$ at $s=0$ vanish, equation 8.4 follows.

8.7. **Conclusion.** As recalled in Section 2.3, if $\partial W=\partial_1 W\sqcup\partial_2 W$ is the union of two disjoint components, and since the boundary term is local,

$$\log T_{\mathrm{rel}\ \partial_1,\mathrm{abs}\ \partial_2}((W,g);\rho)=\log\tau(((W,\partial_1 W),g);\rho)+A_{\mathrm{BM,rel}}(\partial_1 W)+A_{\mathrm{BM,abs}}(\partial_2 W).$$

Applying this formula to the conical frustum we have

$$\log T_{\mathrm{rel}\ \partial_1,\mathrm{abs}\ \partial_2}(C_{[l_1,l_2]}W)=\log\tau(C_{[l_1,l_2]}W,\partial_1 C_{[l_1,l_2]}W)+A_{\mathrm{BM,rel}}(\partial_1)+A_{\mathrm{BM,abs}}(\partial_2).$$

Let X be a manifold of dimension $2p$ with boundary $\partial X=\partial_2 C_{[l_1,l_2]}W$, and assume there is an isometry of a collar neighborhood of the boundary of X onto a collar neighborhood of $\partial_2 C_{[l_1,l_2]}W$. Let Z be the manifold obtained by glueing smoothly X to $C_{[l_1,l_2]}W$ along the boundary $\partial_2 C_{[l_1,l_2]}W$. Applying duality of analytic torsion [16] Proposition 2.10 to Z, and since the anomaly boundary term is local, it follows that $A_{\mathrm{BM,rel}}(\partial_1 C_{[l_1,l_2]}W)=-A_{\mathrm{BM,abs}}(\partial_1 C_{[l_1,l_2]}W)$. Since it follows by the definition that $A_{\mathrm{BM,abs}}(\partial_1 C_{[l_1,l_2]}W)=-A_{\mathrm{BM,abs}}(\partial_2 C_{[l_1,l_2]}W)$, we obtain

$$\log T_{\mathrm{rel}\ \partial_1,\mathrm{abs}\ \partial_2}(C_{[l_1,l_2]}W)=\log\tau(C_{[l_1,l_2]}W,\partial_1 C_{[l_1,l_2]}W)+2A_{\mathrm{BM,abs}}(\partial_2 C_{[l_1,l_2]}W).$$

Considering the exact sequence of chain complex associated to the pair $(C_{[l_1,l_2]}W,\partial_1 C_{[l_1,l_2]}W)$, it is not difficult to see (see for example [18] Section 3) that the Reidemeister torsion of the pair vanishes, and hence

$$\log T_{\mathrm{rel}\ \partial_1,\mathrm{abs}\ \partial_2}(C_{[l_1,l_2]}W)=2A_{\mathrm{BM,abs}}(\partial_2 C_{[l_1,l_2]}W).$$

Since the anomaly boundary term is local $A_{\mathrm{BM,abs}}(\partial_2 C_{[l_1,l_2]}W)=A_{\mathrm{BM,abs}}(\partial C_l W)$, and hence

$$\log T_{\mathrm{rel}\ \partial_1,\mathrm{abs}\ \partial_2}(C_{[l_1,l_2]}W)=2A_{\mathrm{BM,abs}}(\partial C_l W).$$

The general argument presented here deserves a complete proof. This can be found in the new paper of Brüning and Ma [2], where gluing formulas and formulas for the variation of the torsion with mixed boundary conditions are proved. We thanks the authors for making available to us this part of the results of their still unpublished paper. Since by the calculations of the previous subsections

$$\log T_{\mathrm{rel}\ \partial_1,\mathrm{abs}\ \partial_2}(C_{[l_1,l_2]}W)=\log T_{\mathrm{rel}\ \partial_1,\mathrm{abs}\ \partial_2,\mathrm{sing}}(C_{[l_1,l_2]}W)=2\log T_{\mathrm{abs,sing}}(C_l W)=2S(\partial C_l W),$$

this completes the proof of Theorem 1.2.

Appendix

The next two formulas follow from the definition of the Euler Gamma function ($j\in\mathbb{N}$).

$$\operatorname*{Res}_{0}_{s=0}\frac{\Gamma\left(s+\frac{2j+1}{2}\right)}{\Gamma\left(\frac{2j+1}{2}\right)s}=-\gamma-2\log 2+2\sum_{k=1}^{j}\frac{1}{2k-1},\quad \operatorname*{Res}_{1}_{s=0}\frac{\Gamma(s+\frac{2j+1}{2})}{\Gamma(\frac{2j-1}{2})s}=1, \tag{8.6}$$

The next formula is proved in [26] Section 4.2 ($0 < \theta < \pi$, $0 < c < 1$, $a \in \mathbb{R}$).

$$\int_0^\infty t^{s-1}\frac{1}{2\pi i}\int_{\Lambda_{\theta,c}}\frac{\mathrm{e}^{-\lambda t}}{-\lambda}\frac{1}{(1-\lambda)^a}d\lambda dt = \frac{\Gamma(s+a)}{\Gamma(a)s}. \tag{8.7}$$

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