AN IMPROVEMENT OF THE FIVE HALVES THEOREM OF J. BOARDMAN

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Abstract. Let (M^m, T) be a smooth involution on a closed smooth m-dimensional manifold and $F = \bigcup_{j=0}^{n} F^j$ $(n \leq m)$ its fixed point set, where F^j denotes the union of those components of F having dimension j. The famous Five Halves Theorem of J. Boardman, announced in 1967, establishes that, if F is nonbounding, then $m \leq \frac{5}{2}n$. In this paper we obtain an improvement of the Five Halves Theorem when the top dimensional component of F, F^n, is nonbounding. Specifically, let $\omega = (i_1, i_2, ..., i_r)$ be a non-dyadic partition of n and $s_\omega(x_1, x_2, ..., x_n)$ the smallest symmetric polynomial over Z_2 on degree one variables $x_1, x_2, ..., x_n$ containing the monomial $x_1^{i_1} x_2^{i_2} ... x_r^{i_r}$. Write $s_\omega(F^n) \in H^n(F^n, Z_2)$ for the usual cohomology class corresponding to $s_\omega(x_1, x_2, ..., x_n)$, and denote by $\ell(F^n)$ the minimum length of a non-dyadic partition ω with $s_\omega(F^n) \neq 0$ (here, the length of $\omega = (i_1, i_2, ..., i_r)$ is r). We will prove that, if (M^m, T) is an involution for which the top dimensional component of the fixed point set, F^n, is nonbounding, then $m \leq 2n + \ell(F^n)$; roughly speaking, the bound for m depends on the degree of decomposability of the top dimensional component of the fixed point set. Further, we will give examples to show that this bound is best possible.

1. **Introduction**

Let (M^m, T) be a smooth involution on a closed smooth m-dimensional manifold and $F = \bigcup_{j=0}^{n} F^j$ $(n \leq m)$ its fixed point set, where F^j denotes the union of those components of F having dimension j. If F is nonbounding (which means that at least one F^j is nonbounding) then n cannot be too small with respect to m: this intriguing fact was firstly evidenced from Theorem 27.1 of the old book (1964) [5] of P. E. Conner and E. E. Floyd, which stated: for each natural number n, there exists a number $\varphi(n)$ with the property that, if

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(M^m, T) is an involution fixing $F = \bigcup_{j=0}^{n} F^j$ and if $m > \varphi(n)$, then (M^m, T) bounds equivariantly.

Later (1967), in [4], J. Boardman announced the best result in this direction in terms of generality, explicitly confirming the previous result of Conner and Floyd:

Five Halves Theorem of J. Boardman : If (M^m, T) is an involution for which F is nonbounding, then $m \leq \frac{5}{2}n$ (in fact, the first version of the Five Halves Theorem was proved under the hypothesis that M^m is nonbounding; the result in question is a consequence of the following strengthened version of this first version, obtained by C. Kosniowski and R. E. Stong in [8]: if (M^m, T) is a nonbounding involution, then $m \leq \frac{5}{2}n$). Further, this bound is best possible (more detailed proofs for the Five Halves Theorem can be found in [3] and [9]).

The aim of this paper is to obtain improvements of the Five Halves Theorem when the top dimensional component of the fixed point set, F^n, is nonbounding. To precisely state it, first recall from [19] that one has an algebraic scheme to determine the cobordism class of F^n, given by the set of *Stiefel-Whitney numbers* of F^n: write $W(F^n) = 1 + w_1 + ... + w_n$ for the total Stiefel-Whitney class of F^n. A general Stiefel-Whitney number of F^n is a modulo 2 number obtained by evaluating an n-dimensional Z_2-cohomology class of the form $w_{i_1} w_{i_2} ... w_{i_r} \in H^n(F^n, Z_2)$ (that is, with $i_1 + i_2 + ... + i_r = n$) on the fundamental homology class $[F^n] \in H_n(F^n, Z_2)$. There is an useful numerical alternative to analyze the cobordism class of F^n: from a given homogeneous symmetric polynomial over Z_2 of degree n on degree one variables $x_1, x_2, ..., x_n$, $P(x_1, x_2, ..., x_n)$, we get a cohomology class in $H^n(F^n, Z_2)$ by identifying each w_i to the ith elementary symmetric function in the variables $x_1, x_2, ..., x_n$, and next by expressing $P(x_1, x_2, ..., x_n)$ as an n-dimensional polynomial in the $w_{i's}$. For a partition $\omega = (i_1, i_2, ..., i_r)$ of n, $1 \leq r \leq n$, let $s_\omega(x_1, x_2, ..., x_n)$ be the smallest symmetric polynomial containing the monomial $x_1^{i_1} x_2^{i_2} ... x_r^{i_r}$, and denote by $s_\omega(F^n) \in H^n(F^n, Z_2)$ the cohomology class which corresponds to

$s_\omega(x_1, x_2, ..., x_n)$ through the previous procedure. Since every symmetric polynomial $P(x_1, x_2, ..., x_n)$ is a sum of such polynomials, the cobordism class of F^n is determined by the set of numbers of the form $s_\omega(F^n)[F^n]$. In fact, we do not need all of these numbers: we say that a partition $\omega = (i_1, i_2, ..., i_r)$ is *non-dyadic* if none of the i_t is of the form $2^p - 1$. From [8, Section 5], one has the following

Fact. *The cobordism class of F^n is determined by the set of numbers of the form $s_\omega(F^n)[F^n]$, with ω non-dyadic.*

If F^n is nonbounding, denote by $\ell(F^n)$ the minimum length of a non-dyadic partition ω with $s_\omega(F^n)[F^n] \neq 0$ (here, the length of $\omega = (i_1, i_2, ..., i_r)$ is r). We will prove the following

Theorem. *Let (M^m, T) be an involution with fixed point set $F = \bigcup_{j=0}^{n} F^j$, and suppose that the top dimensional component F^n is nonbounding. Then $m \leq 2n + \ell(F^n)$. Further, this bound is best possible.*

Roughly speaking, this result says that the bound for m depends on the degree of decomposability of the top dimensional component of the fixed point set. For example, if F^n is *indecomposable* , which means that its cobordism class cannot be expressed as a sum of products of lower dimensional cobordism classes, then $\ell(F^n) = 1$, and thus the above result complements and generalizes the results of the recent paper [12]. We remark that the precise statement of the Five Halves Theorem is: if (M^m, T) fixes the nonbounding $F = \bigcup_{j=0}^{n} F^j$, then: i) if $n = 2k$ with $k \geq 1$, $m \leq 5k$, and ii) if $n = 2k + 1$ with $k \geq 0$, $m \leq 5k + 2$. Among all non-dyadic partitions of n, the maximum length occurs for the partition $\omega = (2, 2, ..., 2)$ if $n = 2k$, and $\omega = (2, 2, ..., 2, 5)$ if $n = 2k + 1$; that is, $1 \leq \ell(F^{2k}) \leq k$ and $1 \leq \ell(F^{2k+1}) \leq k - 1$. Therefore, generally speaking, our theorem gives the same bound of the Five Halves Theorem for n even; however, for n odd one has the following

Corollary. *If (M^m, T) fixes $F = \bigcup_{j=0}^{n} F^j$, where F^n is nonbounding and $n = 2k + 1$, then $m \leq 5k + 1$.*

In special cases, for $n = 2k$ and $\ell(F^n) < k$, and for $n = 2k+1$ and $\ell(F^n) < k-1$, our theorem gives best possible improvements of the Five Halves Theorem.

Remark. The generality of the Five Halves Theorem, which is valid for every $n \geq 1$ and allows the possibility that fixed components of all dimensions $0 \leq j \leq n$ occur, suggests the question of finding better bounds for m (and ideally the best possible bound) when we omit some components of F and restrict the set of involved dimensions n. In this direction, C. Kosniowski and R. E. Stong showed in [8] that if (M^m, T) is an involution for which F is nonbounding and has constant dimension $= n$, then $m \leq 2n$. This bound is best possible, as can be seen by taking $(F^n \times F^n, twist)$, where F^n is any nonbounding n-dimensional manifold (with the exception of $n = 1$ and $n = 3$). In [18], D. C. Royster showed that if (M^m, T) is an involution for which F has the form $F = F^n \cup \{point\}$, where n is odd, then $m \leq n+1$. This bound is realized by the involution (RP^{n+1}, T), where RP^{n+1} is the $(n+1)$-dimensional real projective space, $T[x_0, x_1, ..., x_{n+1}] = [-x_0, x_1, ..., x_{n+1}]$ and n is odd. Extending the *two components* direction started with this result of Royster, recently some advances have been obtained in the case in which F has the form $F = F^n \cup F^j$, with $n > j$. Specifically, we find best possible bounds for $j = 0$ in [10] and [17], $j = 1$ in [6] and [7], $j = 2$ in [13], [14] and [15], and $j = n-1$ in [16]. In [6] and [7], the case $F = F^n \cup RP^j$, $n \neq j$, was also considered; more generally, in [11] we find results on the case $F = F^n \cup F^j$ with F^j indecomposable.

Remark. In all the above discussion, each j-dimensional part of the fixed point set of an involution can be assumed to be connected, since any involution is equivariantly cobordant to an involution with this property.

2. Proof of the result

Let (M^m, T) be an involution with fixed point set $F = \bigcup_{j=0}^{n} F^j$, where the top dimensional component F^n is nonbounding. Write $\ell(F^n) = r$, and take a non-dyadic partition $\omega = (i_1, i_2, ..., i_r)$ of n for which $s_\omega(F^n)[F^n] \neq 0$. As announced in Section 1, our aim is to show that $m \leq 2n + r$.

For $0 \leq j \leq n$, denote by $\eta_j \to F^j$ the normal bundle of F^j in M^m, with $dim(\eta_j) = m - j$, and by $\lambda_j \to RP(\eta_j)$ the line bundle over the projective space bundle $RP(\eta_j)$, associated to the double cover $S(\eta_j) \to RP(\eta_j)$, $S(\eta_j)$ the sphere bundle.

In general, for a given vector bundle $\eta \to F$, write

$$W(\eta) = 1 + w_1(\eta) + w_2(\eta) + ...$$

for the total Stiefel-Whitney class of η; in particular, if F is a manifold,

$$W(F) = 1 + w_1(F) + w_2(F) + ...$$

means the Stiefel-Whitney class of the tangent bundle of F.

Each F_j has an invariant tubular neighborhood N_j, the disk bundle of η_j. The boundary sphere bundle $\partial(N_j) = S(\eta_j)$ is a manifold with free involution, and the union of the $S(\eta_j)$ bounds the closure of the complement in M^m of the union of all the N_j, which is a manifold with free involution. In other words, $\bigcup_{j=0}^{n} \big(S(\eta_j), T\big)$ bounds as an element of the cobordism group $\mathcal{N}_{m-1}(BZ_2)$ of manifolds with free involution; in this way, $\bigcup_{j=0}^{n} \big(\lambda_j \to RP(\eta_j)\big)$ bounds as a line bundle. From [5], this can be expressed by the following algebraic fact: let $P(w_1, w_2, ..., w_{m-1}, c)$ be any homogeneous polynomial over Z_2 with degree $m-1$, where each variable w_i has degree i and the variable c has degree 1. For each $0 \leq j \leq n$, we can evaluate the cohomology class

$$P\Big(w_1(RP(\eta_j)), w_2(RP(\eta_j)), ..., w_{m-1}(RP(\eta_j)), w_1(\lambda_j)\Big) \in H^{m-1}\Big(RP(\eta_j), Z_2\Big)$$

on the fundamental homology class $\big[RP(\eta_j)\big] \in H_{m-1}\big(RP(\eta_j), Z_2\big)$, thus getting a modulo 2 number (called a *characteristic number*),

$$P\Big(w_1(RP(\eta_j)), w_2(RP(\eta_j)), ..., w_{m-1}(RP(\eta_j)), w_1(\lambda_j)\Big)\Big[RP(\eta_j)\Big] \in Z_2.$$

Then

$$\sum_{j=0}^{n} P\Big(w_1(RP(\eta_j)), w_2(RP(\eta_j)), ..., w_{m-1}(RP(\eta_j)), w_1(\lambda_j)\Big)\Big[RP(\eta_j)\Big] = 0. \quad (***)$$

The key point will consist in applying the above fact choosing a suitable polynomial. To do this, we introduce a special cohomology class associated to line bundles $\lambda \to W$, where W is a closed $(m-1)$-dimensional manifold; we will use the non-dyadic partition $\omega = (i_1, i_2, ..., i_r)$ of n, mentioned in the beginning of the section.

Let $F_\omega(z_1, z_2, ..., z_m)$ be the smallest polynomial on degree one variables z_1, z_2, ..., z_m, which is symmetric in the variables $z_1, z_2, ..., z_{m-1}$ and contains the polynomial

$$z_1^{i_1}(z_m + z_1)^{i_1+1} z_2^{i_2}(z_m + z_2)^{i_2+1} ... z_r^{i_r}(z_m + z_r)^{i_r+1}.$$

As in Section 1, we then identify $w_1(\lambda)$ to z_m and each $w_i(M)$ to the ith elementary symmetric function in the variables $z_1, z_2, ..., z_{m-1}$; next, we express $F_\omega(z_1, z_2, ..., z_m)$ as a polynomial of dimension $2n+r$ in the $w_{i's}(M)$ and $w_1(\lambda)$. This class will be denoted by $F_\omega(\lambda)$.

The crucial point is that $F_\omega(\lambda)$ has a nice behavior with respect to the standard line bundles over projective space bundles. To see this, we will use the *splitting principle,* which allows to write the Stiefel-Whitney class of any k-dimensional vector bundle formally as

$$1 + w_1 + w_2 + ... + w_k = (1 + y_1)(1 + y_2)...(1 + y_k),$$

where each y_i has degree one, and effectively to see each w_i as the ith elementary symmetric function in the variables $y_1, y_2, ..., y_k$.

Let η be a k-dimensional vector bundle over a closed j-dimensional manifold F, where $j + k = m$, and let $\lambda \to RP(\eta)$ be the standard line bundle; set $w_1(\lambda) = c$. From [2; page 517] one has that

$$W\big(RP(\eta)\big) = W(F).W(\eta \otimes \lambda) = \big(1 + w_1(F) + w_2(F) + ... + w_j(F)\big)\big((1+c)^k + (1+c)^{k-1}w_1(\eta) + ... + (1+c)w_{k-1}(\eta) + w_k(\eta)\big),$$

where we are suppressing bundle maps. Using the splitting principle, write

$$W(F) = (1 + x_1)(1 + x_2)...(1 + x_j)$$

and

$$W(\eta) = (1 + y_1)(1 + y_2)...(1 + y_k).$$

Then

$$W\big(RP(\eta)\big)=(1+x_1)(1+x_2)...(1+x_j)(1+c+y_1)(1+c+y_2)...(1+c+y_k).$$

To calculate $F_\omega(\lambda)$, select $z_{e_1},\ z_{e_2},...,z_{e_r}$ from $z_1=x_1,...,z_j=x_j,\ z_{j+1}=c+y_1,...,z_{j+k}=c+y_k$; we must analyse the corresponding term

$$z_{e_1}^{i_1}(c+z_{e_1})^{i_1+1}z_{e_2}^{i_2}(c+z_{e_2})^{i_2+1}...z_{e_r}^{i_r}(c+z_{e_r})^{i_r+1}$$

of $F_\omega(\lambda)$. The product

$$z_{e_h}^{i_h}(c+z_{e_h})^{i_h+1}$$

contains a factor $x_t^{i_h}$ if $z_{e_h}=x_t$, or $y_s^{i_h+1}$ if $z_{e_h}=y_s$. Then every term of $F_\omega(\lambda)$ contains at least n factors x_t or y_s from $H^1(F)$, which implies that $F_\omega(\lambda)=0$ if $j<n$. If $j=n$, the only terms that survive are those where z_{e_h} is taken from $\{x_1,...,x_n\}$, and so $F_\omega(\lambda)=s_\omega(F)c^{n+r}$.

Returning to the line bundles $\lambda_j\to RP(\eta_j)$ coming from the fixed data of (M^m,T), set $w_1(\lambda_j)=c_j$. By contradiction, suppose $m>2n+r$. Then $m-1\geq 2n+r$, and thus it makes sense to consider the polynomial $F_\omega(\lambda_j).c_j^{m-1-2n-r}$ of degree $m-1$, which is zero if $j<n$. Thus equation (***) gives that

$$\sum_{j=0}^{n}F_\omega(\lambda_j).c_j^{m-1-2n-r}\big[RP(\eta_j)\big]=s_\omega(F^n).c_n^{m-1-n}\big[RP(\eta_n)\big]=0.$$

From the Leray-Hirsch Theorem (see [1]; pag. 129) $H^*\big(RP(\eta_n),Z_2\big)$ is the free $H^*(F^n,Z_2)$-module on $1,c_n,c_n^2,...,c_n^{m-1-n}$. Then

$$s_\omega(F^n).c_n^{m-1-n}\big[RP(\eta_n)\big]=s_\omega(F^n)[F^n].$$

Since $s_\omega(F^n)\neq 0$, this gives the desired contradiction.

3. Maximal examples

In this section we will construct involutions to show that the result of Section 2 is best possible. As mentioned in Section 1, our result complements and generalizes the results of the recent paper [12], where it was shown that, if (M^m,T) is an involution for which the top dimensional component of the fixed point set, F^n, is indecomposable, then $m\leq 2n+1$. In fact, in [19, page 79] R. Thom showed that F^n is indecomposable if and only if $s_n(F^n)=s_\omega(F^n)$ is nonzero, where ω is the trivial partition, $\omega = (n)$. In this setting, in [12] we constructed, for each $n \geq 2$ not of the form $2^t - 1$, a special involution (W^{2n+1}, T_n) so that the dimension of the top dimensional component of the fixed point set is n and with this top dimensional component being indecomposable, thus showing that the bound of [12] is best possible (we recall that indecomposable n-dimensional manifolds occur only for these values of n). Here, our examples will be simply products of the examples of [12]. In fact, first suppose that $F_1, F_2, .., F_r$ are indecomposable manifolds, with $dim(F_i) = n_i$ for $i = 1, 2, ..., r$. Then, by using the splitting principle and dimensional considerations, we can see that $s_\omega(F_1 \times F_2 \times ... \times F_r) = s_{n_1}(F_1).s_{n_2}(F_2)...s_{n_r}(F_r) \neq 0$, where $\omega = (n_1, n_2, ..., n_r)$. Further, if $\omega = (j_1, j_2, ..., j_t)$ is a non-dyadic partition of $n_1 + n_2 + ... + n_r$ with $t < r$, then $s_\omega(F_1 \times F_2 \times ... \times F_r) = 0$. This means that $\ell(F_1 \times F_2 \times ... \times F_r) = r$. Now take $\omega = (i_1, i_2, ..., i_r)$ a non-dyadic partition of n. Since each i_j is not of the form $2^t - 1$, one has the example (W^{2i_j+1}, T_{i_j}), with the top dimensional component of the fixed point set, F^{i_j}, being indecomposable. The product involution

$$(W^{2i_1+1}, T_{i_1}) \times (W^{2i_2+1}, T_{i_2}) \times ... \times (W^{2i_r+1}, T_{i_r})$$

has the n-dimensional manifold $F^{i_1} \times F^{i_2} \times ... \times F^{i_r}$ as the top dimensional component of its fixed point set. Since $\ell(F^{i_1} \times F^{i_2} \times ... \times F^{i_r}) = r$ and $(2i_1+1) + (2i_2+1) + ... + (2i_r+1) = 2n+r$, $(W^{2i_1+1}, T_{i_1}) \times (W^{2i_2+1}, T_{i_2}) \times ... \times (W^{2i_r+1}, T_{i_r})$ is the desired maximal example.

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References

[1] Borel, A.: *Sur la cohomologie des espaces fibrés principaux et des espaces homogènes des groupes de Lie compacts*, Ann. of Math. 57, n. 2, (1953), 115-207

[2] Borel, A. and Hirzebruch, F.: *Characteristic classes and homogeneous spaces*, I, Amer. J. Math. 80, (1958), 458-538.

[3] Boardman, J. M.: *Cobordism of involutions revisited*, Proceedings of the Second Conference on Compact Transformation Groups, Univ. Massachusetts, Amherst, Mass., Part I, (1971), 131-151. Lecture Notes in Math., Vol. 298, Springer, Berlin (1972).

[4] Boardman, J. M.: *On manifolds with involution*, Bulletin Amer. Math. Soc. 73, (1967), 136-138.

[5] Conner, P. E. and Floyd, E. E. *Differentiable periodic maps*, Springer-Verlag, Berlin, (1964).
[6] Kelton, S. M.: *Involutions fixing* $\mathbb{R}P^j \cup F^n$, Topology Appl. 142, (2004), 197-203.
[7] Kelton, S. M.: *Involutions fixing* $\mathbb{R}P^j \cup F^n$*, II.*, Topology Appl. 149, n. 1-3, (2005), 217-226.
[8] Kosniowski, C. and Stong, R. E.: *Involutions and characteristic numbers*, Topology 17, (1978), 309-330.
[9] Morava, J.: *Cobordism of involutions revisited, revisited. Homotopy invariant algebraic structures*, Baltimore, MD, (1998), 15-18. Contemp. Math., 239, Amer. Math. Soc., Providence, RI, (1999).
[10] Pergher, P. L. Q.: *Bounds on the dimension of manifolds with certain* Z_2 *fixed sets*, Mat. Contemp. 13, (1996), 269-275.
[11] Pergher, P. L. Q.: *Involutions fixing* $F^n \cup \{Indecomposable\}$, Canadian Math. Bulletin , to appear.
[12] Pergher, P. L. Q.: *Involutions whose top dimensional component of the fixed point set is indecomposable*, Geom. Dedicata , (2009), DOI 10.1007/s10711-009-9419-5.
[13] Pergher, P. L. Q. and Figueira, F. G.: *Dimensions of fixed point sets of involutions* , Arch. Math. (Basel) 87, (2006), n. 3, 280-288.
[14] Pergher, P. L. Q. and Figueira, F. G.: *Bounds on the dimension of manifolds with involution fixing* $F^n \cup F^2$, Glasg. Math. J. 50, n. 3, (2008), 595-604.
[15] Pergher, P. L. Q. and Figueira, F. G.: *Involutions fixing* $F^n \cup F^2$, Topology Appl. 153, (2006), n. 14, 2499-2507.
[16] Pergher, P. L. Q. and Figueira, F. G.: *Two commuting involutions fixing* $F^n \cup F^{n-1}$, Geom. Dedicata 117, (2006), 181-193.
[17] Pergher, P. L. Q. and Stong, R. E.: *Involutions fixing* $\{point\} \cup F^n$, Transformation Groups 6, (2001), 78-85.
[18] Royster, D. C.: *Involutions fixing the disjoint union of two projective spaces*, Indiana Univ. Math. J. 29, n. 2, (1980), 267-276.
[19] Thom, R.: *Quelques propriétés globales des variétés différentiables*, Comment. Math. Helv. 28, (1954), 17-86.

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