ADMISSIBLE WEAK SOLUTION OF A CONSERVATION LAW BY GLIMM METHOD AND THE POLYGONAL APPROXIMATIONS OF SOLUTIONS

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Abstract. In this paper, we propose an alternative construction based on the Glimm's random choice method [G] and the polygonal approximations of solutions by Dafermos [D] to get an admissible weak solution to the single consevation law.

1. Introduction

We consider the Cauchy problem for a scalar conservation law

$$\begin{cases} u_t + f(u)_x = 0, & (x,t) \in \mathbb{R} \times [0,\infty), \\ u(x,0) = u_0(x), & x \in \mathbb{R}, \end{cases} \tag{1}$$

where $f : \mathbb{R} \to \mathbb{R}$ is locally Lipschitz continuous on $\mathbb{R}$ and $u_0(.)$ is bounded, continuous on the left and of locally bounded variation on $\mathbb{R}$.

Since u_0 is bounded and f is locally Lipschitz continuous, we can suppose f is Lipschitz on $[-M, M]$, where $||u_0||_\infty \leq M$. Without loss of generality, we can assume that M is a positive interger.

In general no classical solution of (1) exists even if f and u_0 are smooth. It has been demonstrated that it is possible to establish the existence of weak solutions by many methods, see for example, [B], [B-B], [D], [S] and their references. For more details about the Glimm method see also [G], [L], [H-L] and their references.

The weak solution is not necessarily unique. To attain uniqueness one usually imposes additional restrictions which are motivated by stability arguments or by physical considerations (whenever (1) is studied in connection with a physical model).

In section 2, we will construct a sequence of approximate weak solutions using the Glimm's random choice method [G] and the polygonal approximations of solutions by Dafermos [D]. In section 3, we will use the convergence of the Glimm method to obtain a subsequence convergence and we will show that the limit of this subsequence is an admissible weak solution of (1).

2. Aproximation Solution

An admissible weak solution of the Cauchy problem (1) is defined as in [D]:

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Definition 1. A locally bounded and measurable function $u(x,t)$ on $\mathbb{R}\times[0,\infty)$ is called an admissible weak solution of (1), if for any nondecreasing function $h(u)$ and any smooth nonnegative function $\phi(x,t)$ with compact support in $\mathbb{R}\times[0,\infty)$,

$$\int_0^\infty\int_{-\infty}^\infty (I(u)\phi_t+F(u)\phi_x)\,dx\,dt+\int_{-\infty}^\infty I(u_0(x))\phi(x,0)\,dx\geq 0, \tag{2}$$

where

$$I(u)\equiv\int_{-M}^u h(\xi)\,d\xi$$

and

$$F(u)\equiv\int_{-M}^u h(\xi)\,df(\xi).$$

Remark 1. *An application of (2) for* $h(u)=1$ *and* $h(u)=-1$ *yields*

$$\int_0^\infty\int_{-\infty}^\infty (u\phi_t+f(u)\phi_x)\,dx\,dt+\int_{-\infty}^\infty u_0(x)\phi(x,0)\,dx=0. \tag{3}$$

which is the standard condition satisfied by any weak solution of (1).

Remark 2. *If* $u(x,t)$ *is defined on* $\mathbb{R}\times[0,T)$ *and (2) is satisfied for every nonnegative* $\phi(x,t)$ *with compact support in* $\mathbb{R}\times[0,T)$*, then* $u(x,t)$ *is called a local admissible weak solution of (1) on* $\mathbb{R}\times[0,T)$*, according to* [D].

Initially, we assume that u_0 is of bounded variation on $\mathbb{R}$.

Fix an integer $k\geq 1$. Let $P=\{x_0=-M<x_1<...<x_s=M\}$ be the partition of $[-M,M]$ such that $x_j-x_{j-1}=2^{-k}$, $j=1,2,...,s$, $(s=2^{k+1}M)$. Let f_k be the piecewise linear function which coincides with f at all points $x_j\in P$, i.e.

$$f_k(r)=\frac{r-x_{j-1}}{2^{-k}}f(x_j)+\frac{x_j-r}{2^{-k}}f(x_{j-1}), \tag{4}$$

where $r\in[x_{j-1},x_j]$, $j=1,2,...,s$.

Let K be the Lipschitz constant such that

$$|f(w)-f(z)|\leq K|w-z|,\ \forall\ w,z\in[-M,M], \tag{5}$$

hence

$$|f_k(w)-f_k(z)|\leq K|w-z|,\ \forall\ w,z\in[-M,M], \tag{6}$$

$\forall\ k\in\mathbb{N}$. Without loss of generality, we can assume that $K\geq 1$.

For each fixed $k\geq 1$, we consider the conservation law

$$u_t+f_k(u)_x=0,\quad (x,t)\in\mathbb{R}\times[0,\infty), \tag{7}$$

First, we choose mesh lengths $\triangle x_k$ and $\triangle t_k$ such that the stability condition $\frac{\triangle x_k}{\triangle t_k}=C>K$ holds, where C is constant and independent of k. We assume that $\triangle x_k=2^{-k}$. Let

$$Y\equiv\{(m,n)\in\mathbb{Z}^2: m+n\equiv 0(mod 2), n\geq 0\}$$

and

$$\Phi \equiv \Pi_{(m,n)\in Y}\{[(m-1)\triangle x_k,(m+1)\triangle x_k]\times\{n\triangle t_k\}\}.$$

We choose a point $(m\triangle x_k+\theta_n\triangle x_k, n\triangle t_k)_{(m,n)\in Y}\in\Phi$ where θ_n is randomly chosen in $[-1,1]$. As in [S], there is an isomorphism, $\Phi\approx\prod[0,1]$, of Φ with a countable product of copies of the unit interval, so we can consider the random points θ being defined in a fixed probability space Φ, independent of $\triangle x_k$.

Now, we consider the initial data

$$(8)\qquad u_0^{(k,m)}(x,0)=\begin{cases} u_0((m-1+\theta_0)\triangle x_k), & (m-1)\triangle x_k < x\le m\triangle x_k\\ u_0((m+1+\theta_0)\triangle x_k), & m\triangle x_k < x\le (m+1)\triangle x_k,\end{cases}$$

where $m\in\mathbb{Z}$, such that m is even.

In [D], we have

Lemma 1. *If f is piecewise linear, satisfies (5), and*

$$\bar{u}_0(x)\equiv\begin{cases} u_l, & x\in(-\infty,y],\\ u_r, & x\in(y,\infty),\end{cases}$$

where u_l and u_r are constants in $[-M,M]$ and y is constant in $\mathbb{R}$, then there exists an admissible weak solution u of (1), continuous on the left, that consists of a finite number of constant states separated by shock waves centered at y. Moreover, $TV(u(\,.\,,t))\le|u_r-u_l|$, $\forall\, t\in[0,\infty)$, and $|u(x,t)|\le||\bar{u}_0||_\infty$, $\forall\,(x,t)\in\mathbb{R}\times[0,\infty)$.

Remark 3. *The values intermediaries that the admissible weak solution assume are always vertices of the polygonal f_k.*

Remark 4. *The admissible weak solution is always increasing or decreasing. Moreover, each shock wave $x=\tilde{x}(t)$, $t\in(0,\triangle t_k)$, is a straight line with slope*

$$\frac{d\tilde{x}}{dt}=\frac{f_k(u^+)-f_k(u^-)}{u^+-u^-},$$

and for any u between u^- and u^+,

$$\frac{f_k(u^+)-f_k(u)}{u^+-u}\le\frac{f_k(u^+)-f_k(u^-)}{u^+-u^-},$$

where $u^+=\lim_{x\to\tilde{x}(t)^+}u(x,t)$ and $u^-=\lim_{x\to\tilde{x}(t)^-}u(x,t)$.

For each fixed $k\in\mathbb{N}$ and for each fixed $m\in\mathbb{Z}$, such that m is even, the Riemann problem

$$v_t+f_k(v)_x=0,\quad (x,t)\in\mathbb{R}\times[0,\infty),$$

with initial data

$$v(x,0)=\begin{cases} u_0((m-1+\theta_0)\triangle x_k), & x\in(-\infty,m\triangle x_k]\\ u_0((m+1+\theta_0)\triangle x_k), & x\in(m\triangle x_k,\infty),\end{cases}$$

is solved by Lemma 1. Let $v_{k,m,0}$ be the admissible weak solution this problem, given by Lemma 1.

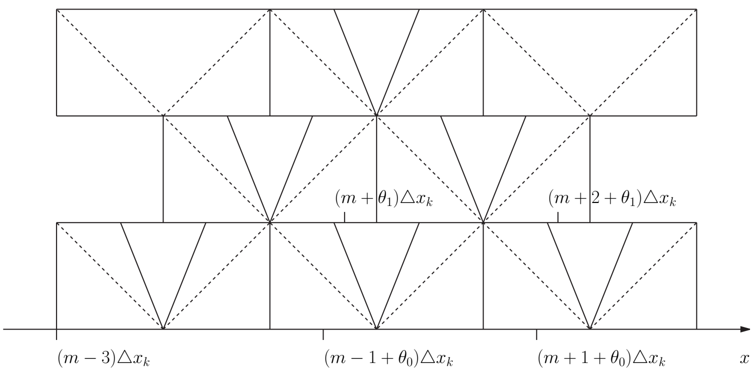

FIGURE 1. Model to construct of the function $u_{k,n}$.

Now, for each fixed $k \in \mathbb{N}$, we define

$$u_{k,1}(x,t) \equiv v_{k,m,0}(x,t), \quad (m-1)\triangle x_k \le x \le (m+1)\triangle x_k, \quad 0 \le t < \triangle t_k,$$

where $m \in \mathbb{Z}$ is even.

In view of our stability condition $\frac{\triangle x_k}{\triangle t_k} = C > K$, the waves don't interact with each other across the lines $x = (m-1)\triangle x_k$.

Thus $u_{k,1}$ is a step function defined in the strip $0 \le t < \triangle t_k$, constant across the lines $x = (m-1)\triangle x_k$, $\forall m \in \mathbb{Z}$ even, and continuous on the left. According to Remark 2, $u_{k,1}$ is a local admissible weak solution of (7) and (8), $\forall m \in \mathbb{Z}$ even.

We also have that $TV(u_{k,1}(\,.\,,t)) \le TV(u_0(.))$, $\forall\, t \in [0, \triangle t_k)$, and $|u_{k,1}(x,t)| \le ||u_0||_\infty$, $\forall\,(x,t) \in \mathbb{R} \times [0, \triangle t_k)$.

Inductively, we assume that $u_{k,n}$ had been defined in the strip $\mathbb{R} \times [(n-1)\triangle t_k, n\triangle t_k)$. (Fig. 1).

In analogous way, we define $u_{k,n+1}$ to solve the Riemann problem

$$v_t + f_k(v)_x = 0, \quad (x,t) \in \mathbb{R} \times [n\triangle t_k, \infty),$$

with initial data

$$v(x, n\triangle t_k) = \begin{cases} u_{k,n}((m-1+\theta_n)\triangle x_k, n\triangle t_k^-), & x \in (-\infty, m\triangle x_k], \\ u_{k.n}((m+1+\theta_n)\triangle x_k, n\triangle t_k^-), & x \in (m\triangle x_k, \infty), \end{cases}$$

where $(m,n) \in Y$ and $\triangle t_k^- = \lim_{t \to \triangle t_k^-} t$. Let $v_{k,m,n}$ the admissible weak solution this problem, given by Lemma (1). (See fig (1)).

We define $u_{k,n+1} \equiv v_{k,m,n}$ in the strip $\mathbb{R} \times [n\triangle t, (n+1)\triangle t_k)$. Hence, we have that $TV(u_{k,n+1}(\,.\,,t)) \le TV(u_0(.))$, $\forall\, t \in [n\triangle t_k, (n+1)\triangle t_k)$, and $|u_{k,n+1}(x,t)| \le ||u_0||_\infty$, $\forall\,(x,t) \in \mathbb{R} \times [n\triangle t_k, (n+1)\triangle t_k)$, $\forall\, n = 0, 1, 2, ...$

In particular, $u_{k,n+1}$ is piecewise constant, continuous on the left, with jumps occurring along of straing lines in the strip $\mathbb{R} \times [n\triangle t_k, (n+1)\triangle t_k)$, for each fixed $n \in \mathbb{N}$.

For each fixed $k \in \mathbb{N}$ and for each fixed $\theta \in \Phi$, we define

$$u_{(\theta,k)}(x,t) \equiv u_{(\theta,\triangle x_k)}(x,t) \equiv u_{k,n}(x,t), \quad (x,t) \in \mathbb{R} \times [(n-1)\triangle t_k, n\triangle t_k),$$

$n \in \mathbb{N}$.

Remark 5. *Since* $\frac{\triangle x_k}{\triangle t_k} = C > K$, *the restriction of* $u_{(\theta,k)}(.,t)$ *on any interval* $[x_1, x_2]$, *is solely determined by the restriction of* $u_0(.)$ *on the interval* $[x_1 - (1+Kt), x_2 + (1+Kt)]$ *(finite domain of dependence) and*

$$TV_{[x_1,x_2]}(u_{(\theta,k)}(.,t)) \leq TV_{[x_1-(1+Kt),x_2+(1+Kt)]}(u_0(.)).$$

Lemma 2. *For each fixed* $k \in \mathbb{N}$, *for each nondecreasing function* $h(u)$ *and each nonnegative function* $\phi \in C_0^1(t \geq 0)$, *we have that, in each strip* $\mathbb{R} \times (l\triangle t_k, (l+1)\triangle t_k)$,

$$\int_{l\triangle t}^{(l+1)\triangle t} \int_{-\infty}^{\infty} (I(u_{(\theta,k)})\phi_t + F_k(u_{(\theta,k)})\phi_x)\, dx\, dt$$
$$+ \int_{-\infty}^{\infty} I(u_{(\theta,k)}(x, l\triangle t_k^+))\phi(x, l\triangle t_k)\, dx$$
$$- \int_{-\infty}^{\infty} I(u_{(\theta,k)}(x,(l+1)\triangle t_k^-))\phi(x,(l+1)\triangle t_k)\, dx \geq 0,$$

where

$$F_k(u) \equiv \int_{-M}^{u} h(\xi)\, df_k(\xi).$$

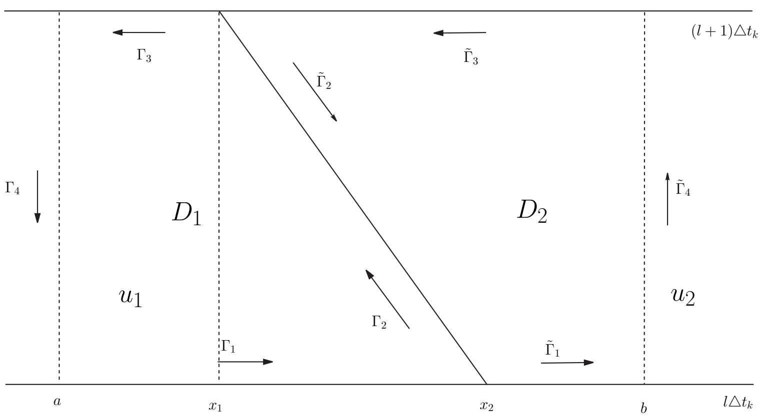

FIGURE 2. Model of local admissible weak solution in the strip $\mathbb{R} \times (l\triangle t_k, (l+1)\triangle t_k)$.

Proof. Indeed, without loss of generality, we assume that in the strip $\mathbb{R} \times (l\triangle t_k, (l+1)\triangle t_k)$, there exists only a discontinuity $\tilde{x}(t)$, with slope $\alpha = \frac{f_k(u_2)-f_k(u_1)}{u_2-u_1}$, according to [D], connecting the states u_1 and u_2, (Fig. 2). In this case, we observe that the line of discontinuity $\tilde{x}(t)$ subdivide the strip $\mathbb{R} \times (l\triangle t_k, (l+1)\triangle t_k)$ into two disjoint regions D_1 and D_2.
We assume that the compact support of ϕ is in $[a,b] \times [0,\infty)$. We notice that $\phi(a,t) = \phi(b,t) = 0$, $\forall\, t \in [0,\infty)$. The oriented boundary of D_1 is parametrized by $\Gamma_1(s) = (s, l\triangle t_k)$ where $a \le s \le x_2$, $\Gamma_2(s) = (\tilde{x}(s), s)$ where $l\triangle t_k \le s \le (l+1)\triangle t_k$ and $\Gamma_3(s) = ((x_1+a)-s, (l+1)\triangle t_k)$ where $a \le s \le x_1$. In analogous way, the oriented boundary of D_2 is parametrized by $\tilde{\Gamma}_1(s) = (s, l\triangle t_k)$ where $x_2 \le s \le b$, $\tilde{\Gamma}_2(s) = (\tilde{x}((2l+1)\triangle t_k - s), (2l+1)\triangle t_k - s)$ where $l\triangle t_k \le s \le (l+1)\triangle t_k$ and $\tilde{\Gamma}_3(s) = ((b+x_1)-s, (l+1)\triangle t_k)$ where $x_1 \le s \le b$.
Integrating, we have

$$\int_{l\triangle t_k}^{(l+1)\triangle t_k} \int_{-\infty}^{\infty} (I(u_{(\theta,k)})\phi_t + F_k(u_{(\theta,k)})\phi_x)\, dx\, dt = \int\int_{D_1} (I(u_1)\phi_t + F_k(u_1)\phi_x)\, dx\, dt + \int\int_{D_2} (I(u_2)\phi_t + F_k(u_2)\phi_x)\, dx\, dt.$$

By Green's theorem, we have

$$\int\int_{D_1} (I(u_1)\phi_t + F_k(u_1)\phi_x)\, dx\, dt = -\int_a^{x_2} I(u_1)\phi(x, l\triangle t_k^+)\, dx + \int_a^{x_1} I(u_1)\phi(x,(l+1)\triangle t_k^-)\, dx + \int_{l\triangle t_k}^{(l+1)\triangle t_k} [-\alpha I(u_1) + F_k(u_1)]\phi(\tilde{x}(s), s)\, ds$$

and

$$\int\int_{D_2} (I(u_1)\phi_t + F_k(u_2)\phi_x)\, dx\, dt = -\int_{x_2}^{b} I(u_2)\phi(x, l\triangle t_k^+)\, dx + \int_{x_1}^{b} I(u_2)\phi(x,(l+1)\triangle t_k^-)\, dx + \int_{l\triangle t_k}^{(l+1)\triangle t_k} [\alpha I(u_2) - F_k(u_2)]\phi(\tilde{x}(s), s)\, ds.$$

Therefore,

$$\int_{l\triangle t_k}^{(l+1)\triangle t_k}\int_{-\infty}^{\infty}(I(u_{(\theta,k)})\phi_t + F_k(u_{(\theta,k)})\phi_x)\,dx\,dt =$$

$$-\int_a^b I(u_{(\theta,k)}(x,l\triangle t_k^+))\phi(x,l\triangle t_k)\,dx$$

$$+\int_a^b I(u_{(\theta,k)}(x,(l+1)\triangle t_k^-))\phi(x,(l+1)\triangle t_k)\,dx$$

$$+\int_{l\triangle t_k}^{(l+1)\triangle t_k}[\alpha(I(u_2)-I(u_1)) + (F_k(u_1)-F_k(u_2))]\phi(\tilde{x}(s),s)\,dt.$$

Since $\alpha = \frac{f_k(u_2)-f_k(u_1)}{u_2-u_1}$, according to [D], so by Lemma 6.1 of [G-R] we have that

$$\frac{f_k(u_2)-f_k(u_1)}{u_2-u_1}\,(I(u_2)-I(u_1)) + (F_k(u_1)-F_k(u_2)) \geq 0.$$

It follows that

$$\int_{l\triangle t_k}^{(l+1)\triangle t_k}\int_{-\infty}^{\infty}(I(u_{(\theta,k)})\phi_t + F_k(u_{(\theta,k)})\phi_x)\,dx\,dt$$

$$+\int_{-\infty}^{\infty} I(u_{(\theta,k)}(x,l\triangle t_k^+))\phi(x,l\triangle t_k)\,dx$$

$$-\int_{-\infty}^{\infty} I(u_{(\theta,k)}(x,(l+1)\triangle t_k^-))\phi(x,(l+1)\triangle t_k)\,dx \geq 0.$$

□

3. Convergence

By constructing, the sequence of functions $(u_{(\theta,k)})$ satisfies

$$||u_{(\theta,k)}(\,.\,,\,.\,)||_\infty \leq ||u_0||_\infty = M_1, \tag{9}$$

$$TV(u_{(\theta,k)}(\,.\,,t)) \leq TV(u_0(.)) = M_2, \tag{10}$$

where M_1 is independent of θ and k, and M_2 is independent of θ, k and t.

Lemma 3. *Let $t_1, t_2 \geq 0$, then*

$$\int_{-\infty}^{\infty}|u_{(\theta,k)}(x,t_1) - u_{(\theta,k)}(x,t_2)|\,dx \leq M_3[|t_1-t_2| + \triangle t_k]. \tag{11}$$

where M_3 is independent of θ, k, t_1 and t_2.

The proof is analogous as the corresponding result in [S], Corollary 19.8. Since $(u_{(\theta,k)})$ satisfies (9), (10) and (11), we have that

Theorem 2. *There is a subsequence $(u_{(\theta,k_j)}) \subset (u_{(\theta,k)})$ convergent in $L^1_{loc}(\mathbb{R}\times[0,\infty);\mathbb{R})$ to a function u_θ, for each fixed $\theta \in \Phi$.*

The proof is carried out in the same way as [S].

Now, we must use the Glimm method to show that the limit function is an admissible weak solution of (1).

Let $\theta \in \Phi$, given any nondecreasing function $h(u)$ and any nonnegative function $\phi \in C_0^1(t \geq 0)$, we define a functional $\mathcal{L}_\phi$ by

(12)
$$\mathcal{L}_\phi(I(u), F(u)) \equiv \int_0^\infty \int_{-\infty}^\infty (I(u)\phi_t + F(u)\phi_x)\, dx\, dt + \int_{-\infty}^\infty I(u_0(x))\phi(x,0)\, dx.$$

Our goal is to produce a u_θ for which $\mathcal{L}_\phi(I(u_\theta), F(u_\theta)) \geq 0$ for each nondecreasing function $h(u)$ and each nonnegative function $\phi \in C_0^1(t \geq 0)$.

For each fixed $k \in \mathbb{N}$, for each nondecreasing function $h(u)$ and each nonnegative function $\phi \in C_0^1(t \geq 0)$, the function $u_{(\theta,k)}$ satisfies

$$\int_{l\triangle t_k}^{(l+1)\triangle t_k} \int_{-\infty}^\infty (I(u_{(\theta,k)})\phi_t + F_k(u_{(\theta,k)})\phi_x)\, dx\, dt + \int_{-\infty}^\infty I(u_{(\theta,k)}(x, l\triangle t_k^+))\phi(x, l\triangle t_k)\, dx - \int_{-\infty}^\infty I(u_{(\theta,k)}(x,(l+1)\triangle t_k^-))\phi(x,(l+1)\triangle t_k)\, dx \geq 0.$$

in the strip $l\triangle t_k \leq t < (l+1)\triangle t_k$, for each $l \in \mathbb{N}$, according to Lemma 2.

If we sum this over l we get

$$\int_0^\infty \int_{-\infty}^\infty (I(u_{(\theta,k)})\phi_t + F_k(u_{(\theta,k)})\phi_x)\, dx\, dt + \int_{-\infty}^\infty I(u_{(\theta,k)}(x,0))\phi(x,0)\, dx + \sum_{l=1}^\infty \int_{-\infty}^\infty [I(u_{(\theta,k)})](x, l\triangle t_k)\phi(x, l\triangle t_k)\, dx \geq 0,$$

where $[I(u_{(\theta,k)})](x, l\triangle t_k) \equiv I(u_{(\theta,k)}(x, l\triangle t_k^+)) - I(u_{(\theta,k)}(x, l\triangle t_k^-))$.

Now, we have that

Lemma 4. *For each nondecreasing function $h(u)$, and each nonnegative function $\phi \in C_0^1(t \geq 0)$, we have*

$$\int\int_{t\geq 0} (I(u_{(\theta,k)})\phi_t + F_k(u_{(\theta,k)})\phi_x)\, dx\, dt \to \int\int_{t\geq 0} (I(u_\theta)\phi_t + F(u_\theta)\phi_x)\, dx\, dt,$$

when $k \to \infty$.

Proof. Indeed, following [D], let $k \in \mathbb{N}$ be fixed, if $u \in [-M, M]$, then there exist $p \in P$ such that $|u - p| \leq 2^{-k}$, so

$$|f(u) - f_k(u)| \leq |f(u) - f_k(p)| + |f_k(p) - f_k(u)| \leq 2K|u-p| \leq \frac{K}{2^{k-1}}. \tag{13}$$

We notice that

$$|F_k(u_{(\theta,k)}) - F(u_\theta)| \leq |F_k(u_{(\theta,k)}) - F_k(u_\theta)| + |F_k(u_\theta) - F(u_\theta)|. \tag{14}$$

We set $H \equiv \max\{|h(-M)|, |h(M)|\}$, so

$$|I(u_{(\theta,k)}) - I(u_\theta)| \leq \left|\int_{u_\theta}^{u_{(\theta,k)}} h(\xi)\, d\xi\right| \leq H|u_{(\theta,k)} - u_\theta|, \tag{15}$$

$$\begin{aligned}
|F(u_\theta) - F_k(u_\theta)| &= \left|\int_{-M}^{u_\theta} h(\xi)\, df(\xi) - \int_{-M}^{u_\theta} h(\xi)\, df_k(\xi)\right| \\
&\leq \left|\int_{-M}^{u_\theta} h(\xi)\, d(f(\xi) - f_k(\xi))\right| \\
&\leq |h(u)(f(u_\theta) - f_k(u_\theta))| + \left|\int_{-M}^{u_\theta} (f(\xi) - f_k(\xi))\, dh(\xi)\right| \\
&\leq \frac{K}{2^{k-1}}(H + 2H) \\
&\leq \frac{3HK}{2^{k-1}}, \qquad (16)
\end{aligned}$$

$$|F_k(u_{(\theta,k)}) - F_k(u_\theta)| = \left|\int_{u_\theta}^{u_{(\theta,k)}} h(\xi)\, df_k(\xi)\right| \leq H\, K\, |u_{(\theta,k)} - u_\theta|. \tag{17}$$

Let $B \subset \mathbb{R} \times [0, \infty)$ be a subset compact. From (13)-(17) we have

$$\int\int_B |I(u_{(\theta,k)}) - I(u_\theta)|\, dx\, dt \leq H \int\int_B |u_{(\theta,k)} - u_\theta|\, dx\, dt \to 0,$$

when $k \to \infty$, and

$$\begin{aligned}
\int\int_B |F_k(u_{(\theta,k)}) - F(u_\theta)|\, dx\, dt &\leq \frac{K}{2^{k-1}} C \\
&\quad + H\, K \int\int_B |u_{(\theta,k)} - u_\theta|\, dx\, dt \to 0,
\end{aligned}$$

when $k \to \infty$, where C is a constant that only dependent of B and of h. Since $u_{(\theta,k)} \to u_\theta$ in $L^1_{loc}(\mathbb{R} \times [0, \infty); \mathbb{R})$. □

Thus our existence theorem will be proved if we can show the next Lemma.

Lemma 5. *Let $\theta \in \Phi$. Then $u_\theta = \lim_{k\to\infty} u_{(\theta,k)}$ is an admisssible weak solution of (1) provided that the following two conditions hold:*

$$I(u_{(\theta,k)}(\,.\,, 0)) \to I(u_0(.)), \tag{18}$$

and

$$\sum_1^\infty [I(u_{(\theta,k)})](\,.\,, l\triangle t_k) \to 0, \tag{19}$$

weakly, when $k \to \infty$.

Since

$$|I(u_{(\theta,k)}(x, 0)) - I(u_0(x))| = \left|\int_{u_0(x)}^{u_{(\theta,k)}(x,0)} h(\xi)\, d\xi\right| \leq H|u_{(\theta,k)}(x, 0) - u_0(x)|,$$

and

$$
\begin{aligned}
|[I(u_{(\theta,k)})](.,l\triangle t_k)| &= |I(u_{(\theta,k)}(x,l\triangle t_k^+)) - I(u_{(\theta,k)}(x,l\triangle t_k^-))| \\
&\le \left| \int_{u_{(\theta,k)}(x,l\triangle t_k^-)}^{u_{(\theta,k)}(x,l\triangle t_k^+)} h(\xi)\, d\xi \right| \\
&\le H|u_{(\theta,k)}(x,l\triangle t_k^+) - u_{(\theta,k)}(x,l\triangle t_k^-)| \\
&\le H|[u_{(\theta,k)}](.,l\triangle t_k)|,
\end{aligned}
$$

so the proof of the Lemma 5 is consequence of the next Lemma.

Lemma 6. *Let $\theta \in \Phi$. Then*

$$u_{(\theta,k)}(\,.\,,0) \to u_0(.) \tag{20}$$

and

$$\sum_{l=1}^{\infty}[u_{(\theta,k)}](\,.\,,l\triangle t_k) \to 0, \tag{21}$$

weakly, when $k \to \infty$.

Since $u_{(\theta,k)}(x,0) \to u_0(x)$, when $k \to \infty$, and $|u_{(\theta,k)}(x,0)|$, $|u_0(x)| \le M$, $\forall\, x \in \mathbb{R}$, we can obtain (20) using the Lebesgue Theorem.

Thus, we need to show that (21) holds. To this end, in [S], we have that

Theorem 3. *There is a null set $N \subset \Phi$ and a subsequence $\triangle x_{k_j} \to 0$ (consequently $\triangle t_{k_j} \to 0$) such that for any $\theta \in \Phi/N$ and any nonnegative function $\phi \in C_0^1(t \ge 0)$, (21) holds.*

The next theorem is our main result.

Theorem 4. *If $u_0(.)$ is bounded, continuous on the left and of locally bounded variation on $\mathbb{R}$, there exists a null set $N \subset \Phi$ and a subsequence $\triangle x_{k_j} \to 0$ such that if $\theta \in \Phi/N$, $h(u)$ is any nondecreasing function and $\phi \in C_0^1(t \ge 0)$ is any nonnegative function, then $u_\theta = lim_{k_j\to\infty} u_{(\theta,k_j)}$ is such that $\mathcal{L}_\phi(I(u_\theta), F(u_\theta)) \ge 0$. Moreover, u_θ satisfies the finite domain of dependence property.*

Proof. Assume first that $u_0(.)$ is of bounded variation, so choose a subquence $(\triangle x_{k_j})$ of $(\triangle x_k)$ such that (20) holds. By Theorem 3, choose a subsequence of $(\triangle x_{k_j})$, for simplicity we write $(\triangle x_{k_j})$, such that for $\theta \in \Phi/N$ (21) holds. By Theorem 2, choose a subsequence $(u_{(\theta,k_j')}) \subset (u_{(\theta,k_j)})$ such that $(u_{(\theta,k_j')})$ converges. Then by Lemma 4 u_θ is an admissible weak solution of (1).
For each $j \in \mathbb{N}$, $u_{(\theta,k_j')}$ has the finite domain of dependence property, according to Remark 5, so u_θ has the same property. Therefore, we can relax the assumption that $u_0(.)$ is of bounded variation on $\mathbb{R}$ and replace it by the condition

that $u_0(.)$ is of locally bounded variation on $\mathbb{R}$. Furthermore,

$$TV_{[x_1,x_2]}(u_\theta(.,t)) \leq TV_{[x_1-(1+Kt),x_2+(1+Kt)]}(u_0(.)).$$

$\square$

References

[B-B] BIANCHINI, S., BRESSAN, A.: **Vanishing viscosity solutions of nonlinear hyperbolic systems**, Ann. of Math., Vol 161, 223-342, 2005.

[B] BRESSAN, A.: **Hiperbolic Systems of Conservation Laws**, Oxford University Press, New York, 2000.

[D] DAFERMOS, C. M.: **Polygonal approximations of solutions of the initial value problem for a conservation law**, J. Math. Anal. and Applic., Vol 38, 33-41, 1972.

[G] GLIMM, J.: **Solutions in the large for nonlinear hyperbolic system of equations**, Comm. Pure Appl. Math., Vol 18, 697-715, 1965.

[G-R] GODLEWSKI, E., RAVIART, P. A.: **Hiperbolic Systems of Conservation Laws**, Number 3/4, Ellipses, Paris, 1990.

[H-L] HONG, J. M., LEFLOCH, P.G.: **A version of the Glimm method based on generalized Riemann problems**, J. Port. Math., 64, 199-236, 2007.

[Hr] HÖRMANDER, L.: **Lectures on Nonlinear Hyperbolic Differential Equation**, Springer-Verlag, Berlin, 1997.

[L] LIU, T.P.: **The deterministic version of Glimm scheme**, Commun. Math. Phys., 57, 135-148, 1977.

[S] SMOLLER, J.: **Shock Waves and Reaction-Diffusion Equations**, Springer-Verlag, New York, 1983.

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