On the Loss of regularity for a Class of Weakly Hyperbolic Operators

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Abstract

In this paper we determine bounds for the optimal loss of regularity in the Sobolev scale for a class of weakly hyperbolic operators.

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1 Introduction

A. Nersesian, in [8], proved C^k estimates for solutions of

$$\partial_t^2 u - \lambda(t)^2 \partial_x^2 u + b(x,t)\partial_x u + c(x,t)\partial_t u + d(x,t)u = f(x,t), \tag{1}$$

$$u(x,0) = u_0(x),\ \partial_t u(x,0) = u_1(x),$$

when the Levi condition

$$\limsup_{t\to 0^+} \frac{|b(x,t)|}{\lambda'(t)} \le Q < \infty \tag{2}$$

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holds, with $\lambda \in C^1, \lambda(t) > 0$ for $t > 0$, $\lambda^{'}(t) > 0$ for $t > 0$ and $\lambda'(0) = \lambda(0) = 0$. The loss of derivatives depends on Q. The Levi condition (2) is sharp in the following sense:

i) If $b(x,t) = o(\lambda'(t))$, then the lower order terms has no influence on the loss of regularity in the Sobolev scale.

ii) If $b(x,t) = o(\lambda'(t)^s)$, $s \in (0,1)$, a distribution solution might not exist, see [5].(to related discussion see also [11])

In another hand, F. Colombini and S. Spagnolo (in [2]), showed that there are real non-negative $a \in C^\infty([0,T])$, having an infinite number of oscillations as $t \to 0^+$, such that for suitable C^∞ initial Cauchy data, the equation

$$\partial_t^2 u - a(t)\partial_x^2 u = 0$$

does not have a distribution solution near $t = 0$.

For the weakly hyperbolic Cauchy problem

$$\partial_t^2 u - \sum_{i,j=1}^{n} \partial_{x_j}(a_{ij}(x,t)\partial_{x_i} u) + \sum_{i=1}^{n} b_i(x,t)\partial_{x_i} u + c(x,t)\partial_t u + d(x,t)u = f(x,t), \tag{3}$$
$$u(x,0) = u_0(x),\ \partial_t u(x,0) = u_1(x),$$

under the Levi condition

$$t\alpha\left(\sum_{i=1}^{n} b_i(x,t)\xi_i\right)^2 \leq \theta\left(\sum_{i,j=1}^{n} a_{ij}(x,t)\xi_i\xi_j\right) + \sum_{i,j=1}^{n} \partial_t a_{ij}(x,t)\xi_i\xi_j, \tag{4}$$

O. Oleinik (see [9]) showed that the loss of regularity in the Sobolev scale increases as $\frac{1}{2\alpha} - 3$. Extensions were proved by M. Ebert (see [4])

Later, V. Ivrii and V. Petkov (see [5]) proved that for

$$\partial_t^2 u - t^{2\ell}\partial_x^2 u + bt^k\partial_x u = 0, \tag{5}$$

with $b \neq 0$, a necessary condition for the Cauchy problem to be C^∞ well posed is $k \geq \ell - 1$.

Observe that for (5) both Levi conditions (2) and (4) are sufficient for C^∞ well-posedness of the Cauchy problem. Nevertheless, in both cases the loss of regularity, in C^k scale, is not sharp, as one can see by the example due to Qi Min-You, see [7], namely

$$\partial_t^2 u - t^2\partial_x^2 u - b\partial_x u = 0,\ t > 0,\ x \in \mathbb{R}$$

with the initial condition

$$u(x,0) = u_0(x), \ \partial_t u(x,0) = 0,$$

where $b = 4n+1$, $n \geq 0$ is an integer. The unique solution has the form

$$u(x,t) = \sum_{j=0}^{n} \frac{\sqrt{\pi} t^{2j}}{j!(n-j)!\Gamma(j+\frac{1}{2})} (\partial_x^j u_0)(x + \frac{1}{2}t^2).$$

The loss of regularity increases on n.

In this paper we determine bounds for the optimal loss of regularity in the Sobolev scale for the Cauchy problem

$$\partial_t^2 u - \lambda^2(t) \sum_{i,j=1}^{n} a_{ij}(t) \partial^2_{x_i x_j} u + \lambda(t) \sum_{i=1}^{n} c_i(t) \partial^2_{t x_i} u = g(x,t,u,\partial_t u, \lambda'(t) \nabla_x u), \tag{6}$$

$$u(x,0) = u_0(x), \ \partial_t u(x,0) = u_1(x). \tag{7}$$

We suppose the following hypothesis:

(H_1) $\lambda, a_{ij}, c_i \in C^1(\mathbb{R})$ are real–valued functions, with $\lambda(t) > 0$ for $t > 0$, $\lambda(0) = 0$ and

$$\sum_{i,j=1}^{n} \left[a_{ij}(t) + \frac{c_i(t) c_j(t)}{4} \right] \xi_i \xi_j \geq \gamma |\xi|^2, \gamma > 0 \ \forall \xi \in \mathbb{R}^n \backslash \{0\} \ and \ t \geq 0,$$

that is, the operator $P = \partial_t^2 u - \sum_{i,j=1}^{n} \left(a_{ij}(t) + \frac{c_i(t) c_j(t)}{4} \right) \partial^2_{x_i x_j} u$ + *lower order terms* is strictly hyperbolic with respect to $\{t = 0\}$.

In Theorem 1.1, we will consider linear models, namely when the function g in (6) is given by

$$g(x,t,u,\partial_t u, \lambda'(t) \nabla_x u) = f(x,t) - \lambda'(t) \sum_{i=1}^{n} b_i(t) \partial_{x_i} u - c(t) \partial_t u - d(x,t) u.$$

In addition we assume:

(H_2) $\lambda(t)$ vanishes of order $\ell \in \mathbb{N}$ at $t = 0$; $\hat{d}(\cdot, t) \in L^1(\mathbb{R}^n)$, $\partial_t^p a_{ij}(0)$, $\partial_t^q b_i(0)$, $\partial_t^r c_i(0)$, $\partial_t^k c(0)$ and $\partial_x^\rho \partial_t^k d(x,0)$ are bounded for $p \leq k_0(\ell+1) - \ell - 2$, $q \leq k_0(\ell+1) - 1$, $r \leq k_0(\ell+1)$, $|\rho| \leq k_0$ and $k \leq (k_0 - |\rho|)(\ell+1) + \ell - 2$, respectively, where k_0 is given by

$$k_0 = \min \left\{ k \in \mathbb{N}; k > \frac{\alpha\ell - \ell}{2(\ell+1)} + \frac{1-\ell}{2(\ell+1)} \right\}, \tag{8}$$

with

$$\alpha = \lim_{t\to 0^+} \sup_{\xi\neq 0} \frac{|\sum_{i=1}^n [2b_i(t)-c_i(t)]\xi_i|}{\sqrt{\sum_{i,j=1}^n [4a_{ij}(t)+c_i(t)c_j(t)]\,\xi_i\xi_j}}. \tag{9}$$

Then we have:

Theorem 1.1 *Suppose that (H_1) and (H_2) hold, $u_0 \in H^{m+k_0+1}(\mathbb{R}^n)$ and $u_1 \in H^{m+k_0+1}(\mathbb{R}^n)$. Then (6) and (7) have an unique solution $u \in C^0([0,T], H^m(\mathbb{R}^n)) \cap C^1([0,T], H^{m-1}(\mathbb{R}^n))$ and the following estimate holds for $0 \le t \le T$, for any $m \ge 1$*

$$\|u(\cdot,t)\|_m^2 + \|\partial_t u(\cdot,t)\|_{m-1}^2 \le C(t)\Big\{\|u_0\|_{m+k_0+1}^2 + \|u_1\|_{m+k_0+1}^2$$
$$+\sum_{|\rho|\le k_0} \sum_{i=0}^{(k_0-|\rho|)(\ell+1)+\ell-2} \|\partial_t^i f(\cdot,0)\|_{m+|\rho|+1}^2 + \sup_{0\le s\le t} \|\partial_s^{k_0(\ell+1)+\ell-1} f(\cdot,s)\|_m^2\Big\}, \tag{10}$$

provided the norms of f on the right of (10) are finite and $d(\cdot,t) \in H^{2m+3+[n/2]}$. Here $C(t)$ is a continuous function depending on $\|\hat{d}(\cdot,t)\|_{L^1}$, $\|d(\cdot,t)\|_{2m+3+[n/2]}$ and on the L^∞ norms of $\partial_t^p a_{ij}(0)$,$\partial_t^q b_i(0)$, $\partial_t^r c_i(0)$, $\partial_t^k c(0)$ and $\partial_x^\rho \partial_t^k d(x,0)$ for $p \le k_0(\ell+1)-\ell-2$, $q \le k_0(\ell+1)-1$, $r \le k_0(\ell+1)$, $|\rho| \le k_0$ and $k \le (k_0-|\rho|)(\ell+1)+\ell-2$, respectively.

Remark 1.1 *For the theorem above we can not take, in general, $u_0 \in H^{m+k_0+1}$ and $u_1 \in (H^{m+k_0} \backslash H^{m+k_0+\epsilon})$, for all $\epsilon > 0$. In fact, consider the Cauchy problem*

$$\partial_t^2 u - t^{2\ell}\partial_x^2 u - \ell t^{\ell-1}\partial_x u = 0,$$
$$u(x,0) = u_0(x),\ \partial_t u(x,0) = u_1(x).$$

If $u_0 = 0$, the solution is

$$u(x,t) = \int_0^t u_1\Big(x + 2\frac{s^{\ell+1}}{\ell+1} - \frac{t^{\ell+1}}{\ell+1}\Big)ds.$$

Its Fourier transform is

$$\hat{u}(\xi,t) = \hat{u}_1(\xi) exp\Big(-i\xi\frac{t^{\ell+1}}{\ell+1}\Big) \int_0^t exp\Big(2i\xi\frac{s^{\ell+1}}{\ell+1}\Big)ds.$$

From the values of the Fresnel integrals $\int_0^\infty \frac{\cos(x)}{x^{\ell/(\ell+1)}}dx$ and $\int_0^\infty \frac{\sin(x)}{x^{\ell/(\ell+1)}}dx$ we get that $|\int_0^t exp\Big(2i\xi\frac{s^{\ell+1}}{\ell+1}\Big)ds| \sim C_\ell|\xi|^{-\frac{1}{\ell+1}}$. Then, $u(\cdot,t) \in H^{m+k_0+\frac{1}{\ell+1}} \backslash H^{m+k_0+\frac{1}{\ell+1}+\epsilon}$, in particular $u(\cdot,t) \notin H^{m+k_0+1}$.

To deal with the semi-linear equation (6), when λ might not vanish of finite order, we suppose:

(H_3) $\lambda'(t) \geq 0$ for $t > 0$ and $\lim_{t \to 0^+} \dfrac{\lambda(t)}{\lambda'(t)} = 0$.

(H_4) The function $g = g(x, t, u, p, r_1, ..., r_n)$, with $p = u_t$, $q_i = \partial_{x_i} u$, $r_i = \lambda'(t) q_i$ satisfies $g \in C([0,T], H^s_x(\mathbb{R}^n) \times C^\infty(\mathbb{R} \times \mathbb{R}_p \times \mathbb{R}^n_r))$, for all $s \in \mathbb{N}$ and there exist a nonnegative constant Q such that

$$Q = \lim_{t \to 0^+} \sup \left(\sum_{i=1}^n \Big| \frac{c_i(t)}{2} + \frac{\partial_{q_i} g(x, t, u_0(x), u_1(x), \lambda'(t) \nabla_x u_0(x))}{\lambda'(t)} \Big|^2 \right)^{1/2}, \quad (11)$$

uniformly in x.

The next two theorems are extensions of Theorem 1 and Theorem 2 of Reissig (see [11]). In fact, the technique of our proof is based on the techniques presented there.

Theorem 1.2 *Suppose that (H_1), (H_3) and (H_4) hold and $u_0 \in H^{s_0}(\mathbb{R}^n)$, $u_1 \in H^{s_0 - 1}(\mathbb{R}^n)$, with $s_0 \geq 2\left(\frac{Q}{\sqrt{\gamma}} + 4\right) + n/2 + 1 + r$, $r, s_0 \in \mathbb{N}$. Then (6)–(7) has a solution $u \in C^0\left([0,T^*], H^N(\mathbb{R}^n)\right) \cap C^1\left([0,T^*], H^{N-1}(\mathbb{R}^n)\right)$ for some $0 < T^* \leq T$, with $N = s_0 - 2(p+1)$ and $p = max\left\{3, \left[\frac{Q}{\sqrt{\gamma}}\right] + 2\right\}$.*

Remark 1.2 *In Theorem 1.2 a lower bound for N is $N \geq 7 - 2\max\left\{3 - \left[\frac{Q}{\sqrt{\gamma}}\right], 2\right\}$ $+ [n/2] + r$.*

Theorem 1.3 *Under the assumptions of Theorem 1.2, if*

$$s_0 > \max\{n/2 + 1, p + 3\}, \text{ with } p = \left[\frac{Q}{\sqrt{\gamma}}\right] + 2, \quad (12)$$

then the problem (6)–(7) has at most one solution $u \in C^0([0,T], H^{s_0}(\mathbb{R}^n)) \cap C^1([0,T], H^{s_0-1}(\mathbb{R}^n)) \cap C^2([0,T], H^{s_0-2}(\mathbb{R}^n))$.

In general it is not possible to improve such loss of regularity in Theorem 1.1, as we can see by the examples below.

Example 1.1 *In [12], M. Reissig and M. Dreher applied a result of K. Taniguchi and Y. Tozaki ([13]) to the Cauchy problem*

$$\partial_t^2 u - a t^{2\ell} \partial_x^2 u + c t^\ell \partial_x \partial_t u + b t^{\ell - 1} \partial_x u = 0, \; t > 0, \; x \in \mathbb{R}$$

$$u(x, 0) = u_0(x), \; \partial_t u(x, 0) = 0,$$

with a, b, c real constants and $c^2+4a > 0$, to prove that there exists an uniquely determined solution $u(\cdot,t) \in H^s(\mathbb{R})$, if $u_0 \in H^{s_0}(\mathbb{R})$ and

$$s = s_0 - \frac{|2b - \ell c|/\sqrt{c^2+4a} - \ell}{2(\ell+1)}.$$

This loss of derivatives is sharp, again by [7]. The integer k_0 given in (8) is

$$k_0 = \min\left\{k \in \mathbb{N}; k > \frac{|2b - \ell c|/\sqrt{c^2+4a} - \ell}{2(\ell+1)} + \frac{1-\ell}{2(\ell+1)}\right\}.$$

If n_0 is an integer, with $n_0 \le \frac{|2b-\ell c|/\sqrt{c^2+4a}-\ell}{2(\ell+1)} < n_0 + \frac{1}{2}$, and ℓ is sufficiently large, we obtain that $k_0 = n_0$. Therefore (10) can not be improved when we measure the loss of regularity by integers.

Example 1.2 *(The non-homogeneous equation) Consider the Cauchy problem*

$$\partial_t^2 u + t^\ell \partial_x \partial_t u - b t^{\ell-1} \partial_x u = f(x),$$

$$u(x,0) = \partial_t u(x,0) = 0,$$

where $b = n(\ell+1)+2$, with ℓ and n positive integers. T. Mandai, in [6], obtained an explicit solution of the form

$$u(x,t) = \sum_{j=0}^{n} A_j t^{j(\ell+1)+2} \partial_x^j f(x),$$

where $A_j, j = 0, 1, ..., n$ are positive constants independent of f. Therefore, the optimal loss of regularity in Sobolev scale is n. From Theorem 1.1, for a more general class of operators, we obtain that the loss is $n+1$ if $\ell > 5$.

Observe that, for both examples above, the Theorem 1.2 does not give a better bound for the loss of regularity than Theorem 1.1. Also, regarding Theorem 1.2, we have the following two examples, due to Alexandrian in [1] (see [11] for a related discussion) and Tarama ([14]), respectively.

Example 1.3 *Consider the Cauchy problem*

$$\partial_t^2 u - \lambda^2(t)\partial_x^2 u - b\frac{\lambda^2(t)}{\Lambda(t)}\partial_x u = 0,$$

$$u(x,0) = u_0(x), \partial_t u(x,0) = u_1(x),$$

where $\Lambda(t) = (sign\ t)\exp(-t^{-1})$, $\lambda(t) = \Lambda'(t)$ and b a constant. One can prove that $u(\cdot,t) \in H^{s_0 - \frac{|b|-1}{2}}(\mathbb{R})$ for all $t > 0$ if $u_0 \in H^{s_0}(\mathbb{R})$ and $u_1 \in H^{s_0-1}(\mathbb{R})$. By Theorem 1.2, for a more general class of operators, we obtain that the loss of regularity is $2 + 2\max\{3, [|b|]+2\}$.

The next example shows that the conclusion of Theorem 1.2 might not be true, for C^∞ well posedness, if the behavior of the coefficient b, of the lower order term $b\partial_x$, is not in a suitable relation with λ.

Example 1.4 *The Cauchy problem*

$$\partial_t^2 u - a^2 e^{-2t^{-n}} \partial_x^2 u + b t^{-n-1-\ell} e^{-t^{-n}} \partial_x u = 0, \quad [0,T] \times \mathbb{R},$$

$$u(x,0) = \partial_t u(x,0) = 0,$$

when $Re(b) = 0$ *(respec.* $Re(b) \neq 0$*) is well-posed if, and only if,* $\ell \leq n$ *(respec.* $\ell \leq 0$*) (see [14]).*

For the convenience of the reader we give an overview of the paper. In Section 2, we present a change of variable which reduces our problem to a normal model. Then, we suppose that $\lambda(t)$ vanishes of finite order and compute the trace at $t = 0$ of the higher order derivatives of u with respect to t. An important property to be used in the proof of Theorem 1.2, relating the vanishing of the Cauchy data with the solution of the linear equation, is proved. In Section 3 we prove an energy estimate which implies Theorem 1.1. We observe that the energy inequality is in part motivated by [3] and [10]. In Section 4 we prove Theorem 1.2, there Proposition 2.1 replaces Corollary 1 of [11]), also we sketch the proof of Theorem 1.3.

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2 Preliminary Results

The first objective will be, by a change of variables, to reduce the problem (6)-(7) to the form

$$Lu = \partial_t^2 u - \lambda^2(t) \sum_{i,j=1}^{n} \tilde{a}_{ij}(t) \partial_{x_i x_j}^2 u = h(x, t, u, \partial_t u, \lambda'(t) \nabla_x u), \tag{13}$$

$$u(x,0) = u_0(x),\ \partial_t u(x,0) = u_1(x). \tag{14}$$

From our change of variables, clearly, $(H1)$ to $(H3)$ remain the same. We will see that $(H4)$ is invariant too.

Consider the change on the $x-$variables given by

$$y_k = x_k - \frac{1}{2}\int_0^t \lambda(s)c_k(s)ds, \ k = 1, ..., n.$$

Therefore, with $u(x, t) = v(y, t)$ and using (H3) we obtain (13) with

$$\tilde{a}_{ij}(t) = a_{ij}(t) + \frac{c_i(t)c_j(t)}{4}$$

and

$$h(x, t, u, \partial_t u, \lambda'(t)\nabla_x u) = \frac{1}{2}\sum_{i=1}^{n}\Big(c_i(t)\lambda(t)\Big)'\partial_{x_i}u$$

$$+ g\Big((x_k + \frac{1}{2}\int_0^t \lambda(s)c_k(s)ds), t, u, \partial_t u - \frac{\lambda(t)}{2}\sum_{i=1}^{n} c_i(t)\partial_{x_i}u, \lambda'(t)\nabla_x u\Big),$$

with $k = 1, ...n$.

Also (11) of (H4) is in this case

$$Q = \lim_{t\to 0^+}\sup\left(\sum_{i=1}^{n}\Big|\partial_{q_i}h(x, t, u_0(x), u_1(x), \lambda'(t)\nabla_x u_0(x))/\lambda'(t)\Big|^2\right)^{1/2}$$

$$= \lim_{t\to 0^+}\sup\left(\sum_{i=1}^{n}\Big|\frac{c_i(t)}{2} + \partial_{q_i}g(x, t, u_0(x), u_1(x), \lambda'(t)\nabla_x u_0(x))/\lambda'(t)\Big|^2\right)^{1/2},$$

uniformly in x. From now on, without loss of generality, we can assume that the Cauchy problem (6)–(7) has the form (13)–(14) with h satisfying

$$Q = \lim_{t\to 0^+}\sup\left(\sum_{i=1}^{n}\Big|\partial_{q_i}h(x, t, u_0(x), u_1(x), \lambda'(t)\nabla_x u_0(x))/\lambda'(t)\Big|^2\right)^{1/2}, \quad (15)$$

uniformly in x. In the linear case

$$h(x, t, u, \partial_t u, \lambda'(t)\nabla_x u) = f\Big(\phi(x, t)\Big) + \frac{1}{2}\sum_{i=1}^{n}\Big(c_i(t)\lambda(t)\Big)'\partial_{x_i}u$$

$$- \lambda'(t)\sum_{i=1}^{n}\left(b_i(t) - \frac{\lambda(t)}{2\lambda'(t)}c(t)c_i(t)\right)\partial_{x_i}u - c(t)\partial_t u - d\Big(\phi(x, t)\Big)$$

with $\phi(x, t) = \Big((x_i + \frac{1}{2}\int_0^t \lambda(s)c_i(s)ds), t\Big), \ i = 1, ...n$. In this case, we can assume that the Cauchy problem (13)–(14) has the form

$$Lu = \quad \partial_t^2 u - \lambda^2(t) \sum_{i,j=1}^{n} a_{ij}(t)\partial_{x_i}\partial_{x_j} u + \lambda'(t) \sum_{i=1}^{n} b_i(t)\partial_{x_i} u$$

$$+ c(t)\partial_t u + d(x,t)u = f(x,t), \tag{16}$$

$$u(x,0) = u_0(x),\ \partial_t u(x,0) = u_1(x). \tag{17}$$

If $\lambda(t)$ vanishes of order $\ell \in \mathbb{N}$ at $t = 0$, we can assume that $\lambda(t) = t^\ell$. As we will see in section 3, to implement our technique for the proof of Theorem 1.1, we need to compute, a-priori, the restrictions to $t = 0$ of higher order derivatives with respect to t of u, in terms of (16) and (17).

Lemma 2.1 *Let $0 \leq j < \ell - 1$, then*

$$\partial_t^{j+2} u|_{t=0} = \partial_t^j f|_{t=0} - \sum_{q=0}^{j} \binom{j}{q} \Big\{ \partial_t^{j-q} c(0) \partial_t^{q+1} u|_{t=0} + \partial_t^{j-q} d|_{t=0} \partial_t^q u|_{t=0} \Big\}. \tag{18}$$

Let $\ell - 1 \leq j < 2\ell$, then

$$\partial_t^{j+2} u|_{t=0} = \partial_t^j f|_{t=0} - \frac{j!}{(j-\ell+1)!} \sum_{i=1}^{n} \sum_{q=0}^{j-\ell+1} \binom{j-\ell+1}{q} b_i^{(j-\ell+1-q)}(0) \partial_t^q \partial_{x_i} u|_{t=0}$$

$$- \sum_{q=0}^{j} \binom{j}{q} \Big\{ \partial_t^{j-q} c(0) \partial_t^{q+1} u|_{t=0} + \partial_t^{j-q} d|_{t=0} \partial_t^q u|_{t=0} \Big\}. \tag{19}$$

Let $2\ell \leq j$, then

$$\partial_t^{j+2} u|_{t=0} = \partial_t^j f|_{t=0} + \frac{j!}{(j-2\ell)!} \sum_{i,k=1}^{n} \sum_{q=0}^{j-2\ell} \binom{j-2\ell}{q} a_{ik}^{(j-2\ell-q)}(0) \partial_t^q \partial_{x_i} \partial_{x_k} u|_{t=0}$$

$$- \frac{j!}{(j-\ell+1)!} \sum_{i=1}^{n} \sum_{q=0}^{j-\ell+1} \binom{j-\ell+1}{q} b_i^{(j-\ell+1-q)}(0) \partial_t^q \partial_{x_i} u|_{t=0} \tag{20}$$

$$- \sum_{q=0}^{j} \binom{j}{q} \Big\{ \partial_t^{j-q} c(0) \partial_t^{q+1} u|_{t=0} + \partial_t^{j-q} d|_{t=0} \partial_t^q u|_{t=0} \Big\}.$$

Proof: By Leibniz' formula and a symmetry argument, it is enough to check it for $j \geq m$

$$[\sum_{k=0}^{j} \binom{j}{k} \partial_t^{j-k} t^m \cdot \partial_t^k h(t)]|_{t=0} = \frac{j!}{(j-m)!} (\partial_t^{j-m} h)|_{t=0}.$$

∎

Then we have:

Lemma 2.2 *If $k_0(\ell+1) \leq j+2 \leq k_0(\ell+1)+\ell$, for $k_0 \in \mathbb{Z}_+$, then*

$$(\partial_t^{j+2}u)|_{t=0} = \partial_t^j f|_{t=0} + \sum_{|\rho|\leq k_0} \Big(\alpha_{j\rho}\partial_x^\rho u_0(x) + \beta_{j\rho}\partial_x^\rho u_1(x)\Big) + \sum_{|\rho|\leq k_0} \sum_{p=0}^{j-|\rho|(\ell+1)} \gamma_{pj\rho}\partial_x^\rho \partial_t^p f|_{t=0}. \tag{21}$$

Here $\alpha_{j\rho}$ and $\beta_{j\rho}$ depends on $\partial_t^p a_{ik}(0)$ (for $p \leq j-2\ell$), $\partial_t^q b_i(0)$ (for $q \leq j-\ell+1$), $\partial_t^k c(0)$ (for $k \leq j$) and $\partial_x^\rho \partial_t^k d(x,0)$ (for $|\rho| \leq k_0$ and $k \leq j-|\rho|(\ell+1)$). Furthermore, $\gamma_{ij\rho}$ depends on $\partial_t^p a_{ik}(0)$ (for $p \leq j-2\ell-2$), $\partial_t^q b_i(0)$ (for $q \leq j-\ell-1$), $\partial_t^k c(0)$ (for $k \leq j-1$) and $\partial_x^\rho \partial_t^k d(x,0)$ (for $|\rho| \leq k_0$ and $k \leq j-|\rho|(\ell+1)-2$).

Proof: We will prove by induction on k_0. First we suppose that $\ell \geq 2$.

If $k_0 = 0$, by (18) we have

$$(\partial_t^{j+2}u)|_{t=0} = \partial_t^j f|_{t=0} - \sum_{q=0}^{j} \binom{j}{q} \Big\{\partial_t^{j-q}c(0)\partial_t^{q+1}u|_{t=0} + \partial_t^{j-q}d|_{t=0}\partial_t^q u|_{t=0}\Big\},$$

for $2 \leq j+2 \leq \ell$ and the statement holds.

If $k_0 = 1$, by (19) we obtain

$$\partial_t^{j+2}u|_{t=0} = \partial_t^j f|_{t=0} - \frac{j!}{(j-\ell+1)!}\sum_{i=1}^{n}\sum_{q=0}^{j-\ell+1} \binom{j-\ell+1}{q} b_i^{(j-\ell+1-q)}(0)\partial_t^q \partial_{x_i} u|_{t=0}$$
$$-\sum_{q=0}^{j}\binom{j}{q}\Big\{\partial_t^{j-q}c(0)\partial_t^{q+1}u|_{t=0} + \partial_t^{j-q}d|_{t=0}\partial_t^q u|_{t=0}\Big\}, \text{ for } \ell+1 \leq j+2 \leq (\ell+1)+\ell.$$

Therefore (21) follows, since $q \leq \ell$ in the first sum.

Assume now that the inequality (21) holds for indices $\kappa \leq k_0$ with $k_0 \geq 1$. If $(k_0+1)(\ell+1) \leq j+2 \leq (k_0+1)(\ell+1)+\ell$, by (20) we have

$$(\partial_t^{j+2}u)|_{t=0} = \partial_t^j f|_{t=0} + \frac{j!}{(j-2\ell)!}\sum_{i,k=1}^{n}\sum_{q=0}^{j-2\ell} \binom{j-2\ell}{q} a_{ik}^{(j-2\ell-q)}(0)(\partial_t^q \partial_{x_i}\partial_{x_k}u)|_{t=0}$$

$$-\frac{j!}{(j-\ell+1)!}\sum_{i=1}^{n}\sum_{q=0}^{j-\ell+1}\binom{j-\ell+1}{q}b_i^{(j-\ell+1-q)}(0)(\partial_t^q\partial_{x_i}u)|_{t=0}$$

$$-\sum_{q=0}^{j}\binom{j}{q}\left\{\partial_t^{j-q}c(0)\partial_t^{q+1}u|_{t=0}+\partial_t^{j-q}d|_{t=0}\partial_t^q u|_{t=0}\right\}.$$

In the first sum $q \leq k_0(\ell+1)-1$, then $\partial_t^q\partial_{x_i}\partial_{x_k}u|_{t=0}$, for $q \leq k_0(\ell+1)-1$ can be computed by applying the induction hypothesis to k_0-1.

For the second sum, we have $q \leq k_0(\ell+1)+\ell$, then $\partial_t^q\partial_{x_i}u|_{t=0}$, for $q \leq k_0(\ell+1)+\ell$ can be computed by applying the induction hypothesis to k_0. The case $\ell=1$ follows from the arguments as above. This concludes the proof. ∎

In this paper, it will be used as Sobolev norm of order $s \in \mathbb{N}$, in the variable x, the one given by

$$\|f(\cdot,t)\|_s = \left(\sum_{|\alpha|\leq s}\int|\xi^\alpha|^2|\hat{f}(\xi,t)|^2d\xi\right)^{1/2}.$$

An important tool for the proof of Theorem 1.2 is the following result for linear equations, which was motivated by Corollary 1 of Reissig (see [11]):

Proposition 2.1 *Suppose that* (H_1) *and* (H_3) *hold. If* $f \in C([0,T],H^{N-1}(\mathbb{R}^n))$ *satisfies*

$$\left\|\frac{f(\cdot,t)}{\lambda(t)^{d-1}\lambda'(t)}\right\|_{N-1} \leq C_N,\ \ N\geq 1, d>2, \tag{22}$$

then the Cauchy problem

$$\partial_t^2 w-\lambda^2(t)\sum_{i,j=1}^{n}a_{ij}(t)\partial^2_{x_ix_j}w = f(x,t),\ \ w(x,0)=\partial_t w(x,0)=0, \tag{23}$$

has an unique solution $w \in C^0\left([0,T],H^N(\mathbb{R}^n)\right)\cap C^1\left([0,T],H^{N-1}(\mathbb{R}^n)\right)$. *In addition, given a* $\epsilon>0$, *there exists* $T_\epsilon>0$ *such that* $\mathcal{E}_N(w)(t) \leq \frac{C_N}{(d-1-\epsilon)}\lambda(t)^d$ *for all* $t\in[0,T_\epsilon]$. *Here*

$$\mathcal{E}_N(w)(t)=\left(\sum_{|\alpha|\leq N-1}\int|\xi^\alpha|^2[|\hat{w}_t|^2+(1+a(\xi,t))|\hat{w}|^2]d\xi\right)^{1/2},$$

with $a(\xi,t)=\lambda^2(t)\sum_{i,j=1}^n a_{ij}(t)\xi_i\xi_j$ *and* $\hat{w}(\xi,t)$ *is the Fourier transform of* $w(x,t)$ *in the variable space.*

Proof: We prove the proposition in two steps. In the first one, we deduce the asymptotical behavior of the solution and in the second we conclude the proof.

STEP 1. Consider the energy function

$$E(w)(t) = \Big(\frac{1}{2}\int K(\xi,t)[|\hat{w}_t|^2 + (1+a(\xi,t))|\hat{w}|^2]d\xi\Big)^{1/2}, \tag{24}$$

with

$$K(\xi,t) = \frac{(1+|\xi|^2)^N}{(1+a(\xi,t))}.$$

By differentiating $[E(w)(t)]^2$, and using the Fourier transform of the equation we get

$$\frac{d}{dt}[E(w)(t)]^2 = \frac{1}{2}\int \Big\{\partial_t K(\xi,t)|\hat{w}_t|^2 + K(\xi,t)2Re[\bar{\hat{w}}_t(\hat{f}+\hat{w})]\Big\}\, d\xi.$$

By hypotheses (H_1) and (H_3) we obtain $\partial_t K(\xi,t) \leq 0$ and using Hölder's inequality

$$\frac{d}{dt}[E(w)(t)]^2 \leq [E(w)(t)]^2 + \sqrt{2}E(w)(t)\Big(\int K(\xi,t)|\hat{f}(\xi,t)|^2 d\xi\Big)^{1/2}.$$

Then

$$E'(w)(t) = \frac{1}{2E(w)(t)}\frac{d}{dt}[E(w)(t)]^2 \leq \frac{1}{2}E(w)(t) + \frac{\sqrt{2}}{2}\Big(\int K(\xi,t)|\hat{f}(\xi,t)|^2 d\xi\Big)^{1/2}.$$

Moreover,

$$\Big(\int K(\xi,t)|\hat{f}(\xi,t)|^2 d\xi\Big)^{1/2} \leq C\left(\int (1+|\xi|^2)^{N-1}\left|\frac{\hat{f}(\xi,t)}{\lambda(t)}\right|^2 d\xi\right)^{1/2}, \tag{25}$$

with $C = max\left\{\gamma^{-1/2}, \lambda(T)\right\}$. Therefore, by (22) and (25) we obtain

$$E'(w)(t) \leq \frac{1}{2}E(w)(t) + \frac{\sqrt{2}}{2}CC_N\lambda(t)^{d-2}\lambda'(t),$$

and Gronwall's inequality yields

$$E(w)(t) \leq \tilde{C}\lambda(t)^{d-1}, \quad \text{with } \tilde{C} = CC_N\frac{\sqrt{2}e^{T/2}}{2(d-1)}.$$

STEP 2. After differentiaton of $\mathcal{E}_N(w)(t)$, using hypotheses (H_1), (H_3) and Hölder's inequality, we obtain

$$\mathcal{E}_N'(w)(t) \leq \frac{\lambda'(t)}{\lambda(t)}\mathcal{E}_N(w)(t) + (M/\gamma+1)\mathcal{E}_N(w)(t) + \left(\sum_{|\alpha|\leq N-1}\int |\xi^\alpha|^2|\hat{f}(\xi,t)|^2 d\xi\right)^{1/2},$$

with $M>0$ satisfying $\sum_{i,j=1}^n a_{ij}'(t)\xi_i\xi_j \leq M|\xi|^2 \leq \frac{M}{\gamma}\sum_{i,j=1}^n a_{ij}(t)\xi_i\xi_j$ for all t and ξ. By (H_3), given a $\epsilon > 0$, there exists a $T_\epsilon > 0$ such that for $t \in [0, T_\epsilon]$ $(M/\gamma+1) \leq \epsilon \frac{\lambda'(t)}{\lambda(t)}$. From this it follows

$$\begin{aligned}\left(\mathcal{E}_N(w)(t)\lambda(t)^{-(1+\epsilon)}\right)' &\leq \lambda(t)^{-(1+\epsilon)}\left(\sum_{|\alpha|\leq N-1}\int |\xi^\alpha|^2|\hat{f}(\xi,t)|^2 d\xi\right)^{1/2} \\ &\leq \lambda(t)^{d-2-\epsilon}\lambda'(t)\left\|\frac{f(\cdot,t)}{\lambda(t)^{d-1}\lambda'(t)}\right\|_{N-1}.\end{aligned}$$

Hence, from Step 1, we can integrate from 0 to t, to find

$$\mathcal{E}_N(w)(t) \leq \frac{\lambda(t)^d}{(d-1-\epsilon)}\sup_{t\in[0,T]}\left\|\frac{f(\cdot,t)}{\lambda(t)^{d-1}\lambda'(t)}\right\|_{N-1}. \tag{26}$$

Of course, by (H_1) we also have

$$\begin{aligned}&\left(\sum_{|\alpha|\leq N-1}\int |\xi^\alpha|^2\gamma|\xi|^2|\hat{w}(\xi,t)|^2 d\xi\right)^{1/2} \\ &\leq \lambda(t)^{-1}\left(\sum_{|\alpha|\leq N-1}\int |\xi^\alpha|^2 a(\xi,t)|\hat{w}(\xi,t)|^2 d\xi\right)^{1/2} \leq \lambda(t)^{-1}\mathcal{E}_N(w)(t).\end{aligned} \tag{27}$$

From (26) and (27) we obtain

$$\|w(\cdot,t)\|_N + \|\partial_t w(\cdot,t)\|_{N-1} \leq C\lambda(t)^{d-1}\sup_{t\in[0,T]}\left\|\frac{f(\cdot,t)}{\lambda(t)^{d-1}\lambda'(t)}\right\|_{N-1}. \tag{28}$$

Now, the proof of the Proposition 2.1 is completed by standard arguments. Let $\{f_k\}$ sequence of test functions that converge to f in H^{N-1}. Then there exists a solution $w_k \in C^\infty$, for all k (see for example [9]). By (28), $\{w_k\}$ is a Cauchy sequence in $C^0\left([0,T], H^N(\mathbb{R}^n)\right) \cap C^1\left([0,T], H^{N-1}(\mathbb{R}^n)\right)$ and therefore the limit function $w(x,t) \in C^0\left([0,T], H^N(\mathbb{R}^n)\right) \cap C^1\left([0,T], H^{N-1}(\mathbb{R}^n)\right)$ is solution of (23) satisfying (28). The uniqueness follows from the estimate (28).

∎

3 Proof of Theorem 1.1

With k_0 given by (8), let us consider the function $v(x,t) = u(x,t) - v_{k_0}(x,t)$, where v_{k_0} is given by

$$v_{k_0}(x,t) = \sum_{j=0}^{k_0(\ell+1)+\ell} \frac{t^j}{j!}\partial_t^j u(x,0). \tag{29}$$

Now we consider the Cauchy problem

$$\partial_t^2 v - \lambda^2(t)\sum_{i,j=1}^{n} a_{ij}(t)\partial_{x_i}\partial_{x_j}v + \lambda'(t)\sum_{i=1}^{n} b_i(t)\partial_{x_i}v + c(t)\partial_t v + d(x,t)v = g, \tag{30}$$

$$v(x,0) = 0 = v_t(x,0), \tag{31}$$

with $g(x,t) = f(x,t) - Lv_{k_0}(x,t)$. Now we obtain an energy estimate to the Cauchy problem (30)-(31). We restrict ourselves to the region $0 \le t \le \delta$, because for $t \ge \delta$, the Cauchy problem at $t = \delta$ is strictly hyperbolic. For $0 \le t \le \delta$ consider the following energy function

$$E(t) = \frac{1}{2}\int K(\xi,t)[|\hat{v}_t|^2 + (1 + a(\xi,t))|\hat{v}|^2]d\xi, \tag{32}$$

where $\hat{v}(\xi,t)$ is the Fourier transform of $v(x,t)$ in the variable space and

$$K(\xi,t) = \frac{(1+|\xi|^2)^m}{(1+a(\xi,t))} \textit{ with } a(\xi,t) = \lambda^2(t)\sum_{i,j=1}^{n} a_{ij}(t)\xi_i\xi_j.$$

By differentiating $E(t)$, and using the Fourier transform of (30) we get

$$E'(t) = \int K(\xi,t) Re\left\{\bar{\hat{v}}_t[\hat{v} - i\hat{v}\lambda'(t)\sum_{j=1}^{n} b_j(t)\xi_j - c(t)\hat{v}_t - (2\pi)^{-n}\hat{d} * \hat{v} + \hat{g}]\right\} d\xi$$

$$+\frac{1}{2}\int |\hat{v}_t|^2\partial_t K(\xi,t)d\xi + \frac{1}{2}\int |\hat{v}|^2\partial_t[K(\xi,t)(1+a(\xi,t))]d\xi. \tag{33}$$

By definition of $K(\xi,t)$, for $t \le \delta$

$$\partial_t[K(\xi,t)(1+a(\xi,t))] = 0 \text{ and } \partial_t K(\xi,t) \le 0. \tag{34}$$

Moreover, for every $0 < t$

$$\begin{aligned} &Re\left\{i\lambda'(t)\sum_{j=1}^n b_j(t)\xi_j\hat{v}\bar{\hat{v}}_t\right\} \le \left|\lambda'(t)\sum_{j=1}^n b_j(t)\xi_j\hat{v}\bar{\hat{v}}_t\right| \\ &\le \frac{\lambda'(t)|\sum_{j=1}^n b_j(t)\xi_j|}{2\lambda(t)\sqrt{\sum_{i,j=1}^n a_{ij}(t)\xi_i\xi_j}}\left\{(1+a(\xi,t))|\hat{v}|^2 + \frac{a(\xi,t)}{(1+a(\xi,t))}|\hat{v}_t|^2\right\}. \end{aligned} \tag{35}$$

It follows by (32)-(35) that

$$E'(t) \le \left(\frac{\alpha\lambda'(t)}{\lambda(t)} + 1\right)E(t) + \int K(\xi,t)Re\Big\{\bar{\hat{v}}_t[-c(t)\hat{v}_t - (2\pi)^{-n}\hat{d}*\hat{v} + \hat{g}]\Big\}d\xi. \tag{36}$$

Here,

$$\alpha = \lim_{t\to 0^+}\sup_{\xi\ne 0}\frac{|\sum_{i=1}^n b_i(t)\xi_i|}{\sqrt{\sum_{i,j=1}^n a_{ij}(t)\xi_i\xi_j}}. \tag{37}$$

Using repeatedly Schwarz inequality and that

$$K(\xi,t) \le C_0 2^m(1+|\eta|^2)^{m+1}K(\xi-\eta,t),$$

with $C_0 = 2 + 2\sup_{|\xi|=1} a(\xi,t)$, we obtain (see [3])

$$\int K(\xi,t)Re\left\{\bar{\hat{v}}_t(d*\hat{v})\right\}d\xi \le \Big(1 + C_0 2^m\sigma_n\|\hat{d}(\cdot,t)\|_{L^1}\|\hat{d}(\cdot,t)\|_{2m+3+[n/2]}\Big)E(t).$$

Therefore

$$E'(t) \le \alpha\frac{\lambda'(t)}{\lambda(t)}E(t) + CE(t) + \frac{1}{2}\int K(\xi,t)|\hat{g}(\xi,t)|^2 d\xi, \tag{38}$$

with $C = (2\pi)^{-n}\Big(2||c||_\infty + 3 + C_0 2^m\sigma_n\|\hat{d}(\cdot,t)\|_{L^1}\|\hat{d}(\cdot,t)\|_{2m+3+[n/2]}\Big)$.

Applying a generalization of Gronwall's lemma (see [8]) to (38), which yields

$$\Big(E(t)\lambda(t)^{-\alpha}e^{-Ct}\Big)' \le \frac{1}{2}\lambda(t)^{-\alpha}e^{-Ct}\int K(\xi,t)|\hat{g}(\xi,t)|^2 d\xi. \tag{39}$$

By Taylor's formula we know that

$$v(x,t) = O(t^{k_0(\ell+1)+\ell+1}) \text{ and } g(x,t) = O(t^{k_0(\ell+1)+\ell-1}). \tag{40}$$

By (21) and (29) we can write

$$\begin{aligned} v_{k_0}(x,t) = &\sum_{q=0}^{k_0} \sum_{j=q(\ell+1)}^{q(\ell+1)+\ell} \frac{t^j}{j!} \Big\{ \partial_t^{j-2} f(x,0) + \sum_{|\rho|\le q} \alpha_{jq\rho} \partial_x^\rho u_0(x) + \beta_{jq\rho} \partial_x^\rho u_1(x) \\ &+ \sum_{|\rho|\le q} \sum_{p=0}^{j-2-|\rho|(\ell+1)} \gamma_{jpq\rho} \partial_x^\rho \partial_t^p f(x,0) \Big\}. \end{aligned} \tag{41}$$

From (40) and (41) it follows

$$\begin{aligned} g(x,t) = &f - \sum_{j=2}^{k_0(\ell+1)+\ell} \frac{t^{j-2}}{(j-2)!} \partial_t^{j-2} f|_{t=0} - t^{\ell-1} \sum_{i=1}^n b_i(t) \partial_{x_i} \sum_{j=k_0(\ell+1)}^{k_0(\ell+1)+\ell} \frac{t^j}{j!} \Big\{ \partial_t^{j-2} f|_{t=0} \\ &+ \sum_{|\rho|\le k_0} \Big(\alpha_{j\rho} \partial_x^\rho u_0(x) + \beta_{j\rho} \partial_x^\rho u_1(x) \Big) + \sum_{|\rho|\le k_0} \sum_{p=0}^{j-2-|\rho|(\ell+1)} \gamma_{jp\rho} \partial_x^\rho \partial_t^p f|_{t=0} \Big\} \\ &+ t^{2\ell} \sum_{i,j=1}^n a_{ij}(t) \partial_{x_i} \partial_{x_j} \sum_{q=k_0-1}^{k_0} \sum_{j=q(\ell+1)}^{q(\ell+1)+\ell} \frac{t^j}{j!} \Big\{ \partial_t^{j-2} f|_{t=0} \\ &+ \sum_{|\rho|\le q} \Big(\alpha_{jq\rho} \partial_x^\rho u_0(x) + \beta_{jq\rho} \partial_x^\rho u_1(x) \Big) + \sum_{|\rho|\le q} \sum_{p=0}^{j-2-|\rho|(\ell+1)} \gamma_{jpq\rho} \partial_x^\rho \partial_t^p f|_{t=0} \Big\} \\ &- c(t) \frac{t^{k_0(\ell+1)+\ell-1}}{(k_0(\ell+1)+\ell-1)!} \Big\{ \partial_t^{k_0(\ell+1)+\ell-2} f|_{t=0} + \sum_{|\rho|\le k_0} \partial_x^\rho \Big(\alpha_\rho u_0(x) + \beta_\rho u_1(x) \Big) \\ &+ \sum_{|\rho|\le k_0} \sum_{p=0}^{(k_0-|\rho|)(\ell+1)+\ell-2} \gamma_{p\rho} \partial_x^\rho \partial_t^p f|_{t=0} \Big\} - d(x,t) \sum_{j=k_0(\ell+1)+\ell-1}^{k_0(\ell+1)+\ell} \frac{t^j}{j!} \Big\{ \partial_t^{j-2} f|_{t=0} \\ &+ \sum_{|\rho|\le k_0} \Big(\alpha_{j\rho} \partial_x^\rho u_0(x) + \beta_{j\rho} \partial_x^\rho u_1(x) \Big) + \sum_{|\rho|\le k_0} \sum_{p=0}^{j-|\rho|(\ell+1)-2} \gamma_{jp\rho} \partial_x^\rho \partial_t^p f|_{t=0} \Big\}. \end{aligned} \tag{42}$$

By Taylor's formula we have

$$\begin{aligned} &f(x,t) - \sum_{j=2}^{k_0(\ell+1)+\ell} \frac{t^{j-2}}{(j-2)!} \partial_t^{j-2} f(x,0) \\ &= \frac{t^{k_0(\ell+1)+\ell-1}}{(k_0(\ell+1)+\ell-2)!} \int_0^1 (1-s)^{k_0(\ell+1)+\ell-2} (\partial_t^{k_0(\ell+1)+\ell-1} f)(x,ts) ds. \end{aligned}$$

Therefore, by (42)

$$K(\xi,t)|\hat{g}(\xi,t)|^2 \leq C(t)t^{2(k_0(\ell+1)+\ell-1)}\Big\{(1+|\xi|^2)^{m+k_0+1}\left(|\hat{u_0}(\xi)|^2+|\hat{u_1}(\xi)|^2\right) + \sum_{|\rho|\leq k_0}\sum_{i=0}^{(k_0-|\rho|)(\ell+1)+\ell-2}\gamma_{i\rho}(1+|\xi|^2)^{m+|\rho|+1}|\partial_t^i\hat{f}(\xi,0)|^2 + (1+|\xi|^2)^m\int_0^1|(\partial_t^{k_0(\ell+1)+\ell-1}f)(\xi,ts)|^2ds\Big\}.$$

By our hypothesis (8) we have

$$2(k_0(\ell+1)+\ell-1)-\ell\alpha > -1,$$

hence we can integrate the inequality (39) from 0 to t, to find

$$\begin{aligned} E(t) \leq& \lambda(t)^\alpha e^{Ct}\int_0^t\lambda(s)^{-\alpha}e^{-Cs}\int K(\xi,s)|\hat{g}(\xi,s)|^2d\xi ds \\ \leq& C(t)\Big\{\|u_0\|^2_{m+k_0+1}+\|u_1\|^2_{m+k_0+1}+\sum_{|\rho|\leq k_0}\sum_{i=0}^{(k_0-|\rho|)(\ell+1)+\ell-2}\|\partial_t^i f(\cdot,0)\|^2_{m+|\rho|+1} \\ &+\sup_{0\leq s\leq t}\|\partial_s^{k_0(\ell+1)+\ell-1}f(\cdot,s)\|^2_m\Big\}, \end{aligned} \tag{43}$$

here $C(t)$ is a continuous function, with $C(0)=0$.

By (43) and $v(x,t)=u(x,t)-v_{k_0}(x,t)$ it follows

$$\|u(\cdot,t)\|^2_m+\|u_t(\cdot,t)\|^2_{m-1}\leq C(t)\Big\{\|u_0\|^2_{m+k_0+1}+\|u_1\|^2_{m+k_0+1} + \sum_{|\rho|\leq k_0}\sum_{i=0}^{(k_0-|\rho|)(\ell+1)+\ell-2}\|\partial_t^i f(\cdot,0)\|^2_{m+|\rho|+1}+\sup_{0\leq s\leq t}\|\partial_s^{k_0(\ell+1)+\ell-1}f(\cdot,s)\|^2_m\Big\}.$$

Then, by standard arguments, as in the end of Proposition 2.1 we conclude the proof of Theorem 1.1.

4 Proof of Theorem 1.2 and of Theorem 1.3

Our proofs are heavily based on those of the Theorem 1 and 2 of ([11]). We recall the reduction process, with a minor modification, and the essential lemmas presented there.

Proof of Theorem 1.2.

First we consider the iterates $u^{(i)}$, $i = 0, \cdots, p$ defined as

$$u_{tt}^{(0)} = h(x, t, u^{(0)}, u_t^{(0)}, 0, \cdots, 0),$$
$$u^{(0)}(x,0) = u_0(x), \quad u_t^{(0)}(x,0) = u_1(x), \tag{44}$$

and for $i = 1, \cdots, p$

$$u_{tt}^{(i)} = h(x, t, \sum_{k=0}^{i} u^{(k)}, \sum_{k=0}^{i} u_t^{(k)}, \lambda'(t) \sum_{k=0}^{i-1} \nabla_x u^{(k)}) \tag{45}$$
$$-h(x, t, \sum_{k=0}^{i-1} u^{(k)}, \sum_{k=0}^{i-1} u_t^{(k)}, \lambda'(t) \sum_{k=0}^{i-2} \nabla_x u^{(k)}) + \lambda^2 \sum_{l,j} a_{lj} \partial^2_{x_l x_j} u^{(i-1)},$$
$$u^{(i)}(x,0) = u_t^{(i)}(x,0) = 0. \tag{46}$$

Reissig had proved the following two results. The first one concerning the regularity of the solutions and the second one, to their asymptotic behaviour.

Lemma 4.1 *(Lemma 2 of [11]) There exists a positive constant T_p such that the system of nonlinear ordinary differential equations (44)–(46) has uniquely determined solutions $u^{(i)} \in C^2([0, T_p], H^{s_0-1-2i}(\mathbb{R}^n))$, $i = 1, \cdots, p$ where $s_0 - 1 - 2p > n/2 + 1$. Moreover, to given positive constants ϵ_i, $i = 0, \cdots, p$ there exists an interval $[0, T_p]$ in which the estimates*

$$\max_{j=1,\cdots,n} \left\{ |u^{(0)} - u_0|, |u_t^{(0)} - u_1|, |\lambda'(t) \partial_{x_j}(u^{(0)} - u_0)| \right\} \le \epsilon_0,$$
$$\max_{j=1,\cdots,n} \left\{ |u^{(i)}|, |u_t^{(i)}|, |\lambda'(t) u_{x_j}^{(i)}| \right\} \le \epsilon_i, \tag{47}$$

hold for all $(x,t) \in \mathbb{R}^n \times [0, T_p]$, $i = 1, \cdots, p$.

Lemma 4.2 *(Lemma 3 of [11]) Choosing data $u_0 \in H^{s_0}(\mathbb{R}^n), u_1 \in H^{s_0-1}(\mathbb{R}^n)$ with $n/2 + 1 < \tilde{N} \le s_0 - 2p$ it holds*

$$\mathcal{F}_{\tilde{N}}(u^{(k)})(t) \le C_{\tilde{N},k} \lambda(t)^k, \tag{48}$$

for all $t \in [0, T_p]$ *and* $k = 0, ..., p$. *Here*

$$\mathcal{F}_{\tilde{N}}(w)(t) = \left(\sum_{|\alpha| \le \tilde{N}-1} \int |\xi^\alpha|^2 [|\hat{w}_t|^2 + |\hat{w}|^2] d\xi \right)^{1/2}.$$

Let us consider the function

$$v(x,t) = u(x,t) - \sum_{k=0}^{p} u^{(k)}(x,t), \tag{49}$$

with p to be chosen later and the functions $u^{(k)} \in C^2([0,T_p], H^{s_0-1-2k}(\mathbb{R}^n))$, $k = 0, 1, ..., p$, are the solutions of (44) to (46). It remains to consider the Cauchy problem

$$\partial_t^2 v - \lambda^2(t) \sum_{i,j=1}^{n} a_{ij}(t) \partial^2_{x_i x_j} v = F(x,t,v,\partial_t v, \lambda'(t)\nabla v), \tag{50}$$

with

$$v(x,0) = \partial_t v(x,0) = 0. \tag{51}$$

Here

$$F(x,t,v,\partial_t v, \lambda'(t)\nabla v) = h\Big(x,t,v+\sum_{k=0}^{p} u^{(k)}, \partial_t(v+\sum_{k=0}^{p} u^{(k)}), \lambda'(t)\nabla(v+\sum_{k=0}^{p} u^{(k)})\Big)$$

$$-h\Big(x,t,\sum_{k=0}^{p} u^{(k)}, \sum_{k=0}^{p} \partial_t u^{(k)}, \lambda'(t)\sum_{k=0}^{p-1} \nabla u^{(k)}\Big) + \lambda^2(t) \sum_{i,j=1}^{n} a_{ij}(t) \partial^2_{x_i x_j} u^{(p)}.$$

Now, to solve the Cauchy problem (50)–(51) we consider the successive approximation scheme

$$\partial_t^2 v^{(q+1)} - \lambda^2(t) \sum_{i,j=1}^{n} a_{ij}(t) \partial^2_{x_i x_j} v^{(q+1)} = F(x,t,v^{(q)}, \partial_t v^{(q)}, \lambda'(t)\nabla v^{(q)}),$$

$$v^{(q+1)}(x,0) = \partial_t v^{(q+1)}(x,0) = 0.$$

We define $v^{(0)} \equiv 0$. Then the differences $w^{(q)} = v^{(q+1)} - v^{(q)}$ satisfy

$$\partial_t^2 w^{(0)} - \lambda^2(t) \sum_{i,j=1}^{n} \tilde{a}_{ij}(t) \partial^2_{x_i x_j} w^{(0)} = F(x,t,0,0,0),$$

$$\partial_t^2 w^{(q)} - \lambda^2(t)\sum_{i,j=1}^{n} \tilde{a}_{ij}(t)\partial^2_{x_i x_j} w^{(q)} = F(x,t,v^{(q)},\partial_t v^{(q)},\lambda'(t)\nabla v^{(q)})$$
$$- F(x,t,v^{(q-1)},\partial_t v^{(q-1)},\lambda'(t)\nabla v^{(q-1)}),$$

with $w^{(q)}(x,0) = \partial_t w^{(q)} = 0$, $q = 0,1,....$ Now, with

$$p = max\left\{3, \left[\frac{Q}{\sqrt{\gamma}}\right] + 2\right\} > \max\left\{2, \frac{Q}{\sqrt{\gamma}} + 1\right\},$$

we will use Proposition 2.1 to estimate $\mathcal{E}_N(w^{(q)})(t)$, with

$$N = s_0 - 2(p+1) \geq 7 - 2\max\left\{3 - \left[\frac{Q}{\sqrt{\gamma}}\right], 2\right\} + [n/2] + r.$$

i) We have that $w^{(0)} \in C([0,T_p], H^N(\mathbb{R}^n)) \cap C^1([0,T_p], H^{N-1}(\mathbb{R}^n))$ and $\mathcal{E}_N(w^{(0)})(t) \leq C_{N,0}\lambda(t)^p$ for all $t \in [0,T_\epsilon]$, for some $0 < T_\epsilon$. As in Reissig ([11]), using (48), one can prove that

$$\|F(x,t,0,0,0)\|_{N-1} \leq C_{N,0}\lambda'(t)\lambda(t)^{p-1},$$

and we conclude the statement by Proposition 2.1.

ii) For a small $T^* > 0$, $w^{(q)} \in C([0,T^*], H^N(\mathbb{R}^n)) \cap C^1([0,T^*], H^{N-1}(\mathbb{R}^n))$ and for some $0 < r < 1$, $\mathcal{E}_N(w^{(q)})(t) \leq C_{N,0} r^q \lambda(t)^p$ for all $q \geq 0$ and $t \in [0,T^*]$. In fact, by Taylor's formula

$$F(x,t,v^{(q)},\partial_t v^{(q)},\lambda'(t)\nabla v^{(q)}) - F(x,t,v^{(q-1)},\partial_t v^{(q-1)},\lambda'(t)\nabla v^{(q-1)})$$
$$= h\Big(x,t,v^{(q)} + \sum_{k=0}^{p} u^{(k)}, \partial_t(v^{(q)} + \sum_{k=0}^{p} u^{(k)}), \lambda'(t)\nabla(v^{(q)} + \sum_{k=0}^{p} u^{(k)})\Big)$$
$$- h\Big(x,t,v^{(q-1)} + \sum_{k=0}^{p} u^{(k)}, \partial_t(v^{(q-1)} + \sum_{k=0}^{p} u^{(k)}), \lambda'(t)\nabla(v^{(q-1)} + \sum_{k=0}^{p} u^{(k)})\Big)$$
$$= \int_0^1 \Big\{(\partial_3 h)(X+\tau H)w^{(q-1)} + (\partial_4 h)(X+\tau H)\partial_t w^{(q-1)}\Big\}\,d\tau$$
$$+ \int_0^1 \left\{\sum_{i=1}^{n}(\partial_{r_i} h)(X+\tau H)\lambda'(t)\partial_{x_i} w^{(q-1)}\right\} d\tau,$$

with $X = \Big(x,t,v^{(q-1)} + \sum_{k=0}^{p} u^{(k)}, \partial_t(v^{(q-1)} + \sum_{k=0}^{p} u^{(k)}), \lambda'(t)\nabla(v^{(q-1)} + \sum_{k=0}^{p} u^{(k)})\Big)$ and $H = \Big(0,0,w^{(q-1)},\partial_t w^{(q-1)}), \lambda'(t)\nabla w^{(q-1)}\Big)$. In order to apply Proposition 2.1 we have to compute, iteratively,

$$\left\| F(x,t,v^{(q)},\partial_t v^{(q)},\lambda^{'}(t)\nabla v^{(q)}) - F(x,t,v^{(q-1)},\partial_t v^{(q-1)},\lambda^{'}(t)\nabla v^{(q-1)})\right\|_{N-1}.$$

To handle the higher order derivative of $w^{(q-1)}$, we must estimate the L^2-norm of

$$\int_0^1 \left\{ \sum_{i=1}^n (\partial_{r_i} h)(X+\tau H) d\tau \lambda^{'}(t) \partial_x^\alpha \partial_{x_i} w^{(q-1)} \right\}, \ |\alpha| = N-1.$$

If the arguments of the integrand are bounded (we will see later that this is possible), using Hölder's inequality, the sum of the square of the $L^2_x(\mathbb{R})$-norm of these terms can be estimated by

$$\begin{aligned}
&\sum_{|\alpha|=N-1} \left\| \sum_{i=1}^n \int_0^1 (\partial_{r_i} h)(X+\tau H) d\tau \lambda^{'}(t) \partial_x^\alpha \partial_{x_i} w^{(q-1)} \right\|_{L^2}^2 \\
\leq & \lambda^{'}(t)^2 \sum_{|\alpha|=N-1} \left\| \Big(\sum_{i=1}^n \Big| \int_0^1 (\partial_{r_i} h)(X+\tau H) d\tau \Big|^2 \Big)^{1/2} \Big(\sum_{i=1}^n |\partial_x^\alpha \partial_{x_i} w^{(q-1)}|^2 \Big)^{1/2} \right\|_{L^2}^2.
\end{aligned}$$

Using (H_4), (47), Plancherel's theorem and (H_1), for a given $\epsilon > 0$, there exist $T_\epsilon > 0$ such that

$$\begin{aligned}
&\left(\sum_{|\alpha|=N-1} \left\| \int_0^1 \left\{ \sum_{i=1}^n (\partial_{r_i} h)(X+\tau H) \lambda^{'}(t) \partial_x^\alpha \partial_{x_i} w^{(q-1)} \right\} d\tau \right\|_{L^2}^2 \right)^{1/2} \\
&\leq (Q+\epsilon)\lambda^{'}(t) \left(\sum_{|\alpha|=N-1} \int \sum_{i=1}^n |\partial_x^\alpha \partial_{x_i} w^{(q-1)}|^2 dx \right)^{1/2} \\
&\leq (Q+\epsilon)\lambda^{'}(t) \left(\sum_{|\alpha|=N-1} \int |\xi^\alpha|^2 |\xi|^2 \left| \widehat{w^{(q-1)}} \right|^2 d\xi \right)^{1/2} \\
&\leq \frac{(Q+\epsilon)}{\sqrt{\gamma}} \frac{\lambda^{'}(t)}{\lambda(t)} \mathcal{E}_N(w^{(q-1)})(t),
\end{aligned}$$

for all $t \in [0, T_\epsilon]$. As in the work of Reissig ([11]), pp. 250, by Gagliardo-Nirenberg's estimate, we obtain

$$\mathcal{E}_N(w^{(q)}) \leq C_{N,q-1} \frac{Q+\epsilon}{\sqrt{\gamma}(p-1-\epsilon)} \lambda(t)^p \Big(1 + 2\phi_N(D_N) D_N \sup_{[0,T]} \lambda'(t) \Big).$$

Here ϕ_N is a positive increasing function in N and D_N depends on $\mathcal{E}_N(v^{(q)} + \sum_{k=0}^p u^{(k)})$, for all $q \geq 0$ and $t \in [0, T_\epsilon]$.

Now taking $D_N = 1 + \max \mathcal{E}_N(\sum_{k=0}^p u^{(k)})(t)$, by (H_3) and $p > \max\Big\{2, \frac{Q}{\sqrt{\gamma}} + 1\Big\}$, we can run the iterative process once we assume that T^* is sufficiently small such that $\frac{Q+\epsilon}{\sqrt{\gamma}(p-1-\epsilon)}\Big(1+2\phi(D_N)D_N\lambda'(T^*)\Big) = r < 1$ and $C_{N,0}\lambda(T^*)^p/(1-r) \leq 1$. Then it follows that $\mathcal{E}_N(v^{(q)})(t) \leq 1$ and $\mathcal{E}_N(w^{(q)}) \leq r^q C_{N,0}\lambda(t)^p$ for all $q \geq 0$ and for all $t \in [0, T^*]$.

This implies that $v^{(q)}$ is a Cauchy sequence in $C([0,T^*], H^N(\mathbb{R}^n)) \cap C^1([0,T^*], H^{N-1}(\mathbb{R}^n))$ and the proof is completed.

Proof of Theorem 1.3

Step 1. By the change of variables presented in section 2, we can assume that the Cauchy problem (6)–(7) has the form (13)–(14) with h satisfying (15).

Let v_1 and v_2 two solutions of (13)–(14), then $w = v_1 - v_2$ solves

$$\partial_t^2 w - \lambda^2(t)\sum_{i,j=1}^{n} a_{ij}(t)\partial^2_{x_i x_j} w = \lambda'(t)\sum_{i=1}^{n} b_i(x,t,v_1,v_2)\partial_{x_i} w$$

$$+ c(x,t,v_1,v_2)\partial_t w + d(x,t,v_1,v_2)w, \tag{52}$$

$$w(x,0) = \partial_t w(x,0) = 0. \tag{53}$$

Now, if $s_0 > n/2+1$, there exist constants C_0, C_1 and C_{0i} such that $|u_0(x)| \leq C_0$, $|u_1(x)| \leq C_1$ and $|\partial_{x_i} u_0(x)| \leq C_{0i}$ for all $x \in \mathbb{R}^n$ and $i = 1, ..., n$. By continuity, for a given $0 < \epsilon$, there exists $T_\epsilon > 0$ such that

$$|v_k(x,t)| \leq C_0 + \epsilon,\ |\partial_t v_k(x,t)| \leq C_1 + \epsilon,\ |\partial_{x_i} v_k(x,t)| \leq C_{0i} + \epsilon,$$

for all $(x,t) \in \mathbb{R}^n \times [0, T_\epsilon]$, $i = 1, ..., n$ and $k = 1, 2$. Now, since

$$b_i(x,t,v_1,v_2) = \int_0^1 (\partial_{r_i} h)\Big((x,t,v_2,\partial_t v_2,\lambda'(t)\nabla v_2) + \tau H\Big)d\tau$$

with $H = (0, 0, v_1 - v_2, \partial_t(v_1 - v_2), \lambda'(t)\nabla(v_1 - v_2))$, by (15) we have

$$Q = \lim_{t\to 0^+} \sup\left(\sum_{i=1}^{n}\Big|b_i(x,t,v_1,v_2)\Big|^2\right)^{1/2}, \tag{54}$$

uniformly in x.

Step 2. Taking as before

$$\mathcal{F}_N(w)(t) = \left(\sum_{|\alpha|\leq N-1}\int |\xi^\alpha|^2[|\hat{w}_t|^2 + |\hat{w}|^2]d\xi\right)^{1/2}.$$

As in Reissig ([11]), pp. 252, after differentiaton of $\mathcal{F}_N(w)(t)$, we obtain

$$\mathcal{F}_N'(w)(t) \leq C_N \mathcal{F}_N(w)(t) + \lambda'(t) C_{N+1} \mathcal{F}_{N+1}(w)(t) + \lambda^2(t) C_{N+2} \mathcal{F}_{N+2}(w)(t),$$

for all $t \in [0,T]$ and $1 \leq N \leq s_0 - 2$. Gronwall's inequality yields

$$\mathcal{F}_N(w)(t) \leq C_{N,1} \lambda(t) \max_{[0,T]} \mathcal{F}_{N+2}(w)(t),$$

for all $t \in [0,T]$ and $1 \leq N \leq s_0 - 2$. It is easy to prove by induction that

$$\mathcal{F}_N(w)(t) \leq C_{N,p} \lambda(t)^p \max_{[0,T]} \mathcal{F}_{N+2(1+[p/2])}(w)(t), \tag{55}$$

for all $t \in [0,T]$ and $1 \leq N \leq s_0 - 2 - p$.

Step 3. As in Proposition 2.1, by Plancherel's theorem, we get

$$\begin{aligned}\mathcal{E}_N'(w)(t) \leq & \frac{\lambda'(t)}{\lambda(t)} \mathcal{E}_N(w)(t) + (M/\gamma + 1) \mathcal{E}_N(w)(t) \\ & + \left(\sum_{|\alpha| \leq N-1} \int |\partial_x^\alpha G(x,t)|^2 dx \right)^{1/2},\end{aligned}$$

with

$$G(x,t) = \lambda'(t) \sum_{i=1}^n b_i(x,t,v_1,v_2) \partial_{x_i} w + c(x,t,v_1,v_2) \partial_t w + d(x,t,v_1,v_2) w.$$

By (54), given a $\epsilon > 0$, there exists $T_\epsilon > 0$ such that for all $t \in [0, T_\epsilon]$

$$\begin{aligned}& \left(\sum_{|\alpha|=N-1} \left\| \lambda'(t) \sum_{i=1}^n b_i(x,t,v_1,v_2) \partial_x^\alpha \partial_{x_i} w \right\|_{L^2}^2 \right)^{1/2} \\ \leq & \lambda'(t) \left(\sum_{|\alpha|=N-1} \left\| \Big(\sum_{i=1}^n |b_i(x,t,v_1,v_2)|^2 \Big)^{1/2} \Big(\sum_{i=1}^n |\partial_x^\alpha \partial_{x_i} w|^2 \Big)^{1/2} \right\|_{L^2}^2 \right)^{1/2} \\ \leq & (Q+\epsilon) \lambda'(t) \left(\sum_{|\alpha|=N-1} \int \sum_{i=1}^n |\partial_x^\alpha \partial_{x_i} w|^2 dx \right)^{1/2} \\ \leq & (Q+\epsilon) \lambda'(t) \left(\sum_{|\alpha|=N-1} \int |\xi^\alpha|^2 |\xi|^2 \, |\hat{w}|^2 \, d\xi \right)^{1/2} \\ \leq & \frac{(Q+\epsilon)}{\sqrt{\gamma}} \frac{\lambda'(t)}{\lambda(t)} \mathcal{E}_N(w)(t).\end{aligned}$$

The other derivatives of $\partial_x^\alpha G(x,t)$ can be computed and estimated by chain rule, Leibniz' formula and Gagliardo-Nirenberg's inequality, which allows us to write

$$\mathcal{E}_N'(w)(t) \leq \left(\frac{Q+\epsilon}{\sqrt{\gamma}}+1\right)\frac{\lambda'(t)}{\lambda(t)}\mathcal{E}_N(w)(t) + C_N\mathcal{E}_N(w)(t). \tag{56}$$

Using $\mathcal{E}_N(w)(t) \leq C_N\mathcal{F}_{N+1}(w)(t)$, it follows from (55)

$$\mathcal{E}_N(w)(t) \leq C_{N+1,p}\lambda(t)^p \max_{[0,T]} \mathcal{F}_{N+1+2(1+[p/2])}(w)(t), \tag{57}$$

for all $t \in [0,T]$. By (56), for $0 < t < T_\epsilon$

$$\left(\mathcal{E}_N(w)(t)\lambda(t)^{-(\frac{Q+\epsilon}{\sqrt{\gamma}}+1)}e^{-C_N t}\right)' \leq 0.$$

Finally, if we choose $p > \frac{Q+\epsilon}{\sqrt{\gamma}}+1$, from (57) we can integrate from 0 to t, to conclude that $\mathcal{E}_N(w)(t) = 0$. Since $s_0 > \max\{n/2+1, 3+p\}$, we have that $\mathcal{F}_{2(2+[p/2])}(w)(t)$ is finite. Therefore, for $N = 1$ (57) holds, concluding that $\mathcal{E}_1(w)(t) = 0$ for $t \in [0,T_\epsilon]$. From the strictly hyperbolic theory we obtain the uniqueness in $[0,T]$.

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