

Continuity Properties on p for p -Laplacian Parabolic Problems

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Abstract

In this work we obtain some continuity properties on the parameter p at $p = 2$ for the Takeuchi-Yamada problem which is a degenerate p -laplacian version of the Chafee-Infante problem. We prove the continuity of the flows and the equilibrium sets, and the upper semicontinuity of the global attractors.

1 Introduction

In 1974 N. Chafee and E. F. Infante completely described the set of stationary solutions of a semilinear parabolic problem like

$$\begin{cases} u_t = \lambda u_{xx} + u - u^3, & (x, t) \in (0, 1) \times (0, +\infty) \\ u(0, t) = u(1, t) = 0, & 0 \leq t < +\infty \\ u(x, 0) = u_0(x), & x \in (0, 1), \end{cases} \quad (1.1)$$

where λ is a positive parameter and the initial data are sufficiently smooth. The set of equilibrium states, E_λ , is taken as function of λ and, roughly speaking, the authors obtain that, for large values

of λ , the only stationary solution is zero, and all nonconstant equilibria bifurcate from zero, two by two, while λ cross the values of a sequence λ_n , obtained from the eigenvalues of the linearized problem. For details see [2].

A similar problem, involving the p -laplacian operator was studied by Takeuchi and Yamada in 2000. They consider the problem

$$\begin{cases} u_t = \lambda(|u_x|^{p-2}u_x)_x + |u|^{q-2}u(1 - |u|^r), & (x, t) \in (0, 1) \times (0, +\infty) \\ u(0, t) = u(1, t) = 0, & 0 \leq t < +\infty \\ u(x, 0) = u_0(x), & x \in (0, 1), \end{cases} \quad (1.2)$$

where $p > 2$, $q \geq 2$, $r > 0$ and $\lambda > 0$. In this case the set of equilibrium points, E_λ , is always infinity if $p > q$ and, if $p = q$ or $p < q$, E_λ is a finite set only for large values of λ . However, in each of the three cases, there is the possibility of existing continuum equilibrium sets, which does not happen in the semilinear case, $p = 2$. Notice that problem (1.1) can be seen as a limit problem of (1.2) taking $p = q = r = 2$.

If we consider only the case $p = q$ in (1.2), there are several similarities between this problem and (1.1). In fact, although there is the possibility of bifurcation of a continuum equilibrium set in (1.2), the numbers of connected components of E_λ is always finite for fixed values of λ , and the scheme of bifurcation of this components is the same of (1.1). The stability properties of equilibria are the same, that is, in both cases the trivial solution is asymptotically stable for large values of the diffusion parameter λ and became unstable when appears the first pair of nontrivial stationary solutions, which are asymptotically stable as long as they exist. Any other stationary solution is unstable, for any p and q .

It is well known that problems (1.1) and (1.2) are globally well-posed in $L^2(0, 1)$ and there is a global attractor $\{\mathcal{A}_2\}$ for (1.1), [2, 9, 4, 6]. The existence of a global attractor $\{\mathcal{A}_p\}$ for (1.2) is easily obtained from the uniform estimates in Section 2. Furthermore, the problems (1.1) and (1.2) generate gradient systems in $C_0^1([0, 1])$ and $W_0^{1,p}(0, 1)$ respectively and therefore, the global attractors are characterized as the union of the unstable set of equilibrium points, [4, 10]. Another interesting similarity we can point out is that, in both problems, the lap-number does not increase through orbits, if the initial conditions are continuous. With this information we can determinate which equilibrium points can belong to the ω -limit set of any initial data. The non increasing property of lap-number was obtained for (1.1) and (1.2) by Matano in 1982 and by Gentile and Bruschi in 2005 respectively, [7, 3].

With all of this it is interesting to investigate in which way the parameter $p > 2$ affects the dynamic of (1.2), analyzing the continuity properties of the flows, the equilibrium sets and the global attractors \mathcal{A}_p , with respect to parameter $p \geq 2$.

This work is organized as follows: in Section 2 we obtain some uniform estimates, with respect to parameter $p > 2$, for solutions of (1.2) on $L^2(0, 1)$, and $W_0^{1,p}(0, 1)$. In Section 3, we show a compactness result that will be fundamental to prove the continuity of the flows in $C([0, T] : L^2(0, 1))$ for each $T > 0$ and the upper semicontinuity of the family of global attractors in $L^2(0, 1)$ which are done in Section 4. Finally, in Section 5, we prove the continuity of the equilibrium sets of (1.2) in $C^1[0, 1]$. Before doing that we first guarantee that, as $p \rightarrow 2$ and for fixed values of λ , any continuum connected component of E_λ reduces to a single point when p is still at a positive

distance of 2. Once we know that the equilibrium sets of (1.2) are discrete for p close enough of 2, we can appeal to results of the ordinary differential equation theory to obtain the continuity of the stationary solutions.

2 Uniform Estimates

In this section we will obtain some uniform estimates for solutions u_p of (1.2) as p goes to 2.

Lemma 2.1 *Let u_p be a solution of (1.2) with $u_p(0) = u_{0,p} \in L^2(0,1)$. Given $T_0 > 0$ there exists $\tilde{K}_1 > 0$ such that $\|u_p(t)\|_2 < \tilde{K}_1$ for $t \geq T_0$ and $p \in (2,3]$. Furthermore, given $B \subset L^2(0,1)$, there exists $K_1 > 0$ such that $\|u_p(t)\|_2 < K_1$ for $t \geq 0$, $p \in (2,3]$ and $u_{0,p} \in B$. The positive constants \tilde{K}_1, K_1 are independent of $q \geq 2$, $r > 0$ and $\lambda > 0$.*

Proof:

Given $s > 0$, multiplying (1.2) by $u_p(s)$ we obtain using Young's Inequality

$$\frac{1}{2} \frac{d}{ds} \|u_p(s)\|_{L^2(0,1)}^2 \leq -\lambda \|u_p(s)\|_{W_0^{1,p}(0,1)}^p + \frac{r}{q+r}. \quad (2.1)$$

Then it follows from Lemma 5.1, [10] that given $T_0 > 0$ there exists a positive constant \tilde{K}_1 such that if $t \geq T_0$,

$$\|u_p(t)\|_{L^2(0,1)} \leq \tilde{K}_1$$

where \tilde{K}_1 is independent of initial data and it can be uniformly chosen for $p \in (2,3]$.

Furthermore, integrating (2.1) from 0 to t we get

$$\|u_p(t)\|_{L^2(0,1)} \leq \left(\|u_{0,p}\|_{L^2(0,1)}^2 + \frac{2rT_0}{(q+r)} \right)^{\frac{1}{2}} \doteq \bar{K}_1, \quad \forall t \in (0, T_0].$$

We define $K_1 \doteq \max \{ \tilde{K}_1, \bar{K}_1 \}$. ■

In order to obtain uniform estimates in $W_0^{1,p}(0,1)$ we consider $\varphi_p^1, \varphi^2 : L^2(0,1) \rightarrow \mathbb{R}$ given by

$$\varphi_p^1 u \doteq \begin{cases} \frac{\lambda}{p} \int_0^1 |u_x(x)|^p dx + \frac{1}{q+r} \int_0^1 |u(x)|^{q+r} dx, & u \in W_0^{1,p}(0,1) \\ +\infty, & \text{in the other case,} \end{cases}$$

and

$$\varphi^2 u \doteq \begin{cases} \frac{1}{q} \int_0^1 |u(x)|^q dx, & u \in L^p(0,1) \\ +\infty, & \text{in the other case.} \end{cases}$$

Then problem (1.2) can be rewritten in the following abstract form

$$\frac{du}{dt}(t) + \partial\varphi_p^1(u(t)) - \partial\varphi^2(u(t)) = 0 \quad (2.2)$$

where $\partial\varphi_p^1$ e $\partial\varphi^2$ are subdifferential of φ_p^1 and φ^2 respectively.

Remark 1 *There are $0 < c_0 < 1$ and $c > 0$ depending only on q and r such that*

$$\varphi^2(u) \leq c_0 \varphi_\lambda^1(u) + c$$

for each $u \in W_0^{1,p}(0,1)$ and $\lambda > 0$. In fact, if $\eta > 0$,

$$\begin{aligned} \varphi^2(u) &= \frac{1}{q} \|u\|_{L^q(0,1)}^q \\ &\leq \frac{r}{(q+r)(\eta q)^{\frac{q+r}{r}}} + q\eta^{\frac{q+r}{q}} \left(\frac{\lambda}{p} \|u\|_{W_0^{1,p}(0,1)}^p + \frac{1}{q+r} \|u\|_{L^{q+r}(0,1)}^{q+r} \right). \end{aligned}$$

It is enough to choose η such that $0 < \eta < \left(\frac{1}{q}\right)^{\frac{q}{q+r}}$, $c_0 \doteq q\eta^{\frac{q+r}{q}}$ and $c \doteq \frac{r}{(q+r)(q\eta)^{\frac{q+r}{r}}}$.

Lemma 2.2 *Let u_p a be solution of (1.2) with $u_p(0) = u_{0,p} \in W_0^{1,p}(0,1)$. There exists a positive constant $K_2 > 0$ such that $\|u_p(t)\|_{W_0^{1,p}(0,1)} < K_2$ for $t \geq 0$ and $p > 2$. Furthermore, the positive constant K_2 can be uniformly chosen for $p \in (2,3]$, $q \in (2,3]$ and initial conditions in bounded subsets of $W_0^{1,p}(0,1)$.*

Proof: Since u_p satisfies (1.2), multiplying this equation by $\frac{d}{dt}u_p$ and integrating from 0 to t we get

$$\int_0^t \left\| \frac{d}{d\tau} u_p \right\|^2 d\tau + (1 - c_0) \varphi_\lambda^1(u_p(t)) \leq \varphi_\lambda^1(u_{0,p}) + c.$$

Then

$$\|u_p(t)\|_{W_0^{1,p}(0,1)} \leq \left\{ \frac{(\varphi_\lambda^1(u_0) + c)p}{\lambda(1 - c_0)} \right\}^{\frac{1}{p}} \doteq K_2. \quad \blacksquare$$

Remark 2 *Let u_p be a solution of (1.2) with $u_p(0) = u_{0,p} \in W_0^{1,p}(0,1)$. From Lemma 2.2 we obtain*

$$\|u_p(t)\|_\infty \leq K_2, \quad t \geq 0.$$

Corollary 2.3 *Let $\{S_p(t)\}_{t \in [0, \infty)}$ be the semigroup defined in $L^2(0,1)$ by (1.2). For each $p > 2$ there exists a global attractor \mathcal{A}_p for $S_p(t)$.*

Proof: It is enough to verify $S_p(t)$ is a bounded dissipative and asymptotically compact semigroup in $L^2(0,1)$, [4]. The dissipativity follows from Lemma 2.1, and the asymptotic compactness follows from Lemma 2.2 and the compact inclusion of $W_0^{1,p}(0,1)$ into $L^2(0,1)$ \blacksquare

3 A Compactness Result

In this section we will adapt the Baras' Theorem, [11], to show that given $T > 0$

$$M^p := \{u_p; p > 2, u_p \text{ is a solution of (1.2)}\},$$

is relatively compact in $C([0, T]; L^2(0, 1))$, with $q \geq 2$, $r > 0$ and $\lambda > 0$.

Definition 3.1 *A subset K in $L^1([a, b], X)$ is called uniformly integrable if for every $\epsilon > 0$ there exists $\gamma(\epsilon) > 0$ such that $\int_E \|f(t)\|_X \leq \epsilon$ for each measurable subset $E \subset [a, b]$ whose Lebesgue measure is less than $\gamma(\epsilon)$, uniformly for $f \in K$.*

The next two lemmas are technical results and their proofs are obtained from the uniform estimates, with respect to $p \in (2, 3]$ proved in Section 2.

Lemma 3.1 *The set $\{u_p; p \in (2, 3], u_p \text{ is a solution of (1.2)}\}$ is a uniformly integrable subset of $L^1([0, T], L^2(0, 1))$.*

Lemma 3.2 *The set $\{f_q(u_p); p \in (2, 3], u_p \text{ is a solution of (1.2) and } f_q(s) = |s|^{q-2}s(1 - |s|^r)\}$ is a uniformly integrable subset of $L^1([0, T], L^2(0, 1))$.*

Proof: In fact, it is enough to note that, from Remark 2 $|u_p| \leq \|u_p\|_{L^\infty(0,1)} \leq K_2$, where K_2 is uniform on $p, q \in (2, 3]$. Therefore we have

$$\begin{aligned} |f_q(u_p)| &= |u_p|^{q-1}|1 - |u_p|^r| \leq |u_p|^{q-1}(1 + |u_p|^r) \\ &\leq K_2^{q-1}(1 + K_2^r) \doteq K_3. \end{aligned}$$

■

Lemma 3.3 *Let $\{S^p(t) : L^2(0, 1) \rightarrow L^2(0, 1), t \geq 0\}$ be the semigroup generated by $\partial\varphi_p$, where $\varphi_p : L^2(0, 1) \rightarrow \mathbb{R}$ is given by*

$$\varphi_p u \doteq \begin{cases} \frac{\lambda}{p} \int_0^1 |u_x(x)|^p dx, & u \in W_0^{1,p}(0, 1) \\ +\infty, & \text{in the other case,} \end{cases}$$

with $p > 2$ and u_p solution of problem (1.2). Then

$$\|S^p(h)u_p(t) - u_p(t)\|_{L^2(0,1)} \rightarrow 0,$$

as $h \rightarrow 0$, uniformly for $p > 2$, for each $t > 0$. Furthermore, if $T > 0$ we obtain that

$$\|S^p(h)u_p(T - h) - u_p(T - h)\|_{L^2(0,1)} \rightarrow 0,$$

as $h \rightarrow 0$, uniformly for $p > 2$.

Proof: For $p > 2$, following [8], we obtain

$$\|S^p(h)u_p(t) - u_p(t)\|_{L^2(0,1)} \leq 3\|u_p(t) - J_h^{\varphi_p}u_p(t)\|_{L^2(0,1)},$$

where $J_h^{\varphi_p} = (I + h\partial\varphi_p)^{-1}$.

Now, since for every $\mu > 0$ and $u \in L^2(0,1)$, Proposition 2.11 in [1] shows that

$$\frac{1}{2\mu}\|J_\mu^{\varphi_p}u - u\|_{L^2(0,1)}^2 + \varphi_p(J_\mu^{\varphi_p}u) = \min_{v \in L^2(0,1)} \left[\frac{1}{2\mu}\|v - u\|_{L^2(0,1)}^2 + \varphi_p(v) \right],$$

we obtain from Lemma 2.2,

$$\frac{1}{2h}\|u_p(t) - J_h^{\varphi_p}u_p(t)\|_{L^2(0,1)}^2 + \varphi_p(J_h^{\varphi_p}u_p(t)) \leq \varphi_p(u_p(t)) \leq pK_2.$$

With the same arguments above, we obtain that

$$\|S^p(h)u_p(T-h) - u_p(T-h)\|_{L^2(0,1)} \leq 3\sqrt{2hpK_2}.$$

■

We are now in conditions to prove the following result

Theorem 3.4 *The set $M^p := \{u_p; p \in (2, 3], u_p \text{ is a solution of (1.2)}\}$, is relatively compact in $C([0, T]; L^2(0, 1))$.*

Proof: From Lemma 2.3.1 in [11], if $t \in [0, T)$ and $h > 0$ is such that $T - h, t + h \in [0, T]$, we obtain that

$$\begin{aligned} \|u_p(t+h) - u_p(t)\|_{L^2(0,1)} &\leq \|u_p(t+h) - S^p(h)u_p(t)\|_{L^2(0,1)} + \|S^p(h)u_p(t) - u_p(t)\|_{L^2(0,1)} \\ &\leq \int_t^{t+h} \|f_q(u_p(s))\|_{L^2(0,1)} ds + \|S^p(h)u_p(t) - u_p(t)\|_{L^2(0,1)} \end{aligned} \quad (3.1)$$

and

$$\|u_p(T-h) - u_p(T)\|_{L^2(0,1)} \leq \int_{T-h}^T \|f_q(u_p(s))\|_{L^2(0,1)} ds + \|u_p(T-h) - S^p(h)u_p(T-h)\|_{L^2(0,1)}. \quad (3.2)$$

From Lemma 3.3, given $\eta > 0$, there exists $\bar{\delta} > 0$ such that

$$\|S^p(h)u_p(t) - u_p(t)\|_{L^2(0,1)} < \frac{\eta}{2},$$

for $0 < |h| < \bar{\delta}$, for every $p \in (2, 3]$, Since $\{f_q(u_p); p \in (2, 3], u_p \text{ is a solution of (1.2)}\}$ is a uniformly integrable subset of $L^1([0, T], L^2(0, 1))$, by Lemma 3.2, there exists $\gamma > 0$ such that

$$\int_t^{t+h} \|f_q(u_p(s))\|_{L^2(0,1)} ds < \frac{\eta}{2},$$

if $|h| < \gamma$. Taking $\delta = \min\{\gamma, \bar{\delta}\}$, it follows from (3.1) that

$$\|u_p(t+h) - u_p(t)\|_{L^2(0,1)} < \eta,$$

for $0 < |h| < \delta$. The same arguments above and (3.2) imply that

$$\|u_p(T-h) - u_p(T)\|_{L^2(0,1)} < \eta,$$

for $0 < |h| < \delta$. Therefore M^p is equicontinuous on $C([0, T]; L^2(0, 1))$.

Now we will show that, for each $t \in (0, T]$,

$$M^p(t) := \{u_p(t); p \in (2, 3], u_p \text{ is a solution of (1.2)}\}$$

is relatively compact in $L^2(\Omega)$.

Fixed an arbitrary $t \in (0, T]$, consider $h > 0$ such that $t - h \in [0, T]$. Again, from Lemma 2.3.1 in [11],

$$\|S^p(h)u_p(t-h) - u_p(t)\|_{L^2(0,1)} \leq \int_{t-h}^t \|f_q(u_p(s))\|_{L^2(0,1)} ds,$$

for each $p \in (2, 3]$. Let $T_h : M^p(t) \subset L^2(0, 1) \rightarrow L^2(0, 1)$ be the operator defined by $T_h u_p(t) = S^p(h)u_p(t-h)$ for $u_p(t) \in M^p(t)$. It follows from Lemma 3.1 that $M^p(t-h)$ is bounded in $L^2(0, 1)$. Since $S^p(h)$ is a compact semigroup we obtain that T_h carries bounded subsets on relatively compact subsets in $L^2(0, 1)$. Moreover, from Lemma 3.2 we obtain that $\lim T_h = I$, as $h \rightarrow 0$, uniformly in $M^p(t)$. Therefore, $I : M^p(t) \rightarrow L^2(0, 1)$ is a compact operator. Since $M^p(t)$ is bounded by Lemma 3.1, $M^p(t)$ is relatively compact in $L^2(0, 1)$, for every $t \in (0, T]$.

Finally, observing that $M^p(0) = \{u_0\}$ is relatively compact, from Ascoli-Arzelà's Theorem we conclude that M^p is relatively compact in $C([0, T], L^2(0, 1))$. ■

4 Continuity of the Flows and Upper Semicontinuity of the Attractors

In this section we will proof that, given $T > 0$, the solutions $\{u_p\}$ of (1.2) go to the solution v of (1.1) in $C([0, T]; L^2(0, 1))$ and, after that, we will obtain the upper semicontinuity on p in $L^2(0, 1)$ of the family of global attractors

$$\{\mathcal{A}_p \subset L^2(0, 1); 2 \leq p \leq 3\}$$

of (1.2) at $p = 2$.

Consider a sequence $\{u_p\} \subset L^2(0, 1)$, where u_p satisfies the problem

$$\begin{cases} \frac{d}{dt} u_p - \lambda(|(u_p)_x|^{p-2} (u_p)_x)_x = |u_p|^{p-2} u_p (1 + |u_p|^2) & \text{a.e. in } (0, 1) \times (0, +\infty) \\ u_p(0) = u_{0,p} \in L^2(0, 1) \end{cases}$$

First of all observe that, from Section 2, there exists a positive constant K , independent of $t \geq 0$ and p such that

$$\|u_p(t)\|_{W_0^{1,2}(0,1)} \leq K$$

for all $t > 0$ and $p \in (2, 3]$. Therefore, for $T > 0$ we obtain that $\{u_p(t)\}$ is uniformly bounded in $L^\infty(0, 1)$ for $p \in (2, 3]$ and $t \in [0, T]$. Furthermore, $|u_p|^{p-2}u_p(1 + |u_p|^2)$ is uniformly integrable in $L^1(0, T; L^2(0, 1))$ and, from Section 3, $\{u_p\}$ converges in $C([0, T]; L^2(0, 1))$ to a function $v : [0, T] \rightarrow L^2(0, 1)$. By continuity, $f_p(u_p(t)) \rightarrow f_p(v(t))$ in $L^2(0, 1)$ for each $t > 0$, where $f : L^\infty(0, 1) \cap L^2(0, 1) \rightarrow L^2(0, 1)$ is given by $f_p(s) = |s|^{p-2}(1 - |s|^2)$. Proposition 3.6 in [1] implies that

$$\frac{1}{2}\|u_p(t) - \phi\|_{L^2(0,1)}^2 \leq \frac{1}{2}\|u_p(s) - \phi\|_{L^2(0,1)}^2 + \int_s^t \langle f_p(u_p(\tau)) + \lambda\Delta_p(\phi), u_p(\tau) - \phi \rangle d\tau, \quad (4.1)$$

for every $\phi \in D(\Delta_p)$ and $0 \leq s \leq t \leq T$.

Now, the idea is take the limit as $p \rightarrow 2$ on the last inequality. Consider initially $\phi \in C_0^2(0, 1)$. Then, for every $p \in (2, 3]$,

$$\Delta_p(\phi) = (|\phi_x|^{p-2}\phi_x)_x = (p-1)|\phi_x|^{p-2}\phi_{xx}.$$

Since $\|\phi_x\|_{L^\infty(0,1)}$ and $\|\phi_{xx}\|_{L^\infty(0,1)}$ are bounded, we obtain that $\|(p-1)|\phi_x|^{p-2}\phi_{xx}\|_{L^2(0,1)}$ is uniformly bounded in $p \in (2, 3]$. Then, $\Delta_p(\phi)$ converges weakly to some w in $L^2(0, 1)$. Let us show that $w = -\lambda\Delta\phi$. In fact,

$$-\lambda\Delta_p\phi = \partial\varphi_p(\phi),$$

where

$$\varphi_p(\phi) = \frac{\lambda}{p}\|\phi_x\|_{L^p(0,1)}^p.$$

Then,

$$\frac{\lambda}{p}\|\xi_x\|_{L^p(0,1)}^p - \frac{\lambda}{p}\|\phi_x\|_{L^p(0,1)}^p \geq \langle -\lambda\Delta_p\phi, \xi - \phi \rangle = \langle \lambda\Delta_p\phi, \phi - \xi \rangle,$$

for every $\xi \in L^2(0, 1)$. Now,

$$\frac{\lambda}{p}\|\phi_x\|_{L^p(0,1)}^p = \frac{\lambda}{p} \int_0^1 |\phi_x(x)|^p dx \longrightarrow \frac{\lambda}{2} \int_0^1 |\phi_x(x)|^2 dx = \frac{\lambda}{2}\|\phi_x\|_{L^2(0,1)}^2.$$

In the same way we get that

$$\frac{\lambda}{p}\|\xi_x\|_{L^p(0,1)}^p \longrightarrow \frac{\lambda}{2}\|\xi_x\|_{L^2(0,1)}^2, \quad \text{as } p \rightarrow 2,$$

for every $\xi \in W_0^{1,3}(0, 1)$. Therefore,

$$\frac{\lambda}{2}\|\xi_x\|_{L^2(0,1)}^2 - \frac{\lambda}{2}\|\phi_x\|_{L^2(0,1)}^2 \geq \langle w, \phi - \xi \rangle \quad (4.2)$$

for $\xi \in W_0^{1,3}(0,1)$. By a density argument we can conclude (4.2) for $\xi \in W_0^{1,2}(0,1)$. Since $\frac{\lambda}{2}\|\xi_x\|_{L^2(0,1)}^2 = +\infty$ if $\xi \notin W_0^{1,2}(0,1)$, then

$$\frac{\lambda}{2}\|\xi_x\|_{L^2(0,1)}^2 - \frac{\lambda}{2}\|\phi_x\|_{L^2(0,1)}^2 \geq \langle w, \phi - \xi \rangle,$$

for $\xi \in L^2(0,1)$. Therefore, $w \in \partial\varphi_2(\phi)$, where

$$\varphi_2(\phi) = \left(\frac{\lambda}{2}\|\phi_x\|_{L^2(0,1)}^2 \right).$$

Then, $w = -\lambda\Delta\phi$, and

$$\Delta_p\phi \rightharpoonup \phi_{xx}, \text{ as } p \rightarrow 2$$

in $L^2(\Omega)$, for every $\phi \in C_0^2(0,1)$.

Taking the limit in (4.1) as p goes to 2, we obtain

$$\frac{1}{2}\|v(t) - \phi\|_{L^2(0,1)}^2 \leq \frac{1}{2}\|v(s) - \phi\|_{L^2(0,1)}^2 + \int_s^t \langle f_p(v(\tau)) + \lambda\phi_{xx}, v(\tau) - \phi \rangle d\tau,$$

for every $\phi \in C_0^2(0,1)$ and $0 \leq s \leq t \leq T$. Again by a density argument we obtain that

$$\frac{1}{2}\|v(t) - \phi\|_{L^2(0,1)}^2 \leq \frac{1}{2}\|v(s) - \phi\|_{L^2(0,1)}^2 + \int_s^t \langle f_p(v(\tau)) + \lambda\phi_{xx}, v(\tau) - \phi \rangle d\tau,$$

for $\phi \in W_0^{1,2}(0,1)$ and $0 \leq s \leq t \leq T$. Therefore v is a weak solution of

$$\begin{cases} \frac{d}{dt}v - \lambda v_{xx} = v - v^3 & \text{a.e. in } (0,1) \times (0, +\infty) \\ v(0) = v_0 \in L^2(0,1). \end{cases}$$

Let us show that the family of global attractors

$$\{\mathcal{A}_p \subset L^2(\Omega); 2 \leq p \leq 3\}$$

of the problem (1.2) is upper semicontinuity in $p = 2$, on $L^2(0,1)$ topology.

For $p \in (2, 3]$, consider $\psi_p \in \mathcal{A}_p$. The results contained in Section 2 imply that there exists a bounded set $\mathcal{B} \subset L^2(0,1)$ such that $\mathcal{A}_p \subset \mathcal{B}$, for every $p \in (2, 3]$. Since \mathcal{A}_2 attracts bounded sets of $L^2(0,1)$, for every $\delta > 0$, there is $T_1 > 0$ in such way that

$$\sup_{\psi_p \in \mathcal{A}_p, p \in [2,3]} \text{dist}_{L^2(\Omega)}(u_2(T_1; \psi_p), \mathcal{A}_2) \leq \frac{\delta}{2},$$

where $u_2(t; \psi_p)$ is a solution of problem (1.1), with initial condition ψ_p .

Now, the previous results in this section imply that there exists $p_\delta \in (2, 3]$ such that

$$\|u_p(t; \psi_p) - u_2(t; \psi_p)\|_{L^2(0,1)} < \frac{\delta}{2},$$

for every $p \in (2, p_\delta]$ and $T \geq t \geq T_1$, (here $u_p(t; \psi_p)$ is a solution of problem (1.2), with initial condition ψ_p).

Thus, for every $p \in (2, p_\delta]$, we obtain that

$$\begin{aligned} \text{dist}_{L^2(0,1)}(u_p(T_1; \psi_p), \mathcal{A}_2) &\leq \|u_p(T_1, \psi_p) - u_2(T_1; \psi_p)\|_{L^2(0,1)} + \text{dist}_{L^2(0,1)}(u_2(T_1, \psi_p), \mathcal{A}_2) \\ &< \delta. \end{aligned}$$

On the other hand, it follows from the invariance of the attractors that

$$\text{dist}_{L^2(0,1)}(\mathcal{A}_p, \mathcal{A}_2) \leq \delta,$$

for every $p \in (2, p_\delta]$, showing the upper semicontinuity desired.

5 Continuity of Equilibrium Sets

In this section, we prove the continuity of the family of equilibrium points of the equation (1.2) when p goes to 2.

First we prove that for λ fixed, $q \geq 2$ and p sufficiently close to 2, there is not flat core in the equilibrium points.

Since we deal with the dependence on the parameter p and there are qualitative changes in the equilibrium sets depending on the relation between p and q , we will exhibit explicitly the parameters p and q .

Following the ideas in [9], let ϕ be a solution of

$$\begin{cases} \lambda(\psi)_x + f_p(\phi) = 0, & \text{in } (0, \infty) \\ \phi(0) = 0, \\ \psi(0) = \alpha \end{cases} \quad (5.1)$$

where α is a parameter, $\psi = |\phi_x|^{p-2}\phi_x$ and $f_q(\phi) = |\phi|^{q-2}\phi(1 - |\phi|^2)$. We observe that, in order to a solution of (5.1) be an equilibrium point of (1.2), it is necessary to find α such that $\phi(1) = 0$.

We denote by $X(\alpha, p, q)$ the function that measure the x -time that the solution ϕ of (5.1) takes to obtain a maximum point. Because of the symmetry we have that $\phi(2X(\alpha, p, q)) = 0$. Then, it is sufficient to get $\alpha_{p,q}$ such that $2X(\alpha_{p,q}, p, q) = 1$ in order to ϕ be an equilibrium point of (1.2).

Since that an equilibrium point has flat core only if its maximum value is 1 or, by symmetry, its minimum value is -1 , we prove that for $\lambda > 0$ fixed, $X(\alpha_{p,q}, p, q)$ goes to infinity when p goes to 2 where $\alpha_{p,q}$ is taken such that the maximum value of the solution of (5.1) for $\alpha = \alpha_{p,q}$ is equal to 1.

The function X is given by

$$X(\alpha, p, q) = \left(\frac{\lambda(p-1)}{p} \right)^{1/p} I_{p,q}(\phi_\alpha),$$

where ϕ_α is the maximum value of ϕ ,

$$I_{p,q}(a) = \int_0^a (F_q(a) - F_q(\phi))^{-1/p} d\phi,$$

and $F_q(\phi) = \int_0^\phi f_q(s) ds = \frac{|\phi|^q}{q} - \frac{|\phi|^{q+2}}{q+2}$. Then, in order to study X , we deal directly with the function $I_{p,q}(a)$.

By [9], we know that $I_{p,q}(\cdot)$ is continuous on $(0, 1]$ and $\lim_{a \rightarrow 1} I_{p,q}(a) = I_{p,q}(1)$ is finite for each $p > 2$ and $q \geq 2$.

Now we analyze the behavior of $I_{p,q}(1)$ with respect the parameter p , when p goes to 2.

Since 1 is a maximum point of F_q and $F_q''(1) = -2$, using Taylor's polynomial for $F_q(\phi)$ in $\phi = 1$, we get

$$\frac{F_q(1) - F_q(\phi)}{(\phi - 1)^2} = 1 + \frac{R(|\phi - 1|, q)}{(\phi - 1)^2}.$$

It is easy to see that $\lim_{\phi \rightarrow 1} \frac{R(|\phi - 1|, q)}{|\phi - 1|^2} = 0$ uniformly in q for q in a bounded subset of $[2, \infty)$.

Thus,

$$\lim_{\phi \rightarrow 1} \frac{F_q(1) - F_q(\phi)}{(\phi - 1)^2} = 1 > 0,$$

and it is uniform for q in bounded subsets of $[2, \infty)$. Therefore, considering $\epsilon = \frac{1}{2}$, there exists $\delta > 0$ independent of q , such that if $|\phi - 1| < \delta$ then $\left\| \frac{F_q(1) - F_q(\phi)}{(\phi - 1)^2} - 1 \right\| < \frac{1}{2}$.

This fact implies that, for $|x - 1| < \delta$

$$\left(\frac{2}{3}\right)^{1/p} (x - 1)^{-2/p} < \frac{1}{(F_p(1) - F_p(x))^{1/p}} < 2^{1/p} (x - 1)^{-2/p}.$$

Now we are in conditions to estimate

$$\begin{aligned} \int_0^1 \frac{1}{(F_q(1) - F_q(x))^{1/p}} dx &\geq \int_{1-\delta}^1 \frac{1}{(F_q(1) - F_q(x))^{1/p}} dx \\ &\geq \int_{1-\delta}^1 \left(\frac{2}{3}\right)^{1/p} (x - 1)^{-2/p} dx \\ &= \left(\frac{2}{3}\right)^{1/p} \frac{(x - 1)^{-2/p+1}}{1 - 2/p} \Big|_{1-\delta}^1 \\ &= \left(\frac{2}{3}\right)^{1/p} \frac{(\delta)^{1-2/p}}{1 - 2/p}. \end{aligned}$$

When p goes to 2, we get $I_{p,q}(1) = \int_0^1 \frac{1}{(F_q(1) - F_q(x))^{1/p}} dx \rightarrow \infty$, uniformly for q in bounded sets.

Theorem 5.1 *For each $\lambda > 0$ fixed, there exist $p_\lambda = p(\lambda)$ such that*

$$\lambda_{lp}(1) = \frac{p}{p-1}(2(l+1)I_{p,q}(1))^{-p} \leq \lambda$$

for $2 \leq p \leq p_\lambda$ and $l = 0, 1, \dots$. Furthermore, the equilibrium solutions of (1.2) do not have flat core.

Proof: Since $\lim_{p \rightarrow 2} I_{p,q}(1) = +\infty$, we get that $\lim_{p \rightarrow 2} \lambda_{lp}(1) = 0$, for $l = 0, 1, \dots$. In [9], the sequence $\lambda_l(1)$ denotes the values where the equilibrium solutions with l zeros in $(0, 1)$ have 1 as maximum value (resp. -1 as minimum value), then for $\lambda > 0$ fixed, and p close to 2 the equilibrium solution does not reach the value necessary to constitute a flat core. ■

We observe that the independence of the parameter q in a bounded subset of $[2, +\infty)$ is essential in the following, because we consider $p = q$ and $p \rightarrow 2$.

Now, we consider $p = q$ and we prove the convergence of equilibrium points when $p \rightarrow 2$.

For $p \geq 2$ and $\alpha > 0$, let $\phi_{\alpha,p}$ be a solution of (5.1), with $\psi = |\phi_x|^{p-2}\phi_x$, $f_p(\phi) = |\phi|^{p-2}\phi(1 - |\phi|^2)$. Thus, $(\phi_{\alpha,p}, \psi)$ satisfies

$$\frac{\lambda(p-1)}{p}|\psi|^{p/p-1} + F_p(\phi) = \frac{\lambda(p-1)}{p}|\alpha|^{p/p-1}.$$

Define $\tilde{\phi}_{\alpha,p}$ by

$$\frac{\lambda(p-1)}{p}|\alpha|^{p/p-1} = F_p(\tilde{\phi}_{\alpha,p}),$$

that means, $\tilde{\phi}_{\alpha,p}$ is a maximum value of $\phi_{\alpha,p}$. We denote by $\tilde{\phi}_{\alpha,p}$ the function that mapping (α, p) to $\tilde{\phi}_{\alpha,p}$. Using the Implicit Function Theorem, we get that the map $\tilde{\phi}_{\alpha,p}$ is C^1 on (α, p) and strictly increasing on α .

For $p = 2$, we have $\phi_{\alpha,2}$ solution of

$$\begin{cases} \lambda\phi_{xx} + f_2(\phi) = 0, & \text{in } (0, \infty) \\ \phi(0) = 0, \\ \phi_x(0) = \alpha \end{cases} \quad (5.2)$$

and $f_2(\phi) = \phi(1 - \phi^2)$. Thus, $(\phi_{\alpha,2}, \phi_x)$ satisfies

$$\frac{\lambda}{2}|\phi_x|^2 + F_2(\phi) = \frac{\lambda}{2}|\alpha|^2.$$

Define $\tilde{\phi}_{\alpha,2}$ by

$$\frac{\lambda}{2}|\alpha|^2 = F_2(\tilde{\phi}_{\alpha,2}),$$

that means, $\tilde{\phi}_{\alpha,2}$ is the maximum value of $\phi_{\alpha,2}$.

The x -time map $X(\alpha, p)$, which measures the x time necessary to the solution reach the first maximum, is given by

$$X(\alpha, p) = \left(\frac{\lambda(p-1)}{p}\right)^{1/p} \int_0^{\tilde{\phi}_{\alpha,p}} (F_p(\tilde{\phi}_{\alpha,p}) - F_p(\phi))^{1/p} d\phi.$$

For each $p \geq 2$, we search $\alpha_i(p)$ such that $X(\alpha_i(p), p) = 1/2i$ (in order to get the maximum point in $x = 1/2i$, that means, an equilibrium points with $(i-1)$ zeros in $(0, 1)$). In order to do this, we prove that $X(\alpha, p)$ satisfies the hypothesis of the Implicit Function Theorem.

We observe that $X(\alpha, p)$ is C^1 . In fact, by [9],

$$I(p, a) = I_p(a) = \int_0^a (F_p(a) - F_p(\phi))^{1/p} d\phi = \int_0^1 \Phi_p(s, a)^{-1/p} ds,$$

where $\Phi_p(s, a) = \frac{1-s^p}{p} - \frac{1-s^{p+2}}{p+2}a^2$. We have that $\frac{\partial I}{\partial p}(p, a)$ and $\frac{\partial I}{\partial a}(p, a)$ are continuous, therefore $I(\cdot)$ is C^1 . Using that $\tilde{\phi}_{\alpha,p}$ is C^1 , by Chain Rule we have that $X(\alpha, p)$ is C^1 . Furthermore, $\frac{\partial X}{\partial \alpha} > 0$. For each $i, i = 1, 2, \dots$ and for each (α, p) satisfying $X(\alpha, p) = 1/2i$, by Implicit Function Theorem, there is an open set $U \subset \mathbb{R}^2$ such that $(\alpha, p) \in U$ and α is a C^1 function of p satisfying $X(\alpha_i(p), p) = 1/2i$.

Theorem 5.2 *Let ϕ_p^l be the equilibrium point of (1.2) for $p > 2$ with l zeros in $(0, 1)$ and ϕ_2^l be the equilibrium point of (1.1) with l zeros in $(0, 1)$. Then, ϕ_p^l converges to ϕ_2^l in $C^1[0, 1]$ when p goes to 2.*

Proof: Since that each equilibrium point ϕ_p^l of (1.2) is a solution of (5.1) with initial data $\phi(0) = 0$ and $\psi(0) = \alpha_{l,p}$, we rewrite (5.1) in the following form

$$\dot{z} = h(z, p), \tag{5.3}$$

where $z = [\phi, \psi]$ and $h((\phi, \psi), p) = (\text{sign}(\psi)|\psi|^{1/(p-1)}, -f_p(\phi)/\lambda)$. We have that the map h depends continuously on p and its local Lipschitz constant with respect to z is independent of p for $p \in [2, p_0]$, where p_0 is close enough to 2, in order to guarantee that ϕ doesn't have a flat core.

Then, by continuous dependence with respect initial data and parameter p (see [5]) we have that the solution z_p of (5.3) depends continuously on p and on $(\phi(0), \psi(0)) = (0, \alpha_{l,p})$.

Thus, we obtain that z_p converges to z_2 when $p \rightarrow 2$, that means, ϕ_p converges to ϕ_2 and $\psi_p \rightarrow \psi_2$ when $p \rightarrow 2$ in $C[0, 1]$. ■

References

- [1] H. Brézis, *Operateurs Maximaux Monotones et Semi-groupes de Contractions dans les Espaces de Hilbert*, North-Holland Publishing Company, Amsterdam, (1973).
- [2] N. Chafee and E. F. Infante, A Bifurcation Problem for a Nonlinear Partial Differential Equation of Parabolic Type. *Applicable Analysis* **4**, (1974), pp 17-37.

- [3] C. B. Gentile and S. M. Bruschi, Lap number properties for p -Laplacian problems investigated by Lyapunov methods. *Nonlinear Anal.* 66 (2007), no. 5, 1005–1015.
- [4] J. K. Hale, *Asymptotic Behaviour of Dissipative Systems*, Mathematical Surveys and Monographs, **25**, American Mathematical Society, (1989).
- [5] Hale, J.K., *Ordinary Differential Equations*, Wiley Interscience, (1969).
- [6] Henry, D., *Geometric theory of semilinear parabolic equations*, Lecture Notes in Mathematics, 840. Springer-Verlag, Berlin-New York, 1981.
- [7] H. Matano, Nonincrease of the Lap-number of a Solution for a One-dimensional Semilinear Parabolic Equation, *J. Fac. Sci. Univ. Tokyo Sect., 1A Math.* **29**, 401-441, (1982).
- [8] A. T. Plant, Four Inequalities for Monotone Gradient Vector Fields, *Arch. Rational Mech. Anal.* **82**, 4 (1983), pp. 377-389.
- [9] S. Takeuchi and Y. Yamada, Asymptotic Properties of a Reaction-Diffusion Equation with Degenerate p -Laplacian, *Nonlinear Analysis*, **42**, 41-61, (2000).
- [10] R. Temam, *Infinite Dimensional Dynamical Systems in Mechanics and Physics*, Applied Mathematical Sciences, vol 68, Springer-Verlag, New York, (1988).
- [11] I. I. Vrabie, *Compactness Methods for Nonlinear Evolutions*, Pitman Monographs and Surveys in Pure and Applied Mathematics, London (1987).