A RUDIN-CARLESON THEOREM FOR PLANAR VECTOR FIELDS

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Abstract. This paper generalizes the Rudin-Carleson theorem for homogeneous solutions of locally solvable real analytic vector fields.

1. Introduction and preliminaries

Consider a holomorphic function f on the unit disc $D = \{z \in \mathbb{C} : |z| < 1\}$ that is continuous on the closure of D. It is well known that if $f = 0$ on a subset of the boundary $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ of positive Lebesgue measure, then $f \equiv 0$. In the converse direction, given a closed subset $E \subseteq \mathbb{T}$ of Lebesgue measure zero, there exists a nonconstant continuous function h on $\overline{D}$ which is holomorphic on D and vanishes on E. In fact, more is known: Rudin [R1] and Carleson [C] independently proved the following result that today is known as the Rudin-Carleson Theorem.

Rudin-Carleson Theorem. *Let E be a closed set of Lebesgue measure zero in the unit circle $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$. If g is a continuous function on $\mathbb{T}$, then there is a continuous h on the closed disc $\overline{D} = \{z : |z| \le 1\}$ which is holomorphic on $D = \{z : |z| < 1\}$ and*

$$h(z) = g(z), \quad z \in E,$$
$$\sup_{z\in\mathbb{T}} |h(z)| \le \sup_{z\in E} |g(z)|.$$

This theorem has been a subject of several papers where among other things, new proofs and refinements have been given (see for example, [B], [D] and [O]).

The Rudin-Carleson theorem may be regarded as a result on the existence of a homogeneous solution of the Cauchy-Riemann operator defined on an open set with prescribed values on a certain subset of the boundary. We wish to explore the validity of the Rudin-Carleson theorem for more general complex vector fields. In general, as the examples below show, we cannot expect the theorem to be true for any vector field. For instance, let

$$M = \frac{\partial}{\partial y} - 2iy\frac{\partial}{\partial x} \tag{1}$$

1991 *Mathematics Subject Classification.* Primary 35F15, 35B30; Secondary 42A38, 30E25.
Key words and phrases. Rudin-Carleson property, Sussmann's orbits, Condition $(\mathcal{P})$.
Work supported in part by NSF DMS 0714696, CNPq and FAPESP.

be the Mizohata vector field. If $h \in C(\overline{D})$ satisfies $Mh = 0$ on D in the sense of distributions, it is known [N] that $h(x, y) = h(x, -y)$, $x + iy \in \overline{D}$. To see this, observe that both $h(x, \sqrt{y})$ and $h(x, -\sqrt{y})$ are holomorphic for $y > 0$, continuous up to $y = 0$, and agree when $y = 0$. Thus $h(x, y) \equiv h(x, -y)$. Hence, if E contains two points $z^+ = x + iy$, $z^- = x - iy$ and $g(z^+) \neq g(z^-)$, no solution h can agree with g on E.

More generally, suppose L is a smooth complex vector field with a solution Z of $LZ = 0$ defined on $\overline{D}$ such that $Z : \mathbb{T} \to \mathbb{C}$ is not one-to-one. Assume that every solution h on D that is continuous on $\overline{D}$ is constant on the fibers $Z^{-1}\{p\}$, $p \in Z(\overline{D})$, of Z. The latter property holds for the Mizohata vector field where we may take $Z(x, y) = x + iy^2$, and also for any locally integrable smooth vector field L if the unit disc is replaced by a disc D_r of sufficiently small radius, as the constancy of h on the fibers of Z is a well known consequence of the Baouendi-Treves approximation formula [BT]. As before, we conclude that there can be no solution if E contains two distinct points of a fiber of Z and g is chosen as above. An example of a slightly different nature is

$$N = \frac{\partial}{\partial y} - 2ix\frac{\partial}{\partial x}. \tag{2}$$

Any $h \in C(\overline{D})$ that satisfies $Nh = 0$ on D in the weak sense will be constant on $(0, y) \in \overline{D}$, so there can be no solution unless $g(0, 1) = g(0, -1)$ if $(0, \pm 1) \in E$. Here the constancy of h comes from the fact that the y-axis is a one-dimensional orbit in the sense of Sussmann ([S]) of the pair of real vector fields $\{\partial_y, 2x\partial_x\}$. We recall that if X and Y are smooth real vector fields defined on an open set U, two points belong to the same orbit of $\{X, Y\}$ in U if they can be joined by a continuous, piecewise differentiable curve such that each piece is an integral curve of $\pm X$ or $\pm Y$. If $L = X + iY$, by an orbit of L we shall always mean an orbit of $\{X, Y\}$.

In examples (1) and (2), homogeneous solutions must be constant on certain sets: fibers of a first integral (namely, pair of symmetric points $x \pm iy$) in the first case, and on one-dimensional orbits in the second case (namely, the y-axis). The main difference is that while the notion of fiber of a first integral is a local notion, orbits are a global notion and thus more adequate for dealing with global existence of a homogeneous solution of a vector field subject to additional conditions.

The main result of this paper is a generalization of the Rudin-Carleson theorem for locally solvable vector fields $L = a(x, y)\frac{\partial}{\partial x} + b(x, y)\frac{\partial}{\partial y}$ with complex-valued, real analytic coefficients. We recall that locally solvable vector fields are characterized by a geometric condition, known as the Nirenberg-Treves condition $(\mathcal{P})$ [NT] (see Section 2 for precise definitions).

Theorem A. *Let D be the unit disc and let L a non-vanishing vector field with complex-valued, real analytic coefficients defined on a neighborhood U of $\overline{D}$ satisfying condition $(\mathcal{P})$. Assume that L does not have a relatively compact orbit in U. Let $E \subset \partial D$ be a closed set with Lebesgue measure zero and assume that $g \in C(\partial D)$ is constant on the intersection $\gamma \cap \partial D$ whenever γ is a one-dimensional orbit of L.*

Then there is $h \in C(\overline{D})$ *satisfying:*

$$\begin{aligned} Lh &= 0 \quad \text{in } D, \\ h(z) &= g(z), \quad z \in E, \\ \sup_{z\in\mathbb{T}} |h(z)| &\le \sup_{z\in E} |g(z)|. \end{aligned}$$

Remark. In Theorem A, the unit disc could be replaced by any simply connected domain bounded by an analytic Jordan curve, as follows from an application of the Riemann mapping theorem.

The proof of the above theorem is given in Section 2. In Section 3 we discuss briefly the connection between the Rudin-Carleson theorem and the F. and M. Riesz theorem. It is shown in [BH2] that the local solvability condition $(\mathcal{P})$ is necessary and sufficient for the validity of the Rudin-Carleson property in arbitrary small neighborhoods of a point in an open set.

2. Proof of the main result

If we write $L = X + iY$ with X and Y real, it is well known that condition $(\mathcal{P})$ can be expressed in terms of the orbits of the pair of vector fields $\{X, Y\}$ in the sense of Sussmann ([S]). For any open subset U of Ω, two points belong to the same orbit of $\{X, Y\}$ in U if they can be joined by a continuous, piecewise differentiable curve such that each piece is an integral curve of $\pm X$ or $\pm Y$. Since X and Y are assumed to have no common zeros, the orbits of L in U are immersed submanifolds of U of dimension one or two; furthermore, the two-dimensional orbits are open subsets of U. Let $\mathcal{O} \subset U$ be a two-dimensional orbit of L in U and consider $X \wedge Y \in C^\infty(U; \bigwedge^2(T(U)))$. Since $\bigwedge^2(T(U))$ has a global nonvanishing section $e_1 \wedge e_2$, $X \wedge Y$ is a real multiple of $e_1 \wedge e_2$ and this gives a meaning to the requirement that $X \wedge Y$ does not change sign on any two-dimensional orbit $\mathcal{O}$ of $\{X, Y\}$ in U. The vector field L satisfies condition $(\mathcal{P})$ at $p \in \Sigma$ if there is a disc $U \subset \Omega$ centered at p such that $X \wedge Y$ does not change sign on any two-dimensional orbit of L in U. The validity of condition $(\mathcal{P})$ for L implies that at each point, the bracket $[X, Y]$ is a linear combination of X and Y. However, in general, this latter property is weaker than condition $(\mathcal{P})$.

Given a vector field $L = X + iY$ as above, a smooth function $Z(x, y)$ is called a first integral of L if $LZ = 0$ and the differential $dZ \neq 0$. We say L is locally integrable on an open set Ω if for each $p \in \Omega$, there is a neighborhood Ω_p of p and a first integral Z of L defined on Ω_p. L is said to be elliptic at a point p if the real vector fields X and Y are linearly independent at p. The characteristic set of L is the set of points where L is not elliptic. L is said to be of finite type at a point p if the Lie algebra generated by X and Y gives a vector space of dimension 2 at p. The vector field L is said to be hypocomplex at a point p if every solution h of $Lh = 0$ near p has the the form $h = H \circ Z$ where Z is a first integral of L near p and H is a holomorphic function in a neighborhood of $Z(p)$.

The following example shows that if L has a compact orbit, L may not have any nonconstant solution on $\overline{D}$. More examples can be found in [BM].

Example. Consider a vector field L expressed in polar coordinates as

$$L = rb(r)\frac{\partial}{\partial r} + ia(r)\frac{\partial}{\partial \theta}$$

where $a(r)$ and $b(r)$ are complex valued real analytic on $[0, 2]$, and the function $a(r)/b(r)$ is even in r away from the set where $b(r) = 0$. Observe that the characteristic set of L is given by

$$\Sigma = \{(r, \theta) : \Re(a(r)\overline{b}(r)) = 0\}.$$

Assume that $\Sigma = \{(r, \theta) : r = r_0\}$ for some $0 < r_0 < 1$, $b(r_0) = 0$ and $a(r_0) \neq 0$. We also assume that the order of the pole r_0 for the function $M(r) = \frac{a(r)}{rb(r)}$ is $k > 1$ and that $M(r)$ has the form

$$M(r) = \sum_{j=0}^{k-1} \frac{c_j}{(r - r_0)^{k-j}} + N(r)$$

where each c_j is purely imaginary and $N(r)$ is a real analytic function. It was shown in [BM] (Theorem 1.1, and part (3) of Remark 1.1) that if f is a continuous solution of L in a neighborhood of any annulus of the form $A_\delta = \{z \in \mathbb{C} : r_0 - \delta < |z| < r_0 + \delta\}$, then it is constant. It follows that if $u \in C(\overline{D})$ and $Lu = 0$ in the unit disc D, then u is constant on $\overline{D}$ since L is elliptic away from Σ. Observe that such an L satisfies condition $(\mathcal{P})$ since on any two-dimensional orbit, it is elliptic.

The general idea in the proof of Theorem A is as follows: we first construct a solution h_Ω on each two-dimensional orbit Ω of L in D that is continuous on the closure $\overline{\Omega}$ and satisfies appropriate boundary conditions. We then obtain the required solution h by patching together the functions h_Ω.

We will first consider the case when L is a real vector field:

Lemma 2.1. *Let L be a real analytic, non-vanishing, real vector field defined on a connected and simply connected neighborhood $\tilde{D}$ of the unit disc D. Let $g \in C(\partial D)$ be constant on the intersection of ∂D with the orbits (i.e., integral curves) of L in $\tilde{D}$, i.e., if γ is an orbit of L*

$$p, q \in \gamma \cap \partial D \Longrightarrow g(p) = g(q).$$

Then there is $h \in C(\overline{D})$ satisfying:

(1) $Lh = 0$ *in* D;
(2) $h(z) = g(z)$ *for all* $z \in \partial D$;
(3) $\sup_{\overline{D}} |h| = \sup_{\partial D} |g|$.

Proof. We start by defining $h(z)$. If $z \in \partial D$, we set $h(z) = g(z)$ (so (2) holds). If $z \in D$, we consider the orbit of L in $\tilde{D}$ that contains z. By the Poincaré-Bendixon theorem, if the ω-limit of z is not empty it must contain a singular point or a cycle. However, singular points are ruled out because L does not vanish and cycles are ruled out because $\tilde{D}$ is simply connected and the complement of a cycle in $\tilde{D}$ would have two components, one of which must contain a singular point of L (this is a standard exercise in Poincaré-Bendixon theory). Hence, the ω-limit of z is empty and the positive half orbit of z, $\gamma_+(z) = \{\exp(tL)z,\ t \geq 0\}$, does not remain in a compact subset of $\tilde{D}$ and must hit ∂D and eventually leave $\overline{D}$. In particular, an orbit of L cannot be contained in ∂D. The intersection $\gamma_+(z) \cap \partial D \neq \emptyset$ has a finite number of points (here we use the analyticity for the first time) and among them we single out one that we call z_1 and has the following special property: for some $\varepsilon > 0$, $\exp(tL)z_1 \in D$, $-\varepsilon < t < 0$, $\exp(tL)z_1 \in \tilde{D} \setminus \overline{D}$, $0 < t < \varepsilon$. We set $h(z) \doteq g(z_1)$. Points like z_1 will be called exit points. Of course, by the hypothesis on g, we could have chosen any other point in $\gamma_+(z) \cap \partial D \neq \emptyset$ to define $h(z)$ but z_1 will play a special role in the proof of the continuity of h.

Having defined $h(z)$ we must prove: (i) $h \in C(\overline{D})$ and (ii) $Lh = 0$ on D. Once (i) is proved, (ii) is easy. Indeed, if (i) holds, in a neighborhood of a given point $z \in D$, we may choose local coordinates (x, t) so that $z = 0$ and $L = \partial/\partial x$. Since h was defined to be constant on the orbits of L and the intersection of an orbit of L with the coordinate neighborhood U given by $|x| < \varepsilon$, $|t| < \varepsilon$ is a union of slices $t =$constant, we see that for some continuous function $h_U \in C(-\varepsilon, \varepsilon)$ we have $h(x(z), t(z)) = h_U(t(z))$, $z \in U$, so $Lh = 0$. Hence the main task is to prove that h is continuous, which we do now.

Let $z_1 \in \partial D$ be as before and choose local coordinates (x, t) in a neighborhood U of z_1 such that $|x| < \varepsilon$, $|t| < \varepsilon$, $x(z_1) = t(z_1) = 0$ and $L = \partial/\partial x$. We will initially assume that the point z is very close to z_1, in particular it is also in the coordinate neighborhood U and may be written as $z = (x_0, 0)$.

We distinguish two cases:

L is transversal to ∂D at z_1. In this case, shrinking ε if necessary, we may assume that $\partial D \cap U$ is given by $x = \Phi(t)$, $|t| < \varepsilon$, for some real analytic function $\Phi(t)$, $\Phi(0) = 0$, and according to our definition of h we have $h(x, t) = g(\Phi(t), t)$, $(x, t) \in U$, showing that h is continuous on $U \cap \overline{D}$.

L is tangent to ∂D at z_1. Now we may assume that $\partial D \cap U$ is given by $t = \Phi(x)$, $|x| < \varepsilon$, for some real analytic function $\Phi(x)$ with $\Phi(0) = \Phi'(0) = 0$. By the key property of $z_1 = 0$ the first integer k such that $\Phi^{(k)}(0) \neq 0$ must be odd (otherwise, $\gamma_+(z)$ could not leave D through the origin). Hence, $\Phi(x)$ is injective in a neighborhood of the origin and shrinking ε we may assume that it has a continuous inverse $x = \Psi(t) = \Phi^{-1}(t)$ defined from $|t| < \delta$ onto $|x| < \varepsilon$. Once again, we have an expression $h(x, t) = g(\Psi(t), t)$ that shows that h is continuous in a small neighborhood of the exit point z_1.

Hence, we have shown that there is a neighborhood V of the subset of exit points of ∂D such that $h(z)$ is continuous on $\overline{D} \cap V$. Similarly, h is continuous on a neighborhood in $\overline{D}$ of any point $z_2 \in \gamma_-(z) \cap \partial D$ that is an entry point of $\gamma_-(z)$ for some $z \in D$. Consider now a point $z \in D \setminus V$. We take an arc of $\gamma_+(z)$ that starts at z and ends at a point $z^\flat \in V$, and a small curve Σ transversal to the flow of L, with $z \in \Sigma$. Let $\Phi(t, p) = \exp(tL)p$ denote the flow map and choose $t_0 > 0$ so that $\Phi(t_0, z) = z^\flat \in V$. Then shrink Σ conveniently and take $\Sigma^\flat$ to be the image of Σ near z under the diffeomorphism $p \mapsto \Phi(t_0, p)$ so that $\Sigma^\flat \subset V$. The map $p \mapsto \Phi(t_0, p)$ is a diffeomorphism from Σ to $\Sigma^\flat$ and $h(p) = h(\Phi((t_0, p))$. Since $\Sigma^\flat \subset V$ and h is continuous on V we conclude that h is continuous on Σ which easily implies, after rectifying the flow around z, that h is continuous on a neighborhood of z. Summing up, we have proved that h is continuous on D and at those points $\in \partial D$ that are exit or entry points of an integral curve.

Non exit and non entry points. We will now consider the case of boundary points z_3 that are neither entry or exit points. Therefore, in convenient local coordinates (x, t) defined in a neighborhood U of such a point $z_3 \in \partial D$ we have $|x| < \varepsilon$, $|t| < \varepsilon$, $x(z_3) = t(z_3) = 0$, $L = \partial/\partial x$ and $U \cap D = \{(x, t) : \ t \geq \Phi(x), \ |x| < \varepsilon\}$ for some real analytic function $\Phi(x)$ that has a zero of even order $k > 0$. Suppose first that $\Phi^{(k)}(0) > 0$ (geometrically, this means that locally the integral curve of z_3 remains in the complement of D). Hence, shirking $\varepsilon > 0$ if necessary, $\Phi(x)$ is injective on $0 \leq x < \varepsilon$ and we have a continuous inverse $\Psi(t) : [0, \delta) \longrightarrow [0, \varepsilon)$ of the restriction of $\Phi(x)$ to $[0, \varepsilon)$. Now, for $t > \Phi(x)$, $-\varepsilon < x < \varepsilon$, we have $h(x, t) = g(\Psi(t), t)$, since the point $z_1 = (\Psi(t), t)$ is an exit point of $\gamma_+(z)$ when $z = (x, t)$ with $t > \Phi(x)$. We also see that $g(x, \Phi(x)) = g(x', \Phi(x'))$ if $\Phi(x) = \Phi(x')$ because in this case the points $(x', \Phi(x'))$ and $(x, \Phi(x))$ are in ∂D and belong to the same orbit of L. It is now easy to see that h is continuous in a neighborhood of z_3.

Assume now that $\Phi^{(k)}(0) < 0$. We may find, for small $\varepsilon^+, \varepsilon^-, \delta > 0$, a continuous inverse $\Psi^+(t) : (-\delta, 0] \longrightarrow [0, \varepsilon^+)$ of the restriction of $\Phi(x)$ to $[0, \varepsilon^+)$ and a continuous inverse $\Psi^-(t) : (-\delta, 0] \longrightarrow (-\varepsilon^-, 0]$ of the restriction of $\Phi(x)$ to $(-\varepsilon^-, 0]$. Then,

$$h(x,t) = \begin{cases} g(\Psi^+(t), t) & \text{for } 0 \leq x \leq \varepsilon^+, \ \Phi(x) \leq t \leq 0, \\ g(\Psi^-(t), t) & \text{for } -\varepsilon^- < x \leq 0, \ \Phi(x) \leq t \leq 0, \\ h(0, t) & \text{on } (-\varepsilon^-, \varepsilon^+) \times [0, \delta). \end{cases} \tag{2.1}$$

Note that $h(0, t) = h(\varepsilon^+/2, t)$, $0 \leq t < \delta$, is continuous because the segment $\{\varepsilon^+/2\} \times [0, \delta) \subset D$ and we already know that h is continuous on D. Thus, the three expressions on the right hand side of (2.1) are continuous on their domains of definition and — since $g(\Psi^+(0), 0) = g(\Psi^-(0), 0) = g(0, 0) = h(0, 0)$— they agree at their intersection, showing that h is continuous in a neighborhood of z_3.

Finally, note that the definition of $h(z)$ shows that the sets $h(\overline{D})$ and $g(\partial D)$ are identical, so (3) holds. $\square$

In the proof of Theorem A, we will also need the following result.

Lemma 2.2. *Assume that L is a smooth vector field defined in a neighborhood U of $\overline{D}$ satisfying condition $(\mathcal{P})$ without any compact orbits in U. Then there is a smooth function Z defined in a neighborhood W of $\overline{D}$ such that $LZ = 0$ on W and $dZ(p) \neq 0$ for each $p \in W$.*

Proof. Let

$$Lu = A(x, t)\frac{\partial u}{\partial t} + B(x, t)\frac{\partial u}{\partial x}$$

where $A, B \in C^\infty(U)$. Define

$$d(x,t) = \frac{\partial A(x,t)}{\partial t} + \frac{\partial B(x,t)}{\partial x}.$$

Since L is locally solvable and its orbits never stay in a compact set, by the results in [H,Thm.7.3], in a simply connected neighborhood W of $\overline{D}$, we can find $u \in C^\infty(W)$ such that $Lu = d$. Note that according to that theorem, in order to solve the equation $Lu = d$ we must pick d orthogonal to

$$N = \{v \in \mathcal{E}'(W),\ {}^tLv = 0\}$$

where ${}^tL = -(L+d)$ is the tranpose of L. However, $N = \{0\}$ because ${}^tL$ has no nontrivial null solutions with compact support by uniqueness in the Cauchy property. For a more detailed discussion of these matters we refer the reader to [Ho]. Then the 1-form

$$\omega = B(x,t)e^{-u(x,t)}\,dt - A(x,t)e^{-u(x,t)}\,dx$$

is closed since

$$\frac{\partial(Be^{-u})}{\partial x} + \frac{\partial(Ae^{-u})}{\partial t} = e^{-u}d - e^{-u}Lu = 0 \quad \text{in } W.$$

Furthermore, ω does not vanish. Since W is simply connected, there exists $Z \in C^\infty(W)$ such that $dZ = \omega$. Thus $dZ \neq 0$ in W and $LZ = \langle L, \omega\rangle = e^{-u}\langle A\partial_t + B\partial_x, Bdt - Adx\rangle = 0$. $\square$

Proof of Theorem A. By Lemma 2.1, we may assume that L is not a real vector field, and so by analyticity, it has only a finite number of orbits in D. Moreover, if Σ is a one-dimensional orbit, it cannot be a closed curve in $\overline{D}$ since L does not have a compact orbit. Let now Ω be a two-dimensional orbit in D and consider $\partial\Omega$. Let $p \in \partial\Omega$. If $p \in D$, then let γ be the one-dimensional orbit through p. Note that since L is real analytic and not a real vector field, near p, γ is the only one-dimensional orbit. Therefore, if U is a sufficiently small disc centered at p, $U\backslash\gamma$ is a disjoint union of two domains U_1 and U_2 with $U \cap \Omega = U_1$. It follows that Ω has a real analytic boundary near p. Suppose next $p \in \partial\Omega \cap \partial D$. We will consider different possibilities. If the orbit of L at p is two-dimensional, by the real analyticity of L, we can find a disc B centered at p such that $B \cap D \subset \Omega$. This means that near p, $\partial\Omega$ consists of $\partial D \cap B$. Assume the orbit at p is one-dimensional. If L is transversal to ∂D at p, then the orbit γ through p divides a disc W centered at p into two connected pieces W_1 and W_2 with $W_1 \cap D = W \cap \Omega$. Thus near p, Ω has a piecewise real analytic boundary consisting of two curves that intersect at p. Suppose now L is tangent to ∂D at p. Let γ continue to denote the one-dimensional orbit through p. By the real analyticity, in a small disc V centered at p, if $\gamma_1 = \gamma \cap V$, there are three possibilities:

Case 1) Assume $\gamma_1 \subset D \cup \{p\}$. Then since $p \in \partial\Omega$, $\gamma_1 \subset \partial\Omega$. Near each $q \in \gamma_1$, Ω lies on one side of γ_1. Hence near p, either $\partial\Omega = \gamma_1$, or $\partial\Omega$ consists of γ_1 and a neighborhood of p in ∂D.

Case 2) Suppose $\gamma_1 \cap \overline{D} = \{p\}$. Near p, each side of γ_1 is contained in distinct two-dimensional orbits. It follows that for some neighborhood V' of p, $V' \cap D \subset \Omega$ and so $\partial\Omega$ near p equals $V' \cap \partial D$.
Case 3) Assume $\gamma_1 = \gamma^+ \cup \gamma^-$, where $\gamma^- \subset D$, and $\gamma^+ \cap D = \emptyset$. Again each side of γ_1 is contained in a two-dimensional orbit and so near p, $\partial\Omega$ consists of γ^- and an arc in ∂D with p as an endpoint.

We have thus shown that the boundary of a two-dimensional orbit is piecewise real analytic consisting of a finite number of curves each of which is either a connected piece of a one-dimensional orbit or an arc in ∂D. Fix a two-dimensional orbit Ω in D. Since $\partial\Omega$ is a Jordan curve, Ω is simply connected. Observe that L is of finite type at each point in Ω. Because it also satisfies condition $(\mathcal{P})$, it is a hypocomplex vector field on Ω. This means that any homogeneous solution of L defined in an open subset of Ω may be locally written as a holomorphic function of a first integral so that any two injective solutions are locally holomorphically related (we refer to [BCH] on the notion of hypocomplexity). In other words, L defines a Riemann surface structure on Ω where the local charts are given by local invertible solutions Z of the equation $LZ = 0$. This structure is equivalent to the standard holomorphic one on the unit disc D since Ω is simply connected and L has a bounded, nonconstant solution, namely, the first integral Z constructed in Lemma 2.2. Let $F : \Omega \longrightarrow \Delta$ be a diffeomorphism (where the differentiable structure on Ω is the one induced by its Riemann surface structure) from Ω onto the unit disc Δ satisfying $LF = 0$ in Ω. We are using the notation Δ for the unit disc here so that D is reserved for the one in the statement of the theorem.

We will next prove that F has a continuous extension to $\overline{\Omega}$ which is injective away from the one-dimensional orbits, and that maps distinct one-dimensional orbits to distinct single points. Let $z_0 \in \partial\Omega$.

Assume z_0 is in a two-dimensional orbit. In this case $z_0 \in \partial D$. At such a point, we will use a modification of the classical argument to show that F extends as a homeomorphism up to the boundary. Suppose z_k is a sequence in Ω that converges to z_0. If the sequence $F(z_k)$ does not have a limit, then it clusters at least at two points on $\partial\Delta$. Without loss of generality we may assume $p_k = F(z_{2k})$ converges to v and $q_k = F(z_{2k+1})$ converges to w where v and w are two points on the boundary of Δ. Let T_1 and T_2 be two continuous arcs in Δ such that T_1 contains the p_k and ends at v while T_2 contains the q_k and ends at w. We may assume that $\mathrm{dist}(T_1, T_2) > c$ for some $c > 0$. Let Z be a first integral which is a homeomorphism from a disc U' about z_0 to a neighborhood of the origin and mapping $U' \cap \Omega$ onto V. The first integral Z is a homeomorphism because z_0 is contained in a two-dimensional orbit where L is hypocomplex. Let $G = F \circ Z^{-1}$. Let $S_j = F^{-1}(T_j),\ j = 1, 2$. For $r > 0$ small, let C_r be the intersection of the circle of radius r centered at 0 with the region V. Observe that if r is small enough, say $r \le r_0$ for some $r_0 > 0$, $Z^{-1}(C_r)$ intersects both S_1 and S_2 since these sets are connected and both accumulate at z_0. Let $C_r = \{re^{i\theta} : \theta_1(r) < \theta < \theta_2(r)\}$. Let $C'_r = G(C_r)$. Observe that C'_r contains points of both T_1 and T_2 since $Z^{-1}(C_r)$

intersects both S_1 and S_2. It follows that

$$c < \ell(C'_r) = \int_{\theta_1(r)}^{\theta_2(r)} |G'(z_0 + re^{i\theta})| r\, d\theta.$$

Applying the Schwarz inequality we get:

$$\frac{c^2}{r} < 2\pi \int_{\theta_1(r)}^{\theta_2(r)} |G'(z_0 + re^{i\theta})|^2 r\, d\theta$$

which in turn leads to the contradiction that

$$\infty = c^2 \int_0^{r_0} \frac{dr}{r}\, dr < 2\pi \int_0^{r_0} \int_{\theta_1(r)}^{\theta_2(r)} |G'(z_0 + re^{i\theta})|^2 r\, d\theta\, dr < \pi.$$

It follows that $F(z_k)$ has a limit and therefore F extends continuously to $\overline{\Omega}$.

We wish to prove the same property for F^{-1} away from the finite number of points which as we will see later are the images of the one-dimensional orbits under F. This would be equivalent to showing that F is locally one-to-one at the boundary points which we will next show.

We claim that at each hypocomplex point $z_0 \in \partial\Omega$ as above, the function F extends to be a homeomorphism up to z_0. To see this, first assume that L is transversal to ∂D at z_0. In this case, after contracting U about z_0, $Z(U \cap \partial D)$ is a real analytic piece of the boundary of V through the origin. The function G is holomorphic on V, continuous up to the boundary near the origin, and sends a real analytic boundary piece of V through 0 into the boundary of the disc Δ. By the Schwarz reflection principle, G extends as a holomorphic function in a neighborhood of the origin which in turn leads to a real analytic extension of F past z_0. Suppose now $z_1 \in \partial\Omega$ is another hypocomplex point where L is transversal to ∂D and assume that $F(z_0) = F(z_1) = w$. Then since F extends as a solution past both z_0 and z_1, and L is hypocomplex at these points, the extended F is an open map and hence there are neighborhoods U_0, U_1, W of z_0, z_1 and w respectively such that $F(U_0) = F(U_1) = W$. Moreover, because F is extended using the reflection principle, we may assume that $F(U_0 \cap \Omega) = W \cap \Delta = F(U_1 \cap \Omega)$. But this contradicts the injectivity of F on Ω. Hence $F(z_0) \neq F(z_1)$. Observe that since ∂D is not an orbit of L, there are only a finite number of points on ∂D where L is not transversal to ∂D. Suppose now z_2, z_3 are two points in $\partial\Omega \cap \partial D$ where $F(z_2) = F(z_3)$ and assume that L is transversal to ∂D at z_2 and tangent to ∂D at z_3. We have seen that there is a neighborhood U_2 of z_2 in $\overline{D}$ where F is one-to-one and such that $F(U_2)$ is a neighborhood of $F(z_2)$ in $\overline{\Delta}$. But then, if $z \in \Omega$ and is sufficiently close to z_3, $F(z) \notin F(U_2)$, contradicting the continuity of F at z_3. Therefore, $F(z_2) \neq F(z_3)$. Finally, suppose z_4 and z_5 are two points in $\partial\Omega \cap \partial D$ where L is hypocomplex and assume L is tangent to ∂D at both points and $F(z_4) = F(z_5) = w_0$. Since there are only a finite number of such points in ∂D, there is an open arc I in ∂D containing z_4 in its interior and consisting of hypocomplex points such that z_4 is the only point where L is not transversal to ∂D. Since F is one-to-one on $I \backslash \{z_4\}$, $F(I)$ is an open arc in $\partial\Delta$ containing w_0 ints interior. There is also a similar arc J with z_5 in its interior and we may assume that I and J are disjoint. But then this would contradict the injectivity of F on $I\cup J\setminus\{z_4, z_5\}$ and so we must have $F(z_4) \neq F(z_5)$. Hence F can be extended as a homeomorphism up to the part of the boundary of Ω that is disjoint from the one-dimensional orbits. It is also real analytic past all but a finite number of the points that don't lie in the one-dimensional orbits.

Assume z_0 is in a one-dimensional orbit Γ. . Write $L = X + iY$ with X and Y real vector fields. Replacing L, if necessary, by a convenient nonvanishing multiple of L we may assume that Γ is a closed integral curve of X joining two points A and B that belong to ∂D. Since Y vanishes on Γ, it vanishes identically on any integral curve of X that contains Γ (by analyticity). We may consider an integral curve Γ_1 of X that extends Γ past both endpoints A and B, so that Γ_1 is a one-dimensional orbit of L in a neighborhood U of $\overline{D}$ with endpoints in $U \setminus \overline{D}$. In a tubular neighborhood V of Γ_1 we may choose coordinates that rectify the flow of X and in which L has a canonical form. More precisely, we may choose local coordinates (x,t), so that V is expressed as $|x| \leq 1$, $|t| \leq 2$, $x(z_0) = t(z_0) = 0$, $x(A) = x(B) = 0$, $t(A) = 1$, $t(B) = -1$ and L has the form

$$L = \frac{\partial}{\partial t} + ib(x,t)\frac{\partial}{\partial x},$$

with $t \mapsto b(x,t) \geq 0$ and not identically zero for $0 < x \leq 1$, and $b(0,t) \equiv 0$, $-2 \leq t \leq 2$. The intersection of Ω with V is described by

$$\Omega \cap V = \{(x,t) :\ 0 < x \leq 1,\ \beta(x) < t < \alpha(x)\}.$$

Here, $\alpha(x)$, $\beta(x)$ are continuous on $[0,1]$ and analytic on $(0,1]$, $\alpha(0) < 1$, $\beta(0) > -1$ and their graphs are contained in $\partial D \cap \partial\Omega$. By restricting F to $\Omega \cap V$, we obtain an injective map $F(x,t)$ from $\Omega \cap V$ into D. We may assume that F has already been extended as a homeomorphism from

$$\{(x,t) :\ 0 < x \leq 1,\ \beta(x) \leq t \leq \alpha(x)\}$$

into $\overline{D}$. Hence, $F(x,t)$ maps the graphs $t = \alpha(x)$, $t = \beta(x)$, $0 < x < 1$, into some open arcs $\widehat{A'C'}$, $\widehat{B'D'} \subset \partial D$. Consider the vertical segment $T_\varepsilon = \{\varepsilon\} \times [\beta(\varepsilon), \alpha(\varepsilon)]$, $0 < \varepsilon < 1$, that is mapped by F into a curve $F(T_\varepsilon)$ contained in D that joins two boundary points $A'_\varepsilon \doteq F(\varepsilon, \alpha(\varepsilon)) \in \widehat{A'C'}$, $B'_\varepsilon \doteq F(\varepsilon, \beta(\varepsilon)) \in \widehat{B'D'}$. Notice that $A'_\varepsilon \to A'$ and $B'_\varepsilon \to B'$ as $\varepsilon \to 0$. The length of $F(T_\varepsilon)$ is

$$\ell(F(T_\varepsilon)) = \int_{\beta(\varepsilon)}^{\alpha(\varepsilon)} |F_t(\varepsilon,t)|\, dt = \int_{\beta(\varepsilon)}^{\alpha(\varepsilon)} \left(U_t^2(\varepsilon,t) + V_t^2(\varepsilon,t)\right)^{1/2} dt$$

with $F = \Re F + i\Im F = U + iV$. Since $LF = 0$, i.e., $U_t = bV_x$, $V_t = -bU_x$, we have

$$\begin{aligned}
\ell(F(T_\varepsilon))^2 &= \left(\int_{\beta(\varepsilon)}^{\alpha(\varepsilon)} b(\varepsilon,t)\big(U_x^2(\varepsilon,t) + V_x^2(\varepsilon,t)\big)^{1/2} dt\right)^2 \\
&\leq \int_{\beta(\varepsilon)}^{\alpha(\varepsilon)} b(\varepsilon,t)\, dt \int_{\beta(\varepsilon)}^{\alpha(\varepsilon)} b(\varepsilon,t)\big(U_x^2(\varepsilon,t) + V_x^2(\varepsilon,t)\big)\, dt \\
&\leq \int_{-2}^{2} b(\varepsilon,t)\, dt\ I(\varepsilon) \leq C\varepsilon\, I(\varepsilon). \qquad (2.2)
\end{aligned}$$

On the other hand,

$$\int_0^1 I(\varepsilon)\,d\varepsilon = \int_0^1 \int_{\beta(\varepsilon)}^{\alpha(\varepsilon)} b(\varepsilon,t)\big(U_x^2(\varepsilon,t)+V_x^2(\varepsilon,t)\big)\,dtd\varepsilon$$
$$= \int_0^1 \int_{\beta(\varepsilon)}^{\alpha(\varepsilon)} \det\frac{\partial(U,V)}{\partial(\varepsilon,t)}\,dtd\varepsilon = \operatorname{area}(F(\Omega\cap V)) < \pi.$$

Since the integral on the left hand side is finite, we see that the product $\varepsilon I(\varepsilon)$ cannot remain bounded below by a positive constant in any neighborhood of the origin. In other words, there is a sequence $\varepsilon_j \searrow 0$ such that $\varepsilon_j I(\varepsilon_j) \searrow 0$ and (2.2) shows that $\ell(F(T_{\varepsilon_j})) \to 0$. Hence, $|A'_{\varepsilon_j} - B'_{\varepsilon_j}| \to 0$ and we conclude that $A' = B'$. Notice that the region

$$F\big(\{(x,t):\ 0<x<\varepsilon_j,\ \beta(x)<t<\alpha(x)\}\big)$$

is bounded by the closed curved made of three arcs, to wit, the circular arc from A'_{ε_j} to A', the circular arc from $B' = A'$ to B'_{ε_j} and the curve $F(T_{\varepsilon_j})$ that joins B'_{ε_j} to A'_{ε_j}. It is therefore easy to see that the diameter of that region tends to zero as $j\to\infty$, so given $r>0$ we may find j_0 such that

$$F\big(\{(x,t):\ 0<x<\varepsilon_j,\ \beta(x)<t<\alpha(x)\}\big) \subset \Delta(A',r), \quad j\ge j_0.$$

This shows that, if we extend F to $\{0\}\times[-1,1]$ by setting $F(0,t) = A'$, $-1\le t\le 1$, we obtain a continuous extension. Note that any one-dimensional orbit is in the boundary of a two-dimensional orbit.

Thus F is continuous on $\overline{\Omega}$, injective on Ω, and for each $p\in\partial\Omega\cap\partial D$ where L is hypocomplex, there is a neighborhood W_p of p in $\overline{\Omega}$ on which F is a homeomorphism onto a neighborhood of $F(p)\in\partial D$ in $\overline{D}$. This implies that away from the one-dimensional orbits in $\overline{\Omega}$, F is a one-to-one function. We will next show that if Γ_1 and Γ_2 are two arcs in $\partial\Omega$ that belong to distinct one-dimensional orbits, then $F(\Gamma_1)\ne F(\Gamma_2)$. To see this, note first that F maps $\partial\Omega$ onto ∂D. Indeed, if $q\in\partial D$ and $q\notin F(\partial\Omega)$, then since $F(\partial\Omega)$ is a closed set, there is a maximal open arc $\Sigma\subset\partial D$ containing q that is disjoint from $F(\partial\Omega)$. Since $F(\partial\Omega)$ is connected, it follows that $F(\partial\Omega) = \partial D\backslash\Sigma = T$ is an arc containing its endpoints. Let $\{A_1,\dots,A_m\}\subset T$ be the image under F of the one-dimensional orbits in $\partial\Omega$. Since $F(\partial\Omega)$ is a closed curve, there exists a point $C\in T\backslash\{A_1,\dots,A_m\}$ such that for some pair of points $B, B'\in\partial\Omega\cap\partial D$, $F(B) = C = F(B')$. This contradicts the injectivity of F on $\partial\Omega\cap\partial D$. Thus F maps $\partial\Omega$ onto ∂D. Let $\Gamma_1,\dots,\Gamma_k$ be the distinct one -dimensional orbits in $\partial\Omega$. Since we have shown that the map $F^{-1}:\partial D\backslash\{A_1,\dots,A_m\}\longrightarrow\partial D\backslash\{\Gamma_1,\dots,\Gamma_k\}$ is a homeomorphism, we must have $k=m$. Hence F maps different one-dimensional orbits to distinct points.

We have thus shown that for each two-dimensional orbit Ω, we have a continuous function $F_\Omega:\overline{\Omega}:\to\overline{\Delta}$ which is onto, $L(F_\Omega)=0$ in Ω, and it is injective except on the one-dimensional orbits which are mapped to distinct points. It is also clear that it takes sets $X\subset\partial\Omega$ of Lebesgue measure zero into subsets of $\partial\Delta$ of Lebesgue measure zero. These properties allow us to define $g_\Omega \in C(\partial\Delta)$ for each two-dimensional orbit Ω such that $g_\Omega \circ F_\Omega = g$ on $\partial\Omega \cap \partial D$. We can then use the Rudin-Carleson theorem to get corresponding functions $h_\Omega \in C(\overline{\Delta})$ which are holomorphic on Δ and satisfy $h_\Omega = g_\Omega$ on $F_\Omega(E \cap \partial\Omega)$ and $\sup|h_\Omega| \leq \sup|g_\Omega|$. This is possible because the sets $F_\Omega(E \cap \partial\Omega)$ are closed and of measure zero. All the functions $\{F_\Omega\}$ as well as the function g are constant when restricted to any one-dimensional orbit. This determines a well defined function h on $\overline{D}$ which agrees with h_Ω on $\overline{\Omega}$ for each Ω. To prove the continuity of h, we may assume without loss of generality that the set E contains the finite number of points on the boundary of D which belong to some one-dimensional orbit of L. It is clear that h is continuous on the two-dimensional orbits and the part of their boundaries in ∂D which are disjoint from the one-dimensional orbits. Suppose $p \in \overline{D} \cap \gamma$ where γ is a one-dimensional orbit of L. We consider two cases:

Case 1: Assume $p \in \gamma \cap D$. In this case, in a neighborhood V of p, $V \backslash \gamma = (\Omega_1 \cap V) \cup (\Omega_2 \cap V)$ where the Ω_j are two-dimensional orbits. We have

$$h_{\Omega_1}(\gamma) = h_{\Omega_2}(\gamma) = a$$

where a is the constant value of g on $\gamma \cap \partial D$. It follows that h is continuous at p.

Case 2: Assume $p \in \gamma \cap \partial D$. In this case we first assume in addition that there is a small neighborhood W of p in which γ intersects D. In that case, near the point p, arcs (or an arc) of γ emanating from p and that stay in D lie in the boundaries of two-dimensional orbits (at most three of them). The constancy of the F_Ω (whenever $\gamma \cap \partial\Omega \neq \emptyset$) on γ as well as the assumption on g then imply as in the first case that h is continuous at p. Finally, if in a neighborhood W of p, $\gamma \cap D = \emptyset$, then for W small enough, $W \cap D$ is contained in a two-dimensional orbit Ω and the continuity of h at such p follows from the fact that $h = \tilde{g_\Omega} \circ F_\Omega \circ Z$ on $\overline{\Omega}$.
It is clear that h is a solution on the two-dimensional orbits in D. Suppose γ is a one-dimensional orbit in D and $p \in \gamma$. We can assume we are in coordinates (x, t) vanishing at p such that in a rectangle $Q = (-a, a) \times (-a, a)$, the vector field

$$L = \frac{\partial}{\partial t} + b(x, t)\frac{\partial}{\partial x}$$

with $b(0, t) \equiv 0$, $Q^+ = \{(x, t) \in Q : x > 0\}$ and $Q^- = \{(x, t) \in Q : x < 0\}$ are two-dimensional orbits of L in Q. Let $\psi(x, t) \in C_0^\infty(Q)$. We wish to show that

$$\int_Q h(x, t) L^t \psi(x, t)\, dxdt = 0.$$

For each $\epsilon > 0$, let $\phi_\epsilon(x) \in C^\infty(-a, a)$ such that

(1) $0 \leq \phi_\epsilon(x) \leq 1$.
(2) $\phi_\epsilon(x) \equiv 1$ when $|x| \geq 2\epsilon$ and $\phi_\epsilon(x) \equiv 0$ for $|x| \leq \epsilon$.
(3) For some $C > 0$, $|\phi_\epsilon'(x)| \leq C\epsilon^{-1}$.

Since h is a solution on $Q^+ \cup Q^-$,

$$\begin{aligned} 0 &= \int_Q h(x,t) L^t(\phi_\epsilon(x)\psi(x,t))\, dxdt \\ &= \int_Q h(x,t)\phi_\epsilon(x) L^t(\psi)(x,t)\, dxdt - \int_Q h(x,t) b(x,t)\psi(x,t)\phi'_\epsilon(x)\, dxdt. \end{aligned}$$

Observe that since $b(0,t) \equiv 0$ and $\phi'_\epsilon(x)$ is supported in the set $\{x : \epsilon \leq |x| \leq 2\epsilon\}$,

$$\lim_{\epsilon \to 0} \int_Q h(x,t) b(x,t) \psi(x,t) \phi'_\epsilon(x)\, dxdt = 0,$$

while

$$\lim_{\epsilon \to 0} \int_Q h(x,t)\phi_\epsilon(x) L^t\psi(x,t)\, dxdt = \int_Q h(x,t) L^t\psi(x,t)\, dxdt.$$

It follows that $\int_Q h(x,t) L^t\psi(x,t)\, dxdt = 0$ and hence h is a solution in D. Finally, from the construction of h, we have:

$$\sup|h|_{\overline{D}} \leq \sup_E |g|.$$

3. The Rudin-Carleson and the F. and M. Riesz theorems

In this section it will be useful to recall a result of Bishop's. In [B] the author proved an abstract theorem which permits a generalization of the Rudin-Carleson theorem to any situation where a version of the F. and M. Riesz theorem is valid. Bishop's theorem has been a key tool in the study of peak-interpolation sets for $A(\Omega)$ where Ω is a bounded domain in $\mathbb{C}^n$ (typically strictly pseudoconvex) and $A(\Omega)$ is the algebra of holomorphic functions on Ω that are continuous up to the boundary (see [Bh], [R2], [Na] and the references therein). We state here a strengthened version from [G] of the theorem proved in [B]:

Theorem 3.1. *(Theorem 12.5 in [G]) Let $C(X)$ be the uniformly-normed Banach space of all continuous complex-valued functions on a compact Hausdorff space X. Let B be a closed subspace of $C(X)$. Let $B^\perp$ consist of all (finite, complex-valued, Baire) measures μ on X such that $\int f\, d\mu = 0$ for all f in B. Let $\hat{\mu}$ be the regular Borel extension of the Baire measure μ. Let S be a closed subset of X with the property that $\hat{\mu}(T) = 0$ for every Borel subset T of S and every μ in $B^\perp$. Let f be a continuous complex-valued function on S and Δ a positive continuous function on X such that $|f(x)| \leq \Delta(x)$ for all x in S. Then there exists F in B with $|F(x)| \leq \Delta(x)$ for all x in X and $F(x) = f(x)$ for all x in S.*

If $X = \mathbb{T}$ equals the unit circle and B denotes the space of continuous functions on $\mathbb{T}$ which are restrictions of functions holomorphic on the unit disc D and continuous on the closure $\overline{D}$, then a measure μ on $\mathbb{T}$ is in $B^\perp$ if and only if it is the boundary value of a holomorphic function on D. By the F. and M. Riesz theorem, it follows that any $\mu \in B^\perp$ is absolutely continuous with respect to Lebesgue measure and so the preceding theorem implies the Rudin-Carleson theorem. The classical F. and M. Riesz theorem was generalized for solutions of locally integrable vector fields in the paper [BH1]. This generalization says that if L is a locally integrable vector field on a smooth domain Ω, $h \in C(\Omega)$ is a solution of $Lh = 0$ and has a distribution trace bh which is a measure on $\partial\Omega$, then bh is absolutely continuous with respect to arclength measure on $\partial\Omega$. We will say that L has the F. and M. Riesz property whenever such a generalization holds for L. It turns out that unlike the holomorphic case, we cannot use the F. and M. Riesz property of a vector field together with Theorem 3.1 to deduce the Rudin-Carleson property. For example, if M is the Mizohata vector field given in (1) and μ is a measure on $\mathbb{T}$ which is of the form $\mu = \delta_{(0,1)} - \delta_{(0,-1)}$ where δ_p denotes the Dirac mass at p, then $\int_{\mathbb{T}} h\, d\mu = 0$ for every $h \in C(\overline{D})$ that satisfies $Mh = 0$ on D. Such a measure cannot be the boundary value of a solution of M by the F. and M. Riesz property for M. Thus for a general vector field, a measure that is orthogonal to the boundary values of continuous solutions may not be a boundary value of a solution and in fact, it may not be absolutely continuous with respect to Lebesgue measure. If a vector field L satisfies the hypotheses of Theorem A, we have the following:

Corollary 3.2. *Suppose L is a vector field as in Theorem A defined on a neighborhood U of $\overline{D}$. Let $\mathcal{A}$ denote the algebra of continuous functions h on $\overline{D}$ satisfying the equation $Lh = 0$ in D. Let μ be a complex Baire measure defined on ∂D with the property that*

$$\int_{\partial D} h\, d\mu = 0$$

for every $h \in \mathcal{A}$. If a closed set $E \subseteq \partial D$ has Lebesgue measure zero and it is disjoint from the one-dimensional orbits of L in U, then $\mu(E) = 0$.

Proof. Let F be a closed subset of E. Let P be a positive continuous function on ∂D such that:

(1) $P \equiv 1$ on F.
(2) For any $y \notin F$, $P(y) < 1$.

An application of Theorem 3.1 in the proof of Theorem A shows that there is $h \in \mathcal{A}$ that equals 1 on F and satisfies $|h(p)| < 1$ for $p \notin F$. By hypothesis, for each positive integer n, we have $\int h^n\, d\mu = 0$. Letting $n \to \infty$, we are led to conclude that $\mu(F) = 0$. By the regularity of the measure μ, it follows that $\mu(E) = 0$.

References

[BT] M. S. Baouendi and F. Treves, *A property of the functions and distributions annihilated by a locally integrable system of complex vector fields*, Ann. of Math. **113** (1981), 387–421.

[BCH] S. Berhanu, P. Cordaro, and J. Hounie, *An introduction to involutive structures*, Cambridge University Press, 2008.

[BH1] S. Berhanu and J. Hounie, *An F. and M. Riesz theorem for planar vector fields*, Math. Ann. **320** (2001), 463–485.

[BH2] S. Berhanu and J. Hounie, *A generalization of the Rudin-Carleson theorem*, Advances in Phase Space Analysis of PDEs, Bove, A., Murthy, M.K., Del Santo, D. (Eds), Birkhauser, to appear.

[BM] S. Berhanu and A. Meziani, *Global properties of a class of planar vector fields of infinite type*, Commun. in Partial Differential Equations **22** (1997), 99–142.

[B] E. Bishop, *A general Rudin-Carleson theorem*, Proceedings of the AMS **13** (1962), 140–143.

[Bh] G. Bharali, *On peak-interpolation manifolds for $A(\Omega)$ for convex domains in $\mathbb{C}^n$*, Transactions of the AMS **356** (2004), 4811–4827.

[C] L. Carleson, *Representations of continuous functions*, Math. Z. **66** (1957), 447–451.

[D] R. Doss, *Elementary proof of the Rudin-Carleson and the F. and M. Riesz theorems*, Proceedings of the AMS **82** (1981), 599–602.

[Du] P. Duren, *Theory of H^p spaces*, Academic Press, 1970.

[G] T. Gamelin, *Uniform algebras (second edition)*, Chelsea Pub. Comp., 1984.

[Ho] J. Hounie, *A note on global solvability of vector fields*, Proc. Amer. Math. Soc. **52** (1985), 61–64.

[H] L. Hormander, *Propagation of singularities and semi-global existence theorems for (pseudo)-differential operators of principal type*, Ann. of Math. **108** (1978), 569–609.

[Na] A. Nagel, *Smooth zero sets and interpolation sets for some algebras of holomorphic functions on strictly pseudoconvex domains*, Duke Math. J. **43** (1976), 323–348.

[N] L. Nirenberg, *Lectures on linear partial differential equations*, Reg. Conf. Series in Math., Amer. Math.Soc.17, 1973.

[NT] L. Nirenberg and F. Treves, *Solvability of a first order linear partial differential equation*, Comm. Pure Appl. Math. **16** (1963), 331–351.

[O] D. Oberlin, *A Rudin-Carleson theorem for uniformly convergent Taylor series*, Michigan Math. J. **27** (1980), 309–313.

[R1] W. Rudin, *Boundary values of continuous analytic functions*, Proceedings of the AMS **7** (1956), 808–811.

[R2] W. Rudin, *Peak-interpolation sets of class C^1*, Pacific J. Math. **75** (1978), 267-279.

[S] H. J. Sussmann, *Orbits of families of vector fields and integrability of distributions*, Transactions of the AMS **180** (1973), 171–188.

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