# HIGHER-ORDER FOR THE GENERALIZED BBM-BURGERS EQUATION: EXISTENCE AND CONVERGENCE RESULTS

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Abstract. We study the global existence of solutions for certain equations of the form

$$u_t + f(u)_x = \delta u_{xxt} + \sum_{n=1}^{N} (-1)^{n+1} \gamma_n \partial_x^{2n} u$$

as $\delta > 0$ and $\gamma_n > 0$, $n = 1, ..., N$ aproach zero, and $f$ is a sufficiently smooth function satisfying certain appropriate assumptions. We consider solutions of hyperbolic conservation laws regularized of this equations. Following a pioneering work by Schonbek and a work by LeFloch and Natalini, we establish the convergence of the regularized solutions toward discontinuous solutions of the hyperbolic conservation law.

## 1. Introduction

In this paper we study the existence and convergence of the smooth solutions $\{u(x, t; \delta, \gamma_1, ..., \gamma_N)\}$ for partial differential equation of the form

$$(1) \qquad u_t + f(u)_x = \delta u_{xxt} + \sum_{n=1}^{N} (-1)^{n+1} \gamma_n \partial_x^{2n} u, \qquad (x, t) \in \mathbb{R} \times \mathbb{R}_+.$$

with initial data

$$(2) \qquad u(x, 0) = u_0(x), \qquad x \in \mathbb{R}.$$

as $\delta > 0$ and $\gamma_n > 0$, $n = 1, ..., N$ aproach zero. We assume that the flux function is a given sufficiently smooth function $f : \mathbb{R} \to \mathbb{R}$ satisfying certain assumptions to be listed in Section 4. The equation of type (1) is related to the well-known BBM equations which were advocated by Benjamin-Bona-Mahony [2] as a refinement of the KdV equation [2],[10], and [1]. Since the viscous term $\gamma_1 u_{xx}$ and the dissipative term $\gamma_2 u_{xxxx}$ (case $N = 2$) are of physical backgrounds [2] and [3] and, as pointed out in [3] and [11], the convergences of the solution sequences $\{u(x, t; \delta, \gamma_1, \gamma_2)\}$ as $\delta \to 0$, $\gamma_1 \to 0$, and $\gamma_2 \to 0$ correspond to some physical processes, such as vanishing viscosity, etc. In [14] Huijiang Zhao and Benjin Xuan obtain an existence and convergence result on (1) for $N = 2$ if the condition $\delta = O(\gamma_1^{\frac{(4+2p)}{(2-p)}})$ and $\gamma_2 = O(\gamma_1^{\frac{(6+3p)}{(2-p)}})$ hold for $0 \le p < 2$. The case $N = 1$, is studied by Schonbek [11], when $f(u)$ satisfies certain growth conditions of infinity. If $f$ is convex, Schonbek discussed the

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strong convergence of the solution sequence $\{u(x,t;\delta,\gamma_1)\}$ as $\delta \to 0$ and $\gamma_1 \to 0$. In [7] we considered the case $N = 2$ with the addition of one nonlinear term for the derivative of second-order, that is,

$$u_t + f(u)_x = \gamma B(u_x)_x + \delta u_{xxt} - \alpha u_{xxxx}, \qquad (x,t) \in \mathbb{R} \times \mathbb{R}_+,$$

with initial data

$$u(x,0) = u_0(x), \qquad x \in \mathbb{R},$$

as $\gamma > 0$, $\delta > 0$, and $\alpha > 0$ are positive constants, $f$ and $B(\lambda) = \beta(\lambda) + \lambda$ are a smooth functions. Now in this paper the equation does not have any nonlinear additional terms, besides of the flux function $f$, and the equation has only even derivatives, by order $2N$, where $N$ is a positive integer . Therefore this paper extends the results of [7], when $\beta(\lambda) = 0$, and [14].

In [6], following a pionering work by Schonbek, Kondo and LeFloch studied the convergence of the solutions of hyperbolic conservation laws regularized with vanishing diffusion and dispersion terms of the form

$$u_t + f(u)_x = \epsilon u_{xx} + \delta u_{xxx}, \qquad u = u(x,t), \qquad x \in \mathbb{R} \times \mathbb{R}_+. \tag{3}$$

A convergence result is also establish for multidimensional conservation laws by relying on Diperna's uniqueness theorem for entropy measure-valued solutions. In [4], Correia and LeFloch obtained convergence results for a classe of multidimensional conservation with vanishing nonlinear diffusion and dispersion terms of the form

$$u_t + div f(u) = div(\epsilon b_j(\nabla u) + \delta \partial^2_{x_j} u)_{1 \le j \le d}, \qquad x \in \mathbb{R}^d \times \mathbb{R}_+. \tag{4}$$

We observe that the equations (3) and (4) are different from the equation (1) because they have the dispersive term $u_{xxx}$ and do not have the term $u_{xxt}$. The two papers ([6] and [4]) address the convergence of approximations conservation laws in the framework of measure-valued solutions. This technique was introduced first by Kondo and LeFloch to handle multidimensional equations and the same technique was used in this work.

The remainder of this paper is divided into five sections. After this introduction, which constitutes Section 1, we consider in Section 2 some results of the [9], [11], [13], [5], [12], and [8]. In section 3 we consider global existence results, in section 4 a priori estimates, and the convergence results are stated in Section 5.

## 2. Preliminaries

This section contains short background material on $H^s(\mathbb{R})$, Young measures, and entropy measure-valued (m.-v.) solutions.

The following result is a consequence from Theorem 4.7 (page 30) of [9]:

**Theorem 1.** *Suppose that $G = G(w)$ is sufficiently smooth. If functions $\overline{w} = \overline{w}(x)$ and $\overline{\overline{w}} = \overline{\overline{w}}(x)$ satisfy $\|w\|_\infty \le M$ ($M$ is a positive constant), and $\overline{w}$, $\overline{\overline{w}} \in H^s(\mathbb{R})$ with $s \ge 1$ then for*

$$w^* = \overline{w} - \overline{\overline{w}}$$

*we have*

$$\|G(\overline{w}) - G(\overline{\overline{w}})\|_{H^s(\mathbb{R})} \le C_s \|w^*\|_{H^s(\mathbb{R})}(|G'(0)| + \|\overline{w}\|_{H^s(\mathbb{R})} + \|\overline{\overline{w}}\|_{H^s(\mathbb{R})})$$

*where $C_s$ is a positive constant depending on $M$ and on $s$.* □

Following Schonbek [11], we state a representation theorem for the Young measures associated with a given sequence of uniformly bounded functions on $L^q$. The corresponding setting in $L^\infty$ was first established by Tartar [13].

**Lemma 2.** *Let $\{u_j\}$ be a uniformly bounded sequence in $L^\infty(\mathbb{R}_+; L^q(\mathbb{R}))$. Then there exists a subsequence $\{u'_j\}$ and a weakly-$\star$ measurable mapping $\nu : \mathbb{R}\times\mathbb{R}_+ \to Prob(\mathbb{R})$ taking its values in the space of nonnegative measures with unit total mass (probability measures) such that, for all functions $g \in C(\mathbb{R})$ satisfying*

$$g(u) = o(|u|^r) \ \ as \ \ |u| \to \infty \tag{5}$$

*for some $r \in [0, q)$, the following limit representation holds*

$$\lim_{j'\to\infty} \int\int_{\mathbb{R}\times\mathbb{R}_+} g(u_{j'}(x,t))\phi(x,t)dxdt = \int\int_{\mathbb{R}\times\mathbb{R}_+}\int_{\mathbb{R}} g(\lambda)d\nu_{(x,t)}(\lambda)\phi(x,t)dxdt \tag{6}$$

*for all $\phi \in C_c^\infty(\mathbb{R}\times\mathbb{R}_+)$.*

The measure-valued function $\nu_{(x,t)}$ is a Young measure associated with the sequence $\{u_j\}$.

Following DiPerna [5] and Szepessy [12], we now define the measure-valued (m.-v.) solutions to the first-order Cauchy problem

$$u_t + f(u)_x = 0 \tag{7}$$

$$u(x,0) = u_0(x), \ \ x \in \mathbb{R}. \tag{8}$$

**Definition 3.** Assume that $f$ satisfies the growth condition (5) and $u_0 \in L^1(\mathbb{R}) \cap L^q(\mathbb{R})$. A Young measure $\nu$ associated with a sequence $\{u_j\}$, which is assumed to be uniformly bounded in $L^\infty(\mathbb{R}_+; L^q(\mathbb{R}))$, is called an entropy m.-v. solution to the problem (7)-(8) if

$$\partial_t < \nu(.), |\lambda - k| > +\partial_x < \nu(.), sgn(\lambda - k)(f(\lambda) - f(k)) > \leq 0 \tag{9}$$

in the distributional sense for all $k \in \mathbb{R}$ and, for all intervals $I \subset \mathbb{R}$,

$$\lim_{T\to 0^+} \frac{1}{T}\int_0^T \int_I < \nu_{(x,t)}, |\lambda - u_0(x)| > dxdt = 0. \tag{10}$$

Following LeFloch and Natalini [8] we state a convergence result:

**Lemma 4.** *Assume that $f$ satisfies Eq. (5) and $u_0 \in L^1(\mathbb{R}) \cap L^q(\mathbb{R})$. Let $\{u_j\}$ be a sequence, uniformly bounded in $L^\infty(\mathbb{R}_+; L^q(\mathbb{R}))$ for $q \geq 1$, and let $\nu$ be a Young measure associated with this sequence. If $\nu$ is an entropy m.-v. solution to the problem (7)-(8), then*

$$\lim_{j\to\infty} u_j = u \ \ in \ \ L^\infty(\mathbb{R}_+; L^r_{loc}(\mathbb{R}))$$

*for all $r \in [1, q)$, where $u \in L^\infty(\mathbb{R}_+; L^q(\mathbb{R}))$ is the unique entropy solution to (7)-(8).*

## 3. Global Existence of Solutions

In this section we study the existence of global smooth solutions to the Cauchy problem (1) and (2). Here $\delta$ and $\gamma_n$, $n = 1, ..., N$, are positive constants, $f$ is a sufficiently smooth function.

Denote $F(u)$ the Fourier transform of $u$ with respect to the spatial variable $x$, $F^{-1}$ is the inverse transform of $F$. Formally, from (1)

$$F(u_t - \delta u_{xxt} + \sum_{n=1}^{N}(-1)^n \gamma_n \partial_x^{2n} u) = -F(f(u)_x)$$

$$(1+\delta\xi^2)F(u)_t + (\sum_{n=1}^{N}\gamma_n \xi^{2n})F(u) = -F(f(u)_x).$$

Then

$$\left[\exp\left\{\frac{(\sum_{n=1}^{N}\gamma_n\xi^{2n})t}{(1+\delta\xi^2)}\right\}F(u)\right]_t = -\exp\left\{\frac{(\sum_{n=1}^{N}\gamma_n\xi^{2n})t}{(1+\delta\xi^2)}\right\}\frac{F(f(u)_x)}{(1+\delta\xi^2)},$$

and integrate on $[0,t]$ we have

$$u(x,t) = F^{-1}\left(\exp\left\{\frac{-(\sum_{n=1}^{N}\gamma_n\xi^{2n})t}{(1+\delta\xi^2)}\right\}F(u_0)\right)$$
$$- \int_0^t F^{-1}\left(\frac{\exp\left\{\frac{-(\sum_{n=1}^{N}\gamma_n\xi^{2n})(t-s)}{(1+\delta\xi^2)}\right\}F(f(u)_x)}{(1+\delta\xi^2)}\right)ds.$$

Hence the integral equation of the solution is

$$u(x,t) = G(t)u_0 - \int_0^t G(t-s)F^{-1}\left(\frac{F(f(u)_x)}{(1+\delta\xi^2)}\right)ds \tag{11}$$

where $G(t)u = F^{-1}\left(\exp\left\{\frac{-(\sum_{n=1}^{N}\gamma_n\xi^{2n})t}{(1+\delta\xi^2)}\right\}F(u)\right)$. The family of linear operators $\{G(t)\}_{t\geq 0}$ satisfies the properties of semigroup.

In the following lemma we give some estimates which will be used in this section:

**Lemma 5.** *For $\theta > 0$, $\delta > 0$ and, $\gamma_n > 0$, $n = 1, ..., N$, we have the following inequalities:*

*i)* $\dfrac{\xi^{2j}}{(1+\delta\xi^2)^2}\exp\left\{\dfrac{-2(\sum_{n=1}^{N}\gamma_n\xi^{2n})\theta}{(1+\delta\xi^2)}\right\} \leq \delta^{-2}$, *if* $\delta \leq 4$, $j = 1, 2$;

*ii)* $\dfrac{\xi^{2j}}{(1+\delta\xi^2)^2}\exp\left\{\dfrac{-2(\sum_{n=1}^{N}\gamma_n\xi^{2n})\theta}{(1+\delta\xi^2)}\right\} \leq (2\gamma_j e\theta)^{-1}$, $j = 1, ..., N$;

*iii)* $\xi^2\exp\left\{\dfrac{-2(\sum_{n=1}^{N}\gamma_n\xi^{2n})\theta}{(1+\delta\xi^2)}\right\} \leq \dfrac{(\gamma_2+\gamma_1\delta)}{2\gamma_1\gamma_2\theta}$.

*Proof.* The proof follows from simple computations. □

To begin, define the operator

$$\mathcal{L}u(t) = G(t)u_0 - \int_0^t G(t-s)F^{-1}\left(\frac{F(f(u)_x)}{(1+\delta\xi^2)}\right) ds$$

on

$$\mathcal{A}_T = \{u \in C([0,T]; H^1(\mathbb{R})); \|u(t) - G(t)u_0\|_{H^1(\mathbb{R})} \leq \|u_0\|_{H^1(\mathbb{R})}, t \in [0,T]\}$$

and the norm in $\mathcal{A}_T$ by $\|u(x,t)\|_{\mathcal{A}_T} = \sup_{0\leq t\leq T} \|u(t)\|_{H^1(\mathbb{R})}$.

Our local existence result will follow from the properties of $\mathcal{L}$ given in the following lemma:

**Lemma 6.** *Suppose that $f$ is sufficiently smooth. Assume that $u(t), u_0 \in H^1(\mathbb{R})$ and that*

$$\|u(t) - G(t)u_0\|_{H^1(\mathbb{R})} \leq \|u_0\|_{H^1(\mathbb{R})}, \qquad \forall t \in [0,T]. \tag{12}$$

*If $T > 0$ is sufficiently small, then the following hold:*
*(i) $\mathcal{L}u(t) \in H^1(\mathbb{R})$ with*

$$\|\mathcal{L}u(t) - G(t)u_0\|_{H^1(\mathbb{R})} \leq \|u_0\|_{H^1(\mathbb{R})}, \qquad \forall t \in [0,T]$$

*and*

$$\|\mathcal{L}u(t)\|_{H^1(\mathbb{R})} \leq 2\|u_0\|_{H^1(\mathbb{R})}, \qquad \forall t \in [0,T];$$

*(ii) $\|\mathcal{L}u(t)\|_\infty \leq 2\sqrt{2}\|u_0\|_{H^1(\mathbb{R})}$;*
*(iii) $\mathcal{L}$ maps $\mathcal{A}_T$ into itself;*
*iv) $\mathcal{L}$ is a contraction on $\mathcal{A}_T$.*

*Proof.* Let $E = \sup_{|v|\leq 2\sqrt{2}\|u_0\|_{H^1(\mathbb{R})}} |f'(v)|$. Without loss of generallity, we take $f(0) = 0$.

(i) Let $u(t) \in H^1(\mathbb{R})$ satisfying (12), then using the properties of Fourier transform on $L^2(\mathbb{R})$, and the Parseval's equality we have

$$\begin{aligned}
\|G(t)u_0\|_{H^1(\mathbb{R})} &= \left[\sum_{k=0}^{1}\int_{\mathbb{R}}\left|\frac{\partial^k G(t)u_0}{\partial x^k}\right|^2 dx\right]^{\frac{1}{2}} \\
&= \left[\sum_{k=0}^{1}\int_{\mathbb{R}} \exp\left\{\frac{-2(\sum_{n=1}^N \gamma_n \xi^{2n})t}{(1+\delta\xi^2)}\right\}\left|F\left(\frac{\partial^k u_0}{\partial x^k}\right)\right|^2 d\xi\right]^{\frac{1}{2}} \\
&\leq \|u_0\|_{H^1(\mathbb{R})}.
\end{aligned}$$

Furthermore, using the Parseval's equality

$$\begin{aligned}
\|\mathcal{L}u(t) - G(t)u_0\|_{H^1(\mathbb{R})} \leq \int_0^t &\left\{\int_{\mathbb{R}} \frac{\exp\left\{\frac{-2(\sum_{n=1}^N \gamma_n\xi^{2n})(t-s)}{(1+\delta\xi^2)}\right\}|F(f(u))|^2\xi^2}{(1+\delta\xi^2)^2} d\xi\right. \\
&\left. + \int_{\mathbb{R}} \frac{\exp\left\{\frac{-2(\sum_{n=1}^N \gamma_n\xi^{2n})(t-s)}{(1+\delta\xi^2)}\right\}|F(f(u))|^2\xi^4}{(1+\delta\xi^2)^2} d\xi\right\}^{\frac{1}{2}} ds
\end{aligned}$$

using the Lemma 5 (ii) and the Parseval's equality

$$\begin{aligned}
&\leq \int_0^t [(2\gamma_1 e(t-s))^{-1} + (2\gamma_2 e(t-s))^{-1}]^{\frac{1}{2}} \left\{ \int_{\mathbb{R}} |f(u)|^2\,dx \right\}^{\frac{1}{2}} ds \\
&\leq \int_0^t [(2\gamma_1 e(t-s))^{-1} + (2\gamma_2 e(t-s))^{-1}]^{\frac{1}{2}} E\|u(s)\|_{L^2(\mathbb{R})} ds \\
&\leq \int_0^t 2[(2\gamma_1 e(t-s))^{-1} + (2\gamma_2 e(t-s))^{-1}]^{\frac{1}{2}} E\|u_0\|_{H^1(\mathbb{R})}\,ds \\
&\leq \|u_0\|_{H^1(\mathbb{R})}
\end{aligned}$$

if $T \leq \dfrac{1}{8[(\gamma_1 e)^{-1} + (\gamma_2 e)^{-1}]E^2}$.

(ii) The estimate (ii) is the consequence from (i) and $\|\mathcal{L}u(t)\|_\infty \leq (2\|\mathcal{L}u(t)\|_{L^2(\mathbb{R})} \|(\mathcal{L}u(t))_x\|_{L^2(\mathbb{R})})^{\frac{1}{2}}$.

(iii) To proof (iii), we only need to show if $u \in C([0,T];H^1(\mathbb{R}))$ then $\mathcal{L}u \in C([0,T];H^1(\mathbb{R}))$. Let $t_0 \in (0,T]$. Let $t \in (0,T]$. Without loss of generallity, we take $t_0 < t$. We have

$$\begin{aligned}
&\|\mathcal{L}u(t) - \mathcal{L}u(t_0)\|_{H^1(\mathbb{R})} \leq \|G(t)u_0 - G(t_0)u_0\|_{H^1(\mathbb{R})} \\
&+ \left\| \int_0^t G(t-s)F^{-1}\left(\frac{F(f(u)_x)}{1+\delta\xi^2}\right) ds - \int_0^{t_0} G(t_0-s)F^{-1}\left(\frac{F(f(u)_x)}{1+\delta\xi^2}\right) ds \right\|_{H^1(\mathbb{R})} \\
&\quad \leq \|G(t)u_0 - G(t_0)u_0\|_{H^1(\mathbb{R})} + \int_{t_0}^t \left\| G(r)F^{-1}\left(\frac{F(f(u(t-r))_x)}{(1+\delta\xi^2)}\right)\right\|_{H^1(\mathbb{R})} dr \\
&+ \int_0^{t_0} \left\| G(r)F^{-1}\left(\frac{F\left[f(u(t-r))_x - f(u(t_0-r))_x\right]}{(1+\delta\xi^2)}\right)\right\|_{H^1(\mathbb{R})} dr = A + B + C.
\end{aligned}$$

For $A$, using Parseval's equality,

$$\begin{aligned}
A &\leq \left\{ \sum_{k=0}^1 \int_{\mathbb{R}} \frac{\exp\left\{\frac{-2(\sum_{n=1}^N \gamma_n \xi^{2n})\theta}{1+\delta\xi^2}\right\} (\sum_{n=1}^N \gamma_n \xi^{2n})^2 |t-t_0|^2}{(1+\delta\xi^2)^2} \left| F\left(\frac{\partial^k u_0}{\partial x^k}\right)\right|^2 d\xi \right\}^{\frac{1}{2}} \\
&\leq \frac{|t-t_0|\|u_0\|_{H^1(\mathbb{R})}}{\sqrt{2}t_0}.
\end{aligned}$$

To estimate $B$, we use the Parseval's equality and the Lemma 5 (i)

$$\begin{aligned}
B &\leq \int_{t_0}^t \left\{ \int_{\mathbb{R}} \frac{\exp\left\{\frac{-2(\sum_{n=1}^N \gamma_n \xi^{2n})r}{1+\delta\xi^2}\right\} |F(f(u(t-r)))|^2(\xi^2+\xi^4)}{(1+\delta\xi^2)^2} d\xi \right. \\
&\leq \int_{t_0}^t \sqrt{2}\delta^{-1} \left\{ \int_{\mathbb{R}} |f(u(t-r))|^2\,dx \right\}^{\frac{1}{2}} dr \\
&\leq \int_{t_0}^t \sqrt{2}\delta^{-1} E\|u(t-r)\|_{L^2(\mathbb{R})} dr \\
&\leq 2\sqrt{2}\delta^{-1} E\|u_0\|_{H^1(\mathbb{R})}|t-t_0|.
\end{aligned}$$

Finally,

$$C \leq \int_0^{t_0} \left\{ \int_{\mathbb{R}} \frac{\exp\left\{\frac{-2(\sum_{n=1}^N \gamma_n \xi^{2n})r}{1+\delta\xi^2}\right\}(\xi^2+\xi^4)}{(1+\delta\xi^2)^2} \cdot \right.$$

$$\left. |F(f(u(t-r)) - f(u(t_0-r)))|^2 \, d\xi \right\}^{\frac{1}{2}} dr$$

we use the Lemma 5 (i)

$$\begin{aligned} &\leq \int_0^{t_0} \sqrt{2}\delta^{-1} \left\{ \int_{\mathbb{R}} |f(u(t-r)) - f(u(t_0-r))|^2 \, dx \right\}^{\frac{1}{2}} dr \\ &\leq \sqrt{2}\delta^{-1} E \int_0^{t_0} \|u(t-r) - u(t_0-r)\|_{H^1(\mathbb{R})} dr \\ &\leq \overline{C} \end{aligned}$$

where $\overline{C} \to 0$ when $|t-t_0| \to 0$ because $u \in C([0,T];H^1(\mathbb{R}))$ and $t_0 - r \in [0,T]$.

iv) Let $u, v \in \mathcal{A}_T$,

$$\|\mathcal{L}u(t) - \mathcal{L}v(t)\|_{H^1(\mathbb{R})} \leq$$

$$\int_0^t \left\{ \int_{\mathbb{R}} \frac{\exp\left\{\frac{-2(\sum_{n=1}^N \gamma_n \xi^{2n})(t-s)}{1+\delta\xi^2}\right\}(\xi^2+\xi^4)}{(1+\delta\xi^2)^2} |F[f(u) - f(v)]|^2 \, d\xi \right\}^{\frac{1}{2}} ds$$

$$\begin{aligned} &\leq \int_0^t \frac{[(2\gamma_1 e)^{-1} + (2\gamma_2 e)^{-1}]^{\frac{1}{2}} E \|u(s) - v(s)\|_{H^1(\mathbb{R})}}{(t-s)^{\frac{1}{2}}} \\ &\leq \frac{1}{2} \|u(x,t) - v(x,t)\|_{\mathcal{A}_T} \end{aligned}$$

if $T \leq \dfrac{1}{8\{((\gamma_1 e)^{-1} + (\gamma_2 e)^{-1})E^2\}}$. □

**Remark 7.** *We can to replace the space $C([0,T];H^1(\mathbb{R}))$ by $L^\infty(0,T;H^1(\mathbb{R}))$ and the properties above also hold.*

We can now obtain the local existence of solutions of (1) and (2). In te case $N \leq 3$, we obtain similar regularity results as we obtain in [7], that is, for each integer $k \geq 1$, we have $u \in C((0,T];H^k(\mathbb{R})) \cap C^1((0,T];H^{k-1}(\mathbb{R}))$. When $N > 3$, the regularity results are different from those that we obtain in [7].

**Theorem 8.** *Suppose that $u_0$ and $f$ satisfy the same assumptions as in Lemma 6 then the Cauchy problem (1) and (2) admits a unique local smooth solution*

$$u \in C([0,T];H^1(\mathbb{R})).$$

*Furthermore, for each integer $k \geq 1$, we have*
*i) $u \in C((0,T];H^k(\mathbb{R}))$;*
*ii) $u \in C((0,T];H^k(\mathbb{R})) \cap C^1((0,T];H^{k-1}(\mathbb{R}))$ if $N \leq 3$;*
*iii) $u \in C((0,T];H^{k+N-3}(\mathbb{R})) \cap C^1((0,T];H^{k-1}(\mathbb{R}))$ if $N > 3$,*
*where $T$ depends on $\gamma_1$, $\gamma_2$, $E$, and $\|u_0\|_{H^1(\mathbb{R})}$.*

*Proof.* Let $u^0 \equiv 0$ and $u^n \equiv \mathcal{L}(u^{n-1})$. Then by induction the estimates of the Lemma 6 $(i)$ and $(ii)$ hold for each $u^n$. Furthermore, by Lemma 6 (iii) and (iv), $\mathcal{L}$ maps $\mathcal{A}_T$ onto itself and is contractive. From Banach's fixed point theorem, the integral equation (11) possesses a unique solution $u \in C([0,T];H^1(\mathbb{R}))$. To prove the regularity results, we only need to show if $u \in C((0,T];H^l(\mathbb{R}))$, $l \geq 1$, then $u \in C((0,T];H^{l+1}(\mathbb{R}))$ (and in the case $N \leq 3$, $u_t \in C((0,T];H^{l-1}(\mathbb{R}))$ or if $N > 3$ and $u \in C((0,T];H^{l+N-3}(\mathbb{R}))$, then $u_t \in C((0,T];H^{l-1}(\mathbb{R}))$). Let $t_1 \in (0,T)$, we need to show that $u \in C([t_1,T];H^{l+1}(\mathbb{R}))$ (and in the case $N \leq 3$, $u_t \in C([t_1,T];H^{l-1}(\mathbb{R}))$ or if $N > 3$ and $u \in C((0,T];H^{l+N-3}(\mathbb{R}))$ then $u_t \in C([t_1,T];H^{l-1}(\mathbb{R}))$). We take $t_2 = \frac{t_1}{2}$. Then the semigroup property of $G$ implies that, for $t > t_2$

$$u(x,t) = G(t-t_2)u(t_2) - \int_{t_2}^{t} G(t-s)F^{-1}\left[\frac{F(f(u)_x)}{1+\delta\xi^2}\right] ds.$$

To begin, using the Parseval's equality and the Lemma 5 (i) and (iii), we have

$$\begin{aligned}
\|u(t)\|_{H^{l+1}(\mathbb{R})} &\leq \|G(t-t_2)u(t_2)\|_{H^{l+1}(\mathbb{R})} \\
&\quad + \int_{t_2}^{t} \left\| G(t-s)F^{-1}\left(\frac{F(f(u)_x)}{1+\delta\xi^2}\right)\right\|_{H^{l+1}(\mathbb{R})} ds \\
&\leq \left\{ \int_{\mathbb{R}} \exp\left\{\frac{-2(\sum_{n=1}^N \gamma_n \xi^{2n})(t-t_2)}{(1+\delta\xi^2)}\right\} |F(u(t_2))|^2 d\xi \right. \\
&\quad \left. + \sum_{k=0}^{l} \int_{\mathbb{R}} \exp\left\{\frac{-2(\sum_{n=1}^N \gamma_n \xi^{2n})(t-t_2)}{(1+\delta\xi^2)}\right\} \xi^2 \left| F\left(\frac{\partial^k u(t_2)}{\partial x^k}\right)\right|^2 d\xi \right\}^{\frac{1}{2}} \\
&\quad + \left\{ \sum_{k=0}^{l+1} \int_{\mathbb{R}} \frac{\exp\left\{\frac{-2(\sum_{n=1}^N \gamma_n \xi^{2n})(t-s)}{(1+\delta\xi^2)}\right\} \xi^{2k}}{(1+\delta\xi^2)^2} |F(f(u)_x)|^2 d\xi \right\}^{\frac{1}{2}} ds \\
&\leq \left\{ \int_{\mathbb{R}} |F(u(t_2))|^2 d\xi + \frac{(\gamma_2+\gamma_1\delta)}{2\gamma_1\gamma_2(t-t_2)} \sum_{k=0}^{l} \left| F\left(\frac{\partial^k u(t_2)}{\partial x^k}\right)\right|^2 d\xi \right\}^{\frac{1}{2}} \\
&\quad + \left\{ \int_{\mathbb{R}} \frac{\exp\left\{\frac{-2(\sum_{n=1}^N \gamma_n \xi^{2n})(t-s)}{(1+\delta\xi^2)}\right\} \xi^2 |F(f(u))|^2}{(1+\delta\xi^2)^2} d\xi \right. \\
&\quad \left. + \sum_{k=0}^{l} \int_{\mathbb{R}} \frac{\exp\left\{\frac{-2(\sum_{n=1}^N \gamma_n \xi^{2n})(t-s)}{(1+\delta\xi^2)}\right\} \xi^{2k+4} |F(f(u))|^2}{(1+\delta\xi^2)^2} d\xi \right\}^{\frac{1}{2}} ds \\
&\leq \left[1 + \frac{(\gamma_2+\gamma_1\delta)}{2\gamma_1\gamma_2(t_1-t_2)}\right]^{\frac{1}{2}} \|u(t_2)\|_{H^l(\mathbb{R})} + \int_{t_2}^{t} \sqrt{2}\delta^{-1} \|f(u(s))\|_{H^l(\mathbb{R})} ds
\end{aligned}$$

using the Theorem 4.3 on [9] (page 22),

$$\leq \left[1 + \frac{(\gamma_2+\gamma_1\delta)}{2\gamma_1\gamma_2(t_1-t_2)}\right]^{\frac{1}{2}} \|u(t_2)\|_{H^l(\mathbb{R})} + \int_{t_2}^{t} \sqrt{2}\delta^{-1} C \sup_{[t_2,T]} \|u(s)\|_{H^l(\mathbb{R})} ds.$$

where $C$ is a positive constant depending on $\|u_0\|_{H^1(\mathbb{R})}$. Thus

$$\sup_{t_1\le t\le T}\|u(t)\|_{H^{l+1}(\mathbb{R})}<\infty.$$

On the other hand, note

$$F(u)_t=-\frac{(\sum_{n=1}^N\gamma_n\xi^{2n})}{(1+\delta\xi^2)}F(u)-\frac{F(f(u)_x)}{(1+\delta\xi^2)}$$

and for $t_2=\frac{t_1}{2}<t_1\le t$

$$\begin{aligned}u_t(t)=&-F^{-1}\left[\frac{(\sum_{n=1}^N\gamma_n\xi^{2n})}{(1+\delta\xi^2)}\exp\left\{\frac{-(\sum_{n=1}^N\gamma_n\xi^{2n})(t-t_2)}{(1+\delta\xi^2)}\right\}F(u(t_2))\right]\\&+\int_{t_2}^t F^{-1}\left[\frac{(\sum_{n=1}^N\gamma_n\xi^{2n})\exp\left\{\frac{-(\sum_{n=1}^N\gamma_n\xi^{2n})(t-s)}{(1+\delta\xi^2)}\right\}F(f(u)_x)}{(1+\delta\xi^2)^2}\right]ds\\&+F^{-1}\left[\frac{F(f(u)_x)}{(1+\delta\xi^2)}\right].\end{aligned}$$

Then (in the case $N\ge 3$)

$$\begin{aligned}\|u_t(t)\|_{H^{l-1}(\mathbb{R})}\le&\left[\sum_{k=0}^{l-1}\int_{\mathbb{R}}\frac{\xi^{2k}|F(u(t_2))|^2}{2(t-t_2)^2}d\xi\right]^{\frac12}\\&+\int_{t_2}^t\left[\sum_{k=0}^{l-1}\int_{\mathbb{R}}\frac{(\sum_{n=1}^N\gamma_n\xi^{2n})\xi^{2k}F(f(u)_x)}{2(1+\delta\xi^2)^3(t-s)}d\xi\right]^{\frac12}ds+\|f(u(t))\|_{H^l(\mathbb{R})}\\\le&\frac{\|u(t_2)\|_{H^l(\mathbb{R})}}{\sqrt2(t_1-t_2)}+\int_{t_2}^t\left\{\left[\sum_{k=0}^{l-1}\int_{\mathbb{R}}\frac{(\sum_{n=1}^3\gamma_n\xi^{2n})\xi^{2k}|F(f(u)_x)|^2}{2(1+\delta\xi^2)^3(t-s)}d\xi\right]^{\frac12}\right.\\&\left.+\left[\sum_{k=0}^{l-1}\int_{\mathbb{R}}\frac{\xi^6(\sum_{n=1}^{N-3}\gamma_{n+3}\xi^{2n})\xi^{2k}|F(f(u)_x)|^2}{2(1+\delta\xi^2)^3(t-s)}d\xi\right]^{\frac12}\right\}ds+C\|u(t)\|_{H^l(\mathbb{R})}\\\le&\frac{\|u(t_2)\|_{H^l(\mathbb{R})}}{\sqrt2(t_1-t_2)}+\overline{C}(T-t_2)^{\frac12}\sup_{[t_2,T]}\|u(t)\|_{H^{l+N-3}(\mathbb{R})}+C\sup_{[t_1,T]}\|u(t)\|_{H^l(\mathbb{R})}\end{aligned}$$

where $\overline{C}=\overline{C}(N,\delta,\gamma_1,...,\gamma_N)$ e $C=C(\|u_0\|_{H^1(\mathbb{R})})$. Thus

$$\sup_{t_1\le t\le T}\|u_t(t)\|_{H^{l-1}(\mathbb{R})}<\infty.$$

To finish, the proof is quite similar to the Lemma 6 (iii), using the Theorem 4.3 of [9] and the Theorem 1, and its proof will be omitted. □

In the order to extend these solutions globally, that is, to all of $t>0$, we first give the following lemma.

**Lemma 9.** *Suppose $u(x,t)=u(x,t;\delta,\gamma_1,...,\gamma_N)$ a solution of (1) and (2) on $\mathbb{R}\times[0,t_2]$. Then we have the following estimate for $0\le t_1\le t_2$:*

$$\int_{\mathbb{R}}u^2(x,t_1)\,dx+\delta\int_{\mathbb{R}}u_x^2(x,t_1)\,dx\le\int_{\mathbb{R}}u_0^2(x)\,dx+\delta\int_{\mathbb{R}}u_{0x}^2(x)\,dx.$$

*Proof.* We multiply (1) by $2u$ and integrate in $\mathbb{R}$ and in $[0,t_1]$, we have

$$\int_{\mathbb{R}} u^2(x,t_1)\,dx + \delta\int_{\mathbb{R}} u_x^2(x,t_1)\,dx + \sum_{n=1}^{N}\gamma_n\int_0^{t_1}\int_{\mathbb{R}} |\partial_x^n u(x,t)|^2\,dxdt = \int_{\mathbb{R}} u_0^2(x)\,dx + \delta\int_{\mathbb{R}} u_{0x}^2(x)\,dx. \tag{13}$$

□

We can now state our global existence result:

**Theorem 10.** *Suppose $f$ and $u_0$ satisfy the same assumptions of the Theorem 8. Then the problem (1) and (2) has a global smooth solution*

$$u \in C([0,\infty); H^1(\mathbb{R})).$$

*Furthermore, for each integer $k \geq 1$, we have*
*i) $u \in C((0,\infty); H^k(\mathbb{R}))$;*
*ii) $u \in C((0,\infty); H^k(\mathbb{R})) \cap C^1((0,\infty); H^{k-1}(\mathbb{R}))$ if $N \leq 3$;*
*iii) $u \in C((0,\infty); H^{k+N-3}(\mathbb{R})) \cap C^1((0,\infty); H^{k-1}(\mathbb{R}))$ if $N > 3$.*

*Proof.* From Theorem 8, there is a unique solution $u(x,t;\delta,\gamma_1,...,\gamma_N) = u(x,t)$ defined up to time $T$ and satisfies (i)-(iii) of the Theorem 8. Furthermore, from Lemma 9, we have for $0 \leq t \leq T$

$$\|u(t)\|^2_{L^2(\mathbb{R})} + \delta\|u_x(t)\|^2_{L^2(\mathbb{R})} \leq \|u_0\|^2_{L^2(\mathbb{R})} + \delta\|u_{0x}\|^2_{L^2(\mathbb{R})}. \tag{14}$$

We consider the problem (1) with initial data $u(x,T) = u_T(x)$. Then $u_T \in H^1(\mathbb{R})$, so that, by Theorem 8, $u(x,t)$ can be extended up to time $2T$. Now suppose that $u(x,t)$ has been defined up to time $kT$ for some integer $k$, and that for each integer $l \geq 1$, we have
a) $u \in C((0,kT]; H^l(\mathbb{R}))$;
b) $u \in C((0,kT]; H^l(\mathbb{R})) \cap C^1((0,kT]; H^{l-1}(\mathbb{R}))$ if $N \leq 3$;
c) $u \in C((0,kT]; H^{l+N-3}(\mathbb{R})) \cap C^1((0,kT]; H^{l-1}(\mathbb{R}))$ if $N > 3$,
and (14) hold for $0 \leq t \leq kT$. Then by Theorem 8, $u(x,t)$ can be extended up to time $(k+1)T$ and (a)-(c) hold for $(k+1)T$. But then, from Lemma 9 (for $kT \leq t \leq (k+1)T$) and (14) (for $kT$) , we have

$$\begin{aligned}\|u(t)\|^2_{L^2(\mathbb{R})} + \delta\|u_x(t)\|^2_{L^2(\mathbb{R})} &\leq \|u(kT)\|^2_{L^2(\mathbb{R})} + \delta\|u_x(kT)\|^2_{L^2(\mathbb{R})}\\ &\leq \|u_0\|^2_{L^2(\mathbb{R})} + \delta\|u_{0x}\|^2_{L^2(\mathbb{R})}.\end{aligned}$$

and $u_{(k+1)T} \in H^1(\mathbb{R})$. Proceeding inductively, we thus establish the existence of the solution $u(x,t)$ in all of $t \geq 0$ and $u(x,t)$ satisfies (i)-(iii).

□

## 4. A Priori Estimates

Assume $f$ smooth satisfy the growth condition $|f'(u)| \leq C(1+|u|^p)$, $0 \leq p < 2$. Let $\{u = u(x,t;\delta,\gamma_1,...,\gamma_N)\}$ be a sequence of solutions of (1) and (2) obtained previously, for $\delta$ and $\gamma_n$ sufficiently smalls ($\delta + \sum_{n=1}^{N}\gamma_n \to 0$) and we take the smooth initial data $u_0 = u(x, 0; \delta, \gamma_1, ..., \gamma_N)$ which has compact support and satisfies

$$\|u_0\|_{H^N(\mathbb{R})} + \|u_0\|_{L^{2(p+1)}(\mathbb{R})} \le C_0, \tag{15}$$

for some $0 \le p < 2$, and $C_0 > 0$ independent of $\delta$, $\gamma_1, ..., \gamma_N$.

**Lemma 11.** *For every $T > 0$, we have*

$$\int_{\mathbb{R}} \frac{u_x^2}{2}\, dx + \frac{\delta}{2}\int_{\mathbb{R}} u_{xx}^2\, dx + \sum_{n=2}^{N} \gamma_n \int_0^T \int_{\mathbb{R}} |\partial_x^{n+1} u|^2\, dxdt \tag{16}$$
$$+\frac{\gamma_1}{2}\int_0^T\int_{\mathbb{R}} u_{xx}^2\, dxdt \le \overline{C}\gamma_1^{-\frac{4}{(2-p)}};$$

$$\gamma_n^2 \int_0^T \int_{\mathbb{R}} |\partial_x^{n+1} u|^2\, dxdt \le \overline{C}\gamma_n\gamma_1^{-\frac{4}{(2-p)}}; \tag{17}$$

$$\sum_{j=2}^{N-1} \frac{1}{2}\left[\gamma_j \int_{\mathbb{R}} |\partial_x^j u|^2\, dx + \delta\gamma_j \int_{\mathbb{R}} |\partial_x^{j+1} u|^2\, dx\right] \tag{18}$$

$$+\sum_{j=2}^{N-1}\left[\frac{\gamma_j^2}{2}\int_0^T\int_{\mathbb{R}} |\partial_x^{2j}u|^2\, dxdt + \sum_{\substack{n=1,\\ n\neq j}}^{N} \gamma_n\gamma_j \int_0^T\int_{\mathbb{R}} |\partial_x^{n+j}u|^2\, dxdt\right] \le C\gamma_1^{-\frac{(2+p)}{(2-p)}}.$$

*If $\delta = O(\gamma_1^{\frac{(4+2p)}{(2-p)}})$, $\gamma_2 = O(\gamma_1^{\frac{(6+p)}{(2-p)}})$, and $\gamma_n = O(\gamma_{n-1}\gamma_1^{\frac{(4+2p)}{(2-p)}})$, $n = 3, ..., N$ then*

$$d\int_{\mathbb{R}} \frac{u^{2p+2}}{2p+2}\, dx + \left[(2p+1)d - C^2 - \frac{1}{2}\right]\gamma_1 \int_0^T\int_{\mathbb{R}} u^{2p}u_x^2\, dxdt \tag{19}$$
$$+\frac{\gamma_1}{2}\int_0^T\int_{\mathbb{R}} u_t^2\, dxdt + \frac{\gamma_1\delta}{2}\int_0^T\int_{\mathbb{R}} u_{xt}^2\, dxdt + \sum_{n=1}^{N}\frac{\gamma_1\gamma_n}{2}\int_{\mathbb{R}} |\partial_x^n u|^2\, dx \le C.$$

*Proof.* First we multiply (1) by $-u_{xx}$ and integrate in $\mathbb{R}$ and in $[0, T]$, we obtain

$$\frac{1}{2}\int_{\mathbb{R}} u_x^2\, dx + \frac{\delta}{2}\int_{\mathbb{R}} u_{xx}^2\, dx + \sum_{n=1}^{N}\gamma_n \int_0^T\int_{\mathbb{R}} |\partial_x^{n+1}u|^2\, dxdt$$
$$= \frac{1}{2}\int_{\mathbb{R}} u_{0x}^2\, dx + \frac{\delta}{2}\int_{\mathbb{R}} u_{0xx}^2\, dx + \int_0^T\int_{\mathbb{R}} f'(u)u_x u_{xx}\, dxdt.$$

The last integral can be estimated using (13)

$$\int_0^T\int_{\mathbb{R}} f'(u)u_x u_{xx}\, dxdt \le \frac{\gamma_1}{2}\int_0^T\int_{\mathbb{R}} u_{xx}^2\, dxdt + \gamma_1^{-1}\int_0^T\int_{\mathbb{R}} |f'(u)|^2 u_x^2\, dxdt$$
$$\le \frac{\gamma_1}{2}\int_0^T\int_{\mathbb{R}} u_{xx}^2\, dxdt + C(1 + \|u\|_\infty^{2p})\gamma_1^{-2}.$$

$$\frac{1}{2}\int_{\mathbb{R}} u_x^2\,dx + \frac{\delta}{2}\int_{\mathbb{R}} u_{xx}^2\,dx + \frac{\gamma_1}{2}\int_0^T\int_{\mathbb{R}} u_{xx}^2\,dxdt + \sum_{n=2}^{N}\gamma_n\int_0^T\int_{\mathbb{R}}|\partial_x^{n+1}u|^2\,dxdt \le C(1+\|u\|_\infty^{2p})\gamma_1^{-2}. \tag{20}$$

Using the Cauchy-Schwarz inequality, (20), and (13) we obtain

$$\begin{aligned}|u(x,t)|^2 &\le 2\int_{-\infty}^{x}|uu_x|\,dx\\ &\le \|u(t)\|_{L^2(\mathbb{R})}\|u_x(t)\|_{L^2(\mathbb{R})}\\ &\le C(1+\|u\|_\infty^{2p})^{\frac{1}{2}}\gamma_1^{-1},\end{aligned}$$

hence

$$\|u\|_\infty \le C\gamma_1^{-\frac{1}{(2-p)}}. \tag{21}$$

Combining (20) and (21) we obtain (16) and (17).

We multiply (1) by $(-1)^j\gamma_j\partial_x^{2j}u$, $j\in\{2,...,N-1\}$ and integrate in $\mathbb{R}$ and in $[0,T]$,

$$\frac{\gamma_j}{2}\int_{\mathbb{R}}|\partial_x^j u|^2\,dx + \frac{\delta\gamma_j}{2}\int_{\mathbb{R}}|\partial_x^{j+1}u|^2\,dx + \sum_{n=1}^{N}\gamma_n\gamma_j\int_0^T\int_{\mathbb{R}}|\partial_x^{n+j}u|^2\,dxdt$$

$$= \frac{\gamma_j}{2}\int_{\mathbb{R}}|\partial_x^j u_0|^2\,dx + \frac{\delta\gamma_j}{2}\int_{\mathbb{R}}|\partial_x^{j+1}u_0|^2\,dx + (-1)^{j+1}\gamma_j\int_0^T\int_{\mathbb{R}} f(u)_x\partial_x^{2j}u\,dxdt.$$

The last integral can be estimated using (13) and (21)

$$\begin{aligned}(-1)^{j+1}\gamma_j\int_0^T\int_{\mathbb{R}} f(u)_x\partial_x^{2j}u\,dxdt &\le \gamma_j\int_0^T\int_{\mathbb{R}} C(1+|u|^p)|u_x\partial_x^{2j}u|\,dxdt\\ &\le \frac{\gamma_j^2}{2}\int_0^T\int_{\mathbb{R}}|\partial_x^{2j}u|^2\,dxdt + C\gamma_1^{-\frac{(p+2)}{(2-p)}}.\end{aligned}$$

This give (18). Finally, we multiply (1) by $du^{2p+1}+\gamma_1 u_t$, and integrate in $\mathbb{R}$ and in $[0,T]$,

$$d\int_{\mathbb{R}}\frac{u^{2p+2}}{2p+2}\,dx + (2p+1)d\gamma_1\int_0^T\int_{\mathbb{R}} u^{2p}u_x^2\,dxdt + \gamma_1\int_0^T\int_{\mathbb{R}} u_t^2\,dxdt$$

$$+\gamma_1\delta\int_0^T\int_{\mathbb{R}} u_{xt}^2\,dxdt + \sum_{n=1}^{N}\frac{\gamma_1\gamma_n}{2}\int_{\mathbb{R}}|\partial_x^n u|^2\,dx = \sum_{n=1}^{N}\frac{\gamma_1\gamma_n}{2}\int_{\mathbb{R}}|\partial_x^n u_0|^2\,dx$$

$$+d\int_{\mathbb{R}}\frac{u_0^{2p+2}}{2p+2}\,dx - \gamma_1\int_0^T\int_{\mathbb{R}} f(u)_x u_t\,dxdt - (2p+1)d\delta\int_0^T\int_{\mathbb{R}} u^{2p}u_x u_{xt}\,dxdt$$

$$+\sum_{n=2}^{N}(-1)^n(2p+1)d\gamma_n\int_0^T\int_{\mathbb{R}} u^{2p}u_x(\partial_x^{2n-1}u)\,dxdt.$$

The last $(N+1)$ integrals can be estimed the following. Using (13)

$$-\gamma_1\int_0^T\int_{\mathbb{R}} f(u)_x u_t\,dxdt \le \frac{\gamma_1}{2}\int_0^T\int_{\mathbb{R}} u_t^2\,dxdt + C + C^2\gamma_1\int_0^T\int_{\mathbb{R}} u^{2p}u_x^2\,dxdt.$$

Using (13) and (21)

$$-(2p+1)d\delta\int_0^T\int_{\mathbb{R}}u^{2p}u_xu_{xt}\,dxdt \le \frac{\gamma_1\delta}{2}\int_0^T\int_{\mathbb{R}}u_{xt}^2\,dxdt$$

$$+(2p+1)^2d^2\delta\|u\|_\infty^{4p}\gamma_1^{-1}\int_0^T\int_{\mathbb{R}}u_x^2\,dxdt \le \frac{\gamma_1\delta}{2}\int_0^T\int_{\mathbb{R}}u_{xt}^2\,dxdt + C\delta\gamma_1^{-\frac{(2p+4)}{(2-p)}}.$$

For $n=2$ we use (21) and (17)

$$(2p+1)d\gamma_2\int_0^T\int_{\mathbb{R}}u^{2p}u_xu_{xxx}\,dxdt \le \frac{\gamma_1}{2(N-1)}\int_0^T\int_{\mathbb{R}}u^{2p}u_x^2\,dxdt$$

$$+C\gamma_1^{-1}\|u\|_\infty^{2p}\left[\gamma_2^2\int_0^T\int_{\mathbb{R}}u_{xxx}^2\,dxdt\right]$$

$$\le \frac{\gamma_1}{2(N-1)}\int_0^T\int_{\mathbb{R}}u^{2p}u_x^2\,dxdt + C\gamma_2\gamma_1^{-\frac{(6+p)}{(2-p)}}.$$

For $n\ge 3$ we use (21) and (18) with $j=(n-1)$

$$(-1)^n(2p+1)d\gamma_n\int_0^T\int_{\mathbb{R}}u^{2p}u_x\partial_x^{2n-1}u\,dxdt \le \frac{\gamma_1}{2(N-1)}\int_0^T\int_{\mathbb{R}}u^{2p}u_x^2\,dxdt$$

$$+C\gamma_n^2\gamma_1^{-1}\|u\|_\infty^{2p}\int_0^T\int_{\mathbb{R}}|\partial_x^{2n-1}u|^2\,dxdt$$

$$\le \frac{\gamma_1}{2(N-1)}\int_0^T\int_{\mathbb{R}}u^{2p}u_x^2\,dxdt + C\gamma_n\gamma_{n-1}^{-1}\gamma_1^{-\frac{(4+2p)}{(2-p)}}.$$

Thus

$$d\int_{\mathbb{R}}\frac{u^{2p+2}}{2p+2}\,dx + \left[(2p+1)d - C^2 - \frac{1}{2}\right]\gamma_1\int_0^T\int_{\mathbb{R}}u^{2p}u_x^2\,dxdt +$$

$$+\frac{\gamma_1}{2}\int_0^T\int_{\mathbb{R}}u_t^2\,dxdt + \frac{\gamma_1\delta}{2}\int_0^T\int_{\mathbb{R}}u_{xt}^2\,dxdt + \sum_{n=1}^N\frac{\gamma_1\gamma_n}{2}\int_{\mathbb{R}}|\partial_x^n u|^2\,dx$$

$$\le C\left[1 + \delta\gamma_1^{-\frac{(2p+4)}{(2-p)}} + \gamma_2\gamma_1^{-\frac{(6+p)}{(2-p)}} + \gamma_n\gamma_{n-1}^{-1}\gamma_1^{-\frac{(4+2p)}{(2-p)}}\right].$$

The proof of Lemma 11 is completed.

□

## 5. Convergence Results

Observe that $u$ depends on $\delta$, and $\gamma_1,...,\gamma_N$, but in theorem below we denote $u=u^\gamma$. By making use of estimates obtained in Lemma 11 we can now state our convergence results.

**Theorem 12.** *Assume $f$ sufficiently smooth satisfy the growth condition $|f'(u)|\le C(1+|u|^p)$, $0\le p<2$. If $\delta = O(\gamma_1^{\frac{(4+2p)}{(2-p)}})$, and $\gamma_n = O(\gamma_{n-1}\gamma_1^{\frac{(4+2p)}{(2-p)}})$, $n=2,...,N$ there exist a subsequence $\{u^{\gamma_k}\}$ such that $u^{\gamma_k}\rightharpoonup\overline{u}$, $f(u^{\gamma_k})\rightharpoonup f(\overline{u})$ in the sense of distributions and $\overline{u}$ is a weak solution of (7). Furthermore, if $f''>0$ then $u^{\gamma_k}\to\overline{u}$ strongly in $L^q_{loc}(\mathbb{R}\times\mathbb{R}_+)$, $1<q<2(p+1)$.*

*Proof.* Let $\Omega = \mathbb{R}\times(0,T)$ for some $T>0$. Following Schonbek [11] (we have $2(p+1)>1$, $f(u)=o(|u|^{(p+\frac{3}{2})})$ as $|u|\to\infty$, and $p+\frac{3}{2}\in[0,2(p+1))$, we only need to show
(i) $\{u^\gamma\}$ lies in a bounded set of $L^{2(p+1)}(\Omega)$ ;
(ii) $\frac{\partial}{\partial t}\eta(u^\gamma)+\frac{\partial}{\partial x}\psi(u^\gamma)\in$ {compact set of $H^{-1}(\Omega)$}+ {bounded set of $M(\Omega)$} (a consequence from Murat's lemma [13]),
where $M(\Omega)$ denotes the space of measures and $\eta(u)$ is a smooth function with linear growth at infinity and, more precicely, such that $\eta'$ and $\eta''$ are uniformly bounded in $\mathbb{R}$ and $\psi'(u)=\eta'(u)f'(u)$.Throughout the calculation and for simplicity, we omit the upper-index $\gamma$.

We replace $T$ by $t*\in(0,T)$ in (19). We integrate once more in $(0,T)$. Hence condition (i) follows. We multiply (1) by $\eta'(u)$ and replacing $\eta'(u)f'(u)$ by $\psi'(u)$

$$\frac{\partial}{\partial t}\eta(u)+\frac{\partial}{\partial x}\psi(u)=\delta[\eta'(u)u_{xt}]_x-\delta\eta''(u)u_xu_{xt}$$

$$+\sum_{n=1}^{N}\left\{(-1)^{n+1}\gamma_n[\eta'(u)\partial_x^{2n-1}u]_x+(-1)^n\gamma_n\eta''(u)u_x\partial_x^{2n-1}u\right\}=\sum_{j=1}^{2N+2}\Gamma_j$$

Let $\theta\in C_0^\infty(\Omega)$. To estimate $\Gamma_1$ we use (19)

$$|<\Gamma_1,\theta>|\le\delta\int_0^T\int_{\mathbb{R}}|\eta'(u)u_{xt}\theta_x|\,dxdt$$
$$\le C\gamma_1^{-\frac{1}{2}}\delta^{\frac{1}{2}}\left[\delta\gamma_1\int_0^T\int_{\mathbb{R}}u_{xt}^2\,dxdt\right]^{\frac{1}{2}}\le C\gamma_1^{\frac{(3p+2)}{2(2-p)}}.$$

To estimate $\Gamma_2$ we use (19) and (13)

$$|<\Gamma_2,\theta>|\le\delta\int_0^T\int_{\mathbb{R}}|\eta''(u)u_xu_{xt}\theta|\,dxdt$$

$$\le C\delta^{\frac{1}{2}}\gamma_1^{-1}\left[\gamma_1\int_0^T\int_{\mathbb{R}}u_x^2\,dxdt\right]^{\frac{1}{2}}\left[\delta\gamma_1\int_0^T\int_{\mathbb{R}}u_{xt}^2\,dxdt\right]^{\frac{1}{2}}\le C\gamma_1^{\frac{2p}{(2-p)}}.$$

To estimate $\Gamma_3$ and $\Gamma_4$ we use (13)

$$|<\Gamma_3,\theta>|\le\gamma_1\int_0^T\int_{\mathbb{R}}|\eta'(u)u_x\theta_x|\,dxdt$$
$$\le C\gamma_1^{\frac{1}{2}}\left[\gamma_1\int_0^T\int_{\mathbb{R}}u_x^2\,dxdt\right]^{\frac{1}{2}}\le C\gamma_1^{\frac{1}{2}};$$

$$|<\Gamma_4,\theta>|\le\gamma_1\int_0^T\int_{\mathbb{R}}|\eta''(u)u_x^2\theta|\,dxdt$$
$$\le C\gamma_1\int_0^T\int_{\mathbb{R}}u_x^2\,dxdt\le C.$$

For $\Gamma_5$ and $\Gamma_6$ we use (17) and (13)

$$|<\Gamma_5,\theta>| \leq \gamma_2 \int_0^T \int_{\mathbb{R}} |\eta'(u) u_{xxx}\theta_x|\, dxdt$$
$$\leq C\left[\gamma_2^2 \int_0^T \int_{\mathbb{R}} u_{xxx}^2\, dxdt\right]^{\frac{1}{2}} \leq C\gamma_1^{\frac{(2+p)}{2(2-p)}};$$

$$|<\Gamma_6,\theta>| \leq \gamma_2 \int_0^T \int_{\mathbb{R}} |\eta''(u) u_x u_{xxx}\theta|\, dxdt$$
$$\leq C\left[\gamma_2^2 \int_0^T \int_{\mathbb{R}} u_{xxx}^2\, dxdt\right]^{\frac{1}{2}} \left[\int_0^T \int_{\mathbb{R}} u_x^2\, dxdt\right]^{\frac{1}{2}}$$
$$\leq C\gamma_2^{\frac{1}{2}}\gamma_1^{-\frac{(6-p)}{2(2-p)}} \leq \gamma_1^{\frac{p}{(2-p)}}.$$

Finally, for $n \geq 3$ we use (18):

$$\gamma_n \left|\int_0^T \int_{\mathbb{R}} \eta'(u)\partial_x^{2n-1} u\theta_x\, dxdt\right| \leq C\gamma_n \left[\int_0^T \int_{\mathbb{R}} |\partial_x^{2n-1}u|^2\, dxdt\right]^{\frac{1}{2}}$$
$$\leq C\gamma_n\gamma_{n-1}^{-\frac{1}{2}}\gamma_n^{-\frac{1}{2}}\left[\gamma_{n-1}\gamma_n \int_0^T \int_{\mathbb{R}} |\partial_x^{2n-1}u|^2\, dxdt\right]^{\frac{1}{2}}.$$
$$\leq C\gamma_n^{\frac{1}{2}}\gamma_{n-1}^{-\frac{1}{2}}\gamma_1^{-\frac{(2+p)}{2(2-p)}} \leq C\gamma_1^{\frac{(2+p)}{2(2-p)}}$$

and

$$\gamma_n \left|\int_0^T \int_{\mathbb{R}} \eta''(u) u_x \partial_x^{2n-1} u\theta\, dxdt\right|$$
$$\leq C\gamma_n \left[\int_0^T \int_{\mathbb{R}} |\partial_x^{2n-1}u|^2\, dxdt\right]^{\frac{1}{2}} \left[\int_0^T \int_{\mathbb{R}} u_x^2\, dxdt\right]^{\frac{1}{2}}$$
$$\leq C\gamma_n^{\frac{1}{2}}\gamma_{n-1}^{-\frac{1}{2}}\gamma_1^{-\frac{4}{2(2-p)}} \leq C\gamma_1^{\frac{p}{(2-p)}}$$

We see that, $\frac{\partial}{\partial t}\eta(u) + \frac{\partial}{\partial x}\psi(u)$ decomposed in the form

$$\frac{\partial}{\partial t}\eta(u) + \frac{\partial}{\partial x}\psi(u) = \widetilde{\Gamma}_1 + \widetilde{\Gamma}_2,$$

where (for $p > 0$) $\widetilde{\Gamma}_1 = \sum_{i\neq 4} \Gamma_i \in$ {compact set in $H^{-1}(\Omega)$} and $\widetilde{\Gamma}_2 = \Gamma_4$ $\in${bounded set of $M(\Omega)$}. When $p = 0$ then $\widetilde{\Gamma}_1 = \sum_{i=1}^{N+1} \Gamma_{2i-1}$ and $\widetilde{\Gamma}_2 = \sum_{i=1}^{N+1} \Gamma_{2i}$ and the proof is completed.

□

Assume again that there exists a limiting function $u_0 \in L^1(\mathbb{R}) \cap L^{2(p+1)}(\mathbb{R})$ such that

$$\lim_{\gamma\to 0} u_0^\gamma = u_0 \quad \in L^1(\mathbb{R}) \cap L^{2(p+1)}(\mathbb{R}). \tag{22}$$

**Theorem 13.** *Suppose that $f$, $\delta$, and $\gamma_n$ satisfy the same assumptions as in Theorem 12. Then*

$$\lim_{\gamma\to 0} u^{\gamma} = u \ \ in \ \ L^{\infty}(\mathbb{R}_+; L^r_{loc}(\mathbb{R}))$$

*for all $r \in [1, 2(p+1))$, where $u \in L^{\infty}(\mathbb{R}_+; L^{2(p+1)}(\mathbb{R}))$ is the unique entropy solution to (7)-(8).*

**Remark 14.** *When $p = 0$ we let $\delta = o(\gamma_1^2)$ and $\gamma_n = o(\gamma_{n-1}\gamma_1^2)$, $n \geq 2$ on Theorem 13.*

*Proof.* First of all we establish that, for any convex function $\eta(u)$ such that $\eta'$ and $\eta''$ are uniformly bounded on $\mathbb{R}$,

$$\Lambda^{\gamma} = \sum_{i=1}^{2N+2} \Gamma_i \textit{ converges to a nonpositive measure in } \mathfrak{D}'(\mathbb{R}\times\mathbb{R}_+), \tag{23}$$

conformes with the proof of Theorem 12. Furthermore, from this proof, for any given $\theta \in C_c^{\infty}(\mathbb{R}\times(0,T))$, $\theta \geq 0$, $< \sum_{i\neq 4} \Gamma_i, \theta > \to 0$ when $\gamma \to 0$. The $\Gamma_4$ is nonpositive:

$$< \Gamma_4, \theta >= -\gamma_1 \int_0^T \int_{\mathbb{R}} \eta''(u) u_x^2 \theta \, dxdt \leq 0.$$

To apply Lemma 4 we show that (9) and (10) are satisfied for a Young measure $\nu$ associated with the sequence $\{u^{\gamma}\}$. It is a standard matter to deduce, for all convex entropy pairs,

$$\partial_t < \nu(.), \eta(\lambda) > + \partial_x < \nu(.), \psi(\lambda) > \leq 0$$

from the convergence property (23). Namely it follows from the definition of the Young measure that the terms in (23) converge (in the sense of distributions) to their "natural" limits,

$$\eta(u^{\gamma}) \rightharpoonup < \nu, \eta >, \quad \psi(u^{\gamma}) \rightharpoonup < \nu, \psi > .$$

Inequality (9) (for all $k \in \mathbb{R}$) then follows by a standard regularization of the function $|u - k|$. To show that (10) is satisfied, we combine the entropy inequalities and the weak consistency property, as was suggested by DiPerna [5]. We follow the detailed argument in Szepessy [12] and in LeFloch and Natalini [8]. Consider the function $g(\lambda) = \lambda^2$ and set

$$G(\lambda, \lambda_0) := g(\lambda) - g(\lambda_0) - g'(\lambda_0)(\lambda - \lambda_0) = \frac{g''(\theta)}{2}(\lambda-\lambda_0)^2 = (\lambda-\lambda_0)^2 \geq 0 \tag{24}$$

Let $I \subset \mathbb{R}$ be a closed and bounded interval. Using the Jensen inequality and (24), it is easily checked that

$$\begin{aligned} &\frac{1}{T}\int_0^T \int_I < \nu_{(x,t)}, \lambda - u_0(x) > dxdt \\ &\leq C_I \left( \frac{1}{T}\int_0^T \int_I < \nu_{(x,t)}, G(\lambda, u_0(x)) > dxdt \right)^{\frac{1}{2}}. \end{aligned} \tag{25}$$

Let $\{\psi_n\} \subset C_c^\infty(\mathbb{R})$ be a sequence of test functions such that

$$\lim_{n\to\infty} \psi_n = g'(u_0) \;\; in \;\; L^2(\mathbb{R}).$$

Using (6) and (2), we get

$$\int_0^T \int_I < \nu_{(x,t)}, G(\lambda, u_0(x)) > dxdt \le \int_0^T \int_I < \nu_{(x,t)}, u_0(x) - \lambda > dxdt \tag{26}$$

$$+T \int_{\mathbb{R}-I} |u_0|^2 dx + 2T \|u_0\|_{L^2(\mathbb{R})} \|g'(u_0) - \psi_n\|_{L^2(\mathbb{R})}.$$

Taking an increasing sequence of compact sets $K_i$ covering $\mathbb{R}$, i.e. such that $I \subset K_1 \subset K_2 \subset ...$ and $\cup_{i=1}^\infty K_i = \mathbb{R}$, we have

$$\int_0^T \int_I < \nu_{(x,t)}, G(\lambda, u_0(x)) > dxdt \le \int_0^T \int_{K_i} < \nu_{(x,t)}, G(\lambda, u_0(x)) > dxdt.$$

This, together with (26) and $I$ replaced by $K_i$, yields

$$\frac{1}{T} \int_0^T \int_I < \nu_{(x,t)}, G(\lambda, u_0(x)) > dxdt \tag{27}$$

$$\le \frac{1}{T} \int_0^T \int_{\mathbb{R}} < \nu_{(x,t)}, u_0(x) - \lambda > \psi_n(x) dxdt + 2\|u_0\|_{L^2(\mathbb{R})} \|g'(u_0) - \psi_n\|_{L^2(\mathbb{R})}$$

since

$$\lim_{i\to\infty} \int_{\mathbb{R}-K_i} |u_0|^2 dx = 0.$$

Therefore, in view of (25) and (27), the strong consistency property (10) will be established if we show

$$\lim_{T\to 0^+} \frac{1}{T} \int_0^T \int_{\mathbb{R}} < \nu_{(x,t)}, u_0(x) - \lambda > \psi_n dxdt \le 0 \tag{28}$$

for all $n$. By definition of the Young measure (Equation (6)), we have

$$\frac{1}{T} \int_0^T \int_{\mathbb{R}} < \nu_{(x,t)}, u_0(x) - \lambda > \psi_n dxdt = \lim_{\gamma\to 0} \frac{1}{T} \int_0^T \int_{\mathbb{R}} [u_0(x) - u^\gamma(x,t)] \psi_n(x) dxdt$$

$$= \lim_{\gamma\to 0} \frac{1}{T} \left[ \int_0^T \int_{\mathbb{R}} [u_0(x) - u_0^\gamma(x)] \psi_n(x) dxdt - \int_0^T \int_{\mathbb{R}} \int_0^t u_s^\gamma(x,s) ds \psi_n(x) dxdt \right]$$

$$:= \lim_{\gamma\to 0} (A + B).$$

In view of the property (22), the term $A$ tends to zero as $\gamma \to 0$. Using (1) we have

$$B = \frac{1}{T} \int_0^T \int_{\mathbb{R}} \int_0^t u_s^\gamma(x,s) ds \psi_n(x) dxdt = \frac{1}{T} \int_0^T \int_{\mathbb{R}} \int_0^t [-f(u^\gamma)_x$$

$$+ \sum_{n=0}^{N} (-1)^{n+1} \gamma_n \partial_x^{2n} u^\gamma + \delta u_{xxt}^\gamma - \alpha u_{xxxx}^\gamma] ds \psi_n(x) dxdt \le C_n T.$$

This shows inequality (28). The proof of Theorem 13 is completed. □

**Remark 15.** *The same convergence result is also established in the case* $p > 0$ *and* $|f'(u)| \le C$ *if* $\delta = O(\gamma_1^{2(p+1)})$ *and* $\gamma_n = O(\gamma_{n-1}\gamma_1^{p+2})$.

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