C-TOTALLY REAL SUBMANIFOLDS WITH PARALLEL MEAN CURVATURE IN λ-SASAKIAN SPACE FORMS

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Dedicated to Professor Manfredo P. do Carmo on his 80th birthday

Abstract. In this paper, we prove a generalized integral inequality for an n-dimensional oriented closed C-totally real submanifold M with parallel mean curvature vector h in a $(2m+1)$-dimensional closed λ-Sasakian space form $\tilde{M}(c)$ of constant φ-sectional curvature c with $0 < c \leq \lambda$, $n \geq 2$ and if a tensor ϕ related to h and the second fundamental form satisfies a certain inequality. As a consequence we obtain that M is totally umbilic or minimal with $S = (n(c+3\lambda) + (c-\lambda))/6$, which generalize the Theorem 3 of [10]. Finally, we prove that if M is f-pseudo-parallel in a $(2n+1)$-dimensional λ-Sasakian space form with $f \geq (n(c+3\lambda) + (c-\lambda))/4n$, then M is totally geodesic, which generalize the Theorem 1 of [13], when $\lambda = 1$.

1. Introduction

Let $\tilde{M}$ be a $(2m+1)$-dimensional manifold and $\Gamma(\tilde{M})$ the Lie algebra of vector fields on $\tilde{M}$. An almost contact structure on $\tilde{M}$ is defined by a (1,1)-tensor φ, a vector field ξ and a 1-form η on $\tilde{M}$ such that for any $p \in \tilde{M}$, we have

$$\varphi_p^2 = -I + \eta_p \otimes \xi_p, \quad \eta_p(\xi_p) = 1,$$

where I denote the identity transformation of the tangent space $T_p\tilde{M}$ at p. Then $\varphi(\xi) = 0$ and $\eta \circ \varphi = 0$. Manifolds equipped whit an almost contact structure are called almost contact manifolds. A Riemannian manifold $\tilde{M}$ with metric tensor $\langle \, , \, \rangle$ and an almost contact structure (φ, ξ, η) such that

$$\langle \varphi X, \varphi Y \rangle = \langle X, Y \rangle - \eta(X)\eta(Y),$$

or equivalently

$$\langle X, \varphi Y \rangle = -\langle \varphi X, Y \rangle \quad \text{and} \quad \langle X, \xi \rangle = \eta(X),$$

for all $X, Y \in \Gamma(\tilde{M})$, is an almost contact metric manifold. The existence of an almost contact metric structure on $\tilde{M}$ is equivalent with the existence of a reduction of the structural group to $\mathcal{U}(m) \times 1$, i. e. all the matrices of $\mathcal{O}(2m+1)$ of the form

$$\begin{pmatrix} A & B & 0 \\ -B & A & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

where A and B are real $(n \times n)$-matrices. The fundamental 2-form Ψ of an almost contact metric manifold $(\tilde{M}, \varphi, \xi, \eta, \langle \, , \, \rangle)$ is defined by

$$\Psi(X, Y) = \langle X, \varphi Y \rangle,$$

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for all $X, Y \in \Gamma(\tilde{M})$, and this form satisfies $\eta \wedge \Psi^m \neq 0$. When $\Psi = \frac{1}{\lambda} d\eta$, $\lambda \neq 0$ the associated structure is a contact structure and $\tilde{M}$ is an almost λ-Sasakian manifold. An almost λ-Sasakian manifold $(\tilde{M}, \varphi, \xi, \eta, \langle\ ,\ \rangle)$ is called a λ-*Sasakian manifold* if

$$[\varphi X, \varphi Y] + \varphi^2[X, Y] - \varphi[X, \varphi Y] - \varphi[\varphi X, Y] = -2d\eta(X, Y)\xi$$

for all $X, Y \in \Gamma(\tilde{M})$. A necessary and sufficient condition for an almost contact metric manifold $(\tilde{M}, \varphi, \xi, \eta, \langle\ ,\ \rangle)$ to be a λ-Sasakian manifold is

$$\left(\tilde{\nabla}_X \varphi\right) Y = \lambda\{\langle X, Y\rangle \xi - \eta(Y)X\}, \tag{1.1}$$

for all $X, Y \in \Gamma(\tilde{M})$, where $\tilde{\nabla}$ is the Levi-Civita connection of the Riemannian metric $\langle\ ,\ \rangle$. Moreover, a λ-Sasakian manifold satisfies:

$$\tilde{\nabla}_X \xi = -\lambda \varphi X, \tag{1.2}$$

see [6]. If $\lambda = 1$ a λ-Sasakian manifold is a Sasakian manifold [4].

An n-dimensional Riemannian manifold M isometrically immersed in $\tilde{M}$ is said to be anti-invariant in $\tilde{M}$ if $\varphi T_pM \subset T_pM^\perp$ for each p of M, where T_pM and $T_pM^\perp$ denote respectively the tangent and the normal space to M at p . Thus, for any vector X tangent to M, φX is normal to M. In this case, φ is necessarily of rank $2m$ and hence $n \leq m + 1$. An n-dimensional Riemannian manifold M isometrically immersed in $\tilde{M}$ is said to be C-totally real if ξ is a normal vector field to M. Recall that a direct consequence of this definition is that M is a anti-invariant submanifold in $\tilde{M}$ and $n \leq m$. A plane section σ in $T_p\tilde{M}$ of a λ-Sasakian manifold is called a φ-section if it is spanned by X and φX, where X is a unit tangent vector field orthogonal to ξ. The sectional curvature $\tilde{k}(\sigma)$ with respect a φ-section σ is called a φ-sectional curvature. In this paper a λ-Sasakian manifold $\tilde{M}$ complete simply connected with constant φ-sectional curvature c is called a λ-Sasakian space form and is denoted by $\tilde{M}(c)$. The curvature tensor $\tilde{R}$ of $\tilde{M}(c)$ is given by [9]:

$$\begin{aligned} \tilde{R}(X, Y)Z = &\frac{c + 3\lambda}{4}(X \wedge Y)Z + \frac{c - \lambda}{4}\{\eta(X)\eta(Z)Y \\ &- \eta(Y)\eta(Z)X + \langle X, Z\rangle \eta(Y)\xi - \langle Y, Z\rangle \eta(X)\xi \\ &+ \langle \varphi Y, Z\rangle \varphi X - \langle \varphi X, Z\rangle \varphi Y - 2\langle \varphi X, Y\rangle \varphi Z\}, \end{aligned} \tag{1.3}$$

where $X \wedge Y$ is the operator defined by $(X \wedge Y)Z = \langle Y, Z\rangle X - \langle X, Z\rangle Y$.

Example 1.1. [4] *Let $\mathbb{R}^{2m+1}$ be a Euclidean space with cartesian coordinates (x^i, y^i, z). Then a 1-Sasakian structure on $\mathbb{R}^{2m+1}$ is defined by $(\varphi_0, \xi, \eta, g)$ such that*

$$\xi = 2\frac{\partial}{\partial z}, \quad \eta = \frac{1}{2}(dz - \sum_{i=1}^{m} y^i dx^i), \quad g = \frac{1}{4}(\eta \otimes \eta + \sum_{i=1}^{m} \left((dx^i)^2 + (dy^i)^2\right))$$

and the tensor field φ_0 is given by matrix

$$\begin{pmatrix} 0 & \delta_{ij} & 0 \\ -\delta_{ij} & 0 & 0 \\ 0 & y^j & 0 \end{pmatrix}.$$

With such a structure, $\mathbb{R}^{2m+1}$ is of constant φ-sectional curvature -3 and denoted by $\mathbb{R}^{2m+1}(-3)$.

Example 1.2. [1] *For $\theta \in (0, \pi/2)$, the immersion*

$$F(u, v, w, s, t) = 2(u, 0, w, 0, v\cos\theta, v\sin\theta, s\cos\theta s\sin\theta, t),$$

defines a 5-dimensional submanifolds M in $\mathbb{R}^9(-3)$. We consider on M the induced almost contact structure (φ, ξ, η, g), where $\varphi = (\sec\theta)T$, T being the tangential component of φ_0. It can be checked that $(\nabla_X \varphi)Y = \cos\theta(g(X, Y)\xi - \eta(Y)X)$, for *any vector fields X, Y tangent to M, which means that M is a λ-Sasakian manifold with $\lambda = \cos\theta \in (0, 1)$.*

For other examples, we refer to [2].

The purpose of present paper is to study n-dimensional C-totally real submanifolds M, with parallel mean curvature in λ-Sasakian space form $\tilde{M}(c)$.

It is we need consider $\Phi : T_pM \times T_pM \to T_pM^\perp$ a bilinear map defined as follows: choose an orthonormal frame $\{e_{n+1}, ..., e_{2m+1}\}$ of $T_pM^\perp$ and for each $\alpha = n+1, \dots, 2m+1$, define maps $\Phi_\alpha : T_pM \to T_pM$ by

$$\Phi_\alpha X = \langle h, e_\alpha \rangle X - A_{e_\alpha} X, \tag{1.4}$$

where h is the mean curvature vector and A_{e_α}'s are the shape operators. Then Φ is given by

$$\Phi(X, Y) = \sum_\alpha \langle \Phi_\alpha X, Y \rangle e_\alpha. \tag{1.5}$$

Therefore both Φ and $|\Phi|$ not depend on the choice of $\{e_\alpha\}$, moreover, if S be the squared norm of the second fundamental form of M, then

$$|\Phi|^2 = \sum_\alpha tr\ (\Phi_\alpha)^2 = S - nH^2, \tag{1.6}$$

where $H = |h|$. We recall that $|\Phi|^2 \equiv 0$ if and only if M is totally umbilic; $H \equiv 0$ if and only if M is minimal; and $S \equiv 0$ if and only if M is totally geodesic. We remark that the immersion F in the example (1.2) defines a 5-dimensional minimal submanifold M in an 1-Sasakian space form $\mathbb{R}^9(-3)$.

Now, for any $H \in \mathbb{R}$, we define the polynomial $P_{H,c,\lambda}$ by

$$P_{H,c,\lambda}(x) = \frac{3}{2}x^2 + \frac{n(n-2)}{\sqrt{n(n-1)}}Hx - \left(\frac{n(c+3\lambda)+c-\lambda}{4} + nH^2\right). \tag{1.7}$$

Denoting by ϑ_H the square of the positive root of $P_{H,c,\lambda}(x) = 0$, our results can be stated as:

Theorem 1.1. *Let M be an n-dimensional oriented complete closed C-totally real submanifold with parallel mean curvature vector in a closed λ-Sasakian space form $\tilde{M}(c)$, $n \geq 2$ and $0 < c \leq \lambda$. If $|\Phi|^2 \leq \vartheta_H$ on M, then*

$$\int_M |\Phi|^2 P_{H,c,\lambda}(|\Phi|) dM \geq 0. \tag{1.8}$$

As a consequence Theorem 1.1 , we get:

Theorem 1.2. *Let M be an n-dimensional oriented complete closed C-totally real submanifold with parallel mean curvature vector in a closed λ-Sasakian space form $\tilde{M}(c)$, $n \geq 2$ and $0 < c \leq \lambda$. If $|\Phi|^2 \leq \vartheta_H$ on M, then either M is totally umbilical or $m = n$ and M is minimal, non-totally geodesic. In this case,*

$$S = \frac{1}{6}\{n(c+3\lambda) + c - \lambda\}.$$

In particular, if $c = \lambda = 1$, then M is either a totally geodesic submanifold or a Veronese surface.

A submanifold M is *f-pseudo-parallel* if its second fundamental form σ satisfies the following condition

$$\overline{R}(X, Y) \cdot \sigma = f\ X \wedge Y \cdot \sigma,$$

for some real valued smooth function f on M and for any X and Y vectors tangent to M, where $\overline{R}(X, Y)$ is the curvature operator of the Van der Waerden-Bortolotti connection $\overline{\nabla}$ of M, which with the operator $X \wedge Y$ act on σ as a derivation [3]. We prove a result that generalize the Theorem 1 of [13].

Theorem 1.3. *Let M be an n-dimensional C-totally real submanifold with parallel mean curvature vector in a $(2n+1)$-dimensional λ-Sasakian space form $\tilde{M}(c)$. If M is f-pseudo-parallel and $f \geq (n(c+3\lambda)+c-\lambda)/4n$, then M is totally geodesic.*

Finally, we get the following results for closed f-pseudo-parallel submanifolds with parallel mean curvature vector in a λ-Sasakian space form.

Theorem 1.4. *Let M be an n-dimensional closed C-totally real submanifold with parallel mean curvature vector in a $(2m+1)$-dimensional λ-Sasakian space form $\tilde{M}(c)$. If M is f-pseudo-parallel and $f \geq 0$, then M is parallel, i.e. $\overline{\nabla}\sigma = 0$.*

Corollary 1.1. *Let M be an n-dimensional closed C-totally real submanifold with parallel mean curvature vector in a $(2n+1)$-dimensional λ-Sasakian space form $\tilde{M}(c)$. If M is f-pseudo-parallel and $f > 0$, then M is totally geodesic.*

2. Preliminaries

Let $\tilde{M}(c)$ be a $(2m+1)$-dimensional λ-Sasakian space form with structure $(\varphi, \xi, \eta, \langle\ ,\ \rangle)$ and M an n-dimensional C-totally real submanifold ($n \leq m$). As usual, $\tilde{\nabla}$ (resp. ∇) be the Riemannian connection with respect to $\langle\ ,\ \rangle$ (resp. $\langle\ ,\ \rangle|_M$) and $\nabla^{\perp}$ the connection in the normal bundle on M. These connections are related by the Gauss and the Weingarten formulas

$$\begin{aligned} \tilde{\nabla}_X Y &= \nabla_X Y + \sigma(X,Y), \\ \tilde{\nabla}_X N &= -A_N X + \nabla^{\perp}_X N, \end{aligned} \tag{2.1}$$

for any X, Y vectors tangent to M and any N vector normal to M, where A_N is the shape operator (which is auto-adjunt) in the direction N and σ is the second fundamental form on M. The shape operator and second fundamental form are related by

$$\langle A_N X, Y\rangle = \langle \sigma(X,Y), N\rangle. \tag{2.2}$$

Let R, $\tilde{R}$ and $R^{\perp}$ the curvature tensors of ∇, $\tilde{\nabla}$ and $\nabla^{\perp}$, respectively. Then, the Gauss and the Ricci equations are given by

$$\begin{aligned} \langle R(X,Y)Z, W\rangle &= \langle \tilde{R}(X,Y)Z, W\rangle + \langle \sigma(X,W), \sigma(Y,Z)\rangle \\ &\quad -\langle \sigma(X,Z), \sigma(Y,W)\rangle, \end{aligned} \tag{2.3}$$

$$\langle R^{\perp}(X,Y)N_1, N_2\rangle = \langle \tilde{R}(X,Y)N_1, N_2\rangle + \langle [A_{N_1}, A_{N_2}], Y\rangle. \tag{2.4}$$

The Codazzi-Mainardi equation is

$$(\overline{\nabla}\sigma)(X,Y,Z) = (\overline{\nabla}\sigma)(X,Z,Y), \tag{2.5}$$

where $\overline{\nabla}\sigma$ is the first covariant derivative of σ is defined by

$$\begin{aligned} (\overline{\nabla}\sigma)(X,Y,Z) &= (\overline{\nabla}_Z\sigma)(X,Y) \\ &= \nabla^{\perp}_Z[\sigma(X,Y)] - \sigma(\nabla_Z Y, X) - \sigma(Y, \nabla_Z X), \end{aligned} \tag{2.6}$$

and the second covariant derivative is defined by

$$\begin{aligned} (\overline{\nabla}^2\sigma)(X,Y,Z,W) &= (\overline{\nabla}_W\overline{\nabla}_Z\sigma)(X,Y) \\ &= \nabla^{\perp}_W[(\overline{\nabla}_Z\sigma)(X,Y)] - (\overline{\nabla}_Z\sigma)(\nabla_W X, Y) \\ &\quad -(\overline{\nabla}_Z\sigma)(X, \nabla_W Y) - (\overline{\nabla}_{\overline{\nabla}_W Z}\sigma)(X,Y). \end{aligned} \tag{2.7}$$

Then, we have

$$R^{\perp}(X,Y)[\sigma(Z,W)] = (\overline{\nabla}_X\overline{\nabla}_Y\sigma)(Z,W) - (\overline{\nabla}_Y\overline{\nabla}_X\sigma)(Z,W) \tag{2.8}$$
$$+\sigma(R(X,Y)Z,W) + \sigma(Z,R(X,Y)W).$$

In this work we use the following convention of index:

$$1 \leq A,B,C,\cdots \leq 2m+1,$$
$$1 \leq i,j,k,\cdots \leq n, \quad i^* = m+i,$$
$$n+1 \leq \alpha,\beta,\gamma,\cdots \leq 2m+1.$$

As M is a C-totally real submanifold, we can choose a local orthonormal frame $\{e_1,...,e_n,e_{n+1},...,e_m,e_{1^*} = \varphi e_1,...,e_{(n+1)^*} = \varphi e_{n+1},...,e_{m^*} = \varphi e_m, e_{2m+1} = \xi\}$ in $\tilde{M}(c)$ such that $\{e_i\}$ at each point of M span the tangent space of M.

Let $\{\omega_A\}$ be the dual of $\{e_A\}$ and let $\{\omega_{AB}\}$ be the connection 1-forms of $\tilde{M}(c)$. Then the structure equations of Cartan are given by

$$d\omega_A = -\sum_B \omega_{AB}\wedge\omega_B, \quad \omega_{AB}+\omega_{BA} = 0, \tag{2.9}$$

$$d\omega_{AB} = \sum_C \omega_{AC}\wedge\omega_{CB} + \frac{1}{2}\sum_{C,D}\tilde{R}_{ABCD}\,\omega_C\wedge\omega_D. \tag{2.10}$$

The (ω_{AB}) is a real representation of a skew-Hermitian matrix. Hence

$$\omega_{i^*j} = \omega_{j^*i}. \tag{2.11}$$

Moreover,

$$\omega_{ij} = \omega_{i^*j^*} \quad \text{and} \quad \omega_{i^*} = -\omega_{i(2m+1)}. \tag{2.12}$$

Thus, we have along M that

$$\omega_\alpha = 0,$$

which implies $0 = d\omega_\alpha = -\sum_i \omega_{\alpha i}\wedge\omega_i$ along M. From Cartan's Lemma, we write

$$\omega_{\alpha i} = \sum_j h^\alpha_{ij}\omega_j, \quad h^\alpha_{ij} = h^\alpha_{ji}, \tag{2.13}$$

where h^α_{ij} denoted the components of second fundamental form σ, that is

$$h^\alpha_{ij} = \langle A_{e_\alpha}e_i, e_j\rangle = \langle\sigma(e_i,e_j),e_\alpha\rangle. \tag{2.14}$$

Therefore, from (2.11) and (2.2) we have

$$h^{i^*}_{jk} = h^{j^*}_{ik} = h^{k^*}_{ij},\ h^{2m+1}_{ij} = 0. \tag{2.15}$$

From (1.3), we get

$$\tilde{R}_{ijkl} = \frac{c+3\lambda}{4}(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}), \tag{2.16}$$

and

$$\tilde{R}_{\alpha\beta kl} = \begin{cases} \dfrac{c-\lambda}{4}(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}), & \text{if} \quad \alpha = i^*, \quad \beta = j^*; \\ 0, \quad \text{otherwise,} \end{cases} \tag{2.17}$$

where $\langle e_i, e_j\rangle = \delta_{ij}$. Using (2.16) in (2.3), we obtain

$$R_{ijkl} = \frac{c+3\lambda}{4}(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) + \sum_\alpha \left(h^\alpha_{ik}h^\alpha_{jl} - h^\alpha_{il}h^\alpha_{jk}\right), \tag{2.18}$$

and subtituting (2.17) in (2.4), we get

$$R^{\perp}_{\alpha\beta kl} = \begin{cases} \frac{c-\lambda}{4}(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) + \sum_r \left(h^{\alpha}_{rk}h^{\beta}_{rl} - h^{\alpha}_{rl}h^{\beta}_{rk}\right), & \text{if } \alpha = i^*, \beta = j^*; \\ \sum_r \left(h^{\alpha}_{rk}h^{\beta}_{rl} - h^{\alpha}_{rl}h^{\beta}_{rk}\right), & \text{otherwise.} \end{cases} \tag{2.19}$$

Let S be the squared norm of second fundamental form, h denote the mean curvature vector field and H the mean curvature of M, that is

$$S = \sum_{\alpha,i,j} (h^{\alpha}_{ij})^2, \quad h = \frac{1}{n}\sum_{\alpha}\left(\sum_i h^{\alpha}_{ii}\right) e_{\alpha}, \quad H = |h|. \tag{2.20}$$

The Ricci curvature tensor $\{R_{kl}\}$ and the scalar curvature K are expressed, respectively, as follows:

$$R_{kl} = \frac{c+3\lambda}{4}(n-1)\delta_{kl} + \sum_{\alpha}\left(\sum_i h^{\alpha}_{ii}\right) h^{\alpha}_{kl} - \sum_{\alpha,i} h^{\alpha}_{ki}h^{\alpha}_{il}, \tag{2.21}$$

$$K = \frac{c+3\lambda}{4}n(n-1) + (n^2H^2 - S). \tag{2.22}$$

The components of the covariant derivative of σ are given by

$$h^{\alpha}_{ijk} = \left\langle \left(\overline{\nabla}_{e_k}\sigma\right)(e_i, e_j), e_{\alpha}\right\rangle = \overline{\nabla}_{e_k} h^{\alpha}_{ij}, \tag{2.23}$$

hence, the square of the length of third fundamental form of M is given

$$|\overline{\nabla}\sigma|^2 = \sum_{\alpha,i,j,k} \left(h^{\alpha}_{ijk}\right)^2. \tag{2.24}$$

The components of the second covariant derivative of σ are given by

$$h^{\alpha}_{ijkl} = \left\langle \left(\overline{\nabla}_{e_l}\overline{\nabla}_{e_k}\sigma\right)(e_i, e_j), e_{\alpha}\right\rangle = \overline{\nabla}_{e_l} h^{\alpha}_{ijk} = \overline{\nabla}_{e_l}\overline{\nabla}_{e_k} h^{\alpha}_{ij}. \tag{2.25}$$

Hence, we get

$$\sum_k h^{\alpha}_{ijk}\omega_k = dh^{\alpha}_{ij} - \sum_r h^{\alpha}_{jr}\omega_{ri} - \sum_r h^{\alpha}_{ir}\omega_{rj} + \sum_{\beta} h^{\beta}_{ij}\omega_{\alpha\beta}, \tag{2.26}$$

$$\begin{aligned} \sum_l h^{\alpha}_{ijkl}\omega_l =& dh^{\alpha}_{ijk} - \sum_r h^{\alpha}_{rjk}\omega_{ri} - \sum_r h^{\alpha}_{irk}\omega_{rj} \\ & - \sum_r h^{\alpha}_{ijr}\omega_{rk} + \sum_{\beta} h^{\alpha}_{ijk}\omega_{\alpha\beta}. \end{aligned} \tag{2.27}$$

From (2.5), we have

$$h^{\alpha}_{ijk} - h^{\alpha}_{ikj} = 0, \tag{2.28}$$

and by (2.8), we obtain the following Ricci formula

$$h^{\alpha}_{ijkl} - h^{\alpha}_{ijlk} = \sum_r h^{\alpha}_{rj}R_{rikl} + \sum_r h^{\alpha}_{ri}R_{rjkl} - \sum_{\beta} h^{\beta}_{ij}R^{\perp}_{\alpha\beta kl}. \tag{2.29}$$

From (2.12), (2.11) and (2.26), we get

$$h^{2m+1}_{ijk} = -h^{k^*}_{ij}. \tag{2.30}$$

The Laplacian $\triangle h^{\alpha}_{ij}$ of h^{α}_{ij} is defined by $\triangle h^{\alpha}_{ij} = \sum_k h^{\alpha}_{ijkk} = \sum_k h^{\alpha}_{kijk}$. Using (2.28) and (2.29), we obtain

$$\begin{aligned} \Delta h^{\alpha}_{ij} =& \sum_{k,r} h^{\alpha}_{kr}R_{rijk} + \sum_{kr} h^{\alpha}_{ri}R_{rkjk} - \sum_{k,\beta} h^{\beta}_{ki}R^{\perp}_{\alpha\beta kj} \\ =& \sum_{k,r}(h^{\alpha}_{kr}\tilde{R}_{rijk} + h^{\alpha}_{ri}\tilde{R}_{rkjk}) + \sum_{k,\beta} h^{\beta}_{ki}\tilde{R}_{\alpha\beta kj} \\ & + \sum_{r,k,\alpha}(h^{\beta}_{ri}h^{\beta}_{rj}h^{\beta}_{kk} + 2h^{\alpha}_{kr}h^{\beta}_{rj}h^{\beta}_{ik} - h^{\alpha}_{kr}h^{\beta}_{kr}h^{\beta}_{ij} \\ & - h^{\alpha}_{ri}h^{\beta}_{kr}h^{\beta}_{kj} - h^{\alpha}_{rj}h^{\beta}_{ki}h^{\beta}_{kr}). \end{aligned} \tag{2.31}$$

Since

$$\frac{1}{2}\Delta S = \sum_{\alpha,i,j} h_{ij}^{\alpha}\Delta h_{ij}^{\alpha} + \sum_{\alpha,i,j,k}\left(h_{ijk}^{\alpha}\right)^2, \tag{2.32}$$

we have

$$\begin{aligned}\frac{1}{2}\Delta S = &\sum_{\alpha,i,j,k}\left(h_{ijk}^{\alpha}\right)^2 + \sum_{\alpha,i,j,k,r}(h_{ij}^{\alpha}h_{kr}^{\alpha}\tilde{R}_{rijk} + h_{ij}^{\alpha}h_{rj}^{\alpha}\tilde{R}_{rkik})\\ &+ \sum_{\alpha,\beta,i,j,k} h_{ij}^{\alpha}h_{ki}^{\beta}\tilde{R}_{\alpha\beta kj} - \sum_{\alpha,\beta,i,j,k,r} h_{ij}^{\alpha}h_{kr}^{\alpha}h_{ij}^{\beta}h_{kr}^{\beta}\\ &+ \sum_{\alpha,\beta,i,j,k,r} h_{ij}^{\alpha}h_{ir}^{\alpha}h_{jr}^{\beta}h_{kk}^{\beta}\\ &- \sum_{\alpha,\beta,i,j,k,r}(h_{rj}^{\alpha}h_{kr}^{\beta} - h_{kr}^{\alpha}h_{rj}^{\beta})(h_{ij}^{\alpha}h_{ki}^{\beta} - h_{ki}^{\alpha}h_{ij}^{\beta}).\end{aligned} \tag{2.33}$$

We remark that 2.31 and 2.33 can be found by specialising the result of [8] to this case.

3. Estimates and proofs of Theorems 1.2 and 1.3

Now, we assume that the mean curvature vector h of M is parallel (i.e., $\nabla^{\perp}h = 0$), and M is a complete submanifold in $\tilde{M}(c)$.

In this section Φ_α denoted the matrix (Φ_{ij}^{α}), where $\Phi_{ij}^{\alpha} = \langle\Phi_\alpha e_i, e_j\rangle$. Note that to $H = 0$ (i.e., M is minimal submanifold), we get $\Phi_\alpha = -H_\alpha$, for all α, where H_α is the matrix (h_{ij}^{α}). If $H \neq 0$, we choose a local orthonormal frame $\{e_1, ..., e_n, e_{n+1}, ..., e_m, ..., e_{2m+1}\}$ such that $e_{n+1} = \frac{h}{H}$. With this choose

$$\Phi_{n+1} = HI - H_{n+1},\ \Phi^{\alpha} = H_\alpha,\ \ \alpha \neq n+1, \tag{3.1}$$

where $I = (\delta_{ij})$. Since e_{n+1} is a parallel direction,

$$H_\alpha H_{n+1} = H_{n+1}H_\alpha, \quad \omega_{\alpha(n+1)} = 0 \quad \text{and} \quad \sum_k h_{kki}^{\alpha} = 0. \tag{3.2}$$

In this case, we obtain

$$\text{tr } H_{n+1} = nH, \quad \text{tr } H_\alpha = 0,\ \ \alpha \neq n+1 \quad \text{and} \quad R^{\perp}_{(n+1)\alpha ij} = 0. \tag{3.3}$$

Furthermore,

$$|\Phi_{n+1}|^2 = \text{tr } H_{n+1}^2 - nH^2, \tag{3.4}$$

$$\sum_{\alpha\neq n+1}|\Phi_\alpha|^2 = \sum_{\beta\neq n+1}\left(h_{ij}^{\beta}\right)^2, \tag{3.5}$$

and

$$\text{tr } \Phi_\alpha = 0, \tag{3.6}$$

for all α. Thus,

$$S = \sum_\alpha |\Phi_\alpha|^2. \tag{3.7}$$

Now, we need the following algebraic lemmas:

Lemma 3.1. [11] *If A and B are two symmetric linear maps of $\mathbb{R}^n$ with $AB - BA = 0$ and $\text{tr } A = \text{tr } B = 0$. Then*

$$|\text{tr } A^2B| \leq \frac{(n-2)}{\sqrt{n(n-1)}}\text{tr } A^2\sqrt{\text{tr } B^2} \tag{3.8}$$

and the equality holds if only if $n-1$ of eigenvalues x_i of A and the corresponding eigenvalues y_i of B satisfy

$$|x_i| = \sqrt{\frac{\text{tr } A^2}{n(n-1)}}, \quad x_ix_j \geq 0,$$

$$y_i = \sqrt{\frac{\text{tr } B^2}{n(n-1)}} \quad \left(\textit{resp. } y_i = -\sqrt{\frac{\text{tr } B^2}{n(n-1)}}\right).$$

Lemma 3.2. [5, 10]. *Let $A_1, A_2, ..., A_k$ be symmetric $(n \times n)$-degree matrices, where $k \geq 2$. Denote $L_{ij} = \text{tr } A_i A_j^t$ and $L = L_{11} + L_{22} + ... + L_{kk}$. Then*

$$\sum \left\{ N(A_i A_j - A_j A_i) + (L_{ij})^2 \right\} \leq \frac{3}{2} L^2, \tag{3.9}$$

where $N(A) = \text{tr } AA^t$, for all matrix A.

The ideas used for proving the following lemmas are analogous to that found in [8].

Lemma 3.3.

$$\sum_{\alpha,i,j,k,r} (h_{ij}^\alpha h_{rk}^\alpha \tilde{R}_{rijk} + h_{ij}^\alpha h_{rj}^\alpha \tilde{R}_{rkik}) = \frac{c+3\lambda}{4} n|\Phi|^2. \tag{3.10}$$

Proof. Fix a vector e_α and let $\{e_i\}$ be a local orthogonal frame on M such that the matrix H_α (resp. Φ_α) takes the diagonal form with $h_{ij}^\alpha = \mu_i^\alpha \delta_{ij}$ (resp. $\Phi_{ij}^\alpha = \lambda_i^\alpha \delta_{ij}$, where $\lambda_i^\alpha = \langle h, e_\alpha \rangle - \mu_i^\alpha$). Then, of (2.16) we get

$$\begin{aligned}
\sum_{i,j,k,r} (h_{ij}^\alpha h_{rk}^\alpha \tilde{R}_{rijk} + h_{ij}^\alpha h_{rj}^\alpha \tilde{R}_{rkik}) &= \sum_{i,k} (\mu_i^\alpha \mu_k^\alpha \tilde{R}_{kiik} + (\mu_i^\alpha)^2 \tilde{R}_{ikik}) \\
&= \sum_{i,k} ((\mu_i^\alpha)^2 - \mu_i^\alpha \mu_k^\alpha) \tilde{R}_{ikik} \\
&= \sum_{i,k} ((\lambda_i^\alpha)^2 - \lambda_i^\alpha \lambda_k^\alpha) \tilde{R}_{ikik} \\
&= \frac{c+3\lambda}{4} n \text{ tr } \Phi_\alpha^2 \\
&= \frac{c+3\lambda}{4} n |\Phi_\alpha|^2.
\end{aligned}$$

Hence

$$\sum_{\alpha,i,j,k,r} (h_{ij}^\alpha h_{rk}^\alpha \tilde{R}_{rijk} + h_{ij}^\alpha h_{rj}^\alpha \tilde{R}_{rkik}) = \frac{c+3\lambda}{4} n|\Phi|^2.$$

□

Lemma 3.4. *If $c \leq \lambda$, then*

$$\sum_{\alpha,\beta,i,j,k} h_{ij}^\alpha h_{ki}^\beta \tilde{R}_{\alpha\beta kj} \geq \frac{c-\lambda}{4} |\Phi|^2.$$

Proof. As M is a C-totally real submanifold, we can choose a local orthonormal frame $\{e_1, ..., e_n, e_{n+1}, ..., e_m, e_{1^*} = \varphi e_1, ..., e_{(n+1)^*} = \varphi e_{n+1}, ..., e_{m^*} = \varphi e_m, e_{2m+1} = \xi\}$ in $\tilde{M}(c)$. If $\alpha \neq r^*$ or $\beta \neq s^*$, then from (2.17) we have

$$\sum_{\alpha,\beta,i,j,k} h_{ij}^\alpha h_{ki}^\beta \tilde{R}_{\alpha\beta kj} = 0.$$

If $\alpha = r^*$ and $\beta = s^*$, from (2.17) we obtain

$$\begin{aligned}
\sum_{r^*,s^*,i,j,k} h_{ij}^{r^*} h_{ki}^{s^*} \tilde{R}_{r^* s^* kj} &= \sum_{r^*,s^*,i,k} h_{jr}^{i^*} h_{ks}^{i^*} \tilde{R}_{r^* s^* kj} \\
&= \sum_{r,s,i} \frac{c-\lambda}{4} \left((h_{sr}^{i^*})^2 - h_{rr}^{i^*} h_{ss}^{i^*} \right) \\
&= \frac{c-\lambda}{4} \sum_i \text{tr } \Phi_{i^*}^2 = \frac{c-\lambda}{4} \sum_i |\Phi_{i^*}|^2 \geq \frac{c-\lambda}{4} |\Phi|^2.
\end{aligned}$$

and the lemma is proved. □

Lemma 3.5.

$$-\sum_{\alpha,\beta,i,j,k,l} h_{ij}^\alpha h_{kl}^\alpha h_{ij}^\beta h_{kl}^\beta = -\sum_{\alpha,\beta} (\text{tr } \Phi_\alpha \Phi_\beta)^2 - n^2 H^4 - 2nH^2 |\Phi_{n+1}|^2.$$

Proof. If $H = 0$, we have $\Phi_\alpha = -H_\alpha$ for all α. Hence,

$$-\sum_{\alpha,\beta,i,j,k,l} h^\alpha_{ij}h^\alpha_{kl}h^\beta_{ij}h^\beta_{kl} = -\sum_{\alpha,\beta}(\text{tr } H_\alpha H_\beta)^2 = -\sum_{\alpha,\beta}(\text{tr } \Phi_\alpha\Phi_\beta)^2,$$

which proves the lemma in this case. If $H \neq 0$, choose a local orthonormal frame $\{e_1, ..., e_n, e_{n+1}, ..e_m, ..., e_{2m+1}\}$ such that $e_{n+1} = \frac{h}{H}$, and thus

$$\begin{aligned}
-\sum_{\alpha,\beta,i,j,k,l} h^\alpha_{ij}h^\alpha_{kl}h^\beta_{ij}h^\beta_{kl} &= -\sum_{\alpha,\beta}(\text{tr } H_\alpha H_\beta)^2 \\
&= -\sum_{\alpha,\beta>n+1}(\text{tr } \Phi_\alpha\Phi_\beta)^2 - 2\sum_{\alpha>n+1}(\text{tr } (HI-\Phi_{n+1})\Phi_\alpha)^2 \\
&\quad -(\text{tr } (HI-\Phi_{n+1})^2)^2 \\
&= -\sum_{\alpha,\beta>n+1}(\text{tr } \Phi_\alpha\Phi_\beta)^2 - 2\sum_{\alpha>n+1}(H\text{tr } (\Phi_\alpha) - \text{tr } \Phi_{n+1}\Phi_\alpha)^2 \\
&\quad -(\text{tr } (H^2I - 2H\Phi_{n+1} + \Phi^2_{n+1}))^2 \\
&= -\sum_{\alpha,\beta>n+1}(\text{tr } \Phi_\alpha\Phi_\beta)^2 - 2\sum_{\alpha>n+1}(\text{tr } \Phi_{n+1}\Phi_\alpha)^2 \\
&\quad -(nH^2 + \text{tr } \Phi^2_{n+1})^2 \\
&= -\sum_{\alpha,\beta}(\text{tr } \Phi_\alpha\Phi_\beta)^2 - n^2H^4 - 2nH^2\text{tr } \Phi^2_{n+1} \\
&= -\sum_{\alpha,\beta}(\text{tr } \Phi_\alpha\Phi_\beta)^2 - n^2H^4 - 2nH^2|\Phi_{n+1}|^2.
\end{aligned}$$

□

Lemma 3.6.

$$\sum_{\alpha,\beta,i,j,k,l} h^\alpha_{ij}h^\alpha_{il}h^\beta_{jl}h^\beta_{kk} \geq -\frac{n(n-2)}{\sqrt{n(n-1)}}H|\Phi|^3 + 2nH^2|\Phi_{n+1}|^2 + nH^2|\Phi|^2 + n^2H^4.$$

Proof. Note that the inequality is obvious if $H = 0$. If $H \neq 0$, we obtain

$$\begin{aligned}
\sum_{\alpha,\beta,i,j,k,l} h^\alpha_{ij}h^\alpha_{il}h^\beta_{jl}h^\beta_{kk} &= \sum_{\alpha,\beta}\text{tr } H_\alpha \text{ tr } H_\alpha H^2_\beta \\
&= nH\sum_\alpha \text{tr } H_{n+1}H^2_\alpha \\
&= nH^2\sum_{\alpha>n+1}\text{tr } (HI-\Phi_{n+1})\Phi^2_\alpha + nH \text{ tr } (HI-\Phi)^3 \\
&= nH^2\sum_{\alpha>n+1}\text{tr } \Phi^2_\alpha - nH\sum_{\alpha>n+1}\text{tr } \Phi_{n+1}\Phi^2_\alpha \\
&\quad + nH \text{ tr } (H^3I - 3H^2\Phi_{n+1} + 3H\Phi^2_{n+1} - \Phi^3_{n+1}) \\
&= nH^2\sum_{\alpha>n+1}\text{tr } \Phi^2_\alpha - nH\sum_\alpha \text{tr } \Phi_{n+1}\Phi^2_\alpha \\
&\quad + n^2H^4 + 3nH^2 \text{ tr } \Phi^2_{n+1} \\
&= nH^2|\Phi|^2 - nH\sum_\alpha \text{tr } \Phi_{n+1}\Phi^2_\alpha + n^2H^4 + 2nH^2|\Phi_{n+1}|^2
\end{aligned}$$

Using lemma 3.1, we have

$$\text{tr } \Phi_{n+1}\Phi^2_\alpha \leq \frac{n-2}{\sqrt{n(n-1)}}|\Phi_{n+1}||\Phi_\alpha|^2, \tag{3.11}$$

and so

$$\sum_\alpha \text{tr } \Phi_{n+1}\Phi^2_\alpha \leq \frac{n-2}{\sqrt{n(n-1)}}|\Phi_{n+1}||\Phi|^2. \tag{3.12}$$

Hence,

$$\sum_{\alpha,\beta,i,j,k,l} h_{ij}^{\alpha} h_{il}^{\alpha} h_{jl}^{\beta} h_{kk}^{\beta} \geq -\frac{n(n-2)}{\sqrt{n(n-1)}} H|\Phi|^3 + 2nH^2|\Phi_{n+1}|^2 + nH^2|\Phi|^2 + n^2H^4.$$

□

Lemma 3.7.

$$\sum_{\alpha,\beta,i,j,k,r} (h_{rj}^{\alpha} h_{kr}^{\beta} - h_{kr}^{\alpha} h_{rj}^{\beta})(h_{ij}^{\alpha} h_{ki}^{\beta} - h_{ki}^{\alpha} h_{ij}^{\beta}) - \sum_{\alpha,\beta,i,j,k,r} h_{ij}^{\alpha} h_{kr}^{\alpha} h_{ij}^{\beta} h_{kr}^{\beta}$$

$$\geq -\frac{3}{2}|\Phi|^4 - n^2H^4 - 2nH^2|\Phi_{n+1}|^2.$$

Proof. Note that

$$\sum_{\alpha,\beta,i,j,k,r} (h_{rj}^{\alpha} h_{kr}^{\beta} - h_{kr}^{\alpha} h_{rj}^{\beta})(h_{ij}^{\alpha} h_{ki}^{\beta} - h_{ki}^{\alpha} h_{ij}^{\beta}) = -\sum_{\alpha,\beta} N(\Phi_\alpha \Phi_\beta - \Phi_\beta \Phi_\alpha),$$

and

$$-\sum_{\alpha,\beta,i,j,k,r} h_{ij}^{\alpha} h_{kr}^{\alpha} h_{ij}^{\beta} h_{kr}^{\beta} = -\sum_{\alpha,\beta} (tr(\Phi_\alpha \Phi_\beta))^2 - n^2H^4 - 2nH^2|\Phi_{n+1}|^2.$$

From lemma 3.2, we have

$$-\sum_{\alpha,\beta} N(\Phi_\alpha \Phi_\beta - \Phi_\beta \Phi_\alpha) - \sum_{\alpha,\beta} (tr(\Phi_\alpha \Phi_\beta))^2 \geq -\frac{3}{4}|\Phi|^4,$$

and so

$$-\sum_{\alpha,\beta,i,j,k,l} (h_{ik}^{\alpha} h_{jk}^{\beta} - h_{jk}^{\alpha} h_{ik}^{\beta})(h_{il}^{\alpha} h_{jl}^{\beta} - h_{jl}^{\alpha} h_{il}^{\beta}) - \sum_{\alpha,\beta,i,j,k,l} h_{ij}^{\alpha} h_{kl}^{\alpha} h_{ij}^{\beta} h_{kl}^{\beta}$$

$$\geq -\frac{3}{2}|\Phi|^4 - n^2H^4 - 2nH^2|\Phi_{n+1}|^2.$$

□

3.1. **Proof of the Theorem 1.1.** Now, using lemmas 3.3, 3.4, 3.5, 3.6 and 3.7, we get the following result:

Proposition 3.1. *Let $\tilde{M}(c)$ an $(2m+1)$-dimensional λ-Sasakian space form with structure $(\varphi, \xi, \eta, \langle\ ,\ \rangle)$ and M an n-dimensional C-totally real submanifold with parallel mean curvature vector in $\tilde{M}(c)$. If $c \leq \lambda$, then*

$$\frac{1}{2}\Delta S \geq |\overline{\nabla}\sigma|^2 - \frac{3}{2}|\Phi|^4 - \frac{n(n-2)}{\sqrt{n(n-1)}} H|\Phi|^3 + \left(\frac{n(c+3\lambda)+c-\lambda}{4} + H^2\right)|\Phi|^2. \tag{3.13}$$

Suppose now that M is a closed n-dimensional C-totally real submanifold with parallel mean curvature vector in $\tilde{M}(c)$. From proposition 3.1, we have

$$0 \leq \int_M |\overline{\nabla}\sigma|^2 dM \leq \int_M |\Phi|^2 P_{H,c,\lambda}(|\Phi|) dM, \tag{3.14}$$

where

$$P_{H,c,\lambda}(x) = \frac{3}{2}x^2 + \frac{n(n-2)}{\sqrt{n(n-1)}} Hx - \left(\frac{n(c+3\lambda)+c-\lambda}{4} + nH^2\right).$$

This proves the Theorem 1.1.

3.2. **Proof of the Theorem 1.2.** If $|\Phi|^2 \leq \vartheta_H$, we have that $P_{H,c}(|\Phi|) \leq 0$. Then, follows from Theorem 1.1 that

$$0 \leq \int_M |\Phi|^2 P_{H,c}(|\Phi|) dM \leq 0. \tag{3.15}$$

Thus, $|\Phi|^2 P_{H,c}(|\Phi|) \equiv 0$. Therefore, $|\Phi|^2 = 0$ and M is totally umbilical or $|\Phi|^2 = \vartheta_H$.

If $|\Phi|^2 = \vartheta_H$, from (3.15) we have that in all the inequalities of the lemmas above become equalities. Then, from lemma 3.4, we obtain $\sum_{i=1}^n |\Phi_{i^*}|^2 = |\Phi|^2$ and $m = n$. Hence M is minimal by Theorem 1.1 given in [12]. Note that, in this case

$$P_{H,c,\lambda}(|\Phi|) = \frac{3}{2}|\Phi|^2 - \frac{n(c+3\lambda)+c-\lambda}{4},$$

and

$$S = |\Phi|^2 = \frac{n(c+3\lambda)+c-\lambda}{6}.$$

In particular, if $c = \lambda = 1$, then $\tilde{M}(c)$ is the Sakakian unit sphere $S^{2n+1}(1) \subset \mathbb{C}^{m+1}$ with contact structure induced and $S = \frac{2n}{3}$. Hence, from Theorem 3 in [10], M is a Veronese surface in $S^4(1) \subset S^{2m+1}(1)$.

4. Proofs of the Theorems 1.3 and 1.4

4.1. **Proof of theorem 1.3.** Let M be a n-dimensional C-totally real submanifold in a $(2n+1)$-dimensional λ-Sasakian space form $\tilde{M}(c)$. We choose a local orthonormal frame $\{e_1, ..., e_n, e_{n+1}, ..., e_n, e_{1^*} = \varphi e_1, ..., e_{(n+1)^*} = \varphi e_{n+1}, ..., e_{n^*} = \varphi e_n, e_{2n+1} = \xi\}$. From [4] follows that

$$\begin{aligned} \frac{1}{2} \triangle S = & \sum_{i,j,\alpha} h_{ij}^\alpha \overline{\nabla}_{e_i} \overline{\nabla}_{e_j} (\text{tr } H_\alpha) + \frac{n(c+3\lambda)+c-\lambda}{4} S \\ & - \sum_{\alpha,\beta} \left[(\text{tr } H_\alpha H_\beta)^2 + |[H_\alpha, H_\beta]|^2 - \text{tr } H_\beta \text{ tr } H_\alpha H_\beta H_\alpha \right] + |\overline{\nabla}\sigma|^2. \end{aligned} \tag{4.1}$$

And the other hand, we have that f is pseudo-parallel if and only if

$$h_{ijkl}^\alpha = h_{ijlk}^\alpha - f \left\{ \delta_{ki} h_{lj}^\alpha - \delta_{li} h_{kj}^\alpha + \delta_{kj} h_{il}^\alpha - \delta_{lj} h_{ik}^\alpha \right\}, \tag{4.2}$$

where $i, j, k, l = 1, ..., n$ and $\alpha = n+1, ..., 2n+1$, see [3]. Using (4.2), (2.16), (2.17), (2.18) and Codazzi equation in (2.33), we get

$$\frac{1}{2} \triangle S = \sum_{i,j,\alpha} h_{ij}^\alpha \overline{\nabla}_{e_i} \overline{\nabla}_{e_j} (\text{tr } H_\alpha) + nf|\Phi|^2 + |\overline{\nabla}\sigma|^2. \tag{4.3}$$

Therefore, for a C-totally real f-pseudo-parallel submanifold of a λ-Sasakian space form of φ-sectional curvature c, we have:

$$0 = \sum_{\alpha,\beta} \left[(\text{tr } H_\alpha H_\beta)^2 + |[H_\alpha, H_\beta]|^2 - \text{tr } H_\beta \text{ tr } H_\alpha H_\beta H_\alpha \right] + nf|\Phi|^2 - \frac{n(c+3\lambda)+c-\lambda}{4} S$$

Now, the condition $\nabla^\perp h = 0$ in an n-dimensional C-totally real submanifold M of a $(2n+1)$-dimensional λ-Sasakian space form $\tilde{M}(c)$ is equivalent to the condition $H = 0$. This follows by taking the trace of (2.30), see also [7] in the special case that $\lambda = 1$. Hence, we have that tr $H_\alpha = 0$, for all α and we get:

$$0 = \left(nf - \frac{n(c+3\lambda)+c-\lambda}{4} \right) S + \sum_{\alpha,\beta} \left[(\text{tr } H_\alpha H_\beta)^2 + |[H_\alpha, H_\beta]|^2 \right].$$

If $f \geq (n(c+3\lambda)+c-\lambda)/4n$, then tr $(H_\alpha H_\beta) = 0$, for all α, β. In particular $|A_\alpha|^2 =$ tr $H_\alpha^2 = 0$, hence $\sigma = 0$. This proves Theorem 1.3.

4.2. **Proof of Theorem 1.4.** If M is f-pseudo-parallel and $\nabla^\perp h = 0$, then we obtain

$$\frac{1}{2} \triangle S = nf|\Phi|^2 + |\overline{\nabla}\sigma|^2.$$

If $f \geq 0$, we get $\frac{1}{2} \triangle S \geq 0$. Hence, if M is compact, then we have $\overline{\nabla}\sigma = 0$. This proves our result.

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