INVOLUTIONS WHOSE TOP DIMENSIONAL COMPONENT OF THE FIXED POINT SET IS INDECOMPOSABLE

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Abstract. Let (M^m, T) be a smooth involution on a closed smooth m-dimensional manifold and $F = \bigcup_{j=0}^{n} F^j$ $(n \leq m)$ its fixed point set, where F^j denotes the union of those components of F having dimension j. In this paper we show that, if the top dimensional component F^n is indecomposable, then $m \leq 2n + 1$. We also give examples to show that this bound is best possible. This gives an improvement for the famous Five Halves Theorem of J. Boardman when the top dimensional component of the fixed point set is indecomposable.

1. Introduction

Let (M^m, T) be a smooth involution on a closed smooth m-dimensional manifold and $F = \displaystyle\bigcup_{j=0}^{n} F^j$ $(n \leq m)$ its fixed point set, where F^j denotes the union of those components of F having dimension j. If either M^m or F is nonbounding (which means that at least one F^j is nounbounding) then n cannot be too small with respect to m: this intriguing fact was firstly evidenced from Theorem 27.1 of the old book (1964) [CF] of P. E. Conner and E. E. Floyd, which stated: for each natural number n, there exists a number $\varphi(n)$ with the property that, if (M^m, T) is a involution fixing $F = \displaystyle\bigcup_{j=0}^{n} F^j$ and if $m > \varphi(n)$, then (M^m, T) bounds equivariantly. In this direction, they obtained in [CF] the following related and explicit result: if (M^m, T) is an involution for which F has odd Euler characteristic, then $m \leq 2n$. This bound is best possible, as can be seen by taking the involution $(RP^{2n} \times RP^{2n}, T)$, where RP^{2n} is the $2n$-dimensional real projective space and T is the twist involution, $T(x, y) = (y, x)$.

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Later (1967), in [BO], J. Boardman announced the best result in this direction in terms of generality, explicitly confirming the previous result of Conner and Floyd:

Five Halves Theorem of J. Boardman : If (M^m, T) is an involution for which F is nonbounding, then $m \leq \frac{5}{2}n$ (in fact, the first version of the Five Halves Theorem was proved under the hypothesis that M^m is nonbounding; the result in question is a consequence of the following strengthened version of this first version, obtained by C. Kosniowski and R. E. Stong in [KS]: if (M^m, T) is a nonbounding involution, then $m \leq \frac{5}{2}n$). Further, this bound is best possible (more detailed proofs for the Five Halves Theorem can be found in [BJ] and [M]).

The generality of the Five Halves Theorem, together with the above mentioned result of Conner and Floyd, which improves the Boardman bound to $m \leq 2n$ for a special nonbounding F, suggests the question of finding better bounds for m for specific nonbounding fixed point sets, and ideally the best possible bound in each considered case. In this direction, in 1978 and 1980 we find in the literature two more related results:

i) *C. Kosniowski and R. E. Stong [KS]* : if (M^m, T) is an involution for which F is nonbounding and has constant dimension $= n$, then $m \leq 2n$;

ii) *D. C. Royster [R]* : if (M^m, T) is an involution for which F has the form $F = F^n \cup \{point\}$, where n is odd, then $m \leq n + 1$.

Again these bounds cannot be improved: for the first, consider $(F^n \times F^n, twist)$, where F^n is any nonbounding n-dimensional manifold (with the exception of $n = 1$ and $n = 3$). For the second, consider (RP^{n+1}, T), where $T[x_0, x_1, ..., x_{n+1}] = [-x_0, x_1, ..., x_{n+1}]$ and n is odd. Extending the *two components* direction started with the above result of D. C. Royster, recently some advances have been obtained in the case in which F has the form $F = F^n \cup F^j$, with $n > j$ (note that the case $n = j$ is covered by the result i) of [KS] above cited). Specifically, we find best possible bounds for $j = 0$ in [P] and [PS], $j = 1$ in [K] and [KE], $j = 2$

in [PFA], [PEF] and [PF], and $j = n - 1$ in [PPF]. In [K] and [KE], the case $F = F^n \cup RP^j$, $n \neq j$, was also considered.

We recall that a closed manifold is called *indecomposable* if its unoriented cobordism class cannot be expressed as a sum of products of lower dimensional cobordism classes. In this paper we contribute to the problem in question, proving the following

Theorem. *Let (M^m, T) be an involution with fixed point set $F = \bigcup_{j=0}^{n} F^j$, and suppose that the top dimensional component F^n is indecomposable. Then $m \leq 2n + 1$.*

Further, we show that this result is best possible. Section 2 will be devoted to the proof of the stated result. In Section 3 we construct, for each $n \geq 2$ not of the form $2^t - 1$, a special involution (M^{2n+1}, T) so that the dimension of the top dimensional component of the fixed point set is n and with this top dimensional component being indecomposable, thus showing that our result is best possible (we recall that indecomposable n-dimensional manifolds occur only for these values of n). To obtain these examples, the key point is a result of Richard L. W. Brown of [BR].

Remark. In all the above discussion, each j-dimensional part of the fixed point set of an involution can be assumed to be connected, since any involution is equivariantly cobordant to an involution with this property.

Remark. In [BO], J. Boardman had also obtained the following parallel result: if M^m is indecomposable, then $m \leq 2n + 1$ (this result was also proved by Stong and Kosniowski in [KS]). This fact is independent of our result, since the indecomposability of a manifold equipped with an involution is independent of the indecomposability of the top dimensional component of the fixed point set; in fact, see the examples (RP^{2n}, T), where $T[x_0, x_1, ..., x_{2n}] = [-x_0, -x_1, ..., -x_n, x_{n+1}, ..., x_{2n}]$ and n is odd, and $(RP^{2n} \times RP^{2n}, twist)$.

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2. **Proof of the result**

Let (M^m, T) be an involution with fixed point set $F = \bigcup_{j=0}^{n} F^j$, where the top dimensional component F^n is indecomposable. As announced in Section 1, our aim is to show that $m \leq 2n+1$. For $0 \leq j \leq n$, denote by $\eta_j \to F^j$ the normal bundle of F^j in M^m, with $dim(\eta_j) = m - j$, and by $\lambda_j \to RP(\eta_j)$ the line bundle over the projective space bundle $RP(\eta_j)$, associated to the double cover $S(\eta_j) \to RP(\eta_j)$, $S(\eta_j)$ the sphere bundle. In general, for a given vector bundle $\eta \to F$, write $W(\eta) = 1 + w_1(\eta) + w_2(\eta) + ...$ for the total Stiefel-Whitney class of η; in particular, if F is a manifold, $W(F) = 1 + w_1(F) + w_2(F) + ...$ means the Stiefel-Whitney class of the tangent bundle of F. The following basic fact from [CF] will be needed to prove our result: the disjoint union of line bundles $\cup_{j=0}^{n}\big(\lambda_j \to RP(\eta_j)\big)$ bounds as an element of the cobordism group of manifolds with line bundles, $\mathcal{N}_{m-1}\big(BO(1)\big)$. This yields the following algebraic fact: let $P(w_1, w_2, ..., w_{m-1}, c)$ be any homogeneous polynomial over Z_2 with degree $m-1$, where each variable w_i has degree i and the variable c has degree 1. For each $0 \leq j \leq n$, we can evaluate the cohomology class

$$P\Big(w_1(RP(\eta_j)), w_2(RP(\eta_j)), ..., w_{m-1}(RP(\eta_j), w_1(\lambda_j)\Big) \in H^{m-1}\Big(RP(\eta_j), Z_2\Big)$$

on the fundamental homology class $\big[RP(\eta_j)\big] \in H_{m-1}\big(RP(\eta_j), Z_2\big)$, thus getting a modulo 2 number (called a *characteristic number*),

$$P\Big(w_1(RP(\eta_j)), w_2(RP(\eta_j)), ..., w_{m-1}(RP(\eta_j), w_1(\lambda_j)\Big)\Big[RP(\eta_j)\Big] \in Z_2.$$

From [CF], one has that

$$\sum_{j=0}^{n} P\Big(w_1(RP(\eta_j)), w_2(RP(\eta_j)), ..., w_{m-1}(RP(\eta_j), w_1(\lambda_j)\Big)\Big[RP(\eta_j)\Big] = 0. \ (***)$$

The key point will consist in applying the above fact choosing a suitable polynomial. In this direction, first consider an arbitrary homogeneous symmetric polynomial over Z_2 of degree p on degree one variables $x_1, x_2, ..., x_n$, $P(x_1, x_2, ..., x_n)$. For a given n-dimensional closed manifold F^n, we then get a cohomology class in $H^p(F^n, Z_2)$ by identifying each $w_i(F^n)$ to the ith elementary symmetric function in the variables $x_1, x_2, ..., x_n$ and next by expressing $P(x_1, x_2, ..., x_n)$ as a p-dimensional polynomial in the $w_{i's}(F^n)$. We recall that in [T, page 79] R. Thom showed that the geometric concept of indecomposability is recognized in the following algebraic way: for a natural number p, call $s_p(F^n) \in H^p(F^n, Z_2)$ the cohomology class which corresponds to the symmetric polynomial $P(x_1, x_2, ..., x_n) = x_1^p + x_2^p + ... + x_n^p$ through the previous procedure. Then F^n is indecomposable if and only if the characteristic number $s_n(F^n)[F^n]$ is nonzero.

Next, for each natural number p, we shall introduce a special cohomology class of dimension $2p+1$ associated to line bundles $\lambda \to N$, where N is a closed q-dimensional manifold. Consider the polynomial on degree one variables $x_1, x_2, ..., x_q, c$ given by

$$\mathcal{S}_{2p+1}(x_1, x_2, ..., x_q, c) = x_1^p(x_1 + c)^{p+1} + x_2^p(x_2 + c)^{p+1} + ... + x_q^p(x_q + c)^{p+1}.$$

This polynomial is symmetric in the variables $x_1, x_2, ..., x_q$. As before, we then identify $w_1(\lambda)$ to c and each $w_i(N)$ to the ith elementary symmetric function in the variables $x_1, x_2, ..., x_q$; next, we express the above polynomial as a polynomial of dimension $2p+1$ in the $w_{i's}(N)$ and $w_1(\lambda)$. This class will be denoted by $\mathcal{S}_{2p+1}(\lambda \to N)$. Our interest is to analyze the behavior of $\mathcal{S}_{2p+1}$ with respect to line bundles over projective space bundles; to do this, we will use the *splitting principle* , which allows to write the Stiefel-Whitney class of any k-dimensional vector bundle ξ formally as

$$W(\xi) = 1 + w_1(\xi) + w_2(\xi) + ... + w_k(\xi) = (1 + x_1)(1 + x_2)...(1 + x_k),$$

where each x_i has degree one. Consider a k-dimensional vector bundle $\xi \to Q$, where Q is a closed q-dimensional manifold, and let $\lambda \to RP(\xi)$ be the usual line bundle. From [BH; page 517] one has that

$$W\big(RP(\xi)\big) = \big(1+w_1(Q)+w_2(Q)+...+w_q(Q)\big)\Big(\big(1+w_1(\lambda)\big)^k+\big(1+w_1(\lambda)\big)^{k-1}w_1(\xi)+ ... + \big(1 + w_1(\lambda)\big)w_{k-1}(\xi) + w_k(\xi)\Big),$$

where here we are suppressing bundle maps. Using the splitting principle, write

$$\begin{aligned} W(Q) &= (1 + x_1)(1 + x_2)...(1 + x_q), \\ W(\xi) &= (1 + y_1)(1 + y_2)...(1 + y_k), \end{aligned}$$

and set $w_1(\lambda) = c$. Then

$$W\big(RP(\xi)\big) = (1 + x_1)(1 + x_2)...(1 + x_q)(1 + c + y_1)(1 + c + y_2)...(1 + c + y_k).$$

It follows that

$$\mathcal{S}_{2p+1}\Big(\lambda \to RP(\xi)\Big) = x_1^p(x_1 + c)^{p+1} + x_2^p(x_2 + c)^{p+1} + ... + x_q^p(x_q + c)^{p+1} + y_1^{p+1}(y_1 + c)^p + y_2^{p+1}(y_2 + c)^p + ... + y_k^{p+1}(y_k + c)^p = x_1^p\Big(c^{p+1} + \sum_{i=0}^p \binom{p+1}{i} x_1^{p+1-i}c^i\Big) + x_2^p\Big(c^{p+1} + \sum_{i=0}^p \binom{p+1}{i} x_2^{p+1-i}c^i\Big) + ... x_q^p\Big(c^{p+1} + \sum_{i=0}^p \binom{p+1}{i} x_q^{p+1-i}c^i\Big) + y_1^{p+1}(y_1 + c)^p + y_2^{p+1}(y_2 + c)^p + ...+ y_k^{p+1}(y_k + c)^p = (x_1^p + x_2^p + ... + x_q^p)c^{p+1} + \text{ terms with smaller powers of } c = s_p(Q)c^{p+1} + \text{ terms with smaller powers of } c.$$

Returning to the line bundles over the projective space bundles coming from the fixed data of (M^m, T), set $w_1\big(\lambda_j \to RP(\eta_j)\big) = c_j$. By contradiction, suppose $m > 2n + 1$. Then $m - 1 \geq 2n + 1$, and thus it makes sense to consider the polynomial of degree $m - 1$ given by $\mathcal{S}_{2n+1}(x_1, x_2, ..., x_n, c).c^{m-1-2n-1}$. For $j < n$ one has

$$\mathcal{S}_{2n+1}\Big(\lambda_j \to RP(\eta_j)\Big).c_j^{m-1-2n-1} = \Big(s_n(F^j)c_j^{n+1} + \text{ terms with smaller powers of } c_j\Big)c_j^{m-1-2n-1} = s_n(F^j)c_j^{m-1-n} + \text{ terms with smaller powers of } c_j = s_n(F^j)c_j^{m-1-n} + \sum A_l c_j^t,$$

where in the above expression each term $A_l c_j^t$ has $t < m-1-n$ and $l+t = m-1$; further, the l-dimensional factor A_l comes from the cohomology of F^j. Thus $l > n > j$, which means that each such term is zero. By the same reason, $s_n(F^j) = 0$, and thus $\mathcal{S}_{2n+1}\big(\lambda_j \to RP(\eta_j)\big).c_j^{m-1-2n-1} = 0$. By using equation (***), we conclude that $\mathrm{S}_{2n+1}\big(\lambda_n \to RP(\eta_n)\big).c_n^{m-1-2n-1}\big[RP(\eta_n)\big] = 0$. On the other hand, one has

$$\mathcal{S}_{2n+1}\Big(\lambda_n \to RP(\eta_n)\Big).c_n^{m-1-2n-1} = \Big(s_n(F^n)c_n^{n+1} + \text{ terms with smaller powers of } c_n\Big)c_n^{m-1-2n-1} = s_n(F^n)c_n^{m-1-n} + \text{ terms with smaller powers of } c_n,$$

where similarly as above the terms involving powers c_n^t with $t < m - 1 - n$ are zero. Therefore $\mathcal{S}_{2n+1}\big(\lambda_n \to RP(\eta_n)\big).c_n^{m-1-2n-1} = s_n(F^n)c_n^{m-1-n}$, and since from the Leray-Hirsch Theorem (see [B]; pag. 129) $H^*\big(RP(\eta_n), Z_2\big)$ is the free $H^*(F^n, Z_2)$-module on $1, c_n, c_n^2, ..., c_n^{m-1-n}$, we get

$$\mathcal{S}_{2n+1}\Big(\lambda_n \to RP(\eta_n)\Big).c_n^{m-1-2n-1}\Big[RP(\eta_n)\Big] =$$
$$s_n(F^n)c_n^{m-1-n}\Big[RP(\eta_n)\Big] = s_n(F^n)[F^n] = 1.$$

This gives the desired contradiction.

3. **Maximal examples**

In this section we will construct, for each $n \geq 2$ not of the form $2^t - 1$, a special involution (M^{2n+1}, T) so that the dimension of the top dimensional component of the fixed point set is n and with this top dimensional component being indecomposable, thus showing that the result of Section 2 is best possible. First we will construct these involutions for $n \geq 2$ even. To do this, we recall the following construction from [CF]: for a given involution (W, T) with fixed point set F and W a boundary, the involution $\Gamma(W,T) = (\dfrac{S^1 \times W}{-Id \times T}, \tau)$ is equivariantly cobordant to an involution fixing F; here, S^1 is the 1-sphere, Id is the identity map and τ is the involution induced by $c \times Id$, where c is complex conjugation. If the orbit space $\dfrac{S^1 \times W}{-Id \times T}$ is a boundary, we can repeat the process taking $\Gamma^2(W,T)$, and so on. Now take $n \geq 2$ even, and begin with the involution (RP^{2n-1}, T_1) given by $T_1[x_0, x_1, ..., x_{2n-1}] = [-x_0, -x_1, ..., -x_n, x_{n+1}, ..., x_{2n-1}]$. Note that T_1 fixes the disjoint union $RP^n \cup RP^{n-2}$ with RP^n indecomposable, and since RP^{2n-1} bounds, $\Gamma(RP^{2n-1}, T_1)$ is equivariantly cobordant to an involution (V^{2n}, T_2) also fixing $RP^n \cup RP^{n-2}$. We claim that V^{2n} bounds. It suffices to see that

$$\frac{RP^{2n-1} \times S^1}{T \times A}$$

bounds. This last manifold is diffeomorphic to the total space of the projective space bundle associated to the vector bundle $(n+1)\mu \oplus \varepsilon \to RP^1$, where $(n+1)\mu$ is the Whitney sum of $n+1$ copies of the canonical line bundle $\mu \to RP^1$ and $\varepsilon \to RP^1$ is the $(n-1)$-dimensional trivial vector bundle over RP^1 (see the proof of the lemma of [PRO, Section 3]). From [CFL; Lemma 2.2], every projective space bundle over RP^1 bounds, which gives the assertion. In this way, $\Gamma(V^{2n}, T_2)$ is equivariantly cobordant to an involution (W^{2n+1}, T_3) fixing $RP^n \cup RP^{n-2}$, which is the desired example.

Now we obtain examples for each $n > 2$ odd and not of the form $2^t - 1$. The crucial point will be the following (nontrivial) result of Richard L. W. Brown ([BR], Proposition 4.1, page 1106)): if F is indecomposable, then the orbit space $\dfrac{S^1 \times F \times F}{-Id \times twist}$ is indecomposable. We introduce the following general construction: for a given involution (W, T) with fixed point set F, consider the involution $Id \times T \times T$ defined on $S^1 \times W \times W$. This involution commutes with $-Id \times twist$ and then induces an involution on the orbit space $\dfrac{S^1 \times W \times W}{-Id \times twist}$; we denote this involution by $\mathit{\Pi}(W, T)$, and it is easy to see that the fixed point set of $\mathit{\Pi}(W, T)$ is $\dfrac{S^1 \times F \times F}{-Id \times twist}$. In this way, $\mathit{\Pi}(W, T)$ satisfies the following three properties:

i) if the dimension of W is m, then the dimension of the underlying manifold of $\mathit{\Pi}(W, T)$ is $2m + 1$;

ii) if the dimension of the top dimensional component of F is n, then the dimension of the top dimensional component of the fixed point set of $\mathit{\Pi}(W, T)$ is $2n + 1$; and

iii) if the top dimensional component of F is indecomposable, then the top dimensional component of the fixed point set of $\mathit{\Pi}(W, T)$ is indecomposable.

Now take $n > 2$ odd and not of the form $2^t - 1$, and let $n = 2^{t_0} + 2^{t_1} + ... + 2^{t_r}$ be the 2-adic expansion of n, where $t_i < t_j$ if $i < j$. Then $t_0 = 0$ and there exists the smallest $0 \leq j < r$ so that $t_{j+1} - t_j \geq 2$. Set $x = t_j + 1$ and consider the even number $e(n) = 2^{t_{j+1}-x} + 2^{t_{j+2}-x} + ... + 2^{t_r - x}$. We then have the previous example $(W^{2e(n)+1}, T)$ with the top dimensional component of the fixed point set being indecomposable and having dimension $e(n)$. Consider the involution obtained by iteratively applying x times the operation $\mathit{\Pi}$ on $(W^{2e(n)+1}, T)$, $\mathit{\Pi}^x(W^{2e(n)+1}, T)$. By using properties i), ii) and iii) we can see that $\mathit{\Pi}^x(W^{2e(n)+1}, T)$ is the desired example.

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