ON $\mathbb{Z}_2$ AND $\mathbb{S}^1$ FREE ACTIONS ON SPACES OF COHOMOLOGY TYPE (a, b)

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ABSTRACT. Let X be a simply connected finite CW complex whose cohomology groups with coefficients in the group of integers, Z, satisfy $H^j(X; Z) = Z$ if $j = 0,\ n,\ 2n$ or $3n$, and $H^j(X; Z) = 0$ otherwise. Let u_i generate $H^{in}(X; Z)$ for $i = 0,\ 1,\ 2$ and 3. We say that X has *type* (a, b), for integers a and b, if $u_1^2 = au_2$ and $u_1u_2 = bu_3$. We show that the group $G = \mathbb{Z}_2$ can not act freely on a space of type (a, b) if a is odd and b is even, and that the group $G = \mathbb{S}^1$ can not act freely on a space of type (a, b) if $a \neq 0$. For the remaining pairs (a, b), we may have free actions, and thus it makes sense to ask for the possible cohomology rings of the corresponding orbit spaces. In this direction, we determine the possible Z_2-cohomology rings of orbit spaces of free actions of $\mathbb{Z}_2$ on spaces of type (a, b), where a and b are even, and of free actions of $\mathbb{S}^1$ on spaces of type $(0, b)$. As a consequence of these cohomological calculations, we also obtain some results of the Borsuk-Ulam type, concerning the existence of equivariant maps $S^m \to X$, where S^m is equipped with standard G-actions ($G = \mathbb{Z}_2$ or $\mathbb{S}^1$) and X are spaces of type (a, b) equipped with arbitrary G-actions.

1. **Introduction**

Let G be a topological group and X a topological space. Associated to the pair (G, X), there is a natural question, which concerns to the existence of a continuous free action of G on X. There is a considerable amount of results of this type in the literature; as a prominent example, we cite the following consequence of the results of the paper [9] of John Milnor: the symmetric group on three elements, S_3, can not act freely on the n-sphere S^n. In the positive case, that is, if such an action exists, other natural question is the study of properties of the orbit space X/G. In particular, we have in this setting the

1991 *Mathematics Subject Classification.* (2000 Revision) Primary 57S17; Secondary 57S25.

Key words and phrases. space of type (a, b), free action, cohomology ring, Leray-Serre spectral sequence, Borel construction, Stiefel-Whitney class.

The first author was partially supported by CNPq and FAPESP.

(usually difficult) question of determining the structure of the cohomology ring of X/G, which is a useful tool; for example, the (known) structures of the cohomology rings of the real, complex and quaternionic projective spaces RP^n, CP^n and KP^n appear as ingredients of many interesting questions in topology. In this case, these spaces are orbit spaces of certain standard free actions of $\mathbb{Z}_2$, $\mathbb{S}^1$ and $\mathbb{S}^3$ on S^n, S^{2n+1} and S^{4n+3}, respectively.

In this paper, we are concerned with these questions, considering $G = \mathbb{Z}_2$ or $\mathbb{S}^1$ and certain special spaces, described as follows: let X be a simply connected finite CW complex with Z-cohomology groups satisfying $H^j(X;Z) = Z$ if $j = 0,\ n,\ 2n$ or $3n$, and $H^j(X;Z) = 0$ otherwise $(n > 1)$. Let u_i generate $H^{in}(X;Z)$ for $i = 0,\ 1,\ 2$ and 3. Then the structure of the cohomology ring of X is determined by the two integers a and b for which $u_1^2 = au_2$ and $u_1u_2 = bu_3$. In this case, X is said to be of *type* (a, b) . These spaces include certain products of spheres and projective spaces, and were first studied by James [7] and Toda [13]. Section 3 of this paper will be devoted to the following non existence result: $G = \mathbb{Z}_2$ can not act freely on a space of type (a, b) if a is odd and b is even. As an example of such spaces, take the one-point union of CP^2 and S^6. In a more simple way, it will be also showed that $\mathbb{S}^1$ can not act freely on a space of type (a, b) if $a \neq 0$. For the remaining pairs (a, b) we may have free actions; for example, $S^3 \times S^6$ is of type $(0, 1)$ and admits free G-actions for $G = \mathbb{Z}_2$ and $\mathbb{S}^1$ (for other examples, see [3]). Therefore it makes sense to ask for the possible cohomology rings of the corresponding orbit spaces. For related results, see [4] ($G = \mathbb{Z}_p$ with p an odd prime, X of type $(0, 0)$), [5] ($G = \mathbb{Z}_p$ or $\mathbb{S}^1$, p a prime, X a finitistic space with $\mathbb{Z}_p$-cohomology ring, or rational cohomology ring, isomorphic to that of $S^m \times S^n$; we recall that a paracompact Hausdorff space is finitistic if every open covering has a finite-dimensional refinement), [10] ($G = \mathbb{Z}_2$, X of type (a, b) where a and b are odd), [11] and [12] ($G = \mathbb{Z}_p$ with p an odd prime (respectively, $G = \mathbb{S}^1$), X a finite-dimensional CW complex with $\mathbb{Z}_p$-cohomology ring isomorphic to that of a lens space $L^{2m-1}(p; q_1, q_2, ..., q_m)$). We contribute to this question, determining the possible Z_2-cohomology rings of orbit spaces of free actions of $\mathbb{Z}_2$ on spaces of type (a, b), where a and b

are even, and of free actions of $\mathbb{S}^1$ on spaces of type $(0, b)$; this will be made in Section 4. The main tool used is the Leray-Serre spectral sequence of the fibration $\pi : X_G \to B_G$ with fiber X, where $X_G = (E_G \times X)/G$ is the Borel construction of X associated to a universal G-bundle $E_G \to B_G$. In Section 2 we give a brief sketch on this topic.

Let X, Y be spaces equipped with free G-actions. Then it makes sense to ask for the existence of equivariant maps $f : X \to Y$. For example, one formulation of the celebrated Borsuk-Ulam theorem is that there is no map from S^m to S^n equivariant with respect to the antipodal action when $m > n$. As a consequence of our cohomological calculations and using some basic facts on characteristic classes, we also obtain some Borsuk-Ulam results concerning the existence of equivariant maps $S^m \to X$, where S^m is equipped with standard G-actions ($G = \mathbb{Z}_2$ or $\mathbb{S}^1$) and X are spaces of type (a, b) equipped with arbitrary G-actions. This is the content of Section 5.

2. **Preliminaries**

In this section, we describe the main tool used to obtain our results, the special Leray-Serre spectral sequence mentioned in Section 1; as a reference, see for example [2; Chapter 7]. Given a G-action on a finitistic space X, there is an associated fibration $\pi : X_G \to B_G$ with fiber X, where B_G is the classifying space for G and $X_G = (E_G \times X)/G$ is the Borel construction of X associated to a universal G-bundle $E_G \to B_G$. There is also a natural map $\eta : X_G \to X/G$. When G acts freely on X, $\eta : X_G \to X/G$ is a homotopy equivalence, and thus the cohomology rings $H^*(X_G)$ and $H^*(X/G)$ are isomorphic. Associated to the fibration $\pi : X_G \to B_G$, one has the cohomological Leray-Serre spectral sequence, which is then a suitable tool to study $H^*(X/G)$ when the action is free. This spectral sequence has $E_2^{k,l} \cong H^k(B_G; \mathcal{H}^l(X))$ as its E_2-term and converges to $H^{k+l}(X_G)$, in the sense of Bredon [2]; here, the coefficients $\mathcal{H}^l(X)$ means $H^l(X)$ twisted by the action of $\pi_1(B_G)$. If $\pi_1(B_G)$ acts trivially on $H^*(X)$, then the E_2-term takes the simple form $E_2^{k,l} \cong H^k(B_G; H^l(X))$. In this case, the product structure in the spectral sequence induces a product in the subalgebras $E_2^{*,0}$ and $E_2^{0,*}$ which coincides with the cup products coming from $H^*(B_G)$ and $H^*(X)$, respectively. Also, the edge homomorphisms

$$H^p(B_G) = E_2^{p,0} \to E_3^{p,0} \to \ldots \to E_{p+1}^{p,0} = E_\infty^{p,0} \subseteq H^p(X_G), \text{ and}$$

$$H^q(X_G) \to E_\infty^{0,q} = E_{q+1}^{0,q} \subset \ldots \subset E_2^{0,q} = H^q(X)$$

are the homomorphisms

$$\pi^* : H^p(B_G) \to H^p(X_G), \text{ and}$$

$$i^* : H^q(X_G) \to H^q(X),$$

respectively; here, $i : X \to X_G$ is the inclusion map. For a more detailed exposition on spectral sequences, see for example [8]. The following well known facts will be also needed:

$$H^*(B_G; \mathbb{Z}_2) = \begin{cases} \mathbb{Z}_2[t], & \deg t = 1, \ G = \mathbb{Z}_2, \\ \mathbb{Z}_2[t], & \deg t = 2, \ G = \mathbb{S}^1. \end{cases}$$

Finally, the following technical result will be useful.

Proposition 2.1. *Suppose $G = \mathbb{Z}_2$ or $\mathbb{S}^1$ acts freely on a finitistic space X with $H^j(X; \mathbb{Z}_2) = 0$ for all $j > n$. Then $H^j(X_G; \mathbb{Z}_2) = 0$ for all $j > n$.*

For $G = \mathbb{Z}_2$, see [2], and for $G = \mathbb{S}^1$, see [12].

3. Non existence of certain $\mathbb{Z}_2$ and $\mathbb{S}^1$ free actions

This section is devoted to the proofs of the results referring to the non existence of free actions of $\mathbb{Z}_2$ and $\mathbb{S}^1$, announced in Section 1. For $G = \mathbb{Z}_2$, one has the following

Theorem 3.1. *Let X be a space of type (a, b), characterized by a natural number $n > 1$, where a is odd and b is even. Then $G = \mathbb{Z}_2$ can not act freely on X.*

Proof. Suppose, on the contrary, one has a free $G = \mathbb{Z}_2$-action on X, and consider the Leray-Serre spectral sequence associated to the corresponding fibration $\pi : X_G \to B_G$. By the Universal Coefficient theorem, $H^{in}(X;\mathbb{Z}_2) = \mathbb{Z}_2$ for $i = 0,\ 1,\ 2$ and 3. Let $v_i \in H^{in}(X;\mathbb{Z}_2)$ be the generators, $i = 1,\ 2,\ 3$. Then we have $v_1^2 = v_2$ and $v_1v_2 = 0$. The additive structure of $H^*(X)$ gives that $\pi_1(B_G) = \mathbb{Z}_2$ acts trivially on $H^*(X)$. In this way, $E_2^{k,l} \cong H^k(B_G) \otimes H^l(X)$ and the spectral sequence does not collapse at the E_2-term, which implies that some differential

$$d_r : E_r^{k,l} \to E_r^{k+r,l-r+1}$$

must be nontrivial. This is only possible for $r = n+1$, $2n+1$ and $3n+1$, which gives that $E_2^{k,l} = E_n^{k,l}$ for every k and l. Note that $E_2^{k,l} = \mathbb{Z}_2$ for every k when $l = 0$, n, $2n$ or $3n$, and $E_2^{k,l} = 0$ otherwise. Our strategy will be to study the effect of the nontrivial differentials on the generators of $E_n^{k,l} \cong H^k(B_G) \otimes H^l(X)$. If $t \in H^1(B_G)$ is the generator, then by dimensional reasons $d_r(t^s \otimes 1) = 0$ for every $s \geq 1$ and $r \geq 2$. We assert that $d_{n+1}(1 \otimes v_1) = 0$. Otherwise, $d_{n+1}(1 \otimes v_1) = t^{n+1} \otimes 1$, and then the multiplicative properties of the spectral sequence give $d_{n+1}(1 \otimes v_2) = d_{n+1}(1 \otimes v_1^2) = d_{n+1}((1 \otimes v_1)(1 \otimes v_1)) = 0$. Thus, $d_{n+1}((1 \otimes v_1)(1 \otimes v_2)) = d_{n+1}(1 \otimes (v_1.v_2)) = 0$ and $d_{n+1}((1 \otimes v_1)(1 \otimes v_2)) = (d_{n+1}(1 \otimes v_1))(1 \otimes v_2) = t^{n+1} \otimes v_2$ gives the contradiction. Now we assert that $d_{n+1}(1 \otimes v_2) = 0$. Otherwise, $d_{n+1}(1 \otimes v_2) = t^{n+1} \otimes v_1$, and then $0 = d_{n+1}((1 \otimes v_1)(1 \otimes v_2)) = t^{n+1} \otimes v_1^2 = t^{n+1} \otimes v_2$, a contradiction. Now, if $d_{n+1}(1 \otimes v_3) \neq 0$, then two lines in the spectral sequence survive to infinity, and this contradicts Proposition 2.1. Thus $d_{n+1}(1 \otimes v_3) = 0$, which means that d_{n+1} is trivial and $E_2^{k,l} = E_n^{k,l} = E_{2n}^{k,l}$ for every k and l. By dimensional reasons, $d_{2n+1}(1 \otimes v_1) = 0$, and then the same argument above shows that $d_{2n+1}(1 \otimes v_2) = 0$. It follows that at least two lines in the spectral sequence survive to infinity, again contradicting Proposition 2.1. This completes the proof. □

Concerning the existence of free $\mathbb{S}^1$-actions on spaces of type (a, b), we have the following

Theorem 3.2. *Let X be a space of type (a, b), characterized by a natural number $n > 1$. If a is nonzero, then $G = \mathbb{S}^1$ can not act freely on X.*

Proof. In [13], Toda showed that the existence of a space of type (a, b) with $a \neq 0$ implies that n must be even, and thus the Euler characteristic of X, $\chi(X)$, is 4. Now if $\mathbb{Z}_p$ with p an odd prime acts on a finite CW complex X and if $F \subset X$ is the fixed point set of the action, then the Floyd formula of [6] says that $\chi(X) = \chi(F) \pmod{p}$. If $\phi : \mathbb{S}^1 \times X \to X$ is an action of $\mathbb{S}^1$ on a space X of type (a, b) with $a \neq 0$, we can choose an odd prime p and to consider the restricted action $\phi' : \mathbb{Z}_p \times X \to X$, to conclude that $\chi(Fix(\phi')) = 4 \pmod{p}$, that is, $\chi(Fix(\phi')) \neq 0 \pmod{p}$. Then $Fix(\phi')$ is nonempty, and so ϕ' is not free. Therefore, ϕ is not free, which gives the result. □

4. The cohomology ring of orbit spaces of $\mathbb{Z}_2$ and $\mathbb{S}^1$ free actions on spaces of type (a, b)

The structure of the Z_2-cohomology ring of the orbit space of a free action of $\mathbb{Z}_2$ on a space of type (a, b), when both a and b are even, is given by the following

Theorem 4.1. *Let $G = \mathbb{Z}_2$ act freely on a space of type (a, b), characterized by a natural number $n > 1$, where both a and b are even. Then, as a graded commutative algebra,*

$$H^*(X/G; \mathbb{Z}_2) = \mathbb{Z}_2[x, z]/\langle x^{3n+1}, z^2, zx^{n+1}\rangle,$$

where $\deg x = 1$ *and* $\deg z = n$.

Proof. In this case, for the generators of $H^*(X)$, we have the relations $v_1^2 = 0$ and $v_1 v_2 = 0$. As in Theorem 3.1, for the corresponding spectral sequence, one has that $E_2^{k,l} \cong H^k(B_G) \otimes H^l(X)$, the sequence does not collapse at the E_2-term and no line can survive to infinity. Arguments as those used in Theorem 3.1, based on the multiplicative properties of the spectral sequence, show that $d_{n+1}(1 \otimes v_1) = 0$, $d_{n+1}(1 \otimes v_3) = 0$ and $d_{n+1}(1 \otimes v_2) \neq 0$. Therefore, we get that $E_{n+2}^{k,l} = \mathbb{Z}_2$ for every k if $l = 0$ or $3n$. Also, we have $E_{n+2}^{k,l} = \mathbb{Z}_2$ for $k = 0, 1, 2, \ldots, n$ if $l = n$. In the remaining cases, $E_{n+2}^{k,l} = 0$. Again, the multiplicative properties show that $d_{2n+1}(1 \otimes v_i) = 0$, $i = 1, 2, 3$, and

$d_{3n+1}(1 \otimes v_3) \neq 0$. Thus $E_{3n+2}^{k,l} = \mathbb{Z}_2$ for $k \leq 3n$ if $l = 0$, and $E_{3n+2}^{k,l} = \mathbb{Z}_2$ for $k \leq n$ if $l = n$. This determines the additive structure of $H^*(X_G)$, given by

$$H^j(X_G) = \begin{cases} \mathbb{Z}_2 \oplus \mathbb{Z}_2 & \text{for } n \leq j \leq 2n, \\ \mathbb{Z}_2 & \text{for } \ 0 \leq j \leq n-1 \text{ and } \ 2n+1 \leq j \leq 3n, \\ 0 & \text{otherwise.} \end{cases}$$

Now, we compute the ring structure of $H^*(X_G)$. The element $1 \otimes v_1 \in E_2^{0,n}$ is a permanent cocycle; since the edge homomorphism

$$H^n(X_G) \to E_\infty^{0,n} = E_{n+1}^{0,n} \subset \ldots \subset E_2^{0,n} = H^n(X)$$

is the homomorphism $i^* : H^n(X_G) \to H^n(X)$, $1 \otimes v_1 \in E_2^{0,n}$ determines an element $z \in E_\infty^{0,n}$ such that $i^*(z) = v_1$, and clearly $z^2 = 0$. Write $x = t \otimes 1 \in E_\infty^{1,0}$. Then $x^{3n+1} = 0$ and $zx^{n+1} = 0$. Thus the total complex Tot $E_\infty^{*,*}$, given by

$$(\text{Tot } E_\infty^{*,*})^m = \bigoplus_{k+l=m} E_\infty^{k,l},$$

is a graded commutative algebra isomorphic to

$$\text{Tot } \ E_\infty^{*,*} \cong \mathbb{Z}_2[x, z]/\langle x^{3n+1}, z^2, zx^{n+1} \rangle,$$

where $\deg x = 1$ and $\deg z = n$.

Thus $H^*(X_G)$ has the same description, which completes the proof. □

For free actions of $G = \mathbb{S}^1$ on spaces of type $(0, b)$, we have the following result.

Theorem 4.2. *Let $G = \mathbb{S}^1$ act freely on a space X of type $(0, b)$, characterized by a natural number $n > 1$. Then $H^*(X/G; \mathbb{Z}_2)$ is isomorphic to one of the following graded commutative algebras:*

$$(i)\, \mathbb{Z}_2[x, z]/\langle x^{\frac{3n+1}{2}}, z^2, zx^{\frac{n+1}{2}} \rangle,$$

where $\deg x = 2$, $\deg z = n$.

$$(ii)\, \mathbb{Z}_2[x, z]/\langle x^{\frac{n+1}{2}}, z^2 \rangle,$$

where $\deg x = 2$, $\deg z = 2n$ *and b is odd.*

Proof. The argument in the proof of Theorem 3.2, based on the Floyd formula, shows that n must be odd. Since $\pi_1(B_G) = 0$, the E_2-term of the corresponding spectral sequence is $E_2^{k,l} = H^k(B_G) \otimes H^l(X)$ and, similarly as before, the sequence does not collapse at the E_2-term and by Proposition 2.1 no line can survive to infinity. We have that $E_2^{k,l} = \mathbb{Z}_2$ for every k even when $l = 0,\ n,\ 2n$ or $3n$, and $E_2^{k,l} = 0$ otherwise. Let $v_i \in H^{in}(X;\mathbb{Z}_2)$ be the generators, $i = 1,\ 2$ and 3; in this case, $v_1^2 = 0$ and $v_1 v_2 = bv_3 \pmod 2$. First, assume that $d_{n+1}(1 \otimes v_1) = 0$. Using the multiplicative properties of the spectral sequence and the fact that no line can survive to infinity, we can easily show that $d_{n+1}(1 \otimes v_2) = t^{\frac{n+1}{2}} \otimes v_1$ and $d_{n+1}(1 \otimes v_3) = 0$, where $t \in H^2(B_G)$ is the generator. By dimensional reasons, $d_{2n+1} = 0$, and so $d_{3n+1}(1 \otimes v_3) \neq 0$. This gives that $E_{3n+2}^{k,l} = \mathbb{Z}_2$ for every k even with $k \leq 3n-1$, if $l = 0$. Also, this gives that $E_{3n+2}^{k,l} = \mathbb{Z}_2$ for every k even with $k \leq n-1$, if $l = n$. In the remaining cases, $E_{3n+2}^{k,l} = 0$. Note that $E_{3n+2} = E_\infty$. Thus the additive structure of $H^*(X_G)$ is given in this case by

$$H^j(X_G) = \begin{cases} 0 & \text{for } j = 2i+1 \ \ (0 \leq i \leq \frac{n-3}{2} \text{ or } n \leq i \leq \frac{3n-3}{2}) \text{ or } j \geq 3n, \\ \mathbb{Z}_2 & \text{otherwise.} \end{cases}$$

Write $\pi^*(t) = x \in H^2(X_G)$. Then $x^{\frac{3n+1}{2}} = 0$, and the multiplication by x,

$$x \cup (\cdot) : E_\infty^{k,l} \to E_\infty^{k+2,l},$$

is an isomorphism for $k < 3n-1$ if $l = 0$ and for $k < n-1$ if $l = n$. Therefore,

$$x \cup (\cdot) : H^k(X_G) \to H^{k+2}(X_G)$$

is an isomorphism for every k even with $k < n-1$. Then the element $1 \otimes v_1 \in E_2^{0,n}$ is a permanent cocyle and so determines an element $z \in E_\infty^{0,n}$ such that $i^*(z) = v_1$ and $z^2 = 0$. Clearly, $zx^{\frac{n+1}{2}} = 0$. If follows that the total complex Tot $E_\infty^{*,*}$ is a graded commutative algebra isomorphic to

$$\text{Tot } E_\infty^{*,*} \cong \mathbb{Z}_2[x,z]/\langle x^{\frac{3n+1}{2}}, z^2, zx^{\frac{n+1}{2}} \rangle,$$

where $\deg x = 2$, $\deg z = n$.

Therefore,

$$H^*(X_G) = \mathbb{Z}_2[x,z]/\langle x^{\frac{3n+1}{2}}, z^2, zx^{\frac{n+1}{2}} \rangle,$$

where $\deg x = 2$, $\deg z = n$.

Now, consider the case in which $d_{n+1}(1 \otimes v_1) = t^{\frac{n+1}{2}} \otimes 1$. In this case, we must then have $d_{n+1}(1 \otimes v_2) = 0$. If b is even, $v_1 v_2 = 0$ implies that $0 = d_{n+1}((1 \otimes v_1)(1 \otimes v_2)) = t^{\frac{n+1}{2}} \otimes v_2$, a contradiction. So, b must be odd. Thus we have $d_{n+1}(1 \otimes v_3) = t^{\frac{n+1}{2}} \otimes v_2$. It follows that $E_\infty^{k,l} = \mathbb{Z}_2$ for $k = 0,\ 2,\ 4, \ldots,\ n-1$ if $l = 0$ or $2n$, and $E_\infty^{k,l} = 0$ otherwise. In this way,

$$H^j(X_G) = \begin{cases} \mathbb{Z}_2 & \text{for } j = 2i \ (0 \leq i \leq \frac{n-1}{2} \ \text{ or } \ n \leq i \leq \frac{3n-1}{2}), \\ 0 & \text{otherwise.} \end{cases}$$

The element $1 \otimes v_2 \in E_2^{0,2n}$ is a permanent cocyle, hence it determines an element $z \in E_\infty^{0,2n}$ such that $i^*(z) = v_2$ and $z^2 = 0$. Write $\pi^*(t) = x$. Then $x \in H^2(X_G)$ comes from $t \otimes 1 \in E_\infty^{2,0}$ and $x^{\frac{n+1}{2}} = 0$. If follows that the total complex Tot $E_\infty^{*,*}$ is a graded commutative algebra given by

$$\mathrm{Tot} E_\infty^{*,*} \cong \mathbb{Z}_2[x, z]/\langle x^{\frac{n+1}{2}}, z^2 \rangle$$

where $\deg x = 2$, $\deg z = 2n$ and b is odd. Since $H^*(X_G)$ has the same description, the proof is ended. □

5. **Some Borsuk-Ulam results**

As mentioned in Section 1, we can obtain some Borsuk-Ulam results by using the results of Section 4 together with some basic facts on characteristic classes. If $\eta \to X$ is a real vector bundle, we write $w_i(\eta)$ to denote its *i-th* Stiefel-Whitney class.

Theorem 5.1.

1) Let X be a space of type (a, b), characterized by a natural number $n > 1$, where a and b are even. Suppose X is equipped with an arbitrary free $\mathbb{Z}_2$-action, and S^m is equipped with the antipodal action. Then there is no equivariant map $S^m \to X$ if $m > 3n$.

2) Let X be a space of type $(0, b)$, characterized by a natural odd number $n > 1$. Suppose X is equipped with an arbitrary free $\mathbb{S}^1$-action, and S^{2m+1} is equipped with the standard (complex multiplication) $\mathbb{S}^1$-action. Then there is no equivariant map $S^{2m+1} \to X$ if $m \geq \frac{3n+1}{2}$.

Proof. To prove 1), suppose, on the contrary, that there exists an equivariant map $f : S^m \to X$. Denote by $\overline{f} : S^m/\mathbb{Z}_2 = RP^m \to X/\mathbb{Z}_2$ the map induced by f, and by $\lambda \to X/\mathbb{Z}_2$ the real line bundle associated to the free $\mathbb{Z}_2$-action on X; we recall that the total space of λ is $X \times \mathbb{R}/\sim$, where $\sim$ identifies (x, r) to $(T(x), -r)$, T being the generator of the action. Denote by $\xi \to B_{\mathbb{Z}_2} = RP^\infty$ the universal real line bundle and by $\xi' \to RP^m$ its restriction to RP^m. By the proof of Theorem 4.1, the element $x = t \otimes 1 \in E_\infty^{1,0}$ of degree 1 comes from $t \otimes 1 \in E_2^{1,0} = H^1(RP^\infty) \otimes H^0(X)$, where $t \in H^1(RP^\infty)$ is the generator. Because the edge homomorphism

$$H^1(RP^\infty) = E_2^{1,0} \to E_3^{1,0} \to \ldots \to E_\infty^{1,0} \subseteq H^1(X/\mathbb{Z}_2)$$

is the homomorphism $\pi^* : H^1(RP^\infty) \to H^1(X/\mathbb{Z}_2)$, we get that $\pi^*(t) = x$. By [14, Chapter III] (alternatively, see [1, Chapter IV]), π is a classifying map for $\lambda \to X/\mathbb{Z}_2$, and thus $x = \pi^*(t) = \pi^*(w_1(\xi)) = w_1(\lambda)$. Since f is equivariant, $\xi' \to RP^m$ is the pullback of λ by $\overline{f}$, and thus by naturality $\overline{f}^*(w_1(\lambda)) = w_1(\xi')$; that is, $\overline{f}^* : H^1(X/\mathbb{Z}_2) \to H^1(RP^m)$ maps x into the generator $t' \in H^1(RP^m)$. By Theorem 4.1, $0 = \overline{f}^*(x^{3n+1}) = t'^{3n+1}$; on the other hand, since $m \geq 3n + 1$, $t'^{3n+1} \in H^{3n+1}(RP^m)$ is nonzero, thus giving the contradiction.

The proof of 2) follows the same lines of 1). Suppose, on the contrary, that there exists a $\mathbb{S}^1$-equivariant map $f : S^{2m+1} \to X$, and let $\overline{f} : S^{2m+1}/\mathbb{S}^1 = CP^m \to X/\mathbb{S}^1$ be the map induced by f. Write $\lambda \to X/\mathbb{S}^1$ for the complex line bundle associated to the free $\mathbb{S}^1$-action on X. In this case, the total space of λ is the orbit space of $X \times \mathbb{C}$ by the diagonal action of $\mathbb{S}^1$, coming from the free action of $\mathbb{S}^1$ on X and the complex multiplication on $\mathbb{C}$; in the subsequent approach, λ will be considered as a 2-dimensional real vector bunde. Denote by $\xi \to B_{\mathbb{S}^1} = CP^\infty$ the universal complex line bundle and by $\xi' \to CP^m$ its restriction to CP^m, both also considered as 2-dimensional real vector bundles. In the proof of Theorem 4.2, $x \in H^2(X/\mathbb{S}^1)$ is the image of t under $\pi^* : H^2(CP^\infty) \to H^2(X/\mathbb{S}^1)$; since also in this case π is a classifying map for $\lambda \to X/\mathbb{S}^1$, this gives that $x = \pi^*(t) = \pi^*(w_2(\xi)) = w_2(\lambda)$. Again, because f is equivariant, $\xi' \to CP^m$ is the pullback of λ by $\overline{f}$, and so naturality gives $\overline{f}^*(w_2(\lambda)) =$

$w_2(\xi') = t' \in H^2(CP^m)$, t' the generator of $H^2(CP^m)$. In the situations i) and ii) of Theorem 4.2, we have $x^{\frac{3n+1}{2}} = 0$; in this way, $0 = \overline{f}^*(x^{\frac{3n+1}{2}}) = t'^{\frac{3n+1}{2}}$. However, because $m \geq \frac{3n+1}{2}$, $2m \geq 3n+1$, and thus $t'^{\frac{3n+1}{2}} \in H^{3n+1}(CP^m)$ is nonzero. This gives the desired contradiction.

□

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