Gevrey micro-regularity for solutions to first order nonlinear PDE

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Abstract

Let $(x,t) \in \mathbb{R}^m \times \mathbb{R}$ and $u \in C^2(\mathbb{R}^m \times \mathbb{R})$. We study the Gevrey micro-regularity of solutions u of the nonlinear equation

$$u_t = f(x,t,u,u_x)$$

where $f(x,t,\zeta_0,\zeta)$ is a Gevrey function of order $s > 1$ and holomorphic in (ζ_0,ζ). We show that the Gevrey wave-front set of any C^2 solution u is contained in the characteristic set of the linearized operator

$$L^u = \frac{\partial}{\partial t} - \sum_{j=1}^{m} \frac{\partial f}{\partial \zeta_j}(x,t,u,u_x)\frac{\partial}{\partial x_j}.$$

To achieve this, we study the notion of *Gevrey approximate solutions*, a concept which we believe is of independent interest and could be applied to much more general situations.

1 Introduction

Let u be a C^2 solution of the nonlinear partial differential equation

$$u_t = f(x,t,u,u_x), \tag{1.1}$$

where $f(x,t,\zeta_0,\zeta)$ is a smooth function for (x,t,ζ_0,ζ) varying in an open subset of $\mathbb{R}^m \times \mathbb{R} \times \mathbb{C} \times \mathbb{C}^m$, holomorphic in (ζ_0,ζ). It is a well known result due to Chemin [Ch] that the C^∞ wave-front set of u is contained in the characteristic set of the linearized operator

$$L^u = \frac{\partial}{\partial t} - \sum_{j=1}^{m} \frac{\partial f}{\partial \zeta_j}(x,t,u,u_x)\frac{\partial}{\partial x_j}.$$

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In the analytic set up a similar result holds. Indeed it was proved by Hanges and Treves [HT] that, under the assumption that f is real-analytic, the analytic wave-front set of u must also be contained in the characteristic set of L^u.

The key point in the Hanges–Treves argument is the construction of certain first integrals by using the Cauchy-Kovalewsky theorem. Although such a device is not available in the smooth case, Asano [As] was able to replace it by the concept of *approximate* first integrals in such a way that the Hanges–Treves argument could be applied in order to derive the Chemin result alluded to above.

Our goal in the present work is to fill the gap between Chemin's and the Hanges–Treves' results. We shall then assume that f is a Gevrey function of order $s > 1$, holomorphic with respect to (ζ_0, ζ), and will prove that *if u is C^2-solution of (1.1) then its Gevrey wave front set of order s is contained in the characteristic set of L^u.*

This is our main result and since our approach is to follow the Asano–Hanges–Treves arguments, we are forced to study the notion of *Gevrey approximate solutions*, a concept which we believe is of independent interest and could be applied to much more general situations.

To explain such a concept we start by recalling a classical question known as the *Carleman problem* (cf. also J. Bruna [Br] and T. Carleman [Ca]): *given a sequence of complex numbers, $\{m_n\}$, satisfying $|m_n| \leq B^{n+1} n^{ns}$, $n = 0, 1, \ldots$, where B is a positive constant and $s > 1$, is there a Gevrey function $f(x)$ of order s, defined on $[-1, 1]$, such that $f^{(n)}(0) = m_n$, $n = 0, 1, \ldots$?* This question has an affirmative answer, as proved by Mityagin [Mi]. In Džanašija [Dz] an explicit construction of such a function f can be found.

In the present work we show that Džanašija's construction can be extended in order to achieve the following result: given a G^s vector field L, a G^s hypersurface Σ which is non-characteristic with respect to L, and a function $u_0 \in G^s(\Sigma)$ it is possible to extend u_0 as a G^s-function u which is an approximate solution (in a very precise sense) of the equation $Lv = 0$. This extension involves, among other things, replacing the sequence $\{m_n\}$ in the Carleman problem by a sequence $\{u_{0n}\} \in G^s(\Sigma)$. After such result is established Asano's arguments can be applied with minor modifications.

We remark that, for a different application, Adwan and Hoepfner [AH] also studied the existence of Gevrey approximate solutions for the same class of vector fields we are interested here. Neverthless, their result is weaker, in the sense that their extension of $u_0 \in G^s(\Sigma)$ is only of Gevrey class $s' > s + 1$, with s' arbitrary. Hence, as a consequence of our statements, the results in [AH] can be improved accordingly.

For more interesting results on analytic regularity of either quasilinear or fully nonlinear first order partial differential equations we refer the reader to S. Berhanu [B], G. Metivier [M] and N. Lerner, Y. Morimoto and C. -J. Xu [LMX] and some of the references therein.

We have organized our exposition in the following way. In Section 2 we recall some standard notation and definitions. In Section 3 we extend the Džanašija-Mityagin's result in order to construct, in Section 4, an s-approximate solution. In Section 5 we extend Asano's arguments to the Gevrey framework and finally, in Section 6, we apply the results obtained in the preceding sections in order to prove our regularity result.

2 Definitions and Notations

In this section we recall some definitions and results about Gevrey functions. Let $\Omega \subset \mathbb{R}^N$ be an open subset and let $s \geq 1$ be a fixed real number.

Definition 2.1 *We say that a function $f(x) \in C^\infty(\Omega)$ is in the Gevrey class $G^s(\Omega)$ if for every compact subset $K \subset \Omega$ there exists a constant $C > 0$ such that $|\partial_x^\alpha f(x)| \leq C^{|\alpha|+1}(\alpha!)^s$, for all $\alpha \in \mathbb{Z}_+^N$ and for all $x \in K$. In particular, $G^1(\Omega)$ is the space of all analytic functions, denoted by $C^\omega(\Omega)$.*

Definition 2.2 *Assume $s > 1$. We shall denote by $G_0^s(\Omega)$ the vector space of all $f \in G^s(\Omega)$ with compact support in Ω.*

For $y = (y_1, \ldots, y_n) \in \mathbb{C}^N$ define $\langle y \rangle = (1 + \sum_{j=1}^N y_j^2)^{1/2}$, which is well defined and holomorphic in a conic neighborhood Γ of $\mathbb{R}^N$, (see M. Christ, [Chr]). Let (x, ξ) be coordinates on $\mathbb{C}^N \times \mathbb{C}^N$. For each $\gamma \in [0, 1]$ define a differential form ω on a conic neighborhood of $\mathbb{R}^N$ in $\mathbb{C}^N$ by $\omega = dx_1 \wedge \cdots \wedge dx_N \wedge d(\xi_1 + ix_1\langle\xi\rangle^\gamma) \wedge \cdots \wedge d(\xi_N + ix_N\langle\xi\rangle^\gamma)$ and define a function α_γ by $\omega = \alpha_\gamma(x, \xi) dx_1 \wedge \cdots \wedge dx_N \wedge d\xi_1 \wedge \cdots \wedge d\xi_N$. The coefficient α_γ is holomorphic with respect to x, and equals $1 + O(\langle\xi\rangle^{\gamma-1})$ for x in any bounded subset of $\mathbb{C}^N$.

We now recall the definition of the FBI transform.

Definition 2.3 *(see M. Christ, [Chr]) For $u \in E'(\mathbb{R}^N)$, $(x, \xi) \in \mathbb{C}^N \times \Gamma$, and for $0 \leq \gamma \leq 1$ define*

$$\mathcal{F}_\gamma u(x, \xi) = \langle u, e^{i(x-x')\cdot\xi - \langle\xi\rangle^\gamma |x-x'|^2} \alpha_\gamma(x - x', \xi) \rangle,$$

where the pairing is that of distributions with test functions, with respect to the variable x'.

For more details on FBI transform we refer the reader to Berhanu, Cordaro and Hounie [BCH].

Definition 2.4 *Let $\Omega \subset \mathbb{R}^N$ be an open neighborhood of a point x_0, and $s \in (1, \infty)$. The distribution u is said to belong to the Gevrey class G^s at x_0 if there exists a neighborhood $U \subset \Omega$ of x_0 such that $u \in G^s(U)$.*

Using the FBI transform one has the following characterization of a Gevrey function:

Theorem 2.5 *(see M. Christ [Chr]) Let $x_0 \in \mathbb{R}^N$, $u \in D'(\mathbb{R}^N)$, and $s > 1$. Then the following conditions are equivalent:*

1. *$u \in G^s$ at x_0.*
2. *There exist $v \in E'(\mathbb{R}^N)$ agreeing with u in some neighborhood of x_0, $\gamma \in [1/s, 1]$, $\delta > 0$, $C > 0$ and a neighborhood Ω of x_0 such that*

$$|\mathcal{F}_\gamma v(x, \xi)| \leq C e^{-\delta\langle\xi\rangle^{1/s}}, \text{ for all } (x, \xi) \in \Omega \times \mathbb{R}^N.$$

In the case that u is non-G^s at x_0 one can obtain more information about its singularities by studying the directions in which the above condition (2) does not hold. This leads to the following definition of Gevrey wave front set.

Definition 2.6 *Let $s > 1$. For fixed $x_0 \in \Omega$ and $\xi_0 \in \mathbb{R}^N$, $\xi_0 \neq 0$, we say that $u \in D'(\Omega)$ is s-micro-regular at (x_0, ξ_0) if there exist $\varphi \in G_0^s(\Omega)$, $\varphi(x) = 1$ in a neighborhood U of x_0, and a conic neighborhood Γ of ξ_0, such that for some positive constants C and δ the condition (2) in Theorem 2.5 holds for $v = \varphi u$ and $(x, \xi) \in U \times \Gamma$. The s-wave-front set of u, $WF_s(u)$, is then defined as the complement in $\Omega \times (\mathbb{R}^N \backslash \{0\})$ of the set of all (x_0, ξ_0) where u is s-micro-regular.*

We now recall that if $\mathcal{N}$ is an open set in $\mathbb{C}^M$ then $\mathcal{O}(\mathcal{N}, G^s(\Omega))$ denote the space of all functions $f(x, \zeta) \in C^\infty(\Omega \times \mathcal{N})$ such that f is Gevrey in x on Ω and holomorphic in ζ on $\mathcal{N}$. Let $V \subset\subset \mathcal{N}$ be an open subset. We denote by $E^s(\Omega, V)$ the set of all functions $f \in \mathcal{O}(\mathcal{N}, G^s(\Omega))$ such that for every compact subset $K \subset \Omega$ there exists $C > 0$ such that,

$$|\partial_x^\alpha u(x, \zeta)| \leq C^{|\alpha|+1} (\alpha!)^s, \quad \forall\ (x, \zeta) \in K \times \bar{V},\ \alpha \in \mathbb{Z}_+^N.$$

3 Carleman's problem for Gevrey functions

We start this section by extending the Džanašija-Mityagin's result (see [Dz] and [Mi]). The proof of our result will follow the lines of the Džanašija's proof.

Lemma 3.1 *Let $s > 1$ be a real number. We assume that $\{v_k(x)\}$, $k = 0, 1, \ldots,$ is a sequence of C^∞ functions defined on an open neighborhood, $\Omega \subset \mathbb{R}^N$ of the origin, such that given a compact subset $K \subset \Omega$, there exists $B > 0$ such that,*

$$|\partial_x^\alpha v_k(x)| \leq B^{|\alpha|+k+1} (\alpha!)^s (k!)^s, \quad \forall x \in K, k = 0, 1, 2, \ldots, \alpha \in \mathbb{Z}_+^N. \tag{3.1}$$

Then, shrinking Ω, there exists $f \in G^s(\Omega \times (-1, 1))$ such that $\frac{\partial^n f}{\partial t^n}(x, 0) = v_n(x), n = 0, 1, \ldots, \forall\, x \in \Omega$.

Proof. For $t \in [-1, 1]$, we define $a_0(t) = 1$ and, for $k \geq 1$, we set

$$b_k(t) = \begin{cases} 0 & , \quad \{-1 \leq t \leq -\sigma_k\} \cup \{0 \leq t \leq 1\} \\ \exp\left(\frac{-k\sigma_k^{4r}}{t^{2r}(\sigma_k + t)^{2r}}\right), & \qquad -\sigma_k < t < 0 \end{cases} \tag{3.2}$$

where r is an integer such that $\frac{1}{2r} < s - 1$ and $\sigma_k = D^{-1} k^{-(s-1)}$, for some $D > 0$ to be chosen later.

For $k \geq 1$ we now define

$$a_k(t) = \int_{-1}^{t} b_k(y) dy \Big/ \int_{-1}^{1} b_k(y) dy, \tag{3.3}$$

for $-1 \leq t \leq 0$ and we put $a_k(t) = a_k(-t)$ for $0 \leq t \leq 1$.

Let $\Omega \subset \mathbb{R}^N$ be an open neighborhood of the origin. We now consider the formal series

$$\sum_{k=0}^{\infty} \frac{v_k(x)}{k!} a_k(t) t^k, \ (x,t) \in \Omega \times [-1,1]. \tag{3.4}$$

Let $K \subset \Omega$ be a compact subset. It follows from (3.1), and from formulas (2), (3) and (4) in [Dz] that there exists a constant $a > 0$ such that

$$\sum_{k=1}^{\infty} \frac{|v_k(x)|}{k!} |a_k(t)||t|^k \leq 2B \sum_{k=1}^{\infty} \left(\frac{Be^{a+1}}{D} \right)^k < \infty, \ \forall \ (x,t) \in K \times [-1,1],$$

if we choose $D \geq Be^{a+1}$. Thus we have proved that the series (3.4) converges uniformly on $K \times [-1,1]$. Shrinking Ω, we set

$$f(x,t) = \sum_{k=0}^{\infty} \frac{v_k(x)}{k!} a_k(t) t^k, \ \text{for } (x,t) \in \Omega \times [-1,1].$$

In order to show that f satisfies the conditions of Lemma 3.1 it suffices to prove that

$$g(x,t) \doteq \sum_{k=1}^{\infty} \frac{v_k(x)}{k!} a_k(t) t^k \tag{3.5}$$

belongs to $G^s(\Omega \times (-1,1))$ and satisfies $\frac{\partial^n g}{\partial t^n}(x,0) = v_n(x), \quad n = 1, 2, \ldots$.

We start by analyzing

$$w_k(x,t) \doteq \partial_x^\alpha \partial_t^n \left(\frac{v_k(x)}{k!} a_k(t) t^k \right) = \frac{\partial_x^\alpha v_k(x)}{k!} \sum_{i=0}^{n} \binom{n}{i} a_k^{(n-i)}(t) (t^k)^{(i)}.$$

For $k \leq n$ it follows from [Dz], p.971, that there exists $T > 1$ such that

$$|w_k(x,t)| \leq 2^{n+1} \frac{|\partial_x^\alpha v_k(x)|}{k!} e^{ak} D^{-k} k^{-k(s-1)} T^n D^n n^{ns}, \ \forall \ (x,t) \in \Omega \times [-1,1].$$

For $x \in K$, $t \in [-1,1]$ and $k \leq n$, it follows from the last inequality, from (3.1) and from the fact that $k! \leq k^k$ and $\frac{k^k}{k!} \leq e^k$ that

$$\begin{aligned} |w_k(x,t)| &\leq \frac{2^{n+1} B^{|\alpha|+k+1} (\alpha!)^s (k!)^s}{k!} e^{ak} D^{-k} k^{-k(s-1)} T^n D^n n^{ns} \\ &\leq B^{|\alpha|+1} (\alpha!)^s 2^{n-k+1} T^n D^n n^{ns}, \end{aligned} \tag{3.6}$$

for $D \geq 2Be^{a+1}$.

We now assume that $k > n$. Once again, it follows from [Dz], p.971, that

$$|w_k(x,t)| \leq 2^{n+1} \frac{|\partial_x^\alpha v_k(x)|}{k!} e^{ak} D^{-k} k^{-k(s-1)} T^n D^n k^{ns}, \ \forall \ (x,t) \in \Omega \times [-1,1].$$

By recalling that $e^{-k} \leq k^{-ns}(ns)^{ns}e^{-ns}$, setting $M = s^s$ and choosing $D \geq 2Be^{a+2}$ it follows from (3.1) and from the last inequality that

$$\sum_{k=n+1}^{\infty} |w_k(x,t)| \leq B^{|\alpha|+1}(\alpha!)^s 2^{n+1} T^n D^n M^n n^{ns},\ \forall\ (x,t) \in K \times [-1,1].$$

For $(x,t) \in K \times [-1,1]$ it follows from the last inequality and from (3.6) that

$$\begin{aligned}\sum_{k=1}^{\infty} |w_k(x,t)| &\leq B^{|\alpha|+1}(\alpha!)^s 2^{n+2} T^n D^n n^{ns} + B^{|\alpha|+1}(\alpha!)^s 2^{n+1} T^n D^n M^n n^{ns} \\ &\leq B^{|\alpha|+1}(\alpha!)^s (8TDM)^n n^{ns} \leq A^{|\alpha|+n+1}(\alpha!)^s (n!)^s,\end{aligned}$$

holds, where $A = \max\{B, 8TDMe^s\}$ is independent of α and n.

Summing up, we have proved that $f \in G^s(\Omega \times (-1,1))$ and satisfies Lemma 3.1. The proof of Lemma 3.1 is complete. □

We shall need the following version of Lemma 3.1 whose proof will be omitted since it is a simple variation of the proof the Lemma 3.1.

Lemma 3.2 *Let $s > 1$ be a real number and let Ω, $\mathcal{N}$, V be as in section two. We assume that $\{v_k(x,\zeta)\}$, $k = 0, 1, \ldots,$ is a sequence of functions such that for each k fixed we have $v_k \in E^s(\Omega, V)$. Furthermore, we assume that for every compact subset $K \subset \Omega$ there exists $C > 0$ such that,*

$$|\partial_x^\alpha v_k(x,\zeta)| \leq C^{|\alpha|+k+1}(\alpha!)^s (k!)^s,\ k = 0, 1, \ldots,\ \forall\ (x,\zeta) \in K \times \bar{V},\ \alpha \in \mathbb{Z}_+^N.$$

Then, shrinking Ω and V if it is necessary, there exists $f \in E^s([\Omega \times (-1,1)], V)$ such that

$$\frac{\partial^n}{\partial t^n} f(x,0,\zeta) = v_n(x,\zeta),\ n = 0, 1, 2, \ldots,\ \forall\ (x,\zeta) \in \Omega \times V.$$

The next result extend Lemma 3.1 for a multi-sequence $\{v_\beta(x)\}_{\beta \in \mathbb{Z}_+^N}$ and $t \in (-1,1)^N$.

Lemma 3.3 *Let $s > 1$ be a real number and $\{v_\beta(x)\}$, $\beta \in \mathbb{Z}_+^N$, be a multi-sequence of C^∞ functions defined on an open neighborhood $\Omega \subset \mathbb{R}^N$ of the origin, such that given a compact subset $K \subset \Omega$, there exists $B > 1$ such that,*

$$|\partial_x^\alpha v_\beta(x)| \leq B^{|\alpha|+|\beta|+1}(\alpha!)^s (\beta!)^s, \quad \forall x \in K,\ \alpha, \beta \in \mathbb{Z}_+^N. \tag{3.7}$$

Then, shrinking Ω, there exists $F \in G^s(\Omega \times (-1,1)^N)$ such that $\partial_y^\gamma F(x,0) = v_\gamma(x)$, $\forall\, x \in \Omega, \gamma \in \mathbb{Z}_+^N$.

Proof. Let Ω and K be as in the statement of Lemma 3.3. For $x \in \Omega$ and $y \in [-1,1]^N$, we now define

$$F(x,y) = \sum_{\beta \in \mathbb{Z}_+^N} \frac{v_\beta(x)}{\beta!} A_\beta(y) y^\beta, \tag{3.8}$$

where $A_\beta(y) = a_{\beta_1}(y_1) \cdot \ldots \cdot a_{\beta_N}(y_N)$ and a_{β_i} are the functions defined in the proof of Lemma 3.1.

Taking, in (3.8), the derivative of order α with respect to x and the derivative of order γ with respect to y we obtain a series whose general term is given by

$$W_\beta(x,y) = \frac{\partial_x^\alpha v_\beta(x)}{\beta!} \sum_{\gamma' \leq \gamma} \binom{\gamma}{\gamma'} \partial_y^{\gamma-\gamma'} A_\beta(y) \partial_y^{\gamma'}(y^\beta).$$

It follows from (3.7) that for $(x,y) \in K \times [-1,1]^N$ there exists a positive constant B such that

$$\begin{aligned} |W_\beta(x,y)| &\leq \frac{B^{|\alpha|+|\beta|+1}}{\beta!} (\alpha!)^s (\beta!)^s \, |G_1(\gamma,\beta,y) \cdot \ldots \cdot G_N(\gamma,\beta,y)| \\ &\leq \prod_{j=1}^N \left(\frac{B^{\alpha_j+\beta_j+1}}{\beta_j!} (\alpha_j!)^s (\beta_j!)^s \, |G_j(\gamma,\beta,y)| \right), \end{aligned}$$

where $G_j(\gamma,\beta,y) = \sum_{\gamma_j' \leq \gamma_j} \binom{\gamma_j}{\gamma_j'} \partial_{y_j}^{\gamma_j - \gamma_j'} a_{\beta_j}(y_j) \partial_{y_j}^{\gamma_j'}(y_j^{\beta_j}),\ j = 1, \ldots, N.$

Applying the techniques we have used in the proof of Lemma 3.1, with $N = 1$, we can conclude that there exist positive constants A_j, independents of α_j and of γ_j, $j = 1, \ldots, N$ such that

$$\begin{aligned} \sum_\beta |W_\beta(x,y)| &\leq \prod_{j=1}^N \sum_{\beta_j=0}^\infty \frac{B^{\alpha_j+\beta_j+1}}{\beta_j!} (\alpha_j!)^s (\beta_j!)^s \, |G_j(\gamma,\beta,y)| \\ &\leq \prod_{j=1}^N A_j^{\alpha_j+\gamma_j+1} (\alpha_j!)^s (\gamma_j)^s \leq A^{|\alpha|+|\gamma|+1} (\alpha!)^s (\gamma!)^s, \end{aligned}$$

for $(x,y) \in K \times [-1,1]^N$, where $A = \max\{A_1, \ldots, A_N\}$.

Therefore, shrinking Ω, $F \in G^s(\Omega \times (-1,1)^N)$ and it is easily seen that

$$\partial_y^\gamma F(x,0) = v_\gamma(x), \quad \forall x \in \Omega,\ \gamma \in \mathbb{Z}_+^N.$$

□

We now apply this result in order to show that there always exists almost analytic extension in the Gevrey class.

Lemma 3.4 *Let $\Omega \subset \mathbb{R}^N$ be an open set and $f \in G^s(\Omega)$. Then, shrinking Ω, there exists $\tilde{f} \in G^s(\Omega \times (-1,1)^N)$, almost analytic extension with exponent s of f, more precisely, $\tilde{f}(x,0) = f(x)$, $\forall\, x \in \Omega$ and there exists $C > 0$ such that for each $j = 1, \ldots, N$ and $(x,y) \in \Omega \times (-1,1)^N$ we have*

$$\left| \frac{\partial \tilde{f}}{\partial \bar{z}_j}(x,y) \right| \leq C^{\nu+1} (\nu!)^{s-1} |y|^\nu, \ \text{for } \nu \in \mathbb{Z}_+,$$

for some C independent of ν.

Proof. For $(x,y) \in \Omega \times [-1,1]^N$ we define, $\tilde{f}(x,y) = \sum_{\alpha\in\mathbb{Z}_+^N} \frac{1}{\alpha!}(iy)^\alpha f^{(\alpha)}(x) A_\alpha(y)$, where A_α is defined as in the proof of Lemma 3.3. Now, the proof is an easy consequence of Lemma 3.3. □

4 Existence of s-approximate solution

In this section we will denote by $\Omega \subset \mathbb{R}^m \times \mathbb{R}$ an open neighborhood of the origin and we start by proving the crucial result:

Lemma 4.1 *Let $U \subset \mathbb{R}^m$ be an open neighborhood of the origin such that $U \times \{0\} \subset \Omega$. Let $f \in G^s(U)$ and $a_k \in G^s(\Omega)$, $k = 1, \dots, m$ be given. Set $u_0(x) = f(x)$ and, for $j \geq 1$,*

$$u_j(x) = -\frac{1}{j} \sum_{i=0}^{j-1} \frac{1}{i!} \sum_{k=1}^{m} \frac{\partial u_{j-1-i}}{\partial x_k}(x) \partial_t^i a_k(x,0). \tag{4.1}$$

Then, given $K \subset U$ a compact subset, there exist constants $M, N > 1$ such that

$$|\partial_x^\alpha u_j(x)| \leq \frac{M^j}{j!} N^{|\alpha|+1} (|\alpha| + j)!^s, \quad \forall x \in K, j = 0, 1, 2, \dots, \alpha \in \mathbb{Z}_+^m. \tag{4.2}$$

Before proving Lemma 4.1 we shall need the following

Lemma 4.2 *Let $A > 0$ be a fixed constant. Given $\alpha \in \mathbb{Z}_+^m$ there exist constants $L > 1$ and $G > 1$, independents of α, such that*

$$\frac{A}{L-1} \sum_{\beta \leq \alpha} G^{1-|\alpha-\beta|} \leq 1. \tag{4.3}$$

Proof. Let $\alpha \in \mathbb{Z}_+^m$ be given and assume that $G > 1$. Thus,

$$\begin{aligned} \sum_{\beta\leq\alpha} G^{-(|\alpha|-|\beta|)} &= \left(\sum_{\beta_1=0}^{\alpha_1} G^{-(\alpha_1-\beta_1)}\right)\left(\sum_{\beta_2=0}^{\alpha_2} G^{-(\alpha_2-\beta_2)}\right)\cdots\left(\sum_{\beta_m=0}^{\alpha_m} G^{-(\alpha_m-\beta_m)}\right) \\ &\leq \left(\sum_{k=0}^{\infty} G^{-k}\right)\left(\sum_{k=0}^{\infty} G^{-k}\right)\cdots\left(\sum_{k=0}^{\infty} G^{-k}\right) = \left(\frac{G}{G-1}\right)^m. \end{aligned}$$

Therefore, choosing $L > 1$ large enough we obtain what we desired. □

Proof of Lemma 4.1. Let U, Ω be as in the statement of Lemma 4.1 and let $K \subset U$ be a compact subset. Since $f \in G^s(U)$ and $a_k \in G^s(\Omega)$ there exists $A > 1$ such that for $n \in \mathbb{Z}_+$, $\alpha \in \mathbb{Z}_+^m$, $x \in K$ and $k = 1, \dots m$, we have

$$|\partial_x^\alpha f(x)| \leq A^{|\alpha|+1} (\alpha!)^s \text{ and } |\partial_x^\alpha \partial_t^n a_k(x,0)| \leq A^{|\alpha|+n+1} (\alpha!)^s (n!)^s. \tag{4.4}$$

We now choose $L, G > 1$ such that (4.3) holds and we define $M = mAL$ and $N = AG$. We are going to prove inequality (4.2) by induction on j.

It follows from (4.4) that for $x \in K$ and $\alpha \in \mathbb{Z}_+^m$ we have

$$|\partial_x^\alpha u_0(x)| = |\partial_x^\alpha f(x)| \leq A^{|\alpha|+1}(\alpha!)^s \leq \frac{M^0}{0!} N^{|\alpha|+1}(|\alpha|+0)!^s$$

and therefore (4.2) holds true for $j = 0$.

Let us now assume that (4.2) is satisfied for all $0 \leq p < j$, and let us prove it for $p = j$. It follows from the definition of u_j, (see (4.1)), that

$$|\partial_x^\alpha u_j(x)| \leq \frac{1}{j} \sum_{i=0}^{j-1} \frac{1}{i!} \sum_{k=1}^{m} \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} |\partial_x^{\beta+e_k} u_{j-1-i}(x)||\partial_x^{\alpha-\beta}\partial_t^i a_k(x,0)| \tag{4.5}$$

where $\{e_1, \ldots, e_m\}$ is the usual basis of $\mathbb{R}^m$. By the induction hypothesis we have

$$|\partial_x^{\beta+e_k} u_{j-1-i}(x)| \leq \frac{M^{j-i-1}}{(j-i-1)!} N^{|\beta|+2}(|\beta|+j-i)!^s \tag{4.6}$$

and from (4.4) we have

$$|\partial_x^{\alpha-\beta}\partial_t^i a_k(x,0)| \leq A^{|\alpha-\beta|+i+1}(\alpha-\beta)!^s(i!)^s, \tag{4.7}$$

for all $x \in K$.

By using the inequalities $p!q! \leq (p+q)!$ and $\binom{\alpha}{\beta} \leq \binom{|\alpha|}{|\beta|}$ for $\alpha, \beta \in \mathbb{Z}_+^m$ with $\beta \leq \alpha$, we obtain, for $i \in \{0, 1, \ldots, j-1\}$,

$$\binom{\alpha}{\beta} \frac{(|\beta|+j-i)!^s(\alpha-\beta)!^s(i!)^s}{(i!)(j-i-1)!} \leq \frac{|\alpha|!(|\beta|+j-i)!}{|\beta|!(j-i-1)!}(|\alpha|+j)!^{s-1} \leq \frac{(|\alpha|+j)!^s}{(j-1)!}.$$

Thanks to (4.3), (4.5), (4.6), (4.7) and the last inequality we obtain

$$\begin{aligned}
|\partial_x^\alpha u_j(x)| &\leq \frac{1}{j} \sum_{i=0}^{j-1} \sum_{k=1}^{m} \sum_{\beta \leq \alpha} M^{j-i-1} N^{|\beta|+2} A^{|\alpha|-|\beta|+i+1} \frac{(|\alpha|+j)!^s}{(j-1)!} \\
&\leq \frac{m}{j!} M^j N^{|\alpha|+1}(|\alpha|+j)!^s \sum_{i=0}^{j-1} \sum_{\beta \leq \alpha} M^{-(i+1)} N^{|\beta|-|\alpha|+1} A^{|\alpha|-|\beta|+i+1} \\
&= \frac{m}{j!} M^j N^{|\alpha|+1}(|\alpha|+j)!^s \sum_{i=0}^{j-1} \sum_{\beta \leq \alpha} \frac{A^{-i-1}}{m^{i+1}L^{i+1}} A^{|\beta|-|\alpha|+1} G^{|\beta|-|\alpha|+1} A^{|\alpha|-|\beta|+i+1} \\
&\leq \frac{m}{j!} M^j N^{|\alpha|+1}(|\alpha|+j)!^s \left(\sum_{i=0}^{\infty} \left(\frac{1}{L}\right)^i\right) \sum_{\beta \leq \alpha} \frac{A}{mL} G^{|\beta|-|\alpha|+1} \\
&= \frac{M^j}{j!} N^{|\alpha|+1}(|\alpha|+j)!^s \frac{A}{L} \frac{L}{L-1} \sum_{\beta \leq \alpha} G^{1-|\alpha-\beta|} \leq \frac{M^j}{j!} N^{|\alpha|+1}(|\alpha|+j)!^s.
\end{aligned}$$

The proof is now complete. □

We shall need the following lemma about linear vector fields.

Proposition 4.3 *Consider the complex vector field*

$$L = \frac{\partial}{\partial t} + \sum_{j=1}^{N} a_j(x,t,\zeta)\frac{\partial}{\partial x_j} + \sum_{k=1}^{M} b_k(x,t,\zeta)\frac{\partial}{\partial \zeta_k},$$

where the coefficients a_j and b_k belongs to the class $E^s(\Omega, V)$, where $V \subset\subset \mathcal{N}$ is an open subset with $\mathcal{N}$ being an open set in $\mathbb{C}^M$. Let $U \subset \mathbb{R}^m$ be an open neighborhood of the origin such that $U \times \{0\} \subset \Omega$ and let $f(x,\zeta) \in E^s(U,V)$. Then, shrinking U, there exists $u(x,t,\zeta) \in E^s([U \times (-1,1)], V)$ which is an s-approximate solution of $Lw = 0$ in the sense that there exists a positive constant A such that

$$|Lu(x,t,\zeta)| \leq A^{\nu+1}(\nu!)^{s-1}|t|^{\nu}, \ (x,t,\zeta) \in U \times (-1,1) \times V, \tag{4.8}$$

and such that $u(x,0,\zeta) = f(x,\zeta)$.

Remark 4.4 *In (4.8), $|.|$ means the norm in $\mathbb{C}^M$ when the function u is valued in $\mathbb{C}^M$ and we note that (4.8) makes sense even if the function u is only C^1.*

Proof of Proposition 4.3. It is standard to look for u as $u(x,t,\zeta) = \sum_{j=0}^{\infty} u_j(x,\zeta)t^j$.

By defining recursively $u_0(x,\zeta) = f(x,\zeta)$ and for $j \geq 1$,

$$\begin{aligned} u_j(x,\zeta) = -\frac{1}{j}\sum_{i=0}^{j-1}\frac{1}{i!}\Bigg[\sum_{k=1}^{N}\frac{\partial u_{j-i-1}}{\partial x_k}(x,\zeta)\partial_t^i a_k(x,0,\zeta) \\ + \sum_{k=1}^{M}\frac{\partial u_{j-i-1}}{\partial \zeta_k}(x,\zeta)\partial_t^i b_k(x,0,\zeta)\Bigg] \end{aligned}$$

we formally obtain $\partial_t^{j-1}(Lu)(x,0,\zeta) = 0$.

It follows from our hypotheses that for each j, fixed, $u_j(x,\zeta) \in E^s(U,V)$. By using this one can prove, as we have proved Lemma 4.1, that for every compact subset $K \subset U$ there exist constants $M, N > 1$ such that

$$|\partial_x^{\alpha} u_j(x,\zeta)| \leq \frac{M^j}{j!}N^{|\alpha|+1}(|\alpha|+j)!^s, \ \ \forall (x,\zeta) \in K \times \bar{V}, j = 0,1,2,\ldots, \ \alpha \in \mathbb{Z}_+^m.$$

We now define $v_j(x,\zeta) = j!u_j(x,\zeta)$. It follows from Lemma 3.2 that, shrinking U, there exists $u(x,t,\zeta) \in E^s([U \times (-1,1)], V)$ such that

$$u_j(x,\zeta) = \frac{1}{j!}\frac{\partial^j}{\partial t^j}u(x,0,\zeta), \ \ \forall (x,\zeta) \in U \times V, j = 0,1,\ldots$$

In particular we have $u(x,0,\zeta) = u_0(x,\zeta) = f(x,\zeta)$. It is now easy to see that u is an s-approximate solution of $Lw = 0$. □

5 Abstract results

Let Ω be as in the section 4. Consider the vector field defined on Ω

$$L=\frac{\partial}{\partial t}+\sum_{k=1}^{m}a_k(x,t)\frac{\partial}{\partial x_k},\ \text{where } a_k\in C^1(\Omega),\ k=1,\dots,m.$$

and let Z be an s-approximate solution of $Lw=0$ such that $Z_x(x,0)=I$ and let $U\subset\subset\Omega$ be an open neighborhood of the origin where $Z_x(x,t)$ is non-singular.

If we define $b=-Z_x^{-1}Z_t$ and $L_1=\frac{\partial}{\partial t}+\sum_{j=1}^{m}b_j(x,t)\frac{\partial}{\partial x_j}$ then L_1 is a complex vector field defined on U that satisfies $L_1Z=0$.

It is easily seen that if $h\in C^1(\Omega)$ is an s-approximate solution of $Lw=0$ then h is an s-approximate solution of $L_1w=0$ in U. We shall need the following key result.

Lemma 5.1 *Suppose that there exist $\Psi_1,\dots,\Psi_m\in C^1(\Omega)$ such that $Z=x+t\Psi(x,t)\doteq x+t(\Psi_1(x,t),\dots,\Psi_m(x,t))$ is an s-approximate solution of $Lw=0$. Let $\xi^0\in\mathbb{R}^m\backslash\{0\}$ be such that $\xi^0\cdot\mathrm{Im}\ \Psi(0,0)<0$. If $h\in C^1(\Omega)$ is an s-approximate solution of $Lw=0$ then, $(0,\xi^0)\notin WF_s(h(\cdot,0))$.*

Proof. Consider an open ball $B\subset\mathbb{R}^m$ and an interval $I\subset\mathbb{R}$, both centered at the origin and small enough such that $\overline{B\times I}\subset U$.

If $H(x,t)$ is a C^1 function defined on U, then $d(HdZ)=(L_1H)dt\wedge dZ$.

Let $0<t_1\in I$ and $\phi\in G_0^s(\mathbb{R}^m)$ be such that $\mathrm{supp}(\phi)\subset B$ and $\phi\equiv 1$ in a neighborhood of the origin. We set

$$H(y,\xi,x,t)=e^{i\xi\cdot(y-Z(x,t))-\langle\xi\rangle[y-Z(x,t)]^2}\alpha(y-Z(x,t),\xi)\phi(x)h(x,t),$$

where for $z\in\mathbb{C}^m$, we write $[z]^2=\sum_{j=1}^{m}z_j^2$, $h\in C^1(\Omega)$ is an s-approximate solution of $Lw=0$, α comes from the definition 2.3, with $\gamma=1$, and (y,ξ) are acting as parameters.

It follows from Stokes' Theorem that

$$\int_B H(y,\xi,x,0)dx=\int_B H(y,\xi,x,t_1)d_xZ(x,t_1)+\int_0^{t_1}\int_B(L_1H)dt\wedge dZ. \tag{5.1}$$

By setting $Q(y,\xi,x,t)=i\xi\cdot(y-Z(x,t))-\langle\xi\rangle[y-Z(x,t)]^2$ it follows, as in [As] page 3013, shrinking B and I if necessary, that there exist a conic neighborhood Γ of $\xi^0\in\mathbb{R}^m\backslash\{0\}$, $\delta>0$ such that for $C_0=-\xi^0\cdot\mathrm{Im}\,\Psi(0,0)>0$ we have

$$\mathrm{Re}\,Q(y,\xi,x,t)\le\frac{C_0t|\xi|}{2},\ \forall\ (x,t)\in B\times I,\ \xi\in\Gamma,\ y\in\mathbb{R}^m,\ 0<t<\delta. \tag{5.2}$$

Thus, for any open, bounded neighborhood of the origin in $\mathbb{R}^m$ there exists a positive constant C such that

$$\left|\int_B H(y,\xi,x,t_1)d_xZ(x,t_1)\right| \tag{5.3}$$
$$\le\int_B e^{\mathrm{Re}\,Q(y,\xi,x,t_1)}\left|\alpha(y-Z(x,t_1),\xi)\phi(x)h(x,t_1)d_xZ(x,t_1)\right|dx\le C\ e^{-\frac{C_0t_1|\xi|}{2}},$$

and therefore, this term has exponential decay in $\xi \in \Gamma$ and y in an open, bounded neighborhood of the origin in $\mathbb{R}^m$.

Since $L_1 Z(x,t) = 0$ we notice that

$$L_1 H(y,\xi,x,t) = e^{Q(y,\xi,x,t)}\alpha(y - Z(x,t),\xi) L_1(\phi(x)h(x,t)).$$

Thus, we have

$$\begin{aligned} &\left|\int_0^{t_1}\int_B (L_1 H) dt \wedge dZ\right| \\ &\leq \left|\int_0^{t_1}\int_B e^{Q(y,\xi,x,t)}\alpha(y - Z(x,t),\xi)\phi(x) L_1 h(x,t) d_x Z(x,t) dt\right| \\ &+ \left|\int_0^{t_1}\int_B e^{Q(y,\xi,x,t)}\alpha(y - Z(x,t),\xi)(L_1\phi(x)) h(x,t) d_x Z(x,t) dt\right|. \end{aligned} \tag{5.4}$$

We now will analyze the second term in the right side of (5.4). Since $\phi(x)$ is constant in an neighborhood of the origin then there exists $r > 0$ such that $L_1\phi(x) \equiv 0$ on $B(0,r) \subset B$. It follows from [As], page 3013, that $\operatorname{Re} Q(y,\xi,x,t) \leq -\frac{|\xi| r^2}{2}$, for $\xi \in \mathbb{R}^m \backslash \{0\}$, $x \in B \backslash \{B(0,r)\}$, $t \in [0,t_1]$, $t_1 < 1$, and y in an open, bounded neighborhood V of the origin in $\mathbb{R}^m$.

From now on we shall use the letter C to represent a positive constant, which may change a finite number of times.

Now, it follows from the last inequality that there exists $C > 0$ such that

$$\begin{aligned} &\left|\int_0^{t_1}\int_B e^{Q(y,\xi,x,t)}\alpha(y - Z(x,t),\xi)(L_1\phi(x)) h(x,t) d_x Z(x,t) dt\right| \\ &\leq \int_0^{t_1}\int_B e^{\operatorname{Re} Q(y,\xi,x,t)} \left|\alpha(y - Z(x,t),\xi)(L_1\phi(x)) h(x,t) d_x Z(x,t)\right| dt \\ &\leq C\ e^{-\frac{|\xi| r^2}{2}} \end{aligned} \tag{5.5}$$

and therefore this term also has exponential decay in $\xi \in \mathbb{R}^m \backslash \{0\}$ and y in an open, bounded neighborhood V of the origin in $\mathbb{R}^m$.

We now are going to estimate the first term of the right side of (5.4). Given $\nu \in \mathbb{N}$ we denote by $M = [\frac{\nu}{s}]$ the least integer such that $M \geq \frac{\nu}{s}$. For $(x,t) \in B \times I$, $\xi \in \Gamma$ and y in an open, bounded neighborhood V of the origin in $\mathbb{R}^m$, $0 < t < \delta$ and $|\xi| > 1$, it follows from (5.2) that there exists a constant $C > 0$ such that

$$\begin{aligned} &|\xi|^{\frac{\nu}{s}}\left|\int_0^{t_1}\int_B e^{Q(y,\xi,x,t)}\alpha(y - Z(x,t),\xi)\phi(x) L_1 h(x,t) d_x Z(x,t) dt\right| \\ &\leq C\int_0^{t_1}\int_B e^{-ct|\xi|}|L_1 h(x,t)||\xi|^M dx dt. \end{aligned}$$

By using the fact that h is an s-approximate solution of $L_1 w = 0$ it follows from the last inequality that there exists $C > 0$, independent of ν, such that

$$\begin{aligned} |\xi|^{\frac{\nu}{s}} I &\doteq |\xi|^{\frac{\nu}{s}} \left| \int_0^{t_1} \int_B e^{Q(y,\xi,x,t)} \alpha(y - Z(x,t)\phi(x) L_1 h(x,t) d_x Z(x,t) dt \right| \\ &\leq C^M (M-1)!^{s-1} \int_0^{t_1} e^{-ct|\xi|} t^{M-1} |\xi|^M dt \\ &\leq C^M (M-1)!^{s-1} \int_0^{\infty} e^{-c\lambda} (\lambda)^{M-1} d\lambda \\ &= C^M (M-1)!^{s-1} \frac{(M-1)!}{c^M} \leq C^M (M-1)!^s \leq C^{M+1} M^{sM}. \end{aligned}$$

In order to complete the analysis of this term we need recall the following

Lemma 5.2 *(see Rodino [R]) Let $\nu \in \mathbb{N}$, $s > 1$ and M the least integer such that $M \geq \frac{\nu}{s}$. Then, given $C > 0$, there exists $C^{'} > 0$, independent of ν, but it depend on C such that $C^{M+1} M^{sM} \leq C^{'} (C^{'} \nu)^{\nu}$.*

It follows from the last inequality and from Lemma 5.2 that there exists $C > 0$, independent of ν, such that $I \leq C(C\nu)^{\nu} |\xi|^{-\frac{\nu}{s}}$, for all $\xi \in \Gamma$, $|\xi| > 1$ and y in an open, bounded neighborhood V of the origin in $\mathbb{R}^m$.

Note that thanks to the inequality $r^N \leq N! e^r$, $N = 1, 2, \ldots$, it follows from the last inequality that there exists $C > 0$, independent of ν, such that

$$I \leq C(C\nu)^{\nu} |\xi|^{-\frac{\nu}{s}} \leq C^{\nu+1} e^{\nu} \nu! |\xi|^{-\frac{\nu}{s}} \leq C^{\nu+1} \nu! |\xi|^{-\frac{\nu}{s}}.$$

Setting $\epsilon = \frac{1}{2C}$ we obtain $2^{\nu} \epsilon^{\nu} (\nu!)^{-1} |\xi|^{\frac{\nu}{s}} I \leq C, \quad \forall\ \nu = 1, 2, \ldots$ and therefore, $I \leq C e^{-\epsilon |\xi|^{\frac{1}{s}}}$.

We would like to point out that the above estimate holds true for $|\xi| > 1$, but for $|\xi| \leq 1$ one can easily estimate this term. Summing up, we have proved that

$$\left| \int_0^{t_1} \int_B e^{Q(y,\xi,x,t)} \alpha(y - Z(x,t)\phi(x) L_1 h(x,t) d_x Z(x,t) dt \right| \leq C_2 e^{-\varepsilon |\xi|^{\frac{1}{s}}}, \tag{5.6}$$

for all $\xi \in \Gamma$ and y in an open, bounded neighborhood V of the origin in $\mathbb{R}^m$.

Thanks to (5.1), (5.3), (5.4), (5.5) and (5.6) we have proved that there exist a conic neighborhood Γ of $\xi^0 \in \mathbb{R}^m \backslash \{0\}$, a neighborhood $V \subset \mathbb{R}^m$ of the origin, $\varepsilon > 0$ and $C > 0$ such that $\left| \int_B H(y, \xi, x, 0) dx \right| \leq C e^{-\varepsilon |\xi|^{\frac{1}{s}}}$, for all $\xi \in \Gamma$ and $y \in V$.

The last estimate is saying that the FBI transform of $\phi(x) h(x, 0)$, see definition 2.3, which is given by

$$\mathcal{F}_1(\phi h(\cdot, 0))(y, \xi) = \int_B e^{Q(y,\xi,x,0)} \alpha(y - x, \xi) \phi(x) h(x, 0)) dx = \int_B H(y, \xi, x, 0) dx,$$

has an s-exponential decay for $y \in V$ and $\xi \in \Gamma$. Since $\phi \in G_0^s(\Omega)$ and $\phi(x) = 1$ for x near to the origin we can conclude from definition 2.6 that $(0, \xi^0) \notin WF_s(h(\cdot, 0))$. Finally the proof of Lemma 5.1 is complete. □

The proof of the next result follows the lines of the proof of Lemma 2.2 in [As].

Lemma 5.3 *Let L and Z be as in Lemma 5.1 and let $h \in C^1(\Omega)$. For each $\theta \in [0, 2\pi)$ define $L^\theta = \frac{\partial}{\partial r} - e^{-i\theta} L$ and $\Psi^\theta_{m+1} = e^{-i\theta}$ and suppose that there exist $\Psi^\theta_1, \ldots, \Psi^\theta_m \in C^1(\Omega \times J)$ such that $Z^\theta_j(x,t,r) = x_j + r\Psi^\theta_j(x,t,r)$, $j = 1, \ldots, m$ are s-approximate solution of $L^\theta w = 0$, in the variable $r \in J$, with J being an interval centered at the origin. The function $Z^\theta_{m+1}(x,t,r) = t + e^{-i\theta} r = t + r\Psi_{m+1}(x,t,r)$ satisfies $L^\theta Z^\theta_{m+1} = 0$. Suppose moreover that there exists $h^\theta \in C^1(\Omega \times J)$ such that $h^\theta(x,t,0) = h(x,t)$ and h^θ is an s-approximate solution of $L^\theta w = 0$, in $r \in J$. If $(0,0,\xi^0,\tau^0) \notin \mathrm{Char}(L)$, then $(0,0,\xi^0,\tau^0) \notin WF_s(h)$, i.e., $WF_s(h)_{|_0} \subset (\mathrm{Char}(L))_{|_0}$.*

6 Application

Let $s > 1$ be a real number and let $\Omega = U \times J \subset \mathbb{R}^{m+1}$ be an open neighborhood of the origin in $\mathbb{R}^m \times \mathbb{R}$. Suppose that $u \in C^2(\Omega)$ is a solution of

$$u_t = f(x,t,u,u_x), \tag{6.1}$$

where $f(x,t,\zeta_0,\zeta)$ is a complex-valued function defined for $(x,t) \in \Omega$ and $(\zeta_0,\zeta) \in \mathcal{N} \subset \mathbb{C} \times \mathbb{C}^m$, with $(a,w) = (u(0,0), u_x(0,0)) \in \mathcal{N}$ and $\mathcal{N}$ is an open set. Let $V \subset\subset \mathcal{N}$ be an open subset. We assume that $f(x,t,\zeta_0,\zeta) \in E^s(\Omega, V)$.

Consider

$$\mathcal{L} = \frac{\partial}{\partial t} - \sum_{j=1}^{m} \frac{\partial f}{\partial \zeta_j}(x,t,\zeta_0,\zeta)\frac{\partial}{\partial x_j} \text{ and } L^u = \frac{\partial}{\partial t} - \sum_{j=1}^{m} \frac{\partial f}{\partial \zeta_j}(x,t,u,u_x)\frac{\partial}{\partial x_j}$$

where L^u is the linearized operator.

Notice that the coefficients of L^u are in $C^1(\Omega)$. Our goal in this section is to use the results obtained in the previous sections in order to prove that $WF_s(u) \subset \mathrm{Char}(L^u)$.

Let $v = (u, u_x)$. It follows as in [As] that

$$L^u v = g(x,t,v), \text{ where } g = (g_0, g_1, \ldots, g_m) \tag{6.2}$$

is given by

$$g_0(x,t,\zeta_0,\zeta) = f(x,t,\zeta_0,\zeta) - \sum_{j=1}^{m} \zeta_j \frac{\partial f}{\partial \zeta_j}(x,t,\zeta_0,\zeta)$$

$$g_i(x,t,\zeta_0,\zeta) = f_{x_i}(x,t,\zeta_0,\zeta) + \zeta_i \frac{\partial f}{\partial \zeta_0}(x,t,\zeta_0,\zeta), \ i = 1, \ldots, m.$$

Consider now the principal part of the holomorphic Hamiltonian of (6.2)

$$H = \mathcal{L} + g_0 \frac{\partial}{\partial \zeta_0} + \sum_{j=1}^{m} g_j \frac{\partial}{\partial \zeta_j}.$$

It follows from the equation (6.2), as in [As], that

$$(H\Phi)^v = \mathcal{L}^v \Phi^v = L^u \Phi^v, \tag{6.3}$$

where $\Phi(x,t,\zeta_0,\zeta) \in E^s(\Omega,V)$, with the notation $\Phi^v(x,t) = \Phi(x,t,v(x,t))$ and $\mathcal{L}^v$ the vector field obtained from $\mathcal{L}$ when we replace (ζ_0,ζ) by $v(x,t)$ in its coefficients.

Since the coefficients of H are in $E^s(\Omega,V)$, it follows from Proposition 4.3 that there exist functions $Z_j(x,t,\zeta_0,\zeta)$, $j = 1,\cdots,m$ and $\Xi_k(x,t,\zeta_0,\zeta)$, $k = 0,\cdots,m$, that are $E^s([U\times(-1,1)],V)$ and are s-approximate solutions of $Hw = 0$. Furthermore, these functions satisfy $Z_j(x,0,\zeta_0,\zeta) = x_j$, and $\Xi_k(x,0,\zeta_0,\zeta) = \zeta_k$.

It is easy to see that we can write $Z_j(x,t,\zeta_0,\zeta) = x_j + t\Psi_j(x,t,\zeta_0,\zeta)$, $j = 1,\cdots,m$ and therefore we have $Z_j^v(x,t) = Z_j(x,t,u,u_x) = x_j + t\Psi_j^v(x,t)$, where $\Psi_j^v \in C^1(\Omega)$, shrinking Ω if it is necessary. Therefore, setting $Z = (Z_1,\ldots,Z_m)$ and $\Xi = (\Xi_0,\ldots,\Xi_m)$ we have $Z^v(x,t) = (Z_1^v(x,t),\cdots,Z_m^v(x,t))$, $Z^v(x,0) = x$ and $\Xi^v(x,t) = (\Xi_0^v(x,t),\cdots,\Xi_m^v(x,t))$, $\Xi^v(x,0) = v(x,0)$.

We now are going to construct an s-approximate solution $h \in C^1(\Omega)$ of $L^u w = 0$ such that $h(x,0) = u(x,0)$. Since Z_j, $j = 1,\cdots,m$ and Ξ_k, $k = 0,\cdots,m$ are G^s in the variable $x \in U$ it follows as in Lemma 3.4 that there exist functions $\tilde{Z}(z,t,\zeta_0,\zeta)$ and $\tilde{\Xi}(z,t,\zeta_0,\zeta)$ ($z = x+iy \in U+i(-1,1)^m$) almost analytic extensions with exponent s of $Z(x,t,\zeta_0,\zeta)$ and $\Xi(x,t,\zeta_0,\zeta)$, i.e., $\tilde{Z}(x,t,\zeta_0,\zeta) = Z(x,t,\zeta_0,\zeta)$, $\tilde{\Xi}(x,t,\zeta_0,\zeta) = \Xi(x,t,\zeta_0,\zeta)$ and there exists a positive constant C such that for $\tilde{W} = \tilde{Z}$ or $\tilde{W} = \tilde{\Xi}$ we have

$$\left|\frac{\partial}{\partial \bar{z}_j}\tilde{W}(z,t,\zeta_0,\zeta)\right| \le C^{\nu+1}(\nu!)^{s-1}|\operatorname{Im} z|^{\nu} \tag{6.4}$$

for all $\nu \in \mathbb{Z}_+$, $(x,t) \in \Omega$, $(\zeta_0,\zeta) \in V$, and $\operatorname{Im} z \in (-1,1)^m$.

Thanks to the fact that $\tilde{Z}(x,0,\zeta_0,\zeta) = Z(x,0,\zeta_0,\zeta) = x$ and $\tilde{\Xi}(x,0,\zeta_0,\zeta) = \Xi(x,0,\zeta_0,\zeta) = (\zeta_0,\zeta)$ we may solve the system

$$\tilde{Z}(z,t,\zeta_0,\zeta) = \tilde{Z},\ \tilde{\Xi}(z,t,\zeta_0,\zeta) = \tilde{\Xi}$$

with respect to (z,ζ_0,ζ) in a neighborhood of $(0,a,w)$ and obtain $z = P(\tilde{Z},t,\tilde{\Xi})$, $(\zeta_0,\zeta) = Q(\tilde{Z},t,\tilde{\Xi})$ with $P(0,0,a,w) = 0$ and $Q(0,0,a,w) = (a,w)$.

As in [As] we have

$$\begin{aligned}
&\frac{\partial(\tilde{Z},\tilde{\Xi})}{\partial(z,\zeta_0,\zeta)}(P(\tilde{Z},t,\tilde{\Xi}),t,Q(\tilde{Z},t,\tilde{\Xi}))\frac{\partial(P,Q)}{\partial\bar{\tilde{Z}}}(\tilde{Z},t,\tilde{\Xi})\\
&+\frac{\partial(\tilde{Z},\tilde{\Xi})}{\partial(\bar{z},\bar{\zeta}_0,\bar{\zeta})}(P(\tilde{Z},t,\tilde{\Xi}),t,Q(\tilde{Z},t,\tilde{\Xi}))\frac{\partial(\bar{P},\bar{Q})}{\partial\bar{\tilde{Z}}}(\tilde{Z},t,\tilde{\Xi}) = 0.
\end{aligned} \tag{6.5}$$

Thus, thanks to (6.4), (6.5) and the fact that $\tilde{Z}$ and $\tilde{\Xi}$ are holomorphic in the variables (ζ_0,ζ) , if $B(\tilde{Z},t,\tilde{\Xi})$ denotes a generic entry of the matrix $\frac{\partial(P,Q)}{\partial\bar{\tilde{Z}}}(\tilde{Z},t,\tilde{\Xi})$ then, there exists $C>0$, independent of ν, such that

$$|B(\tilde{Z},t,\tilde{\Xi})| \le (C)^{\nu+1}(\nu!)^{s-1}|\operatorname{Im}\ P(\tilde{Z},t,\tilde{\Xi})|^{\nu},\ \text{for any } \nu\in\mathbb{Z}_+. \tag{6.6}$$

In particular, for any $\nu \in \mathbb{Z}_+$, there exists a constant $C>0$, independent of ν, such that

$$\left|\frac{\partial Q_0}{\partial\bar{\tilde{Z}}_j}(\tilde{Z},t,\tilde{\Xi})\right| \le (C)^{\nu+1}(\nu!)^{s-1}|\operatorname{Im}\ P(\tilde{Z},t,\tilde{\Xi})|^{\nu},\ j = 1,\ldots,m \tag{6.7}$$

We now define

$$\Theta(z,t,\zeta_0,\zeta) = Q_0(\tilde{Z}(z,t,\zeta_0,\zeta), 0, \tilde{\Xi}(z,t,\zeta_0,\zeta))$$

and we notice that $\Theta^v(x,0) = u(x,0)$, since $Q_0(\tilde{Z}(z,t,\zeta_0,\zeta), t, \tilde{\Xi}(z,t,\zeta_0,\zeta)) = \zeta_0$.

We are going to show that Θ is an s-approximate solution of $Hw = 0$. We have

$$H\Theta = \sum_{j=1}^{m}\left(\frac{\partial Q_0}{\partial \tilde{Z}_j} H\tilde{Z}_j + \frac{\partial Q_0}{\partial \bar{\tilde{Z}}_j} H\bar{\tilde{Z}}_j\right) + \sum_{k=0}^{m}\left(\frac{\partial Q_0}{\partial \tilde{\Xi}_k} H\tilde{\Xi}_k + \frac{\partial Q_0}{\partial \bar{\tilde{\Xi}}_k} H\bar{\tilde{\Xi}}_k\right). \tag{6.8}$$

We notice that H has no derivation on $y = \operatorname{Im} z$ and therefore we have

$$(H\tilde{Z})(x,t,\zeta_0,\zeta) = H[\tilde{Z}(x,t,\zeta_0,\zeta)] = H[Z(x,t,\zeta_0,\zeta)],$$

and

$$(H\tilde{\Xi})(x,t,\zeta_0,\zeta) = H[\tilde{\Xi}(x,t,\zeta_0,\zeta)] = H[\Xi(x,t,\zeta_0,\zeta)].$$

This implies that $(H\tilde{Z})(x,t,\zeta_0,\zeta)$ and $(H\tilde{\Xi})(x,t,\zeta_0,\zeta)$ satisfy the inequality (4.8) with Lu replaced by $H\tilde{Z}$ or $H\tilde{\Xi}$, since $Z(x,t,\zeta_0,\zeta)$ and $\Xi(x,t,\zeta_0,\zeta)$ are s-approximate solution of $Hw = 0$. Therefore, in order to prove that Θ is an s-approximate solution of $Hw = 0$ it suffices to analyze the terms $\dfrac{\partial Q_0}{\partial \bar{\tilde{Z}}_j}$ and $\dfrac{\partial Q_0}{\partial \bar{\tilde{\Xi}}_k}$.

By using the mean value inequality it follows, as in [As], that there exists a positive constant C such that

$$|\operatorname{Im}\, P(\tilde{Z}(x,t,\zeta_0,\zeta), 0, \tilde{\Xi}(x,t,\zeta_0,\zeta)| \le C|t|, \tag{6.9}$$

$\forall\ (x,t) \in \Omega,\ (\zeta_0,\zeta) \in V$, shrinking Ω and V if it is necessary.

It follows from (6.7) and (6.9), that there exists a constant $C > 0$, independent of ν, such that

$$\left|\frac{\partial Q_0}{\partial \bar{\tilde{Z}}_j}(\tilde{Z}(x,t,\zeta_0,\zeta), 0, \tilde{\Xi}(x,t,\zeta_0,\zeta))\right| \le C^{\nu+1}(\nu!)^{s-1}|t|^{\nu}$$

for any $\nu \in \mathbb{Z}_+$. Analogously one can prove that for $\nu \in \mathbb{Z}_+$ there exists a constant $C > 0$, independent of ν, such that

$$\left|\frac{\partial Q_0}{\partial \bar{\tilde{\Xi}}_k}(\tilde{Z}(x,t,\zeta_0,\zeta), 0, \tilde{\Xi}(x,t,\zeta_0,\zeta))\right| \le C^{\nu+1}(\nu!)^{s-1}|t|^{\nu}.$$

Summing up, we have proved that $\Theta(x,t,\zeta_0,\zeta)$ is an s-approximate solution of $Hw = 0$.

Since $L^u\Theta^v = \mathcal{L}^v\Theta^v = (H\Theta)^v$ (see (6.3)) it follows that Θ^v is an s-approximate solution of $L^u w = 0$ and therefore, defining $h(x,t) = \Theta^v(x,t)$, we have constructed a C^1 function h which is an s-approximate solution of $L^u w = 0$. Furthermore, we have $h(x,0) = u(x,0)$.

Here we can apply Lemma 5.1 and conclude that if $\xi^0 \cdot \operatorname{Im}\,\Psi(0,0) < 0$, then $(0,\xi^0) \notin WF_s(u(x,0))$, but it is not enough for what we want. In order to reach our main goal we will use Lemma 5.3.

We start by setting

$$\tilde{u}(x,t,r) = u(x,t),$$

where r is a new real variable. Since u is a solution of the equation $u_t = f(x,t,u,u_x)$, it follows that $\tilde{u}$ is a solution of the equation

$$\tilde{u}_r = f^\theta(x,t,r,\tilde{u},\tilde{u}_x,\tilde{u}_t)$$

where $f^\theta(x,t,r,\zeta_0,\zeta,\eta) = e^{-i\theta}(\eta - f(x,t,\zeta_0,\zeta))$, and $\theta \in [0,2\pi)$.

Note that this last equation is of the same kind as (6.1) with r replacing t and (x,t) replacing x. The associated vector field $\mathcal{L}^\theta$ to the above equation is given by $\mathcal{L}^\theta = \frac{\partial}{\partial r} - e^{-i\theta}\mathcal{L}$, with $\mathcal{L}$ as before.

We also notice that $\left(\mathcal{L}^\theta\right)^v = \frac{\partial}{\partial r} - e^{-i\theta}\mathcal{L}^v = \frac{\partial}{\partial r} - e^{-i\theta}L^u = (L^u)^\theta$.

Thus, repeating our arguments with $\tilde{u}$ replacing u, (x,t) replacing x and r replacing t we conclude that there exists a C^1 function $h^\theta(x,t,r)$ such that h^θ is an s-approximate solution of $(L^u)^\theta w = 0$ and $h(x,t,0) = \tilde{u}(x,t,0) = u(x,t)$. We apply Lemma 5.3 and conclude that $WF_s(u)|_0 \subset \mathrm{Char}(L^u)|_0$.

By translation we may apply the same argument for all points of Ω and state

Theorem 6.1 *Let $u \in C^2(\Omega)$ be a solution of* (6.1)*. Then the s-wave-front set of u is contained in the characteristic set of the linearized operator L^u.*

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References

[AH] Z. Adwan and G. Hoepfner, *A generalization of Borel's theorem and microlocal Gevrey regularity in involutive structures*, J. Diff. Equations, **245, No. 10** (2008), 2846–2870.

[As] C. H. Asano, *On the C^∞ wave-front set of solutions of first-order nonlinear PDE's*, Proc. Amer. Math. Soc., **123, No. 10** (1995), 3009–3019.

[B] S. Berhanu, *On microlocal analyticity of solutions of first-order nonlinear PDE*, Ann. Inst. Fourier, to appear.

[BCH] S. Berhanu, P. Cordaro and J. Hounie, An introduction to involutive structures, Cambridge University Press, 2008.

[Br] J. Bruna, *An extension of Whitney type for non quasi-analytic classes of functions*, J. London Math. Soc., **2** (1980), 495–505.

[Ca] T. Carleman, Léçons sur les fonctions quasi-analytiques, Paris, 1926.

[Ch] J. Y. Chemin, *Calcul paradifférentiel précisé et applications à des équations aux derivées partielles non semilinéaires*, Duke Math. J., **56** (1988), 431–469.

[Chr] M. Christ, *Intermediate optimal Gevrey exponents occur*, Comm. PDE, **22(3-4)** (1997), 359–379.

[Dz] G. A. Džanašija, *Carleman's problem for funtions of the Gevrey class*, Soviet Math. Dokl., **3** (1962), 969–972.

[HT] N. Hanges and F. Treves, *On the analyticity of solutions of first-order nonlinear PDE*, Trans. Amer. Math. Soc., **331** (1992), 627–638.

[LMX] N. Lerner, Y. Morimoto and C. -J. Xu, *Instability of the Cauchy-Kovalevskaya solution for a class of nonlinear system*, preprint.

[M] G. Metivier, *Remarks on the well-posedness of the nonlinear Cauchy problem*, Contemp. Math., **368** (2005), 337-356.

[Mi] B. S. Mityagin, *An indefinitely differentiable function with the values of its derivatives given at a point*, Soviet Math. Dokl., **2** (1961), 594–597.

[R] L. Rodino, Linear Partial Differential operators in Gevrey spaces, World Scientific, (1993).

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