HIGHER-ORDER FOR THE MULTIDIMENSIONAL GENERALIZED BBM-BURGERS EQUATION: EXISTENCE AND CONVERGENCE RESULTS

CEZAR I. KONDO[1] AND CLAUDETE M. WEBLER[2]

ABSTRACT. We study the global existence of solutions for the multidimensional generalized BBM-Burgers equations of the form

$$u_t + \sum_{j=1}^{d} f_j(u)_{x_j} = \delta \sum_{j=1}^{d} u_{x_j x_j t} + \sum_{j=1}^{d} \left(\sum_{n=1}^{N} (-1)^{n+1} \gamma_n \partial_{x_j}^{2n} u \right),$$

for $(x,t) \in \mathbb{R}^d \times \mathbb{R}_+$, with initial data $u(x,0) = u_0(x)$, $x \in \mathbb{R}^d$, as $\alpha > 0$, $\gamma_n > 0$, $n = 1, ..., N$ aproach zero, and f is a sufficiently smooth function. We also deal with the convergence of solutions of this Cauchy problem, and the proofs are based instead on DiPerna's uniqueness theory for entropy measure-valued solutions.

1. Introduction

We study the existence and convergence of the smooth solutions $\{u(x,t;\delta,\gamma_1,...,\gamma_n)\}$ to the escalar multidimensional generalized BBM-Burgers equations of the form

$$u_t + \sum_{j=1}^{d} f_j(u)_{x_j} = \delta \sum_{j=1}^{d} u_{x_j x_j t} + \sum_{j=1}^{d} \left(\sum_{n=1}^{N} (-1)^{n+1} \gamma_n \partial_{x_j}^{2n} u \right), \qquad (x,t) \in \mathbb{R}^d \times \mathbb{R}_+ \tag{1}$$

with initial data

$$u(x,0) = u_0(x), \qquad x \in \mathbb{R}^d. \tag{2}$$

as $\delta > 0$ and $\gamma_n > 0$, $n = 1, ..., N$ aproach zero. Here we assume that the flux function f, is a sufficiently smooth function satisfying certain assumptions to be listed in Section 2 and in Section 3.

The model under study is motived by physical considerations from fluid dynamics.The equations of type (1) are related to the well known BBM equations which were advocated by Benjamin-Bona-Mahony [2] as a refinement of the KdV equation [2], [12], and [1]. The KdV equation was originaly derived for water waves and it is similary justifiable as a model for long waves in many other physical systems. It has been used to account adequately for observable phenomena such as the interaction of solitary waves and dissipationless, undular shocks. The BBM equation equation [2] is useful in that it describes

2000 *Mathematics Subject Classification.* Primary: 35L65. Secondary: 76N10.

Key words and phrases. existence and convergence of the smooth solutions, partial differential equations, entropy measure-valued solutions, hyperbolic conservation law.

approximately the unidirectional propagation of long waves in certain nonlinear dispersive systems. In [9] we studied the existence and convergence of the smooth solutions $\{u(x,t;\delta,\gamma_1,...,\gamma_N)\}$ for partial differential equation of the form

$$u_t + f(u)_x = \delta u_{xxt} + \sum_{n=1}^{N} (-1)^{n+1} \gamma_n \partial_x^{2n} u, \qquad (x,t) \in \mathbb{R} \times \mathbb{R}_+. \tag{3}$$

with initial data

$$u(x,0) = u_0(x), \qquad x \in \mathbb{R}. \tag{4}$$

as $\delta > 0$ and $\gamma_n > 0$, $n = 1,...,N$ aproach zero. We assumed that the flux function is a given sufficiently smooth function $f : \mathbb{R} \to \mathbb{R}$. We considered solutions of hyperbolic conservation laws regularized of this equations. Following a pioneering work by Schonbek and a work by LeFloch and Natalini, we established the convergence of the regularized solutions toward discontinuous solutions of the hyperbolic conservation law. Since the viscous term $\gamma_1 u_{xx}$ and the dissipative term $\gamma_2 u_{xxxx}$ (case $N = 2$) are of physical backgrounds [2] and [3] and, as pointed out in [3] and [13], the convergences of the solution sequences $\{u(x,t;\delta,\gamma_1,\gamma_2)\}$ as $\delta \to 0$, $\gamma_1 \to 0$, and $\gamma_2 \to 0$ correspond to some physical processes, such as vanishing viscosity, etc. To get the strong convergence result, for multidimensional equations, the proofs are based on DiPerna's uniqueness theory for entropy measure-valued solutions. See [5], [14], [7], [8], and [4].

The case $N = 2$, was studied by Hwang [6]. First the author obtained some a priori bounds on the solution and then established its strong convergence toward the entropy solution of the associated conservation law. The basic strategy was to use entropy inequalities to control the limiting behavior of the solution.

In [8], following a pionering work by Schonbek, Kondo and LeFloch studied the convergence of the solutions of hyperbolic conservation laws regularized with vanishing diffusion and dispersion terms of the form

$$u_t + f(u)_x = \epsilon u_{xx} + \delta u_{xxx}, \qquad u = u(x,t), \qquad x \in \mathbb{R} \times \mathbb{R}_+. \tag{5}$$

A convergence result is also established for multidimensional conservation laws by relying on Diperna's uniqueness theorem for entropy measure-valued solutions. In [4], Correia and LeFloch obtained convergence results for a classe of multidimensional conservation with vanishing nonlinear diffusion and dispersion terms of the form

$$u_t + div f(u) = div(\epsilon b_j(\nabla u) + \delta \partial_{x_j}^2 u)_{1 \le j \le d}, \qquad x \in \mathbb{R}^d \times \mathbb{R}_+. \tag{6}$$

We observe that the equations (5) and (6) are different from the equation (3) because they have the dispersive term u_{xxx} and have not the term u_{xxt}. The two papers ([8] and [4]) address the convergence of approximations conservation laws in the framework of measure-valued solutions. This technique was used in [9] in the prove from one the convergence results.

The remainder of this paper is divided into four sections. After this introduction, which constitutes Section 1, we consider in Section 2 the preliminaries, we consider in section 3 the global existence results, and the convergence results are stated in Section 4.

2. Preliminaries

This section contains short background material on $H^s(\mathbb{R}^n)$, Young measures, and entropy measure-valued (m.-v.) solutions.

The following result is a consequence from Theorem 4.7 (page 30) of [11]:

Theorem 1. *Suppose that $G = G(w)$ is sufficiently smooth. If functions $\overline{w} = \overline{w}(x)$ and $\overline{\overline{w}} = \overline{\overline{w}}(x)$ satisfy $\|w\|_\infty \leq N$ (N is a positive constant), and $\overline{w}$, $\overline{\overline{w}} \in H^s(\mathbb{R}^n)$ with $s \geq \left[\frac{n}{2}\right] + 1$ then for*

$$w^* = \overline{w} - \overline{\overline{w}}$$

we have

$$\|G(\overline{w}) - G(\overline{\overline{w}})\|_{H^s(\mathbb{R}^n)} \leq C_s \|w^*\|_{H^s(\mathbb{R}^n)}(|G'(0)| + \|\overline{w}\|_{H^s(\mathbb{R}^n)} + \|\overline{\overline{w}}\|_{H^s(\mathbb{R}^n)})$$

where C_s is a positive constant depending on N and on s. □

The following result is the Theorem 4.3 on [11]:

Theorem 2. *Suppose that $F = F(w)$ is a sufficiently smooth function of $w = (w_1, ..., w_n)$ such that*

$$F(0) = 0. \tag{7}$$

For any given integer $s \geq 0$, if a vector $w = w(x,t)$ satisfies

$$w = w(x) \in W^{s,p}(\mathbb{R}^n), \qquad 1 \leq p \leq +\infty \tag{8}$$

e

$$\|w\|_{L^\infty(\mathbb{R}^n)} \leq M, \tag{9}$$

where M is a positive constant, then the composite function

$$F(w) \in W^{s,p}(\mathbb{R}^n) \tag{10}$$

e

$$|F(w)\|_{W^{s,p}(\mathbb{R}^n)} \leq C(M)|w\|_{W^{s,p}(\mathbb{R}^n)}, \tag{11}$$

where $C(M)$ is a positive constant depenging on M.

In [13], we have a representation theorem for the Young measures associated with a given sequence of uniformly bounded functions on L^p. The corresponding setting in L^∞ was first established by Tartar [15].

Lemma 3. *Let $\{u_j\}$ be a uniformly bounded sequence in $L^\infty(\mathbb{R}_+; L^p(\mathbb{R}^d))$, that is,*

$$\|u_j(t)\|_{L^p(\mathbb{R}^d)} \leq k, \qquad for \qquad a.e. \qquad t \in \mathbb{R}_+. \tag{12}$$

Then there exists a subsequence $\{u_{j'}\}$ and a measurable measure valued mapping $\nu : \mathbb{R}^d \times \mathbb{R}_+ \to Prob(\mathbb{R})$ (probability measures) such that, for all functions $g \in C(\mathbb{R})$ satisfying

$$g(u) = o(1 + |u|^r) \;\; as \;\; |u| \to \infty \tag{13}$$

for some $r \in [0,p)$, the following limit representation holds

(14)
$$\lim_{j'\to\infty}\int\int_{\mathbb{R}^d\times\mathbb{R}_+} g(u_{j'}(x,t))\phi(x,t)dxdt = \int\int_{\mathbb{R}^d\times\mathbb{R}_+}\int_{\mathbb{R}} g(\lambda)d\nu_{(x,t)}(\lambda)\phi(x,t)dxdt$$

for all $\phi \in C_c^\infty(\mathbb{R}^d\times\mathbb{R}_+)$. □

The measure-valued function $\nu_{(x,t)}$ is a Young measure associated with the sequence $\{u_j\}$.

Following DiPerna [5] and Szepessy [14], we now define the measure-valued (m.-v.) solution to the problem

$$u_t + \sum_{j=1}^d f_j(u)_{x_j} = 0 \qquad (x,t)\in\mathbb{R}^d\times\mathbb{R}_+, \tag{15}$$

$$u(x,0) = u_0(x) \qquad x\in\mathbb{R}^d. \tag{16}$$

Definition 4. Let $f \in [C(\mathbb{R})]^d$ satisfy

$$f(\lambda) = o(1+|\lambda|^r) \qquad for \qquad some \qquad r, \qquad 0\le r<p, \tag{17}$$

$$\limsup_{|\lambda|\to 0}\frac{|f(\lambda)-f(0)|}{|\lambda|^\alpha} < \infty \qquad for \qquad some \qquad \alpha, \qquad \frac{d-1}{d} < \alpha \le 1 \tag{18}$$

and

$$u_0 \in L^1(\mathbb{R}^d)\cap L^p(\mathbb{R}^d). \tag{19}$$

A Young measure ν associated to a sequence satisfying (12) is then called a mv entropy solution of (15) and (16) if

$$\partial_t < \nu(.), |\lambda-k| > + div_x < \nu(.), sgn(\lambda-k)(f(\lambda)-f(k)) > \le 0 \tag{20}$$

in $\mathfrak{D}'(\mathbb{R}^d\times\mathbb{R}_+)$ for all $k\in\mathbb{R}$ (i.e. in distribution sense), and if for all compact sets $K\subset\mathbb{R}^d$

$$\lim_{T\to 0^+}\frac{1}{T}\int_0^T\int_K < \nu_{(x,t)}, |\lambda-u_0(x)| > dxdt = 0. \tag{21}$$

Analogously a function $u\in L^\infty(\mathbb{R}_+, L^1(\mathbb{R}^d))\cap L^\infty(\mathbb{R}_+, L^p(\mathbb{R}^d))$ is called a L^P entropy solution of (15) and (16) if $\delta_{u(.)}$ is a mv entropy solution.

In [14], we have the following uniqueness and existence results of entropy solution of (15) and (16).

Theorem 5. *(Uniqueness) Suppose that ν and σ are mv entropy solutions of (15) and (16), then there exists a function*

$$w\in L^\infty(\mathbb{R}_+, L^1(\mathbb{R}^d))\cap L^\infty(\mathbb{R}_+, L^p(\mathbb{R}^d)) \tag{22}$$

such that

$$\nu_y = \sigma_y = \delta_{w(y)} \qquad for \qquad a.e \qquad y\in\mathbb{R}^d\times\mathbb{R}_+, \tag{23}$$

i.e. $\nu(.)$ and $\sigma(.)$ are the Dirac measure concentrated at $w(.)$. □

Lemma 6. *Suppose that ν is a Young measure associated to a sequence $\{u_j\}$ satisfying (12). Then*

$$u_j \to u \qquad in \qquad L^r_{loc}(\mathbb{R}^d \times \mathbb{R}_+), \qquad 1 \le r < p \tag{24}$$

if and only if

$$\nu_y = \delta_{u(y)} \qquad a.e.. \tag{25}$$

□

Theorem 7. *(Existence) Let f and u_0 satisfy the assumptions in Definition 4. Then exists a unique entropy solution $u \in L^\infty(\mathbb{R}_+, L^1(\mathbb{R}^d)) \cap L^\infty(\mathbb{R}_+, L^p(\mathbb{R}^d))$ of (15) and (16), which, moreover, satisfies*

$$\|u(t)\|_{L^r(\mathbb{R}^d)} \le \|u_0\|_{L^r(\mathbb{R}^d)},$$

for almost every $t \in \mathbb{R}_+$ and $1 \le r \le p$. The measure-valued mapping $\nu_{(x,t)} = \delta_{u(x,t)}$ is the unique entropy m.-v. solution of the same problem. □

3. Global Existence of Solutions

In this section, first we study the existence of global smooth solutions to the Cauchy problem (1)-(2). Where δ and γ_n, $n = 1, ..., N$ are positive constants, f is a sufficiently smooth function.

Denote $F(u)$ the Fourier transform of u with respect to the variable x, F^{-1} is the inverse transform of F. As a routine matter, the solutions of the Cauchy problem (1)-(2) have the following integral representation

$$u(x,t) = G(t)u_0 - \sum_{j=1}^{d} \int_0^t G(t-s)F^{-1}\left(\frac{F(f_j(u)_{x_j})}{(1+\delta|\xi|^2)}\right) ds \tag{26}$$

where $G(t)u = F^{-1}\left(\exp\left\{\frac{-[\sum_{j=1}^d(\sum_{n=1}^N \gamma_n \xi_j^{2n})]t}{(1+\delta|\xi|^2)}\right\} F(u)\right)$. The family of linear operators $\{G(t)\}_{t\ge 0}$ satisfies the properties of semigroup.

In the following lemma we give some estimates which will be used in this section:

Lemma 8. *For $\theta > 0$, $\delta > 0$ and, $\gamma_n > 0$, $n = 1, ..., N$, we have the following inequalities:*

i) $\dfrac{\xi_j^{2n}}{(1+\delta|\xi|^2)^2} \exp\left\{\dfrac{-2\sum_{j=1}^d(\sum_{n=1}^N \gamma_n \xi_j^{2n})\theta}{(1+\delta|\xi|^2)}\right\} \le \delta^{-2}$, *if $\delta \le 4$, $n = 1, 2$ and $j = 1, ..., d$;*

ii) $\dfrac{\xi_j^{2}}{(1+\delta|\xi|^2)^2} \exp\left\{\dfrac{-2\sum_{j=1}^d(\sum_{n=1}^N \gamma_n \xi_j^{2n})\theta}{(1+\delta|\xi|^2)}\right\} \le (2\gamma_1 e\theta)^{-1}$, *$j = 1, ..., d$;*

iii) $|\xi|^2 \exp\left\{\dfrac{-2\sum_{j=1}^d(\sum_{n=1}^N \gamma_n \xi_j^{2n})\theta}{(1+\delta|\xi|^2)}\right\} \le \dfrac{(\gamma_2 + 2^{d-1}\delta\gamma_1)}{2\gamma_1\gamma_2}$;

iv) $\dfrac{\xi_j^{2}|\xi|^2}{(1+\delta|\xi|^2)^2} \exp\left\{\dfrac{-2\sum_{j=1}^d(\sum_{n=1}^N \gamma_n \xi_j^{2n})\theta}{(1+\delta|\xi|^2)}\right\} \le (2\delta\gamma_1 e\theta)^{-1}$.

Proof. The proof follows from simple computations. □

To begin, define the operator

$$\mathcal{L}u(t) = G(t)u_0 - \sum_{j=1}^{d}\int_0^t G(t-s)F^{-1}\left(\frac{F(f_j(u)_{x_j})}{(1+\delta|\xi|^2)}\right)ds$$

on

$\mathcal{A}_T = \Big\{u \in C([0,T]; H^{\left[\frac{d}{2}\right]+1}(\mathbb{R}^d)); \|u(t) - G(t)u_0\|_{H^{\left[\frac{d}{2}\right]+1}(\mathbb{R}^d)} \le \|u_0\|_{H^{\left[\frac{d}{2}\right]+1}(\mathbb{R}^d)},$
$t \in [0,T]\Big\}$ and the norm in $\mathcal{A}_T$ by $\|u(x,t)\|_{\mathcal{A}_T} = \sup_{0\le t\le T}\|u(t)\|_{H^{\left[\frac{d}{2}\right]+1}(\mathbb{R}^d)}$.

Our local existence result will follow from the properties of $\mathcal{L}$ given in the following lemma:

Lemma 9. *Suppose that f_j, $j = 1, ..., d$, are sufficiently smooth funcions. Assume that $u(t), u_0 \in H^{\left[\frac{d}{2}\right]+1}(\mathbb{R}^d)$ and that*

$$\|u(t) - G(t)u_0\|_{H^{\left[\frac{d}{2}\right]+1}(\mathbb{R}^d)} \le \|u_0\|_{H^{\left[\frac{d}{2}\right]+1}(\mathbb{R}^d)}, \qquad \forall t \in [0,T]. \tag{27}$$

If $T > 0$ is sufficiently small, then the following hold:
(i) $\mathcal{L}u(t) \in H^{\left[\frac{d}{2}\right]+1}(\mathbb{R}^d)$ with

$$\|\mathcal{L}u(t) - G(t)u_0\|_{H^{\left[\frac{d}{2}\right]+1}(\mathbb{R}^d)} \le \|u_0\|_{H^{\left[\frac{d}{2}\right]+1}(\mathbb{R}^d)}, \qquad \forall t \in [0,T]$$

and

$$\|\mathcal{L}u(t)\|_{H^{\left[\frac{d}{2}\right]+1}(\mathbb{R}^d)} \le 2\|u_0\|_{H^{\left[\frac{d}{2}\right]+1}(\mathbb{R}^d)}, \qquad \forall t \in [0,T];$$

(ii) $\|\mathcal{L}u(t)\|_\infty \le C\|u_0\|_{H^{\left[\frac{d}{2}\right]+1}(\mathbb{R}^d)}$;
(iii) $\mathcal{L}$ maps $\mathcal{A}_T$ into itself;
iv) $\mathcal{L}$ is a contraction on $\mathcal{A}_T$.

Proof. Without loss of generallity, we take $f_j(0) = 0$, $j = 1, ..., d$.
(i) Let $u(t) \in H^{\left[\frac{d}{2}\right]+1}(\mathbb{R}^d)$ satisfying (27), then using the properties of Fourier transform on $L^2(\mathbb{R}^d)$, and the Parseval's equality we have

$$\|\mathcal{L}u(t) - G(t)u_0\|_{H^{\left[\frac{d}{2}\right]+1}(\mathbb{R}^d)} \le$$

$$\sum_{j=1}^{d}\int_0^t \left\{\sum_{|\alpha|\le\left[\frac{d}{2}\right]+1}\int_{\mathbb{R}^d}\frac{\exp\left\{\frac{-2[\sum_{j=1}^{d}(\sum_{n=1}^{N}\gamma_n\xi_j^{2n})](t-s)}{(1+\delta|\xi|^2)}\right\}\xi_j^2|(i\xi)^\alpha F(f_j(u))|^2}{(1+\delta|\xi|^2)^2}d\xi\right\}^{\frac{1}{2}} ds$$

using the Lemma 8 (ii) and the Parseval's equality

$$\le \sum_{j=1}^{d}\int_0^t \frac{1}{[2\gamma_1 e(t-s)]^{\frac{1}{2}}}\left\{\sum_{|\alpha|\le\left[\frac{d}{2}\right]+1}\int_{\mathbb{R}^d}|(i\xi)^\alpha F(f_j(u))|^2 d\xi\right\}^{\frac{1}{2}} ds$$

$$= \sum_{j=1}^{d}\int_0^t \frac{\|f_j(u(s))\|_{H^{\left[\frac{d}{2}\right]+1}(\mathbb{R}^d)}}{[2\gamma_1 e(t-s)]^{\frac{1}{2}}}ds$$

using the Theorem 4.3 on [11] (page 22)

$$\leq \sum_{j=1}^{d} \int_0^t \frac{C\|u(s)\|_{H^{[\frac{d}{2}]+1}(\mathbb{R}^d)}}{[2\gamma_1 e(t-s)]^{\frac{1}{2}}} ds$$
$$\leq \|u_0\|_{H^{[\frac{d}{2}]+1}(\mathbb{R}^d)}$$

if

$$T \leq C_1\gamma_1$$

where C_1 is a positive constant depending on $\|u_0\|_{H^{[\frac{d}{2}]+1}(\mathbb{R}^d)}$.

(ii) The estimate (ii) is the consequence from (i) and for $s \geq \left[\frac{n}{2}\right]+1$ we have the continuous inclusion $H^{[\frac{d}{2}]+1}(\mathbb{R}^d) \subset L^\infty(\mathbb{R}^d)$.

(iii) To proof (iii), we only need to show if $u \in C([0,T]; H^{[\frac{d}{2}]+1}(\mathbb{R}^d))$ then $\mathcal{L}u \in C([0,T]; H^{[\frac{d}{2}]+1}(\mathbb{R}^d))$. Let $t_0 \in (0,T]$. Let $t \in (0,T]$. Without loss of generallity, we take $t_0 < t$. We have

$$\|\mathcal{L}u(t) - \mathcal{L}u(t_0)\|_{H^{[\frac{d}{2}]+1}(\mathbb{R}^d)} \leq \|G(t)u_0 - G(t_0)u_0\|_{H^{[\frac{d}{2}]+1}(\mathbb{R}^d)}$$
$$+ \sum_{j=1}^{d} \int_{t_0}^{t} \left\| G(r)F^{-1}\left(\frac{F(f_j(u(t-r))_{x_j})}{(1+\delta|\xi|^2)}\right)\right\|_{H^{[\frac{d}{2}]+1}(\mathbb{R}^d)} dr$$
$$+ \sum_{j=1}^{d} \int_{0}^{t_0} \left\| G(r)F^{-1}\left(\frac{F(f_j(u(t-r))_{x_j} - f_j(u(t_0-r))_{x_j})}{(1+\delta|\xi|^2)}\right)\right\|_{H^{[\frac{d}{2}]+1}(\mathbb{R}^d)} dr$$
$$= A + B + C.$$

For A, using Parseval's equality,

$$A \leq \frac{|t-t_0|\|u_0\|_{H^{[\frac{d}{2}]+1}(\mathbb{R}^d)}}{2t_0}.$$

To estimate B, we use the Parseval's equality, the Lemma 8 (i), and the Theorem 4.3 on [11]

$$B =$$
$$\int_{t_0}^{t} \left\{ \sum_{|\alpha|\leq[\frac{d}{2}]+1} \int_{\mathbb{R}^d} \frac{\exp\left\{\frac{-2[\sum_{j=1}^{d}(\sum_{n=1}^{N}\gamma_n\xi_j^{2n})]r}{(1+\delta|\xi|^2)^2}\right\}\xi_j^2|(i\xi)^\alpha F(f_j(u(t-r))|^2)}{(1+\delta|\xi|^2)} d\xi \right\}^{\frac{1}{2}} dr$$
$$\leq \int_{t_0}^{t} \delta^{-1}\|f_j(u(t-r))\|_{H^{[\frac{d}{2}]+1}(\mathbb{R}^d)} dr$$
$$\leq C|t-t_0|\delta^{-1},$$

where C depending on $\|u_0\|_{H^{[\frac{d}{2}]+1}(\mathbb{R}^d)}$.

Finally for C, we use the Parseval's equality, the Lemma 8 (i), we have

$$C \leq \delta^{-1} \int_0^{t_0} \|f_j(u(t-r)) - f_j(u(t_0-r))\|_{H^{[\frac{d}{2}]+1}(\mathbb{R}^d)} dr$$

and from Theorem 1

$$\leq C\delta^{-1}\int_0^{t_0}\|u(t-r)-u(t_0-r)\|_{H^{[\frac{d}{2}]+1}(\mathbb{R}^d)}dr$$
$$\leq \overline{C}$$

where $\overline{C}\to 0$ when $|t-t_0|\to 0$ because $u\in C([0,T];H^1(\mathbb{R}))$ and $u\in C([0,T];H^{[\frac{d}{2}]+1}(\mathbb{R}^d))$ and $t_0-r\in[0,T]$.

iv) Let $u,\, v\in\mathcal{A}_T$,

$$\|\mathcal{L}u(t)-\mathcal{L}v(t)\|_{H^{[\frac{d}{2}]+1}(\mathbb{R}^d)}\leq$$

$$\sum_{j=1}^d\int_0^t\left\{\sum_{|\alpha|\leq[\frac{d}{2}]+1}\int_{\mathbb{R}^d}\frac{\exp\left\{\frac{-2[\sum_{j=1}^d(\sum_{n=1}^N\gamma_n\xi_j^{2n})](t-s)}{(1+\delta|\xi|^2)^2}\right\}\xi_j^2|(i\xi)^\alpha F(f_j(u)-f_j(v))|^2}{(1+\delta|\xi|^2)}d\xi\right\}^{\frac{1}{2}}ds$$

$$\leq\sum_{j=1}^d\int_0^t\frac{\|f_j(u)-f_j(v)\|_{H^{[\frac{d}{2}]+1}(\mathbb{R}^d)}}{[2\gamma_1 e(t-s)]^{\frac{1}{2}}}ds$$

and from Theorem 1 we obtain

$$\leq\sum_{j=1}^d\int_0^t\frac{C\|u(s)-v(s)\|_{H^{[\frac{d}{2}]+1}(\mathbb{R}^d)}}{[2\gamma_1 e(t-s)]^{\frac{1}{2}}}ds$$
$$\leq\frac{\|u(s)-v(s)\|_{A_T}}{2}$$

if $T\leq C\gamma_1$, where C is a positive constant depending on $\|u_0\|_{H^{[\frac{d}{2}]+1}(\mathbb{R}^d)}$. □

We can now obtain the local existence of solutions of (1)-(2):

Theorem 10. *Suppose that u_0 and f_j, $j=1,...,d$, satisfy the same assumptions as in Lemma 9 then the Cauchy problem (1)-(2) admits a unique local smooth solution*

$$u\in C([0,T];H^{[\frac{d}{2}]+1}(\mathbb{R}^d)).$$

Furthermore, for each integer $k\geq\left[\frac{d}{2}\right]+1$, we have i) $u\in C((0,T];H^k(\mathbb{R}^d))$;
ii) $u\in C((0,T];H^k(\mathbb{R}^d))\cap C^1((0,T];H^{k-1}(\mathbb{R}^d))$ if $N\leq 3$;
iii) $u\in C((0,T];H^{k+N-3}(\mathbb{R}^d))\cap C^1((0,T];H^{k-1}(\mathbb{R}^d))$ if $N>3$,
where T depends on γ_1 and $\|u_0\|_{H^{[\frac{d}{2}]+1}(\mathbb{R}^d)}$.

Proof. Let $u^0\equiv 0$ and $u^n\equiv\mathcal{L}(u^{n-1})$. Then by induction the estimates of the Lemma 9 (i) and (ii) hold for each u^n. Furthermore, by Lemma 9 (iii) and (iv), $\mathcal{L}$ maps $\mathcal{A}_T$ onto itself and is contractive. From Banach's fixed point theorem, the integral equation (26) possesses a unique solution $u\in C([0,T];H^{[\frac{d}{2}]+1}(\mathbb{R}^d))$. To prove the regularity results, we only need to show if $u\in C((0,T];H^l(\mathbb{R}^d))$, $l\geq\left[\frac{d}{2}\right]+1$, then $u\in C((0,T];H^{l+1}(\mathbb{R}^d))\cap C^1((0,T];H^l(\mathbb{R}^d))$. Let $t_1\in(0,T)$, we only need to show that $u\in$

$C([t_1,T];H^{l+1}(\mathbb{R}^d))$ and $u_t \in C([t_1,T];H^l(\mathbb{R}^d))$. We take $t_2 = \frac{t_1}{2}$. Then the semigroup property of G implies that, for $t > t_2$

$$u(x,t) = G(t-t_2)u(t_2) - \sum_{j=1}^{d} \int_{t_2}^{t} G(t-s)F^{-1}\left[\frac{F(f_j(u)_{x_j})}{1+\delta|\xi|^2}\right] ds. \tag{28}$$

To begin, using the Parseval's equality and the Lemma 8 (iii) and (iv) we have

$$\begin{aligned}
&\|u(t)\|_{H^{l+1}(\mathbb{R}^d)} \leq \\
&\left\{\sum_{|\alpha|\leq l+1}\int_{\mathbb{R}^d} \exp\left\{\frac{-2[\sum_{j=1}^{d}(\sum_{n=1}^{N}\gamma_n\xi_j^{2n})](t-t_2)}{(1+\delta|\xi|^2)}\right\}|(i\xi)^\alpha F(u(t_2))|^2 d\xi\right\}^{\frac{1}{2}} \\
&+\sum_{j=1}^{d}\int_{t_2}^{t}\left\{\sum_{|\alpha|\leq l+1}\int_{\mathbb{R}^d}\frac{\exp\left\{\frac{-2[\sum_{j=1}^{d}(\sum_{n=1}^{N}\gamma_n\xi_j^{2n})](t-s)}{(1+\delta|\xi|^2)}\right\}\xi_j^2|(i\xi)^\alpha F(f_j(u))|^2}{(1+\delta|\xi|^2)^2}d\xi\right\}^{\frac{1}{2}} ds \\
&\leq \left\{\sum_{|\alpha|\leq l}\int_{\mathbb{R}^d} \exp\left\{\frac{-2[\sum_{j=1}^{d}(\sum_{n=1}^{N}\gamma_n\xi_j^{2n})](t-t_2)}{(1+\delta|\xi|^2)}\right\}|\xi|^2|(i\xi)^\alpha F(u(t_2))|^2 d\xi\right\}^{\frac{1}{2}} \\
&+\sum_{j=1}^{d}\int_{t_2}^{t}\left\{\sum_{|\alpha|\leq l}\int_{\mathbb{R}^d}\frac{\exp\left\{\frac{-2[\sum_{j=1}^{d}(\sum_{n=1}^{N}\gamma_n\xi_j^{2n})](t-s)}{(1+\delta|\xi|^2)}\right\}\xi_j^2|\xi|^2|(i\xi)^\alpha F(f_j(u))|^2}{(1+\delta|\xi|^2)^2}d\xi\right\}^{\frac{1}{2}} ds \\
&\qquad\leq \left(\frac{\gamma_2+2^{d-1}\delta\gamma_1}{2\gamma_1\gamma_2}\right)^{\frac{1}{2}}\|u(t_2)\|_{H^l(\mathbb{R}^d)} + \sum_{j=1}^{d}\int_{t_2}^{t}\frac{\|f_j(u(s))\|_{H^l(\mathbb{R}^d)}}{[2\delta\gamma_1 e(t-s)]^{\frac{1}{2}}}ds
\end{aligned}$$

we use the Theorem 4.3 on [11]

$$\leq \left(\frac{\gamma_2+2^{d-1}\delta\gamma_1}{2\gamma_1\gamma_2}\right)^{\frac{1}{2}}\|u(t_2)\|_{H^l(\mathbb{R}^d)} + dC\int_{t_2}^{t}\frac{\|(u(s)\|_{H^l(\mathbb{R}^d)}}{[2\delta\gamma_1 e(t-s)]^{\frac{1}{2}}}ds.$$

On the other hand, note

$$F(u)_t = -\sum_{j=1}^{d}\frac{F(f_j(u)_{x_j})}{(1+\delta|\xi|^2)} - \frac{\sum_{j=1}^{d}(\sum_{n=1}^{N}\gamma_n\xi_j^{2n})}{(1+\delta|\xi|^2)}F(u)$$

and for $t_2 = \frac{t_1}{2} < t_1 \leq t$ we have from (28)

$$\begin{aligned}
u_t(x,t) = &-\sum_{j=1}^{d}F^{-1}\left[\frac{F(f_j(u)_{x_j})}{(1+\delta|\xi|^2)}\right] \\
&-F^{-1}\left[\frac{\sum_{j=1}^{d}(\sum_{n=1}^{N}\gamma_n\xi_j^{2n})}{(1+\delta|\xi|^2)}\exp\left\{\frac{-(\sum_{j=1}^{d}\sum_{n=1}^{N}\gamma_n\xi_j^{2n})(t-t_2)}{(1+\delta|\xi|^2)}\right\}F(u(t_2))\right] \\
&+\sum_{j=1}^{d}\int_{t_2}^{t}F^{-1}\left[\frac{\sum_{j=1}^{d}(\sum_{n=1}^{N}\gamma_n\xi_j^{2n})\exp\left\{\frac{-\sum_{j=1}^{d}(\sum_{n=1}^{N}\gamma_n\xi_j^{2n})(t-s)}{(1+\delta|\xi|^2)}\right\}F(f_j(u)_{x_j})}{(1+\delta|\xi|^2)^2}\right] ds.
\end{aligned}$$

Then

$$\|u_t(t)\|_{H^{l-1}(\mathbb{R}^d)} \leq \sum_{j=1}^{d} \left\{ \sum_{|\alpha|\leq l-1} \int_{\mathbb{R}^d} \frac{|(i\xi)^\alpha F(f_j(u)_{x_j})|^2}{(1+\delta|\xi|^2)^2} d\xi \right\}^{\frac{1}{2}} +$$

$$\left\{ \sum_{|\alpha|\leq l-1} \int_{\mathbb{R}^d} \frac{[\sum_{j=1}^d (\sum_{n=1}^N \gamma_n \xi_j^{2n})]^2 \exp\left\{ \frac{-2(\sum_{j=1}^d \sum_{n=1}^N \gamma_n \xi_j^{2n})(t-t_2)}{(1+\delta|\xi|^2)} \right\} |(i\xi)^\alpha F(u(t_2))|^2}{(1+\delta|\xi|^2)^2} d\xi \right\}^{\frac{1}{2}}$$

$$+ \sum_{j=1}^{d} \int_{t_2}^{t} \left\{ \sum_{|\alpha|\leq l-1} \int_{\mathbb{R}^d} \frac{[\sum_{j=1}^d (\sum_{n=1}^N \gamma_n \xi_j^{2n})]^2 \exp\left\{ \frac{-2\sum_{j=1}^d (\sum_{n=1}^N \gamma_n \xi_j^{2n})(t-s)}{(1+\delta|\xi|^2)} \right\}}{(1+\delta|\xi|^2)^4} \right. .$$

$$\left. |(i\xi)^\alpha F(f_j(u)_{x_j})|^2 d\xi \right\}^{\frac{1}{2}} ds = A + B + C.$$

For A, from Theorem 4.3 on [11]

$$\begin{aligned} A &= \sum_{j=1}^{d} \left\{ \sum_{|\alpha|\leq l-1} \int_{\mathbb{R}^d} \frac{\xi_j^2 |(i\xi)^\alpha F(f_j(u))|^2}{(1+\delta|\xi|^2)^2} d\xi \right\}^{\frac{1}{2}} \\ &\leq \frac{d\overline{C}\|u(t)\|_{H^{l-1}(\mathbb{R}^d)}}{\delta} \end{aligned}$$

and

$$B \leq \frac{\|u(t_2)\|_{H^{l-1}(\mathbb{R}^d)}}{\sqrt{2}(t-t_2)}.$$

For C,

$$
\begin{aligned}
C \leq & \sum_{j=1}^{d}\int_{t_2}^{t}\left\{\sum_{|\alpha|\leq l-1}\int_{\mathbb{R}^d}\frac{\sum_{j=1}^{d}(\sum_{n=1}^{N}\gamma_n\xi_j^{2n})|(i\xi)^\alpha F(f_j(u)_{x_j})|^2}{2(1+\delta|\xi|^2)^3(t-s)}d\xi\right\}^{\frac{1}{2}}ds\\
\leq & \sum_{j=1}^{d}\int_{t_2}^{t}\left\{\sum_{|\alpha|\leq l-1}\left[\int_{\mathbb{R}^d}\frac{\sum_{j=1}^{d}(\sum_{n=1}^{3}\gamma_n\xi_j^{2n})|(i\xi)^\alpha F(f_j(u)_{x_j})|^2}{2(1+\delta|\xi|^2)^3(t-s)}d\xi\right.\right.\\
& \left.\left.+\int_{\mathbb{R}^d}\frac{\sum_{j=1}^{d}(\sum_{n=4}^{N}\gamma_n\xi_j^{2n})|(i\xi)^\alpha F(f_j(u)_{x_j})|^2}{2(1+\delta|\xi|^2)^3(t-s)}d\xi\right]\right\}^{\frac{1}{2}}ds\\
\leq & \sum_{j=1}^{d}\int_{t_2}^{t}\left\{\sum_{|\alpha|\leq l-1}\left[\int_{\mathbb{R}^d}\frac{\overline{C}|(i\xi)^\alpha F(f_j(u)_{x_j})|^2}{(t-s)}d\xi\right.\right.\\
& \left.\left.+\int_{\mathbb{R}^d}\frac{\sum_{j=1}^{d}(\sum_{n=4}^{N}\gamma_n\xi_j^{2(n-3)})|(i\xi)^\alpha F(f_j(u)_{x_j})|^2}{2\delta^3(t-s)}d\xi\right]\right\}^{\frac{1}{2}}ds\\
\leq & \sum_{j=1}^{d}\int_{t_2}^{t}\frac{\overline{\overline{C}}\|f_j(u)\|_{H^{l+N-3}(\mathbb{R}^d)}}{(t-s)^{\frac{1}{2}}}ds\\
\leq & d\int_{t_2}^{t}\frac{\widetilde{C}\|u(s)\|_{H^{l+N-3}(\mathbb{R}^d)}}{(t-s)^{\frac{1}{2}}}ds
\end{aligned}
$$

where $\widetilde{C}=\widetilde{C}(\delta,\gamma_1,...,\gamma_N,\|u_0\|_{H^{[\frac{d}{2}]+1}(\mathbb{R}^d)})$.

To finish, the proof is quite similar to the Lemma 9 (iii), using the Theorem 4.3 of [11] and the Theorem 1, and its proof will be omitted. □

In the order to extend these solutions globally, that is, to all of $t>0$, we first give the following lemma.

Lemma 11. *Suppose $u(x,t)=u(x,t;\delta,\gamma_1,...,\gamma_N)$ a solution of (1) and (2) on $\mathbb{R}^d\times[0,t_1]$ $(t_1\geq T)$. Then we have the following estimates:*
i)

$$
\int_{\mathbb{R}^d}\left[|u|^2+\delta\sum_{j=1}^{d}|u_{x_j}|^2\right]dx+2\sum_{j=1}^{d}\left[\sum_{n=1}^{N}\gamma_n\int_0^{t_1}\int_{\mathbb{R}^d}|\partial_{x_j}^n u|^2dxdt\right] = \int_{\mathbb{R}^d}\left[|u_0|^2+\delta\sum_{j=1}^{d}|u_{0x_j}|^2\right]dx. \tag{29}
$$

ii) If $\|u(t)\|_{L^\infty(\mathbb{R}^d)}\leq C(\gamma_1,T)$, for all $0\leq t\leq t_1$, then for each multiindex $\alpha=(\alpha_1,...,\alpha_d)$ with $\alpha_k\geq 0$, $k=1,...,d$ and $|\alpha|>0$ we have

$$
\frac{1}{2}\int_{\mathbb{R}^d}\left[|D^\alpha u|^2+\delta\sum_{j=1}^{d}|D^{\beta_1^j}u|^2\right]dx+\sum_{j=1}^{d}\frac{\gamma_1}{2}\int_0^{t_1}\int_{\mathbb{R}^d}|D^{\beta_1^j}u|^2dxdt \tag{30}
$$

$$+\sum_{j=1}^{d}\left[\sum_{n=2}^{N}\gamma_n\int_0^{t_1}\int_{\mathbb{R}^d}|D^{\beta_n^j}u|^2dxdt\right]$$

$$\leq \frac{1}{2}\int_{\mathbb{R}^d}\left[|D^{\alpha}u_0|^2+\delta\sum_{j=1}^{d}|D^{\beta_1^j}u_0|^2\right]dx+\overline{C}(\gamma_1,T)\sum_{|\gamma|=|\alpha|}\int_0^{t_1}\int_{\mathbb{R}^d}|D^{\gamma}u|^2dxdt$$

where $\beta_n^j=(\alpha_1,...,\alpha_{j-1},\alpha_j+n,\alpha_{j+1},...,\alpha_d)$.

Proof. We multiply (1) by $2u$ and integrate in $\mathbb{R}^d$ and in $[0,t_1]$, we obtain (29).

Let $\alpha=(\alpha_1,...,\alpha_d)$ such that $\alpha_k\geq 0$, $k=1,...,d$ and $|\alpha|>0$. We multiply (1) by $(-1)^{|\alpha|}D^{2\alpha}u$, and integrate in $\mathbb{R}^d$ and in $[0,t_1]$, we have

$$\frac{1}{2}\int_{\mathbb{R}^d}\left[|D^{\alpha}u|^2+\delta\sum_{j=1}^{d}|D^{\beta_1^j}u|^2\right]dx+\sum_{j=1}^{d}\left[\sum_{n=1}^{N}\gamma_n\int_0^{t_1}\int_{\mathbb{R}^d}|D^{\beta_n^j}u|^2dxdt\right] \tag{31}$$

$$=\frac{1}{2}\int_{\mathbb{R}^d}\left[|D^{\alpha}u_0|^2+\delta\sum_{j=1}^{d}|D^{\beta_1^j}u_0|^2\right]dx+(-1)^{|\alpha|}\sum_{j=1}^{d}\int_0^{t_1}\int_{\mathbb{R}^d}f_j(u)_{x_j}D^{2\alpha}udxdt.$$

The last integral can be estimated using the Theorem 4.3 on [11].

$$(-1)^{|\alpha|}\int_0^{t_1}\int_{\mathbb{R}^d}f_j(u)_{x_j}D^{2\alpha}udxdt\leq\frac{\gamma_1^{-1}}{2}\int_0^{t_1}\int_{\mathbb{R}^d}|D^{\alpha}f_j(u)|^2dxdt$$
$$+\frac{\gamma_1}{2}\int_0^{t_1}\int_{\mathbb{R}^d}|D^{\beta_1^j}u|^2dxdt.$$

Since $\|u(t)\|_{L^\infty(\mathbb{R}^d)}\leq C(\gamma_1,T)$, we have from Theorem 4.3 on [11] that

$$\frac{\gamma_1^{-1}}{2}\int_0^{t_1}\int_{\mathbb{R}^d}|D^{\alpha}f_j(u)|^2dxdt\leq\frac{C(\gamma_1,T)\gamma_1^{-1}}{2}\sum_{|\gamma|=|\alpha|}\int_0^{t_1}\int_{\mathbb{R}^d}|D^{\gamma}u|^2dxdt.$$

This estimates and (31) we give (30).

□

We can now state our global existence result:

Theorem 12. *Suppose that* u_0 *and* f_j, $j=1,...,d$, *satisfy the same assumptions of the Theorem 10. Then the problem (1)-(2) has a global smooth solution*

$$u\in C([0,\infty);H^{\left[\frac{d}{2}\right]+1}(\mathbb{R}^d)).$$

Furthermore, for each integer $k\geq\left[\frac{d}{2}\right]+1$, *we have*
i) $u\in C((0,\infty);H^k(\mathbb{R}^d))$;
ii) $u\in C((0,\infty);H^k(\mathbb{R}^d))\cap C^1((0,\infty);H^{k-1}(\mathbb{R}^d))$ *if* $N\leq 3$;
iii) $u\in C((0,\infty);H^{k+N-3}(\mathbb{R}^d))\cap C^1((0,\infty);H^{k-1}(\mathbb{R}^d))$ *if* $N>3$.

Proof. From Theorem 10, there is a unique solution $u(x,t;\delta,\gamma_1,...,\gamma_N) = u(x,t)$ defined up to time T and satisfies (i)-(iii) of the Theorem 10. Furthermore, from Lemma 9, we have for $0 \le t \le T$

$$\|u(t)\|_\infty \le C\|u_0\|_{H^{\left[\frac{d}{2}\right]+1}(\mathbb{R}^d)},$$

and consequently holds the Lemma 11.

We consider the problem (1) with initial data $u(x,T) = u_T(x)$. From Lemma 11, we obtain that $u_T \in H^{\left[\frac{d}{2}\right]+1}(\mathbb{R}^d)$ and $\|u_T\|_{H^{\left[\frac{d}{2}\right]+1}(\mathbb{R}^d)} \le C(\gamma_1,T)$. Then, by Theorem 10, $u(x,t)$ can be extended up to time $2T$. Now suppose that $u(x,t)$ been defined up to time kT for some integer k, and that for each integer $l \ge \left[\frac{d}{2}\right]+1$, we have

a) $u \in C((0,kT];H^l(\mathbb{R}^d))$;

b) $u \in C((0,kT];H^l(\mathbb{R}^d)) \cap C^1((0,kT];H^{l-1}(\mathbb{R}^d))$ if $N \le 3$;

c) $u \in C((0,kT];H^{l+N-3}(\mathbb{R}^d)) \cap C^1((0,kT];H^{l-1}(\mathbb{R}^d))$ if $N > 3$,

and the Lemma 11 holds for $0 \le t \le kT$. From Lemma 11 we have $u_{kT} \in H^{\left[\frac{d}{2}\right]+1}(\mathbb{R}^d)$ and $\|u_{kT}\|_{H^{\left[\frac{d}{2}\right]+1}(\mathbb{R}^d)} \le C(\gamma_1,T)$. Proceeding inductively, we thus establish the existence of the solution $u(x,t)$ in all of $t \ge 0$ and $u(x,t)$ satisfies (i)-(iii).

□

4. Convergence Results

Let $\{u^\gamma = u(x,t;\delta,\gamma_1,...,\gamma_N)\}$ be a sequence of solutions of (1)-(2) obtained previously, for δ and γ_n sufficiently smalls ($\delta + \sum_{n=1}^{N}\gamma_n \to 0$) and we take the smooth initial data $u_0^\gamma = u(x,0;\delta,\gamma_1,...,\gamma_N)$. We assume also that $u_0 \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$ on (16) satisfies

$$\lim_{\gamma\to 0} u_0^\gamma = u_0 \qquad in \qquad L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d). \tag{32}$$

We will now prove that the solutions of (1)-(2) converge toward the entropy solution of the associated hiperbolic problem (15) and (16). For each $u_0 \in L^2(\mathbb{R}^d)$, the Cauchy problem (15)-(16) admits a unique entropy solution $u \in L^\infty(\mathbb{R}_+, L^2(\mathbb{R}^d))$ in the sense of Kruzkov. See again [10], [5],[14], and [7]. Assume that, for some constant $C_0 > 0$ independent of δ, $\gamma_1,...,\gamma_N$,

$$\|u_0\|_{H^{\left[\frac{d}{2}\right]+1}(\mathbb{R}^d)} + \|u_0\|_{H^{N+1}(\mathbb{R}^d)} \le C_0.$$

We can now state our convergence results:

Theorem 13. *Assume f_j, $j = 1,2,...,d$ sufficiently smooth satisfy the growth condition $\|f_j'(\lambda)\|_\infty \le C$, $\forall\, \lambda \in \mathbb{R}$. If $\delta = o(\gamma_1^2)$, and $\gamma_n = o(\gamma_{n-1}\gamma_1^2)$ for $n \ge 2$, then, when $\gamma_1 \to 0$, the solution u^γ of (1)-(2) converges in $L^r_{loc}(\mathbb{R}^d \times \mathbb{R}_+)$ (for $1 \le r < 2$) toward the unique entropy solution in the sense of Kruzkov of the Cauchy problem (15)-(16).*

Proof. We will rely on the convergence framework proposed by DiPerna [5] for L^∞ solutions and generalized to L^p solutions by Szepessy [14] and Kondo and LeFloch in [7]. Consider a Young measure ν associated with the sequence $\{u^\gamma\}$ and based on the uniform L^2 bound (29).

First for all we establish that ν is a mv entropy solution of (15) and (16). We must check entropy inequalities associated with (15), that is,

$$< \nu, \eta >_t + \sum_{j=1}^{d} < \nu, \psi_j >_{x_j} \leq 0, \tag{33}$$

where $\eta : \mathbb{R} \to \mathbb{R}$ is a convex function with (at most) linear growth at infinity and the entropy flux $\psi_j'(u^\gamma) = \eta'(u^\gamma) f_j'(u^\gamma)$ is normalized so that $\psi_j(0) = 0$. By the definition of the Young measure, we only need to establish that, in the decomposition

$$\frac{\partial}{\partial t}\eta(u) + \sum_{j=1}^{d} \frac{\partial}{\partial x_j}\psi_j(u) = \sum_{j=1}^{d}[\delta(\eta'(u)u_{x_j t})_{x_j} - \delta\eta''(u)u_{x_j}u_{x_j t}] \tag{34}$$

$$+\sum_{j=1}^{d}\left\{\sum_{n=1}^{N}(-1)^{n+1}\gamma_n[\eta'(u)\partial_{x_j}^{2n-1}u]_{x_j} + (-1)^n\gamma_n\eta''(u)u_{x_j}\partial_{x_j}^{2n-1}u\right\} = \Gamma_1 + \Gamma_2$$

where

$$\Gamma_1 = \sum_{j=1}^{d}[\delta(\eta'(u)u_{x_j t})_{x_j} - \delta\eta''(u)u_{x_j}u_{x_j t}] + \sum_{j=1}^{d}\left\{\sum_{n=1}^{N}(-1)^{n+1}\gamma_n[\eta'(u)\partial_{x_j}^{2n-1}u]_{x_j}\right.$$

$$\left.+\sum_{n=2}^{N}(-1)^n\gamma_n\eta''(u)u_{x_j}\partial_{x_j}^{2n-1}u\right\} \text{ and } \Gamma_2 = -\gamma_1\sum_{j=1}^{d}\eta''(u)u_{x_j}^2, \text{ we have}$$

$$\Gamma_1 \to 0 \tag{35}$$

and

$$\Gamma_2 \leq 0. \tag{36}$$

(It is a standard matter to deduce, for all convex entropy pairs, holds (33) from (35) and (36). Namely it follows because $\frac{\partial}{\partial t}\eta(u^\gamma) + \sum_{j=1}^{d}\frac{\partial}{\partial x_j}\psi_j(u^\gamma)$ converges to a nonpositive measure in $\mathfrak{D}'(\mathbb{R}^d \times \mathbb{R}_+)$, from (35) and (36). On the other hand, from the definition of te Young measure that the terms in (34) converge (in the sense of distibutions) to their "natural" limits,

$$\eta(u^\gamma) \rightharpoonup < \nu, \eta >, \quad \psi_j(u^\gamma) \rightharpoonup < \nu, \psi_j > .$$

Inequality (20) (for all $k \in \mathbb{R}$) then follows by a standard regularization of the function $|u - k|$.)

Before proving (35) and (36), we first obtain some estimates.

We multiply (1) by $(-1)^i\gamma_i\partial_{x_k}^{2i}u$ for $k \in \{1, ..., d\}$ and $i \in \{1, ..., N-1\}$, and integrate in $\mathbb{R}^d$ and in $[0, T]$,

$$\frac{\gamma_i}{2}\int_{\mathbb{R}^d}\left[|\partial_{x_k}^i u|^2 + \delta\sum_{j=1}^{d}|\partial_{x_k}^i(u_{x_j})|^2\right]dx + \sum_{j=1}^{d}\left[\sum_{n=1}^{N}\gamma_i\gamma_n\int_0^T\int_{\mathbb{R}^d}|\partial_{x_k}^i(\partial_{x_j}^n u)|^2 dxdt\right]$$

$$= \frac{\gamma_i}{2}\int_{\mathbb{R}^d}\left[|\partial^i_{x_k}u_0|^2+\delta\sum_{j=1}^d|\partial^i_{x_k}(u_{0x_j})|^2\right]dx+(-1)^i\gamma_i\sum_{j=1}^d\int_0^T\int_{\mathbb{R}^d}f_j(u)_{x_j}\partial^{2i}_{x_k}udxdt$$

and the last integral can be estimated using (29):

$$(-1)^i\gamma_i\sum_{j=1}^d\int_0^T\int_{\mathbb{R}^d}f_j(u)_{x_j}\partial^{2i}_{x_k}udxdt \le C\sum_{j=1}^d\int_0^T\int_{\mathbb{R}^d}|u_{x_j}|^2dxdt$$

$$+\frac{\gamma_i^2}{2d}\sum_{j=1}^d\int_0^T\int_{\mathbb{R}^d}|\partial^{2i}_{x_k}u|^2dxdt \le C\gamma_1^{-1}+\frac{\gamma_i^2}{2}\int_0^T\int_{\mathbb{R}^d}|\partial^{2i}_{x_k}u|^2dxdt$$

and we obtain

$$\frac{\gamma_i}{2}\int_{\mathbb{R}^d}\left[|\partial^i_{x_k}u|^2+\delta\sum_{j=1}^d|\partial^i_{x_k}(u_{x_j})|^2\right]dx+\frac{\gamma_i^2}{2}\int_0^T\int_{\mathbb{R}^d}|\partial^{2i}_{x_k}u|^2dxdt \tag{37}$$

$$+\sum_{j=1}^d\left[\sum_{n^*=1}^N\gamma_n\gamma_i\int_0^T\int_{\mathbb{R}^d}|\partial^i_{x_k}(\partial^n_{x_j}u)|^2dxdt\right]\le C\gamma_1^{-1}.$$

where n^* means that $n \neq i$ if $j = k$.

We multiply (1) by $\gamma_1 u_t$, and integrate in $\mathbb{R}^d$ and in $[0,T]$,

$$\gamma_1\int_0^T\int_{\mathbb{R}^d}u_t^2dxdt+\frac{\delta\gamma_1}{2}\sum_{j=1}^d\int_0^T\int_{\mathbb{R}^d}|u_{x_jt}|^2dxdt+\sum_{j=1}^d\left[\sum_{n=1}^N\frac{\gamma_n\gamma_1}{2}\int_{\mathbb{R}^d}|\partial^n_{x_j}(u)|^2dx\right]$$

$$=\sum_{j=1}^d\left[\sum_{n=1}^N\frac{\gamma_1\gamma_n}{2}\int_{\mathbb{R}^d}|\partial^n_{x_j}(u_0)|^2dx\right]-\gamma_1\sum_{j=1}^d\int_0^T\int_{\mathbb{R}^d}f_j(u)_{x_j}u_tdxdt.$$

The last integral can be estimated using (29)

$$-\gamma_1\int_0^T\int_{\mathbb{R}^d}f_j(u)_{x_j}u_tdxdt\le C\gamma_1\int_0^T\int_{\mathbb{R}^d}|u_{x_j}|^2dxdt+\frac{\gamma_1}{2}\int_0^T\int_{\mathbb{R}^d}u_t^2dxdt$$

$$\le C+\frac{\gamma_1}{2}\int_0^T\int_{\mathbb{R}^d}u_t^2dxdt.$$

We obtain

$$\gamma_1\int_0^T\int_{\mathbb{R}^d}u_t^2dxdt+\delta\gamma_1\sum_{j=1}^d\int_0^T\int_{\mathbb{R}^d}|u_{x_jt}|^2dxdt+\sum_{j=1}^d\left[\sum_{n=1}^N\gamma_n\gamma_1\int_{\mathbb{R}^d}|\partial^n_{x_j}(u)|^2dx\right]\le C. \tag{38}$$

Now we can deduce that (35) and (36) hold. Let $\theta \in C_0^\infty(\mathbb{R}^d\times(0,T))$ such that $\theta \ge 0$.

We have for each $j \in \{1,...,d\}$, from (38) that

$$\delta\left|\int_0^T\int_{\mathbb{R}^d}\eta'(u)u_{x_jt}\theta_{x_j}dxdt\right|\le C\delta\|\theta_{x_j}\|_{L^2(\mathbb{R}^d\times(0,T))}\left[\int_0^T\int_{\mathbb{R}^d}|u_{x_jt}|^2dxdt\right]^{\frac{1}{2}}$$

$$\le C\delta^{\frac{1}{2}}\gamma_1^{-\frac{1}{2}}$$

and from (38) and (29) (with $n = 1$)

$$\delta \left| \int_0^T \int_{\mathbb{R}^d} \eta''(u) u_{x_j} u_{x_j t} \theta dx dt \right| \leq C\delta \left[\int_0^T \int_{\mathbb{R}^d} |u_{x_j}|^2 dx dt \right]^{\frac{1}{2}} \left[\int_0^T \int_{\mathbb{R}^d} |u_{x_j t}|^2 dx dt \right]^{\frac{1}{2}}$$

$$\leq C\delta^{\frac{1}{2}} \gamma_1^{-1}.$$

When $n = 1$, we have from (29)

$$\gamma_1 \left| \int_0^T \int_{\mathbb{R}^d} \eta'(u) u_{x_j} \theta_{x_j} dx dt \right| \leq C\gamma_1 \left[\int_0^T \int_{\mathbb{R}^d} |u_{x_j}|^2 dx dt \right]^{\frac{1}{2}} \leq C\gamma_1^{\frac{1}{2}}$$

and

$$< \Gamma_2, \theta >= -\gamma_1 \sum_{j=1}^d \int_0^T \int_{\mathbb{R}^d} \eta''(u) u_{x_j}^2 \theta dx dt \leq 0.$$

For $n \geq 2$ we use (37) (with $i = n - 1$ and $j = k$)

$$\gamma_n \left| \int_0^T \int_{\mathbb{R}^d} \eta'(u) \partial_{x_j}^{2n-1} u \theta_{x_j} dx dt \right| \leq C\gamma_n \left[\int_0^T \int_{\mathbb{R}^d} |\partial_{x_j}^{2n-1} u|^2 dx dt \right]^{\frac{1}{2}}$$

$$\leq C\gamma_n^{\frac{1}{2}} \gamma_{n-1}^{-\frac{1}{2}} \gamma_1^{-\frac{1}{2}}$$

and finally, we use (37) and (29)

$$\gamma_n \left| \int_0^T \int_{\mathbb{R}^d} \eta''(u) u_{x_j} \partial_{x_j}^{2n-1} u \theta dx dt \right| \leq C\gamma_n \left[\int_0^T \int_{\mathbb{R}^d} |u_{x_j}|^2 dx dt \right]^{\frac{1}{2}} \left[\int_0^T \int_{\mathbb{R}^d} |\partial_{x_j}^{2n-1} u|^2 dx dt \right]^{\frac{1}{2}}$$

$$\leq C\gamma_n^{\frac{1}{2}} \gamma_{n-1}^{-\frac{1}{2}} \gamma_1^{-1}.$$

This proves (33).

To show that (21) is satisfied for ν we follow the detailed argument in Kondo and LeFloch [7].

When $\gamma \to 0$, from (33) and (32), we obtain

(39)

$$- \int_{\mathbb{R}^d} \eta(u_0(x)) \theta(x, 0) dx - \int_{\mathbb{R}_+} \int_{\mathbb{R}^d} < \nu_{(x,t)}, \eta > \theta_t dx dt - \int_{\mathbb{R}_+} \int_{\mathbb{R}^d} \sum_{j=1}^d < \nu_{(x,t)}, \psi_j > \theta_{x_j} dx dt \leq 0,$$

in the sense of distributions. Let $K \subset \mathbb{R}^d$ be a compact. Let $\Omega \subset \mathbb{R}^d$ be a open and bounded subset such that $K \subset \Omega$. Using in the formulation (39) a function $\theta(x, t) = \theta_1(x)\theta_2(t)$, compactly supported in $\Omega \times [0, T)$ and having $\theta_1, \theta_2 \geq 0$, we obtain

(40) $$-\theta_2(0) \int_\Omega \eta(u_0(x)) \theta_1(x) dx - \int_{\mathbb{R}_+} \theta_{2t}(t) \int_\Omega < \nu_{(x,t)}, \eta > \theta_1(x) dx dt \leq$$

$$\int_{\mathbb{R}_+} \theta_2(t) \int_\Omega \sum_{j=1}^d < \nu_{(x,t)}, \psi_j > \theta_{1x_j}(x) dx dt.$$

The estimate of right side can be estimated using the definition of the Young measure. For that, let $I \subset \mathbb{R}$ be a compact such that in $\mathbb{R} - I$ we have

$$\eta(u) = au + b, \qquad a, b \in \mathbb{R}$$

and consequently, $\eta'(u)=a$. Since $\psi'(u)=\eta'(u)\nabla f(u)$ then in $\mathbb{R}-I$ we have $\psi_j(u)=af_j(u)$. Thus

$$\int_{\mathbb{R}_+}\theta_2(t)\int_\Omega\sum_{j=1}^d <\nu_{(x,t)},\psi_j>\theta_{1x_j}(x)dxdt$$
$$\leq\int_{\mathbb{R}_+}\theta_2(t)\sum_{j=1}^d\left[\int_\Omega\|\psi_j\|_{L^\infty(I)}|\theta_{1x_j}(x)|dx+\int_\Omega\int_{\mathbb{R}-I}|af_j(\lambda)|d\nu|\theta_{1x_j}(x)|dx\right]dt,$$
$$\leq C\int_{\mathbb{R}_+}\theta_2(t)dt+\sum_{j=1}^d\lim_{\gamma\to 0}\int_{\mathbb{R}_+}\theta_2(t)\int_{\mathbb{R}^d}|a|C|u^\gamma(x,t))||\theta_{1x_j}(x)|dxdt$$
$$\leq C\int_{\mathbb{R}_+}\theta_2(t)dt+\sum_{j=1}^d\lim_{\gamma\to 0}\int_{\mathbb{R}_+}\theta_2(t)|a|C\|u^\gamma(t)\|_{L^2(\mathbb{R}^d)}\|\theta_{1x_j}\|_{L^2(\mathbb{R}^d)}dt$$
$$\leq C_1\int_{\mathbb{R}_+}\theta_2(t)dt.$$

Therefore
(41)
$$-\theta_2(0)\int_\Omega\eta(u_0(x))\theta_1(x)dx-\int_{\mathbb{R}_+}\theta_{2t}(t)\int_\Omega<\nu_{(x,t)},\eta>\theta_1(x)dxdt\leq C_1\int_{\mathbb{R}_+}\theta_2(t)dt.$$

Thus the function

$$V(t)=-C_1t+\int_\Omega<\nu_{(x,t)},\eta>\theta_1(x)dx$$

satisfies the inequality

$$-\int_{\mathbb{R}_+}V(t)\frac{d\theta_2(t)}{dt}dt\leq\theta_2(0)\int_\Omega\eta(u_0(x))\theta_1(x)dx. \quad (42)$$

Using in (41) a test function $\theta_2\geq 0$ compactly supported in $(0,T)$, we find

$$-\int_{\mathbb{R}_+}V(t)\frac{d\theta_2(t)}{dt}dt\leq 0,$$

that is (in the sense of distributions) the function $V(t)$ is decreasing and, therefore, has locally bounded total variation. Since it is uniformly bounded, $V(t)$ has a limit as $t\to 0^+$, that is,
a) for every convex and tame entropy $\eta=\eta(u)$ and for every smooth function $\theta=\theta(x)$ with compact support in Ω, the function

$$t\mapsto\int_\Omega<\nu_{(x,t)},\eta>\theta_1(x)dx$$

has locally bounded total variation and admits a trace as $t\to 0^+$.
Now fix a time $t_0>0$ and consider the sequence of continuous functions

$$\theta_2^\epsilon(t)=\begin{cases}1 & for \quad t\in[0,t_0]\\ \frac{(t_0+\epsilon-t)}{\epsilon} & for \quad t\in[t_0,t_0+\epsilon]\\ 0 & for \quad t\geq t_0+\epsilon.\end{cases}$$

From the regularity property (a) above, we see easily that

$$-\int_{\mathbb{R}_+} V(t)\frac{d\theta_2^\epsilon(t)}{dt}dt \to V(t_0^+), \qquad when \qquad \epsilon \to 0.$$

Since $\theta_2^\epsilon(0) = 1$ and t_0 is arbitrary, (42) yelds

$$V(t_0) = -C_1 t_0 + \int_\Omega < \nu_{(x,t_0)}, \eta > \theta_1(x)dx \leq \int_\Omega \eta(u_0(x))\theta_1(x)dx$$

for all $t_0 > 0$ and, in particular,

(43)

$$\lim_{t\to 0^+} \int_\Omega < \nu_{(x,t)}, \eta > \theta_1(x)dx \leq \int_\Omega \eta(u_0(x))\theta_1(x)dx, \qquad \forall \theta_1 = \theta_1(x) \geq 0.$$

Note that the left-hand limit exists, in view of (a).

Consider the set of all linear, convex and finite combinations of the form $\sum_j \alpha_j \theta_{1,j}(x)U_j(u)$, where $\alpha_j \geq 0$, $\sum_j \alpha_j = 1$, the functions U_j are smooth and convex in u and the functions $\theta_{1,j}(x) \geq 0$ are smooth and compactly support, with moreover

$$|U_j(u)\theta_{1,j}(x)| \leq C|u| + |\widetilde{U}_j(x)|$$

with $c \geq 0$ and $\widetilde{U}_j \in L^1(\Omega)$. This set is dense (for the uniform topology in u and the L^1 topology in x) in the set of all functions $U = U(u,x)$ that are convex in u and mensurable in x and satisfy

$$|U(u,x)| \leq C|u| + |\widetilde{U}(x)|$$

for some $C > 0$ and $\widetilde{U} \in L^1(\Omega)$. Therefore by density we can deduce that

$$\limsup_{t\to 0^+} \int_\Omega < \nu_{(x,t)}, U(.,x) > dx \leq \int_\Omega U(u_0(x),x)dx \tag{44}$$

for all $U = U(u,x)$ that are convex in u and mensurable in x such that $|U(u,x)| \leq C|u| + |\widetilde{U}(x)|$ with $\widetilde{U} \in L^1(\Omega)$ and $C \geq 0$. Then, by choosing $U(u,x) = |u - u_0(x)|$ in (44) we obtain

$$\limsup_{t\to 0^+} \int_\Omega < \nu_{(x,t)}, |u - u_0(x)| > dx = 0,$$

and, consequently, we obtain

$$\lim_{t\to 0^+} \int_K < \nu_{(x,t)}, |u - u_0(x)| > dx = 0. \tag{45}$$

Now suppose that (21) is not true, that is, exist a $\epsilon_0 > 0$ and a sequence $\{T_n\}$ such that $T_n > 0$, $T_n \to 0$, and

$$\frac{1}{T_n}\int_0^{T_n}\int_K < \nu_{(x,t)}, |u - u_0(x)| > dxdt > \epsilon_0, \qquad \forall T_n. \tag{46}$$

From (45), we have a $\delta = \delta(\epsilon_0)$ such that

$$\int_K < \nu_{(x,t)}, |u - u_0(x)| > dx < \frac{\epsilon_0}{2}, \qquad if \qquad 0 < t < \delta. \tag{47}$$

Since $T_n \to 0$, exist a $T_{n_0} < \delta$. From (46) and (47) we obtain then

$$\epsilon_0 < \frac{1}{T_{n_0}} \int_0^{T_{n_0}} \int_K < \nu_{(x,t)}, |u - u_0(x)| > dxdt < \frac{1}{T_{n_0}} \int_0^{T_{n_0}} \frac{\epsilon_0}{2} dt = \frac{\epsilon_0}{2}$$

and that is a contradition.

Hence ν is a mv entropy solution of (15) and (16). It follows then from Theorem 5 that there exists a function $u \in L^\infty(\mathbb{R}_+, L^1(\mathbb{R}^d)) \cap L^\infty(\mathbb{R}_+, L^p(\mathbb{R}^d))$ such that

$$\nu_y = \delta_{u(y)} \qquad for \qquad a.e. \qquad y \in \mathbb{R}^d \times \mathbb{R}_+,$$

that is, $\nu(.)$ is the Dirac measure concentrated at $u(.)$ and u is the unique entropy solution of (15) and (16). Finally, from Lemma 6, $u_j \to u$ in $L^r_{loc}(\mathbb{R}^d \times \mathbb{R}_+)$, for $1 \leq r < 2$.

□

Acknowledgement

The second author was supported by Capes, Brazil.

References

[1] Avrin J., The generalized Benjamin-Bona-Mahony equation in $\mathbb{R}^n$ with singular initial data, *Nonlinear Analysis* **11**, (1987), 139-147.

[2] T. B. Benjamin, J. L. Bona and J.J. Mahony, Model equations for long waves in nonlinear dispersive system, *Phil. Trans. R. Soc. London.*, Ser A **272**, (1972), 47-78 .

[3] Boling Guo, *The Vanishing Viscosity Method and the Viscosity of the Difference Scheme*, Science Press, China (1992). (In Chinese.)

[4] J. M. Correia and P. G. LeFloch, Nonlinear diffusive-dispersive limits for multidimensional conservation laws, *Advances in Nonlinear P.D.E.'s and Related Areas, (Beijing, 1997), World Sci. Publ., River Edge, NJ*, (1998) ,103-123.

[5] R. J. DiPerna, Measure-valued solutions to conservation laws, *Arch. Rat. Mech. Anal.*, **88** (1985), 223-270.

[6] S. Hwang,Kinetic decomposition for the generalized BBMBurgers equations with dissipative term, *Proc. Roy. Soc. Edinburgh Sect. A*, **134**, no. 6, 11491162, 2004.

[7] C. Kondo and P. G. LeFloch, Measure-valued solutions and well-posedness of multidimensional conservation laws in a bounded domain, *Portugal. Math.*, **58** (2001), 171-194.

[8] C. Kondo and P. LeFloch, Zero diffusion-dispersion limits for hyperbolic conservation laws, *SIAM Math. Anal.*, **33** (2002), 1320-1329.

[9] C. I. Kondo and C. M. Webler, Higher-order for the generalized BBM-Burgers equation: existence and convergence results, *to appear in Applicable Analysis.*

[10] S. N. Kruzkov, First order quasilinear equations in several independent variables, *Math. USSR-Sb*, **10**, (1970), 217-243.

[11] T.-T. Li and Y.-M. Chen , *Global Classical Solutions for Nonlinear Evolution Equations*, Longman Scientific and Technical, New York, (1992).

[12] L. A. Medeiros and P. G. Menzala , Existence and uniqueness for periodic solutions of the Benjamin-Bona-Mahony equation, *SIAM J. math. Analysis* **8**(5), (1977), 792-799.

[13] M. E. Schonbek, Convergence of solutions to nonlinear dispersive equations, *Comm. Partial Differential Equations*, **7** (1982), 959-1000.

[14] A. Szepessy, An existence result for scalar conservation laws using measure-valued solutions, *Comm. Part. Diff. Equa.*, **14**, (1989), 1329-1350.

[15] L. Tartar , Compensated compactness and aplications to partial differential equations, *Nonlinear Analisys and Mechanics: Heriot-Watt Symposium*, IV. Research Notes in Math., (1979), 136-210.

[16] L. Tartar,*The compensated compactness method aplied to systems of conservation laws*, in "Systems of Nonlinear Partial Differential Equations" (J.M. Ball, Ed.), NATO ASI Series, C. Reidel publishing Col., (1983), 263-285.

Federal University of Sao Carlos P. O. Box 676, 13565-905, Sao Carlos-SP, Brazil
E-mail address: [1]dcik@dm.ufscar.br

Department of Mathematics, State University of Londrina-UEL, P. O. Box 6001, 86051-990, Londrina-PR, Brazil
E-mail address: [2]claudetewebler@uel.br