PSEUDODIFFERENTIAL OPERATORS ON LOCAL HARDY SPACES

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ABSTRACT. In this work we study the continuity of pseudodifferential operators on local Hardy spaces $h^p(\mathbb{R}^n)$ —generalizing results due to Goldberg and Taylor— and show that operators with symbols in $S^0_{1,\delta}(\mathbb{R}^n)$, $0 \leq \delta < 1$, and in some subclasses of $S^0_{1,1}(\mathbb{R}^n)$ are bounded on $h^p(\mathbb{R}^n)$ $(0 < p \leq 1)$. As an application, we study the local solvability of the planar vector field $L = \partial_t + ib(x,t)\partial_x$, $b(x,t) \geq 0$, in spaces of mixed norm involving Hardy spaces.

1. Introduction

In 1915, Hardy [Har] considered the averages

$$M_p(r,F) = \left\{ \frac{1}{2\pi} \int_0^{2\pi} |F(re^{i\theta})|^p d\theta \right\}^{1/p}, \quad 0 < p < \infty,$$
$$M_\infty(r,F) = \max_{0 \leq \theta < 2\pi} |F(re^{i\theta})|,$$

for $0 < r < 1$, to study the behavior of holomorphic functions on the unit disk. That seminal paper was the starting point for the theory of Hardy spaces, as they were named by F. Riesz in his fundamental work [Ri] where he dealt with Blaschke products and showed the existence of boundary values. Later Krylov [Kr] developed the holomorphic theory of Hardy spaces on the half-plane. To pass from one to higher dimensions, the theory needed to free itself from its dependence on holomorphic functions and the approach followed by Stein and Weiss [SW] was to consider boundary values of certain non-determined systems of harmonic functions to define Hardy spaces $H^p(\mathbb{R}^n)$ for $p > n/(n-1)$. Then,

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departing from ideas of Burkholder, Gundy and Silverstein [BGS], Fefferman and Stein developed a purely real maximal theory of Hardy spaces $H^p(\mathbb{R}^n)$, $0 < p \leq \infty$, that does not depend on the notion of holomorphic or harmonic functions. A tempered distribution $f \in \mathcal{S}'(\mathbb{R}^n)$ belongs to $H^p(\mathbb{R}^n)$ if and only if $\|M_\phi f\|_{L^p(\mathbb{R}^n)} < \infty$, where

$$M_\phi f = \sup_{0<t<\infty} |\phi_t * f(x)|, \quad \phi_t(x) = t^{-n}\phi(x/t),$$

and $\phi \in \mathcal{S}(\mathbb{R}^n)$ is a rapidly decreasing function in Schwartz space satisfying $\int \phi(x)\,dx = 1$. For $p > 1$, $H^p(\mathbb{R}^n) = L^p(\mathbb{R}^n)$, for $p = 1$, $H^1(\mathbb{R}^n)$ is contained in $L^1(\mathbb{R}^n)$ and is stable under important classes of singular integrals that do not preserve $L^1(\mathbb{R}^n)$. Finally, for $p < 1$, —in spite of being a non locally convex topological vector space— $H^p(\mathbb{R}^n)$ has far better functional properties than the space $L^p(\mathbb{R}^n)$ [CW], which is virtually useless in the applications when $p < 1$. While they are well suited as functional spaces for the applications to PDE's with constant coefficients, the Hardy spaces $H^p(\mathbb{R}^n)$ are not stable under multiplication by test functions, a fact that seriously hinders their role when it comes to PDE's with variable coefficients. To circumvent this drawback, Goldberg [G] introduced a class of localizable Hardy spaces $h^p(\mathbb{R}^n)$, $0 < p \leq \infty$. Among many other properties, he proved that $h^p(\mathbb{R}^n)$ is preserved by pseudodifferential operators of order 0 and type $(1, 0)$ in the sense of Hörmander [Hor1] (see Section 3 for precise definitions), although his arguments do not seem to be easily adapted to treat pseudodifferential operators of other types. Other results concerning the continuity of pseudodifferential on Hardy spaces, mainly for $p = 1$ —which certainly is the most important case— can be found in [Ta, Ch1 §2] and [AH1].

In this work, we carry out a more systematic study of the continuity of pseudodifferential operators with symbols in Hörmander classes $S^m_{\rho,\delta}(\mathbb{R}^n)$, discussing sufficient and necessary conditions. One of our motivations was to deal with some parametrices of some locally solvable planar vector fields that involve pseudodifferential operators of type $(1, 1/2)$ (see Section 6). The organization of the paper is as follows. In sections 2 and 3 we recall the basic facts concerning Hardy spaces and pseudodifferential operators that will be used in the rest of the paper. In section 4 we discuss the continuity in Hardy spaces for pseudodifferential operators with symbols in several classes, in particular, we prove that

(1) operators with symbols in Hörmander class $S^m_{\rho,\delta}(\mathbb{R}^n)$, $\delta \leq \rho$, $\delta < 1$, $m = -n(1-\rho)/2$ map continuously $h^1(\mathbb{R}^n)$ into itself and $\mathrm{bmo}(\mathbb{R}^n)$ into itself;
(2) operators with symbols in Hörmander class $S^0_{1,\delta}(\mathbb{R}^n)$, $\delta < 1$, map continuously $h^1(\mathbb{R}^n)$ into itself and $\mathrm{bmo}(\mathbb{R}^n)$ into itself;
(3) operators with symbols in Bony class $\mathcal{B}S^0_{1,1}(\mathbb{R}^n)$ map continuously $h^p(\mathbb{R}^n)$ into itself for any $0 < p \leq 1$;
(4) operators with symbols in Hörmander class $\tilde{S}^0_{1,1}(\mathbb{R}^n)$ map continuously $h^p(\mathbb{R}^n)$ into itself for any $0 < p \leq 1$.

Since $\mathcal{B}S^0_{1,1}(\mathbb{R}^n) \subset \tilde{S}^0_{1,1}(\mathbb{R}^n)$ the last statement implies the previous one, however, for the class $\mathcal{B}S^0_{1,1}(\mathbb{R}^n)$ we have a sharper control on the number of derivatives of the symbol that guarantee boundedness of the operator (Theorem 4.4). In section 5 we show by means of examples that the conditions on order and type assumed in section 4 to prove boundedness of pseudodifferential operators in Hardy spaces cannot be relaxed. In section 6 we discuss the local solvability of the planar vector field with smooth coefficients $L = \partial_t + ib(x,t)\partial_x$, $b(x,t) \geq 0$, in the spaces of mixed norm $L^q\big(\mathbb{R}_t; h^p(\mathbb{R}_x)\big)$, $0 < p < \infty$, $1 \leq q \leq \infty$. In particular, we give a complete picture of the solvability in $L^\infty\big(\mathbb{R}_t; h^p(\mathbb{R}_x)\big)$ for the whole spectrum of values of p. It is already known that L is solvable in $L^\infty\big(\mathbb{R}_t; h^p(\mathbb{R}_x)\big)$ for $1 < p < \infty$ and, although it is not solvable, in general, in $L^\infty\big(\mathbb{R}_t; h^\infty(\mathbb{R}_x)\big) = L^\infty\big(\mathbb{R}_t; L^\infty(\mathbb{R}_x)\big)$ it is solvable in $L^\infty\big(\mathbb{R}_t; \mathrm{bmo}(\mathbb{R}_x)\big)$. Concerning the range $0 < p \leq 1$, we show here that L is solvable in $L^\infty\big(\mathbb{R}_t; h^p(\mathbb{R}_x)\big)$ for $p = 1$ and not solvable, in general, for $0 < p < 1$. Finally, the appendix contains an L^2 continuity result for pseudodifferential operators in the Bony class with sharp control on the number of derivatives of the symbol that we needed for our proof of Theorem 4.4.

Unless otherwise specified, the functions we will consider are defined on some Euclidean space $\mathbb{R}^N$ with complex values. The letter C will denote positive constants that may vary at different occurrences. The Lebesgue measure of a measurable set $A \subset \mathbb{R}^n$ is written as $|A|$ while the absolute value of a complex number z and the Euclidean norm of a vector $\xi \in \mathbb{R}^n$ are denoted respectively by $|z|$ and $|\xi|$; the meaning will be clear from context. We will also write $\langle \xi \rangle = \left(1 + |\xi|^2\right)^{1/2}$ for $\xi \in \mathbb{R}^n$. As usual, $\mathcal{S}$ denotes the Schwartz space of rapidly decreasing functions and $\mathcal{S}'$ its dual, the space of tempered distributions. The Fourier transform of a tempered distribution $f \in \mathcal{S}'(\mathbb{R}^n)$ is denoted by $\widehat{f}$. The space of pseudodifferential operators with symbols in Hörmander classes $S^m_{\rho,\delta}(\mathbb{R}^n)$ will be denoted by $\operatorname{Op} S^m_{\rho,\delta}$ and $W^{s,2}(\mathbb{R}^n)$, $s \in \mathbb{R}$, will denote the Sobolev space $\left\{f \in \mathcal{S}'(\mathbb{R}^n) : \ \langle \xi \rangle^s \widehat{f} \in L^2(\mathbb{R}^n)\right\}$.

2. Local Hardy spaces

We recall some facts about the real Hardy spaces $H^p(\mathbb{R}^n)$ of Fefferman and Stein [FS] and its semi-local version $h^p(\mathbb{R}^n)$ introduced by Goldberg [G] (recall that a space of distributions is said to be semi-local if it is stable under multiplication by test functions). In many situations $H^1(\mathbb{R}^n)$ is an advantageous substitute for $L^1(\mathbb{R}^n)$ [S2], as the latter does not behave well in many respects, for instance, concerning the continuity of singular integral operators, while for $p < 1$, $H^p(\mathbb{R}^n)$ has far better functional properties than the corresponding $L^p(\mathbb{R}^n)$ space. Let us choose a function $\Phi(x) \geq 0 \in \mathcal{S}(\mathbb{R}^n)$ —often one takes $\Phi(x) \in C^\infty_c(B)$ where $B \subset \mathbb{R}^n$ the unit ball— with $\int \Phi(x)\, dx = 1$. Write $\Phi_\varepsilon(x) = \varepsilon^{-n}\Phi(x/\varepsilon)$, $x \in \mathbb{R}^n$, and set

$$M_\Phi f(x) = \sup_{0<\varepsilon<\infty} |(\Phi_\varepsilon * f)(x)|.$$

Then [S2]

$$H^p(\mathbb{R}^n) = \{f \in \mathcal{S}'(\mathbb{R}^n) : \quad M_\Phi f \in L^p(\mathbb{R}^n)\}.$$

The space $H^p(\mathbb{R}^n)$ is not semi-local: ψu may not belong to $H^p(\mathbb{R}^n)$ for $\psi \in C^\infty_c(\mathbb{R}^n)$ and $u \in H^p(\mathbb{R}^n)$. A way around this is the definition of the semi-local (or localizable) Hardy space $h^p(\mathbb{R}^n)$ ([G], [S2]) by means of the truncated maximal function

$$m_\Phi f(x) = \sup_{0<\varepsilon\leq 1} |(\Phi_\varepsilon * f)(x)|;$$

$$h^p(\mathbb{R}^n) = \{f \in \mathcal{S}'(\mathbb{R}^n) : \quad m_\Phi f \in L^p(\mathbb{R}^n)\},$$

which is stable under multiplication by test functions (we denote by $\mathcal{S}(\mathbb{R}^n)$ the Schwartz space of rapidly decreasing functions on $\mathbb{R}^n$ and by $\mathcal{S}'(\mathbb{R}^n)$ its dual, the space of tempered distributions). It turns out that if Φ is substituted in the definition of $h^p(\mathbb{R}^n)$ by any other function $\Phi \in \mathcal{S}(\mathbb{R}^n)$ only subjected to $\int \Phi \neq 0$, this will not change the space $h^p(\mathbb{R}^n)$. Moreover, $h^p(\mathbb{R}^n)$ is a complete metric vector space with the distance

$$d(f, g) = \|f - g\|_{h^p}^p = \|m_\Phi(f - g)\|_{L^p}^p,$$

which for $p = 1$ is a Banach space. Of course, the "norm" $f \mapsto \|f\|_{h^p}$ depends on the choice of Φ but different Φ's will give equivalent "norms", moreover, if $\mathcal{A} \subset \mathcal{S}$ is a bounded subset, there is a constant $C = C(\mathcal{A}) > 0$ such that $\|m_\phi f\|_{L^p} \leq C\|m_\Phi f\|_{L^p}$ for all $f \in \mathcal{S}$ and $\phi \in \mathcal{A}$. In fact more is true: denoting by $\mathcal{M}f(x) = \sup_{\phi\in\mathcal{A}} m_\phi f(x)$ the grand maximal function associated to $\mathcal{A}$ it follows that $\|\mathcal{M}f\|_{L^p} \leq C\|m_\Phi f\|_{L^p}$.

2.1. **Atomic decompositions.** We now describe the atomic decomposition of $h^p(\mathbb{R}^n)$ ([G], [S2]). An $h^p(\mathbb{R}^n)$ atom is a bounded, compactly supported function $a(z)$ satisfying the following properties: there exists an cube Q with sides parallel to the coordinate axes containing the support of a such that

(1) $|a(z)| \leq |Q|^{-1/p}$, a.e., with $|Q|$ denoting the Lebesgue measure of Q;
(2) if $\|a\|_{L^\infty} > 1$, we further require that $\int z^\alpha a(z)\, dz = 0$, $\alpha \in \mathbb{N}^n$, $|\alpha| \leq n(p^{-1} - 1)$.

Any $f \in h^p$ can be written as an infinite linear combination of h^p-atoms, more precisely, there exist scalars λ_j and h^p-atoms a_j such that $\sum_j |\lambda_j|^p < \infty$ and the series $\sum_j \lambda_j a_j$ converges to f both in h^p and in $\mathcal{S}'$. Furthermore, $\|f\|_{h^p}^p \sim \inf \sum_j |\lambda_j|^p$, where the infimum is taken over all atomic representations. Another useful fact is that the atoms may be assumed to be smooth functions, in particular the inclusion $\mathcal{S}(\mathbb{R}^n) \subset h^p(\mathbb{R}^n)$ is dense. The atomic decomposition of $h^p(\mathbb{R}^n)$ is thus quite similar to the atomic decomposition of $H^p(\mathbb{R}^n)$ in terms of H^p-atoms ([S2]), the difference being that the notion of h^p-atom is less restrictive than that of H^p-atom, as an H^p-atom a must satisfy (1) and a stronger form of (2): moments are required to vanish regardless of the size of $\|a\|_{L^\infty}$. In order to prove that a linear operator $T : \mathcal{S}(\mathbb{R}^n) \longrightarrow h^p(\mathbb{R}^n)$ can be extended as a bounded operator on $h^p(\mathbb{R}^n)$ it is enough to check that

(2.1)
$$\text{there exists } C > 0 \text{ such that } \|m_\Phi T a\|_{L^p} \leq C, \text{ for all } h^p\text{-atoms } a.$$

2.2. **Duality.** Given a function $f \in L^1_{\text{loc}}(\mathbb{R}^n)$ and a cube $Q \subset \mathbb{R}^n$ with sides parallel to the axes, denote by f_Q the mean of f over Q:

$$f_Q = \frac{1}{|Q|}\int_Q f(x)\,dx, \quad |Q| = \text{Lebesgue measure of } Q.$$

The space $\text{bmo}(\mathbb{R}^n)$ is defined as the set of $f \in L^1_{\text{loc}}(\mathbb{R}^n)$ such that

$$S_1(f) = \sup_{|Q|<1} \frac{1}{|Q|}\int_Q |f(x) - f_Q|\,dx < \infty,$$
$$S_2(f) = \sup_{|Q|\geq 1} \frac{1}{|Q|}\int_Q |f(x)|\,dx < \infty,$$

equipped with the norm $\|f\|_{\text{bmo}} = \max(S_1(f), S_2(f))$. Then the dual of $h^1(\mathbb{R}^n)$ may be identified ([G]) with $\text{bmo}(\mathbb{R}^n)$ by means of the unique continuous extension to $h^1(\mathbb{R}^n) \times \text{bmo}(\mathbb{R}^n)$ of the bilinear form

$$\mathcal{S}(\mathbb{R}^n) \times \mathcal{S}(\mathbb{R}^n) \ni (\phi, \psi) \mapsto \int \phi(x)\psi(x)\,dx.$$

For $0 < r < 1$, we say that $f(x) \in \dot{\Lambda}^r(\mathbb{R}^n)$ if $f(x)$ is bounded, continuous and

$$\|f\|_{\dot{\Lambda}^r} \doteq \|f\|_{L^\infty} + \sup_{x\neq y} \frac{|f(x) - f(y)|}{|x - y|^r} < \infty.$$

For $r = 1$, we say that $f(x) \in \dot{\Lambda}^1(\mathbb{R}^n)$ if $f(x)$ is bounded, continuous and

$$\|f\|_{\dot{\Lambda}^1} \doteq \|f\|_{L^\infty} + \sup_{y\neq 0} \frac{|f(x+y) + f(x-y) - 2f(x)|}{|y|} < \infty.$$

If k is a positive integer and $k < r \leq k+1$, set $r' = r - k$, so $0 < r' \leq 1$. We say that $f(x) \in \dot{\Lambda}^r(\mathbb{R}^n)$ if $f(x)$ is of class C^k with bounded derivatives up to order k and

$$\|f\|_{\dot{\Lambda}^r} \doteq \max_{|\alpha| \leq k} \|D^\alpha f\|_{\Lambda^{r'}} < \infty.$$

The dual of $h^p(\mathbb{R}^n)$, $0 < p < 1$ may be identified ([G]) with $\dot{\Lambda}^{r_p}(\mathbb{R}^n)$, $r_p = n/p - 1/p$, by means of the unique continuous extension to $h^p(\mathbb{R}^n) \times \dot{\Lambda}^{r_p}(\mathbb{R}^n)$ of the bilinear form

$$\mathcal{S}(\mathbb{R}^n) \times \mathcal{S}(\mathbb{R}^n) \ni (\phi, \psi) \mapsto \int \phi(x)\psi(x)\, dx.$$

3. Pseudodifferential operators

We recall the Hörmander class of pseudodifferential operators [Hor1]. For $m \in \mathbb{R}$ and $\rho, \delta \in [0,1]$, a symbol $a \in S^m_{\rho,\delta}(\mathbb{R}^n)$ of order m and type (ρ, δ) is a smooth function defined on $\mathbb{R}^n \times \mathbb{R}^n$ satisfying the estimates

$$|\partial_x^\alpha \partial_\xi^\beta a(x,\xi)| \leq C_{\alpha,\beta} \langle \xi \rangle^{m - \rho|\beta| + \delta|\alpha|}|, \quad \alpha, \beta \in \mathbb{N}^n.$$

The space $S^m_{\rho,\delta}(\mathbb{R})$ becomes a Fréchet space with the seminorms

$$p_{\alpha,\beta}(a) = \sup_{(x,\xi) \in \mathbb{R}^{2n}} \langle \xi \rangle^{-m + \rho|\beta| - \delta|\alpha|} |\partial_x^\alpha \partial_\xi^\beta a(x,\xi)|$$

The pseudodifferential operator $a(x, D) \in \operatorname{Op} S^m_{\rho,\delta}$ associated to the symbol $a(x,\xi) \in S^m_{\rho,\delta}(\mathbb{R}^n)$ is given by

$$a(x,D)u(x) = (2\pi)^{-n} \int e^{ix\cdot\xi} a(x,\xi)\, \widehat{u}(\xi)\, d\xi, \quad u \in \mathcal{S}(\mathbb{R}^n), \tag{3.1}$$

where $\widehat{u}(\xi)$ denotes the Fourier transform of $u(x)$. Pseudodifferential operators are bounded from $\mathcal{S}(\mathbb{R}^n)$ to $\mathcal{S}(\mathbb{R}^n)$ ([H2]) and as such possess distribution kernels $K(x,y) \in \mathcal{S}'(\mathbb{R}^n \times \mathbb{R}^n)$. A well known and useful formula for the kernel is given in the following proposition (see, e.g., [AH1]):

Proposition 3.1. *Let $a(x,\xi) \in S^m_{\rho,\delta}(\mathbb{R}^n)$, $0 < \rho \leq 1$, $0 \leq \delta \leq 1$. The distribution kernel $K(x,y)$ of $a(x,D)$ is smooth off the diagonal $\{(x,x)\} \subset \mathbb{R}^n \times \mathbb{R}^n$ and is given by*

$$K(x,y) = (2\pi)^{-n} \lim_{\epsilon \to 0} \int e^{i(x-y)\cdot\xi} a(x,\xi) \psi(\epsilon\xi)\, d\xi \tag{3.2}$$

where $\psi \in C_c^\infty(\mathbb{R}^n)$ *satisfies* $\psi \equiv 1$ *for* $|\xi| \leq 1$ *and the limit is taken in* $\mathcal{S}'(\mathbb{R}^n)$*. If* $M \in \mathbb{N}$ *and* $M + m + n > 0$*,* $K(x, y)$ *satisfies the estimates*

$$\sup_{|\alpha|+|\beta|=M} |D_x^\alpha D_y^\beta K(x, y)| \leq \frac{C_{\alpha\beta}}{|x - y|^{\frac{M+m+n}{\rho}}}, \quad x \neq y. \tag{3.3}$$

Furthermore, for any $N \in \mathbb{N}$ *and* α*,* $\beta \in \mathbb{N}^n$*,*

$$\sup_{|x-y|\geq 1/2} |x - y|^N |D_x^\alpha D_y^\beta K(x, y)| \leq C_{\alpha\beta N}. \tag{3.4}$$

From now on we will always assume that $\rho > 0$.

Remark 3.1: In order to study the continuity of a pseudodifferential operator $a(x, D)$ in the Hardy space $h^p(\mathbb{R}^n)$ for $n/(n + 1) < p \leq 1$, we will need below a particular instance of (3.3) with $m = 0$, $\rho = 1$, $\alpha = 0$ and $\beta = 1$. In that case, inspection of the proof shows that the constants $C_{\alpha\beta}$ on the right hand side of (3.3) only depend on the size of the derivatives $D_\xi^\beta a(x, \xi)$ of the symbol for $|\beta| \leq n + 2$.

3.1. Bounded operators in $\mathbf{L^2}$.

To study the continuity of pseudodifferential operators on Hardy spaces, we recall that a common strategy in order to prove (2.1) for operators T which are bounded in $L^2(\mathbb{R}^n)$ is as follows. Assume that T is a linear operator bounded in $L^2(\mathbb{R}^n)$ and let $a(x)$ be an h^p-atom supported in a cube Q centered at x_0 with side length ℓ and denote by Q^* the cube with the same center and side length 2ℓ. To estimate $\|m_\Phi Ta\|_{L^p}$ we may write $\int (m_\Phi Ta(x))^p \, dx = I_1 + I_2$ with

$$I_1 = \int_{Q^*} (m_\Phi Ta(x))^p \, dx, \qquad I_2 = \int_{\mathbb{R}^n \setminus Q^*} (m_\Phi Ta(x))^p \, dx.$$

Since $m_\Phi f$ is majorized by the Hardy-Littlewood maximal operator Mf and in particular is bounded in L^2, we have by Hölder's inequality

$$\begin{aligned} I_1 &\leq \left(\int_{Q^*} (m_\Phi Ta(x))^2 \, dx \right)^{p/2} |Q^*|^{1-p/2} \\ &\leq C \left(\int_{\mathbb{R}^n} |a(x)|^2 \, dx \right)^{p/2} |Q^*|^{1-p/2} \leq 2^n C \end{aligned}$$

with C independent of the atom $a(x)$. On the other hand, to estimate I_2 it will be enough to show that

$$m_\Phi Ta(x) \leq C|Q|^{-1/p} \frac{\ell^L}{|x-x_0|^L} \quad \text{if } x \notin Q^*, \tag{3.5}$$

where $C > 0$ and $L > n/p$ are fixed constants independent of the atom $a(x)$. Indeed, a simple computation shows that (3.5) implies that $I_2 \leq Ck_n$ with k_n a dimensional constant. Equivalently, we must show that

$$|\Phi_\varepsilon * Ta(x)| \leq C|Q|^{-1/p} \frac{\ell^L}{|x-x_0|^L} \quad \text{if } x \notin Q^*, \tag{3.5$'$}$$

where $C > 0$ and $L > n/p$ are independent of $0 < \varepsilon \leq 1$.

3.2. **The case p = 1.** Take pseudodifferential operators $r_j(x, D)$, $j = 1, \dots, n$, with symbols

$$r_j(\xi) = \chi(\xi)\frac{i\xi_j}{|\xi|}, \quad \xi \in \mathbb{R}^n,$$

where $\chi(\xi) \in C^\infty(\mathbb{R})$ satisfies $\chi(\xi) = 0$ for $|\xi| \leq 1/2$ and $\chi(\xi) = 1$ for $|\xi| \geq 1$. Hence, $r_j(\xi) \in S^0_{1,0}(\mathbb{R}^n)$ and $r_j(x, D)$ may also be regarded as a convolution operator. It is known ([G]) that $r_j(x, D)$ maps $h^1(\mathbb{R}^n)$ into itself and that

$$c_1\|f\|_{h^1} \leq \|f\|_{L^1} + \sum_{j=1}^n \|r_j(x, D)f\|_{L^1} \leq c_2\|f\|_{h^1}, \quad f \in h^1(\mathbb{R}^n),$$

providing a norm for $h^1(\mathbb{R}^n)$ which is equivalent to the standard one and does not involve maximal functions. Thus, in order to prove that a linear operator $T : \mathcal{S}(\mathbb{R}^n) \longrightarrow h^1(\mathbb{R}^n)$ can be extended as a bounded operator in $h^1(\mathbb{R}^n)$, it is enough to prove the L^1 estimate

$$\|Tf\|_{L^1} + \sum_{j=1}^n \|r_j(x, D)Tf\|_{L^1} \leq C\|f\|_{h^1}, \quad f \in \mathcal{S}(\mathbb{R}^n). \tag{3.6}$$

4. Sufficient conditions for $\mathbf{h^p}$ continuity

In this section we study the continuity in Hardy spaces for classes of L^2-bounded pseudodifferential operators. We first consider operators of order $m = 0$ and type (ρ, δ), with $\delta \leq \rho$. If $\rho < 1$, as we point out in the next section, there exists pseudodifferential operators with symbols in $S^0_{\rho,0} \subset S^0_{\rho,\delta}$ which are not bounded in $h^p(\mathbb{R}^n)$, $0 < p \leq 1$. Thus, we must restrict ourselves to the classes $S^0_{1,\delta}$. When $\delta < 1$, symbols in the class $S^0_{1,\delta}$ give rise to L^2-bounded pseudodifferential operators while Ching [Ch] gave a now famous example of an unbounded operator with symbol in the class $S^0_{1,1}$. Thus, for $\rho = \delta = 1$ we will work with subclasses of $S^0_{1,1}$, namely, the Bony class $\mathcal{B}S^0_{1,1}(\mathbb{R}^n)$ ([Bo1], [Bo2]) and the Hörmander classes $\mathcal{H}S^0_{1,1}(\mathbb{R}^n)$ and $\tilde{S}^0_{1,1}(\mathbb{R}^n)$ ([Bou], [Hor2]) that we describe more precisely later in this section.

Let $A = a(x,D) \in \operatorname{Op} S^0_{1,\delta}$ be a pseudodifferential operator with symbol $a(x,\xi) \in S^0_{1,\delta}$, $0 \le \delta \le 1$, which is bounded in $L^2(\mathbb{R}^n)$ (note that the latter condition is automatic for $\delta < 1$) and denote by $A^{(\varepsilon)}$ the operator $\mathcal{S}(\mathbb{R}^n) \ni f \mapsto \Phi_\varepsilon * Af$. Observe that $\mathcal{S}(\mathbb{R}^n) \ni f \mapsto \Phi_\varepsilon * f$ may be written as

$$\Phi_\varepsilon * f(x) = \frac{1}{2\pi)^n} \int e^{ix\cdot\xi} \widehat{\Phi}(\varepsilon\xi)\, \widehat{f}(\xi)\, d\xi, \quad f \in \mathcal{S}(\mathbb{R}^n),$$

and regarded as a pseudodifferential operator with symbol $\widehat{\Phi}(\varepsilon\xi)$. Since $\widehat{\Phi}$ is rapidly decreasing,

$$\sup_{0<\varepsilon\le 1} \left| D^\alpha_\xi [\widehat{\Phi}(\varepsilon\xi)] \right| = \sup_{0<\varepsilon\le 1} \varepsilon^{|\alpha|} \big| (D^\alpha_\xi \widehat{\Phi})(\varepsilon\xi) \big| \le C_\alpha \langle \xi \rangle^{-|\alpha|}, \quad \xi \in \mathbb{R}^n,$$

so we see that the symbols $\xi \mapsto \widehat{\Phi}(\varepsilon\xi)$ belong to $S^0_{1,0}$ uniformly in $0 < \varepsilon \le 1$ and $A^{(\varepsilon)}$ is obtained by composing on the left the pseudodifferential operator $f \mapsto \Phi_\varepsilon * f$ with $a(x,D)$. In all the cases we consider, this composition will be a pseudodifferential operator $A^{(\varepsilon)}$ with symbol $a^{(\varepsilon)}(x,\xi)$ and the key point will be that $\{a^{(\varepsilon)}\}_{0<\varepsilon\le 1}$ is a bounded subset of $S^0_{1,1}$. In particular, we will be able to invoke the estimates for its kernel $K^{(\varepsilon)}(x,y)$ granted by Proposition 3.1 with $m = 0$ and $\rho = 1$ and bounds independent of ε. Let $a(x)$ be an h^p-atom with $\|a\|_{L^\infty} \le 1$, so we may assume that it is supported in a cube Q centered at x_0 with length side $\ell \ge 1$. For $x \notin Q^*$ and $y \in Q$ we have $|x-y| \ge \ell/2 \ge 1/2$ and so, using (3.4) with $N = L \doteq [n/p] + 1$ we obtain

$$\begin{aligned} |A^{(\varepsilon)} a(x)| = \left| \int_Q K^{(\varepsilon)}(x,y) a(y)\, dy \right| &\le C_L |x - x_0|^{-L} |Q|^{1-1/p}, \quad x \notin Q^*, \\ &\le C_L |Q|^{-1/p} \frac{\ell^n}{|x-x_0|^L} \le C_L |Q|^{-1/p} \frac{\ell^L}{|x-x_0|^L} \end{aligned}$$

because $L > n$ and $\ell \geq 1$. Hence, (3.5′) holds for $T = a(x, D)$ for h^p-atoms satisfying $\|a\|_{L^\infty} \leq 1$. Let us now consider an h^p-atom —that we assume supported in a cube Q centered at x_0— with $\|a\|_{L^\infty} > 1$, in particular, it has vanishing moments up to order $N_p = [n(p^{-1} - 1)]$. We may write, using the moment conditions,

$$A^{(\varepsilon)}a(x) = \int_Q \Big(K^{(\varepsilon)}(x, y) - \sum_{|\alpha| \leq N_p} \partial_y^\alpha K^{(\varepsilon)}(x, x_0) \frac{(y - x_0)^\alpha}{\alpha!} \Big) a(y)\, dy$$

$$= \int_Q R^{(\varepsilon)}(x, y) a(y)\, dy. \tag{4.1}$$

Standard estimates for the remainder in Taylor's formula and use of (3.3) with $m = 0$, $\rho = 1$ and $M = N_p + 1$ give

$$\begin{aligned} \sup_{y \in Q} |R^{(\varepsilon)}(x, y)| &\leq C \ell^{N_p+1} \sum_{|\alpha| = N_p+1} \sup_{y \in Q} |\partial_y^\alpha K^{(\varepsilon)}(x, y)| \\ &\leq C \frac{\ell^{N_p+1}}{|x - x_0|^{N_p+n+1}}, \quad x \notin Q^*, \end{aligned}$$

since $|x - x_0| \simeq |y - x|$ for $y \in Q$ and $x \notin Q^*$. Plugging this estimate into (4.1) yields

$$\begin{aligned} |A^{(\varepsilon)}a(x)| &\leq C|Q|^{-1/p}|Q| \frac{\ell^{N_p+1}}{|x - x_0|^{N_p+n+1}} \\ &= C|Q|^{-1/p} \frac{\ell^{N_p+n+1}}{|x - x_0|^{N_p+n+1}}, \quad x \notin Q^*. \end{aligned}$$

Thus, (3.5′) holds for $T = a(x, D)$ and $L = N_p + n + 1$ for h^p-atoms satisfying $\|a\|_{L^\infty} > 1$. Summing up, we have proved that $a(x, D)$ is bounded on $h^p(\mathbb{R}^n)$, $0 < p \leq 1$, if

(i) $a(x, D)$ is bounded in $L^2(\mathbb{R}^n)$ and
(ii) $\{a^{(\varepsilon)}\}_{0<\varepsilon\leq 1}$ is a bounded subset of $S^0_{1,1}(\mathbb{R}^n)$.

There is an extensive literature on the L^2 boundedness of pseudodifferential operators and many classes of pseudodifferential operators are known to satisfy condition (i). On the other hand, condition (ii) will need closer attention most of the time and will be addressed in the next sections for specific classes of pseudodifferential operators.

4.1. **The Hörmander class $\mathbf{S^m_{\rho,\delta}}$, $\mathbf{m = -n(1-\rho)/2}$, $\boldsymbol{\delta \le \rho}$, $\boldsymbol{\delta < 1}$.** When $\rho < 1$, there are operators with symbols in $S^0_{\rho,0}(\mathbb{R}^n)$ which are unbounded in $h^p(\mathbb{R}^n)$ for any $0 < p \le 1$. On the other hand, the fact that ρ is less than one may be compensated for by taking pseudodifferential operators $A = a(x, D)$ of appropriate negative order m.

Proposition 4.1. *Assume that $b(x,\xi) \in S^m_{\rho,\delta}(\mathbb{R}^n)$, $\delta \le \rho$, $\delta < 1$, $m = -n(1-\rho)/2$. Then, $b(x, D)$ maps continuously $h^1(\mathbb{R}^n)$ into $L^1(\mathbb{R}^n)$.*

Proof: If $h^1(\mathbb{R}^n)$ is replaced by the smaller space $H^1(\mathbb{R}^n)$ the analogous result is known ([AH1, Thm 3.2]), in particular we have the estimate $\|b(x, D)a\|_{L^1} \le C$ for all H^1 atoms $a(x)$. On the other hand, the corresponding estimate for h^1 atoms $a(x)$ which are not H^1 atoms, i.e., those that do not have vanishing mean but are supported in a cube with length side $\ell \ge 1$ and satisfy $\|a\|_{L^\infty} \le 1$, is implied by the estimate $\|b(x, D)a\|_{h^1} \le C$ that can be easily proved with the method described before in this section to check that $(3.5')$ holds for that class of atoms. Thus, the atomic decomposition of $h^1(\mathbb{R}^n)$ yields the result. □

Theorem 4.1. *Let $b(x,\xi) \in S^m_{\rho,\delta}(\mathbb{R}^n)$, $\delta \le \rho$, $\delta < 1$, $m = -n(1-\rho)/2$. Then $b(x, D)$ maps continuously $h^1(\mathbb{R}^n)$ into itself and* $\mathrm{bmo}(\mathbb{R}^n)$ *into itself.*

Proof: To show the continuity in $h^1(\mathbb{R}^n)$ it is enough to prove that (3.6) holds with $T = b(x, D)$ and this follows from Proposition 4.1 applied to $b(x, D)$ and $b_j(x, D) = r_j(x, D) \circ b(x, D)$, $j = 1, \ldots, n$. An operator $b(x, D)$ satisfies the hypothesis of Theorem 4.1 if an only if its adjoint does. Thus, we obtain the boundedness in $\mathrm{bmo}(\mathbb{R}^n)$ by duality. □

Corollary 4.1. *Let $b(x,\xi) \in S^0_{1,\delta}(\mathbb{R}^n)$, $\delta < 1$. Then $b(x, D)$ maps continuously $h^1(\mathbb{R}^n)$ into $h^1(\mathbb{R}^n)$ and* $\mathrm{bmo}(\mathbb{R}^n)$ *into* $\mathrm{bmo}(\mathbb{R}^n)$.

4.2. **The Bony class.** Let $0 < r < 1$. We say that symbol $a(x,\xi)$ belongs to $\mathcal{B}_r S^m_{1,1}(\mathbb{R}^n)$ if $a(x,\xi) \in S^m_{1,1}(\mathbb{R}^n)$ and for some $c > 0$ it satisfies

$$\hat{a}(\eta,\xi) = 0 \quad \text{whenever } |\eta| > r|\xi| \text{ and } |\xi| \geq c, \tag{4.2}$$

where $\hat{a}(\eta,\xi)$ denotes the Fourier transform of $x \mapsto a(x,\xi)$ in the sense of distributions. Thus, if $x \mapsto a(x,\xi)$ is integrable

$$\hat{a}(\eta,\xi) = \int_{\mathbb{R}^n} e^{-ix\cdot\eta} a(x,\xi)\, dx.$$

The Bony class $\mathcal{B}S^m_{1,1}(\mathbb{R}^n)$ is defined as

$$\mathcal{B}S^m_{1,1}(\mathbb{R}^n) = \bigcup_{0<r<1} \mathcal{B}_r S^m_{1,1}(\mathbb{R}^n).$$

Pseudodifferential operators $a(x,D)$ with symbols $a(x,\xi) \in \mathcal{B}S^0_{1,1}(\mathbb{R}^n)$ are bounded in $L^2(\mathbb{R}^n)$ and the operator norm is controlled by a sharp number of seminorms of $S^0_{1,1}(\mathbb{R}^n)$. In fact,

Theorem 4.2. *Let $a(x,\xi)$ be a function defined on $\mathbb{R}^n \times \mathbb{R}^n$ with continuous derivatives $\partial_x^\alpha \partial_\xi^\beta a(x,\xi)$ up to order $|\alpha|, |\beta| \leq [n/2]+1$, satisfying the estimates*

$$|\partial_x^\alpha \partial_\xi^\beta a(x,\xi)| \leq C\langle\xi\rangle^{|\alpha|-|\beta|}, \quad |\alpha|, |\beta| \leq [n/2]+1. \tag{4.3}$$

If for some fixed $0 < r < 1$, $|\eta| \leq r|\xi|$ whenever (η,ξ) is in the support of $\hat{a}(\eta,\xi)$ then the operator $a(x,D)$ is bounded in $L^2(\mathbb{R}^n)$ with operator norm proportional to C.

This theorem can be proved adapting the arguments of Coifman-Meyer [CM] as we indicate in the appendix. We now discuss the boundedness in $S^0_{1,1}(\mathbb{R}^n)$ of the set $\{a^{(\varepsilon)}\}_{0<\varepsilon\leq 1}$ when $a(x,\xi) \in \mathcal{B}S^m_{1,1}(\mathbb{R}^n)$. Actually, we will do so for a larger class of symbols ([Hor2]): we say that $a(x,\xi) \in S^m_{1,1}(\mathbb{R}^n)$ belongs to $\mathcal{H}S^m_{1,1}(\mathbb{R}^n)$ if for some positive constant B

$$\hat{a}(\eta,\xi) = 0 \quad \text{if } 1 + |\eta+\xi| < |\xi|/B. \tag{4.4}$$

If we assume that (4.2) holds for some $c > 0$ and $0 < r < 1$ then (4.4) must hold for some large B. Indeed, if $1 + |\eta+\xi| < |\xi|/B$, we see, by taking B large enough that $|\xi| > B > c$ and $|\eta| > (1-1/B)|\xi| > r|\xi|$

and this implies that $\hat{a}(\eta,\xi)=0$ by (4.2), so (4.4) holds as well. This shows that $\mathcal{B}S^m_{1,1}(\mathbb{R}^n)\subset\mathcal{H}S^m_{1,1}(\mathbb{R}^n)$.

Given $a(x,\xi)\in\mathcal{H}S^m_{1,1}(\mathbb{R}^n)$ satisfying (4.4) we may write

$$a^{(\varepsilon)}(x,\xi)=\int e^{-iy.\xi}\varepsilon^{-n}\Phi(y/\varepsilon)a(x-y,\xi)\,dy$$

and assume without loss of generality that $\widehat{\Phi}(\xi)$ is supported in $|\xi|\le 1$. By means of a cut-off function $\chi(\xi)$ we may always write $a(x,\xi)=a_1(x,\xi)+a_2(x,\xi)$ with $a_1(x,\xi)$ and $a_2(x,\xi)\in\mathcal{H}S^m_{1,1}(\mathbb{R}^n)$, $a_1(x,\xi)\equiv a(x,\xi)$ for $|\xi|\le c$ and $a_1(x,\xi)\equiv 0$ for $|\xi|\ge c+1$. It follows that the set $\{a_1^{(\varepsilon)}\}_{0<\varepsilon\le 1}$ is bounded in $S^0_{1,1}(\mathbb{R}^n)$ and so the boundedness of $\{a^{(\varepsilon)}\}_{0<\varepsilon\le 1}$ is reduced to the boundedness of $\{a_2^{(\varepsilon)}\}_{0<\varepsilon\le 1}$. In other words, we may assume from the start that $a(x,\xi)=0$ (therefore also $\hat{a}(\eta,\xi)=0$) for $|\xi|\le 2(B+1)$ and we will do so. Since $D^\alpha_x a^{(\varepsilon)}(x,\xi)=(D^\alpha_x a)^{(\varepsilon)}(x,\xi)$ and $\|\Phi\|_{L^1}=1$, we have

$$\sup_x |D^\alpha_x a^{(\varepsilon)}(x,\xi)|\le\sup_x |D^\alpha_x a(x,\xi)|,\quad \xi\in\mathbb{R}^n,$$

we need only worry with derivatives with respect to ξ. Consider, for instance,

$$\begin{aligned}D_{\xi_j}a^{(\varepsilon)}(x,\xi)=\int e^{-iy.\xi}\varepsilon^{-n}\Phi(y/\varepsilon)D_{\xi_j}a(x-y,\xi)\,dy\\ +\int e^{-iy.\xi}y_j\varepsilon^{-n}\Phi(y/\varepsilon)a(x-y,\xi)\,dy\\ =I_1^\varepsilon(x,\xi)+I_2^\varepsilon(x,\xi).\end{aligned}$$

It is clear that

$$|I_1^\varepsilon(x,\xi)|\le\sup_x |D_{\xi_j}a(x,\xi)|\le C\langle\xi\rangle^{m-1}.$$

To estimate the second term we notice that, using the inversion formula for the Fourier transform, we may write

$$\text{(a)}\qquad (2\pi)^n I_2^\varepsilon(x,\xi)=-\varepsilon\int e^{ix\cdot(\theta-\xi)}(D_j\widehat{\Phi})(\varepsilon\theta)\,\hat{a}(\theta-\xi,\xi)\,d\theta.$$

Indeed, set

$$a_1(x)=e^{ix\cdot\xi_0}a(x,\xi_0),\qquad \Psi(y)=y_j\varepsilon^{-n}\Phi(y/\varepsilon)$$

and rewrite $I_2^\varepsilon(x,\xi_0)$ as

$$I_2^\varepsilon(x,\xi_0) = e^{-ix\cdot\xi_0}\int \Psi(y)\, a_1(x-y)\, dy = e^{-ix\cdot\xi_0}\,(\Psi * a_1)(x).$$

The Fourier transform of the right hand side is $\widehat{\Psi}(\theta+\xi_0)\,\widehat{a}_1(\theta+\xi_0)$. Using the inversion formula for the Fourier transform we may write

$$\begin{aligned}(2\pi)^n I_2^\varepsilon(x,\xi_0) &= \int e^{ix\cdot\theta}\widehat{\Psi}(\theta+\xi_0)\,\widehat{a}_1(\theta+\xi_0)\, d\theta \\ \text{(b)}\qquad &= \int e^{ix\cdot(\theta-\xi_0)}\widehat{\Psi}(\theta)\,\widehat{a}_1(\theta)\, d\theta.\end{aligned}$$

On the other hand we have

$$\text{(c)}\qquad \widehat{\Psi}(\theta) = -\varepsilon(D_j\widehat{\Phi})(\varepsilon\theta), \qquad \widehat{a}_1(\theta) = \widehat{a}(\theta-\xi_0,\xi_0),$$

where in the case of $a(x,\xi_0)$ the hat means Fourier transform in the first variable. Plugging (c) into (b) we get the desired expression (a). Notice next that $|\varepsilon\theta|\le 1$ on the support of $D_j\widehat{\Phi}$. Since $1+|\theta|\ge|\xi|/B$ for (θ,ξ) on the support of $\hat{a}(\theta-\xi,\xi)$, we conclude that $I_2^\varepsilon(x,\xi)=0$ for $\varepsilon\ge B/(|\xi|-B)$. Since we already know that $I_2^\varepsilon(x,\xi)=0$ for $|\xi|\le 2(B+1)$ this implies that $I_2^\varepsilon(x,\xi)=0$ for $\varepsilon\ge 2B\langle\xi\rangle^{-1}$. Set $\Psi(y)=y_j\Phi(y)$, $\Psi_\varepsilon(y)=\varepsilon^{-n}\Psi(y/\varepsilon)$. We have

$$\begin{aligned}(2\pi)^n|I_2^\varepsilon(x,\xi)| &= \varepsilon\left|\int e^{-iy.\xi}\Psi_\varepsilon(y)a(x-y,\xi)\,dy\right| \\ &\le 2B\langle\xi\rangle^{-1}\,\|\Psi\|_{L^1}\sup_x|a(x,\xi)| \le (2B)C\langle\xi\rangle^{m-1}.\end{aligned}$$

Similarly, we obtain

$$|D_{\xi_j}D_x^\alpha a^{(\varepsilon)}(x,\xi)| \le C\big(2B\langle\xi\rangle^{-1}\sup_x|D_x^\alpha a(x,\xi)| + \sup_x|D_{\xi_j}D_x^\alpha a(x,\xi)|\big).$$

This process can be continued to estimate $D_x^\alpha D_\xi^\beta a^{(\varepsilon)}(x,\xi)$ for any $\beta\in\mathbb{N}^n$. Expressing the estimates in terms of the seminorms in $S^m_{1,1}(\mathbb{R}^n)$

$$p_{\alpha,\beta}(a) = \sup_{(x,\xi)\in\mathbb{R}^{2n}}\langle\xi\rangle^{-m+|\beta|-|\alpha|}|\partial_x^\alpha\partial_\xi^\beta a(x,\xi)|,$$

we may state

Proposition 4.2. *Let $a(x,\xi)\in\mathcal{H}S^m_{1,1}(\mathbb{R}^n)$ satisfy* (4.4). *Then*

$$\text{(4.5)}\qquad |D_x^\alpha D_\xi^\beta a^{(\varepsilon)}(x,\xi)| \le C_{\alpha\beta}(2B)^{|\beta|}\sum_{|\gamma|\le|\beta|}p_{\alpha\gamma}(a)\,\langle\xi\rangle^{m+|\alpha|-|\beta|}, \quad 0<\varepsilon\le 1,$$

with constants $C_{\alpha\beta}$ independent of B and ε.

An immediate consequence is

Theorem 4.3. *If $a(x,\xi) \in \mathcal{H}S^0_{1,1}(\mathbb{R}^n)$ then $a(x,D)$ maps continuously $h^p(\mathbb{R}^n)$ to $h^p(\mathbb{R}^n)$ for any $0 < p \leq 1$.*

Proof: Proposition 4.2 shows that $\{a^{(\varepsilon)}\}_{0<\varepsilon\leq 1}$ is a bounded subset of $S^0_{1,1}(\mathbb{R}^n)$. Furthermore, $a(x,D)$ is bounded in $L^2(\mathbb{R}^n)$ by Proposition 9.3.1 in [Hor2]. Thus, conditions (i) and (ii) discussed at the beginning of this section are satisfied by $a(x,D)$ and it maps continuously $h^p(\mathbb{R}^n)$ to itself for any $0 < p \leq 1$. □

The following theorem gives a sharper result for the class $\mathcal{B}S^m_{1,1}(\mathbb{R}^n)$.

Theorem 4.4. *Let $a(x,\xi)$ be a function defined on $\mathbb{R}^n \times \mathbb{R}^n$ with continuous derivatives $\partial^\alpha_x \partial^\beta_\xi a(x,\xi)$ up to order $|\alpha| \leq [n/2]+1$, $|\beta| \leq n+2$ satisfying the estimates*

$$|\partial^\alpha_x \partial^\beta_\xi a(x,\xi)| \leq C\langle\xi\rangle^{|\alpha|-|\beta|}, \qquad \text{if } |\alpha|, |\beta| \leq [n/2]+1, \text{ or } \alpha = 0 \text{ and } |\beta| \leq n+2. \tag{4.6}$$

If for some fixed $0 < r < 1$, $|\eta| \leq r|\xi|$ whenever (η,ξ) is in the support of $\hat{a}(\eta,\xi)$ then the operator $a(x,D)$ is bounded in $h^p(\mathbb{R}^n)$, $n/(n+1) < p \leq 1$.

Proof: The main point is to show that (3.5') holds for $T = a(x,D)$ when Q is a cube of side length ≤ 1. This may be proved reasoning as in (4.1). Since we are assuming that $n/(n+1) < p \leq 1$ it follows that $N_p = 0$ so $R^{(\varepsilon)}(x,y) = K(x,y) - K(x,x_0)$ which, in view of the mean value theorem, may be estimated by

$$\sup_{|\beta|=1} \left| D^\beta_y K(x,y) \right| \leq \frac{C}{|x-y|^{n+1}}, \quad x \neq y,$$

with C proportional to

$$\sup_{x,\xi\in\mathbb{R}^n} \max_{|\beta|\leq n+2} \langle\xi\rangle^{|\beta|} \left| D^\beta a(x,\xi) \right|$$

(see Remark 1). □

4.3. **The Hörmander classes $\tilde{\mathbf{S}}^{\mathbf{0}}_{\mathbf{1,1}}$ and $\mathbf{S}^{\mathbf{0}}_{\mathbf{1},\boldsymbol{\delta}}$, $\boldsymbol{\delta < 1}$.** Choose a function $\chi \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ such that

$$\begin{aligned} &\chi(t\xi, t\eta) = \chi(\xi,\eta) \quad \text{if } t \geq 1,\ |\eta| \geq 2 \\ &\operatorname{supp}\chi \subset \{(\xi,\eta): \quad |\xi| \leq |\eta|,\ |\eta| \geq 1\} \\ &\chi \equiv 1 \quad \text{on the set } \{2|\xi| \leq |\eta|, |\eta| \geq 2\}, \end{aligned} \tag{4.7}$$

and for arbitrary $a(x,\eta) \in S^m_{1,1}(\mathbb{R}^n)$ set

$$\hat{a}_{\chi,\epsilon}(\xi,\eta) = \chi(\xi+\eta, \epsilon\eta)\hat{a}(\xi,\eta) \tag{4.8}$$

where $\hat{a}(\eta,\xi)$ denotes the Fourier transform of $x \mapsto a(x,\xi)$.

Definition 4.1. *A symbol $a(x,\xi) \in S^m_{1,1}(\mathbb{R}^n)$ belongs to $\tilde{S}^m_{1,1}(\mathbb{R}^n)$ if and only if for any positive integer N and multi-indices α, β there exists a constant $C_{\alpha\beta N}$ such that*

$$\left|D^\alpha_\xi D^\beta_x a_{\chi,\epsilon}(x,\xi)\right| \leq C_{\alpha\beta N}\epsilon^N(1+|\xi|)^{m+|\beta|-|\alpha|}, \quad 0 < \epsilon < 1. \tag{4.9}$$

As proved in [Hor2, Thm.9.4.2], the class $\tilde{S}^m_{1,1}(\mathbb{R}^n)$ coincides with the following classes:

(1) the class of $a \in S^m_{1,1}$ such that $a(x,D)^* \in Op(S^m_{1,1})$;
(2) the class of $a \in S^m_{1,1}$ such that for any $s \in \mathbb{R}$, $a(x,D)$ maps continuously the Sobolev space $W^{s+m,2}(\mathbb{R}^n)$ to the Sobolev space $W^{s,2}(\mathbb{R}^n)$.

Note that any symbol $a(x,\xi) \in S^m_{1,1}$ may be written as

$$a = a - a_{\chi,1} + \sum_{\nu=0}^{\infty} a_{\chi,2^{-\nu}} - a_{\chi,2^{-\nu+1}} \tag{4.10}$$

where each term

$$b_\nu(x,\xi) = a_{\chi,2^{-\nu}}(x,\xi) - a_{\chi,2^{-\nu+1}}(x,\xi)$$

satisfies

$$\hat{b}_\nu(\xi,\eta) = 0, \quad \text{for } |\xi+\eta| + 1 < |\eta|/B_\nu,$$

for some $B_\nu > 0$ so $b_\nu \in \mathcal{H}S^m_{1,1}$. Furthermore, if $a(x,\xi) \in \tilde{S}^m_{1,1}(\mathbb{R}^n)$, estimates (4.9) yield convergence of (4.10) in the space of symbols.

Theorem 4.5. *If $a(x,\xi) \in \tilde{S}^0_{1,1}(\mathbb{R}^n)$ then $a(x,D)$ maps continuously $h^p(\mathbb{R}^n)$ to $h^p(\mathbb{R}^n)$ for any $0 < p \leq 1$.*

Proof: Write $A = a(x, D)$, denote by $A^{(\varepsilon)}$ the operator $\mathcal{S} \ni f \mapsto \Phi_\varepsilon * Af$ and set $a^{(\varepsilon)}(x, \xi)$ is the symbol of $A^{(\varepsilon)}$. The arguments in Section 4 reduce the proof to showing that

(i) $a(x, D)$ is bounded in $L^2(\mathbb{R}^n)$ and
(ii) $\{a^{(\varepsilon)}\}_{0<\varepsilon\leq 1}$ is a bounded subset of $S^0_{1,1}(\mathbb{R}^n)$.

That (i) holds follows from [Hor2, Thm.9.4.2]. Furthermore, if $b_\varepsilon(\xi)$ is the Fourier transform of $\Phi_\varepsilon(x)$, it follows that $b_\varepsilon(\xi) \in S^0_{1,0}$ uniformly in $0 < \varepsilon < 1$ and $A^{(\varepsilon)} = b_\varepsilon(D) \circ a(x, D)$. Hence, the calculus of pseudodifferential operators with symbols in the class $\tilde{S}^0_{1,1}(\mathbb{R}^n)$ [Hor2, Thm.9.5.2] shows that $\{a^{(\varepsilon)}\}_{0<\varepsilon\leq 1}$ is a bounded subset of $\tilde{S}^0_{1,1}(\mathbb{R}^n) \subset S^0_{1,1}(\mathbb{R}^n)$. $\square$

Corollary 4.2. *If* $a(x, \xi) \in \tilde{S}^0_{1,1}(\mathbb{R}^n)$ *then* $a(x, D)$ *maps continuosly* $\mathrm{bmo}(\mathbb{R}^n)$ *to* $\mathrm{bmo}(\mathbb{R}^n)$.

Remark 4.2: A duality consequence of Theorem 4.5 is that operators with symbols in $S^0_{1,1}(\mathbb{R}^n)$ map continuously $\dot{\Lambda}^r(\mathbb{R}^n)$ into itself. However, better continuity results are known, since Stein proved that the whole class $S^0_{1,1}(\mathbb{R}^n)$ already enjoys this property [S2]. Furthermore, operators of order zero and type $(1, 1)$ preserve the Sobolev spaces $W^{s,p}(\mathbb{R}^n)$, $1 < p < \infty$, for all $s > 0$ ([Me], [Bou], [Hor2]). As a matter of fact, this continuity property together with that of their commutators of all orders with the operators of partial differentiation D_j, $j = 1, \ldots, n$, and multiplication by the coordinate functions x_j, $j = 1, \ldots, n$, characterizes operators of type $(1, 1)$ ([Hou2]).

As shown in [Hor2, Thm.9.6.5], $S^0_{1,\delta}(\mathbb{R}^n) \subset \tilde{S}^0_{1,1}(\mathbb{R}^n)$, $0 \leq \delta < 1$, so Theorem 4.5 contains the following result

Theorem 4.6. *If* $a(x, \xi) \in S^0_{1,\delta}(\mathbb{R}^n)$, $0 \leq \delta < 1$, *then* $a(x, D)$ *maps continuosly* $h^p(\mathbb{R}^n)$ *to* $h^p(\mathbb{R}^n)$ *for any* $0 < p \leq 1$.

5. Necessary conditions for continuity

In this section we discuss order and type restrictions that are imposed by demanding that all pseudodifferential operators of order m and type (ρ, δ) be continuous on $h^p(\mathbb{R}^n)$. Since $S^m_{1,0}(\mathbb{R}^n) \subset S^m_{\rho,\delta}(\mathbb{R}^n)$, to conclude that $m \leq 0$ is a necessary condition, it is enough to exhibit, for every $m > 0$, an operator $a(x, D)$ with symbol $a(x, \xi) \in S^m_{1,0}(\mathbb{R}^n)$ which is unbounded on $h^p(\mathbb{R}^n)$.

Example 5.1: Denote by (x_1, x'), $x' = (x_2, \ldots, x_n)$, a generic point in $\mathbb{R}^n \simeq \mathbb{R} \times \mathbb{R}^{n-1}$. Choose $\psi(x') \in C_c^\infty(\mathbb{R}^{n-1})$ supported in the unit cube $Q' \subset \mathbb{R}^{n-1}$ with $\widehat{\psi}(0) = 1$, define

$$\alpha(t) = \begin{cases} 1/2 & \text{for } 0 \leq t \leq 1, \\ -1/2 & \text{for } -1 \leq t < 0, \\ 0 & \text{for } |t| > 1, \end{cases}$$

and set $f(x) = \alpha(x_1)\psi(x')$, $f_\varepsilon(x) = \varepsilon^{-n/p} f(x/\varepsilon)$, for some fixed $n/(n+1) < p \leq 1$. Then $f_\varepsilon(x)/\|\psi\|_{L^\infty}$ is an h^p-atom and we see that $\|f_\varepsilon\|_{h^p} \simeq \|f_\varepsilon\|_{H^p} \leq C\|\psi\|_{L^\infty} \leq C'$, $0 < \varepsilon \leq 1$. For any $m > 0$, consider the pseudodifferential operator $\langle D\rangle^m$ with symbol $\langle\xi\rangle^m \in S^m_{1,0}(\mathbb{R}^n)$ and set $F_\varepsilon(x) = \langle D\rangle^m f_\varepsilon(x)$. To conclude that $\langle D\rangle^m$ is unbounded on $h^p(\mathbb{R}^n)$ we will show that $\|F_\varepsilon\|_{h^p} \to \infty$ as $\varepsilon \searrow 0$. Notice that

$$\widehat{F}_\varepsilon(\xi_1, \xi') = \langle\xi\rangle^m\, \widehat{\alpha}(\varepsilon\xi_1)\, \widehat{\psi}(\varepsilon\xi_1') = \varepsilon^{n(1-p^{-1})}\langle\xi\rangle^m\, \frac{1 - \cos(\varepsilon\xi_1)}{\varepsilon\xi_1}\, \widehat{\psi}(\varepsilon\xi').$$

Hence, $\widehat{F}_\varepsilon(\pi/2\varepsilon, 0) = \varepsilon^{n(1-p^{-1})}(2/\pi)(1+\pi^2\varepsilon^{-2}/4)^{m/2} \simeq \varepsilon^{n(1-p^{-1})-m}$ and we conclude that

$$\sup_{\xi\in\mathbb{R}^n} \frac{|\widehat{F}_\varepsilon(\xi)|}{\langle\xi\rangle^{n(p^{-1}-1)}} \to \infty \quad \text{as } \varepsilon \searrow 0.$$

Then (5.1) in Proposition 5.1 below shows that $\|F_\varepsilon\|_{h^p} \to \infty$ as $\varepsilon \searrow 0$, as we wished to show. Similar examples can be given when $0 < p \leq n/(n+1)$.

Proposition 5.1. *Let $0 < p \leq 1$. The Fourier transform of a distribution in $h^p(\mathbb{R}^n)$ is a continuous function and there exists a positive constant $C > 0$ such that*

$$|\widehat{F}(\xi)| \leq C\langle\xi\rangle^{n(p^{-1}-1)}\|F\|_{h^p}, \quad F \in h^p(\mathbb{R}^n). \tag{5.1}$$

PROOF: By the atomic decomposition we may write $F \in h^p(\mathbb{R}^n)$ as a series

$$F(x) = \sum_j \lambda_j a_j(x) + \sum_k \mu_k b_k(x)$$

that converges in $\mathcal{S}'$, where the a_j's and b_k's are h^p-atoms such that $\|a_j\|_{L^\infty} > 1$ and $\|b_j\|_{L^\infty} \leq 1$ and $\sum_j |\lambda_j|^p + \sum_k |\mu_j|^p \simeq \|F\|_{h^p}^p$. The atoms $\{a_j\}$ are also H^p-atoms and $\|a_j\|_{H^p}^p \leq C$ where the constant only depends on the dimension n and the function Φ that has been picked to define the maximal function M_Φ. In particular, it is known [S2, p.128] that

$$|\widehat{a}_j(\xi)| \leq C|\xi|^{n(p^{-1}-1)} \leq C\langle\xi\rangle^{n(p^{-1}-1)}, \quad \xi \in \mathbb{R}^n, \quad j = 1, 2, \ldots .$$

On the other hand, we may assume that each b_k is supported in a cube Q_k of side $\ell_k \geq 1$ such that $\|b_k\|_{L^\infty} \leq |Q_k|^{-1/p}$, so that $\|b_k\|_{L^1} \leq |Q_k|^{1-1/p} = \ell^{n(1-1/p)} \leq 1$ and

$$|\widehat{b}_k(\xi)| \leq 1, \quad \xi \in \mathbb{R}^n, \quad k = 1, 2, \ldots .$$

Hence,

$$\begin{aligned} \left| \sum_j \lambda_j \widehat{a}_j(\xi) + \sum_k \mu_k \widehat{b}_k(\xi) \right|^p &\leq C \sum_j |\lambda_j|^p \langle\xi\rangle^{pn(1-p^{-1})} + \sum_k |\mu_k|^p \\ &\leq C\langle\xi\rangle^{pn(1-p^{-1})} \|F\|_{h^p}^p, \quad \xi \in \mathbb{R}^n. \end{aligned}$$

The estimate shows that the series $\sum_j \lambda_j \widehat{a}_j(\xi) + \sum_k \mu_k \widehat{b}_k(\xi)$ converges uniformly over compact sets of $\mathbb{R}^n$ to a continuous function and since it also converges to $\widehat{F}(\xi)$ in $\mathcal{S}'(\mathbb{R}^n)$ we see that $\widehat{F}(\xi)$ is continuous and (5.1) holds. $\square$

EXAMPLE 5.2: Let $\psi(\xi) \in C^\infty(\mathbb{R}^n)$ satisfy $\psi(\xi) \equiv 0$ for $|\xi| \leq 1$ and $\psi(\xi) \equiv 1$ for $|\xi| \geq 2$. Fix $0 < \rho < 1$, $n(\rho - 1)/2 < m \leq 0$, and consider the pseudodifferential operator $a(D)$ with symbol $a(\xi) = \psi(\xi)\langle\xi\rangle^m \exp\big(i|\xi|(1-\rho)\big)$. Then $a(\xi) \in S^m_{\rho,0}(\mathbb{R}^n)$ but $a(D)$ cannot be bounded on $h^1(\mathbb{R}^n)$. Indeed, since $m \leq 0$, $a(D)$ is bounded in $L^2(\mathbb{R}^n) = h^2(\mathbb{R}^n)$. If $a(D)$ were also bounded in $h^1(\mathbb{R}^n)$ we would have, by the interpolation properties of localizable Hardy spaces (*cf.*, e.g., [Tr, p.66]), that $a(D)$ would be also bounded in $h^q(\mathbb{R}^n)$ for any $1 < q < 2$. This means that $a(D)$ would be bounded in $L^q(\mathbb{R}^n)$,

$1 < q < 2$. However, it is known [F] that the convolution operator $a(D)$ is not bounded in $L^q(\mathbb{R}^n)$ if $|q^{-1} - 2^{-1}| > |m|/(1-\rho)n$. By the choice of m, the right hand side of this inequality is $< 1/2$ while the left hand side tends to $1/2$ as $q \to 1$, so we see that any $1 < q < 2$ sufficiently close to 1 will fulfill the inequality. This contradicts the fact that $a(D)$ should be bounded in $L^q(\mathbb{R}^n)$. Thus, $a(D)$ is not bounded in $h^1(\mathbb{R}^n)$. This shows that the order restriction in Theorem 4.1 is sharp.

6. Solvability of planar vector fields

Consider the first-order linear differential operator in two variables

$$L = \frac{\partial}{\partial t} + ib(x,t)\frac{\partial}{\partial x}, \qquad x \in \mathbb{R}, \quad |t| < T. \tag{6.1}$$

We write $\Omega_T = \mathbb{R} \times (-T, T)$ and assume that

(i) $b(x,t)$ is smooth, real and nonnegative,
(ii) all derivatives of $b(x,t)$ are uniformly bounded.

Definition 6.1. *Let $1 \le q \le \infty$, $0 < p \le \infty$. We say that L is locally solvable in $L^q\big([-T,T]; h^p(\mathbb{R})\big)$ if every $(x_0, t_0) \in \Omega_T$ is contained in an open set $U \subset \Omega_T$ such that for every $f \in L^q\big([-T,T]; h^p(\mathbb{R})\big)$ there exists $u \in L^q\big([-T,T]; h^p(\mathbb{R})\big)$ such that*

$$Lu = f \quad \text{in } U$$

in the sense of distributions.

Theorem 6.1. *Assume L is given by* (6.1) *and satisfies* (i) *and* (ii). *Then L is locally solvable in $L^q\big([-T,T]; h^p(\mathbb{R})\big)$ for $1 \le q \le \infty$, $1 \le p < \infty$.*

Corollary 6.1. *Under the hypotheses of the theorem, L is locally solvable in $L^p\big([-T,T]; h^p(\mathbb{R})\big)$, $1 \le p < \infty$.*

The requirement $p \ge 1$ in Theorem 6.1 and Corollary 6.1 is essential, in fact we have

Theorem 6.2. *The vector field ∂_t is not solvable in $L^\infty\big([-T,T]; h^p(\mathbb{R})\big)$ for $0 < p < 1$.*

Remark 6.3: For $1 < p < \infty$, we have $h^p(\mathbb{R}) = L^p(\mathbb{R})$ and it turns out that for $1 < q, p < \infty$ Theorem 6.1 can be proved by a simple adaptation of the arguments in [HL] where the local solvability in $L^p\big([-T,T]; L^p(\mathbb{R})\big)$ was proved. These results were later extended under the weaker hypothesis that L satisfies the Nirenberg-Treves condition $(\mathcal{P})$ ([HM1], [HM2]). On the other hand, local solvability in $L^\infty\big([-T,T]; L^\infty(\mathbb{R})\big)$ fails in general ([HT], [BCH, p.161]) although local solvability in $L^\infty\big([-T,T]; \mathrm{bmo}(\mathbb{R})\big)$ —that may be considered as an alternative limiting case of the solvability in $L^p\big([-T,T]; L^p(\mathbb{R})\big)$ as $p \to \infty$— is also known under condition $(\mathcal{P})$ [HdaS]. Hence, the novelty of Corollary 6.1 lies in the endpoint case $p = 1$. A natural open question is whether the solvability for the case $p = 1$ still remains true when replacing the assumption $b(x,t) \geq 0$ by the weaker Nirenberg-Treves condition $(\mathcal{P})$, i.e., the assumption that $t \mapsto b(x,t)$ does not change sign. Presently, it does not seem to be known whether the simple vector field $L = \partial_t + ix\partial_x$ is locally solvable in $L^\infty\big([-T,T]; h^1(\mathbb{R})\big)$ at the origin.

6.1. **Proof of Theorem 6.1.** It will be enough to prove the local solvability at the origin $(x_0, t_0) = (0,0)$. We will restrict ourselves to the case $p = 1$. The main tool in the proof is a parametrix that we describe next, details of its construction may be found in [Hou3], [AH2] and [HL]. There exist operators K and R such that

$$LKf = f + Rf, \quad f(x,t) \in C^\infty\big([-T,T]; \mathcal{S}(\mathbb{R}_x)\big), \tag{6.2}$$

where $K = K^+ + K^-$, $R = R^+ + R^-$ and

$$K^+f(x,t) = \frac{1}{2\pi}\int_{-T}^{t} \big(A^+_{t,t'} f(\cdot, t')\big) \circ \psi(x,t,t'))\, dt'$$

$$K^-f(x,t) = \frac{1}{2\pi}\int_{T}^{t} \big(A^-_{t,t'} f(\cdot, t')\big) \circ \psi(x,t,t')\, dt'$$

$$R^+f(x,t) = \frac{1}{2\pi}\int_{-T}^{t} \big(B^+_{t,t'} f(\cdot, t')\big) \circ \psi(x,t,t')\, dt'$$

$$R^-f(x,t) = \frac{1}{2\pi}\int_{T}^{t} \big(B^-_{t,t'} f(\cdot, t')\big) \circ \psi(x,t,t')\, dt'.$$

For t, $t' \in [-T, T]$ fixed, the operators $A^{\pm}_{t,t'}$, $B^{\pm}_{t,t'}$ belong to $\operatorname{Op} S^0_{1,1/2}(\mathbb{R})$ uniformly in t, t', in particular, they map $h^1(\mathbb{R})$ into itself with bounds independent of $t, t' \in [-T, T]$ in view of Theorem 4.6. In fact, the map

$$[-T, T] \times [-T, T] \ni (t, t') \mapsto A^{\pm}_{t,t'} \in \operatorname{Op} S^0_{1,1/2}(\mathbb{R})$$

is smooth and the same holds for $B^{\pm}_{t,t'}$, so the uniform bounds of their symbols $a^{\pm}_{t,t'}(x, \xi)$ and $b^{\pm}_{t,t'}(x, \xi)$ in $S^0_{1,1/2}(\mathbb{R})$ are a consequence of the compactness of $[-T, T] \times [-T, T]$. The real function $\psi : \mathbb{R} \times [-T, T] \times [-T, T] \longrightarrow \mathbb{R}$ is smooth and all its derivatives of positive order are bounded, furthermore, for fixed $(t, t') \in [-T, T] \times [-T, T]$, the map $\mathbb{R} \ni x \mapsto \psi(x, t, t')$ is a diffeomorphism of the real line with bounded derivatives. Thus, for fixed $(t, t') \in [-T, T] \times [-T, T]$, the map $g(x) \mapsto g \circ \psi(\cdot, t, t')$ is bounded from $h^1(\mathbb{R})$ into itself with a bound that can be taken independently of t, t'. It follows that

$$\|K^+ f(\cdot, t)\|_{h^1} \leq C \int_{-T}^{t} \|f(\cdot, t')\|_{h^1}\, dt'.$$

Taking L^q norms with respect to t and using Hölder inequality we get

$$\begin{aligned} \|K^+ f\|_{L^q([-T,T];h^1)} &\leq C \left\| \int_{-T}^{t} \|f(\cdot, t')\|_{h^1}\, dt' \right\|_{L^q([-T,T];h^1)} \\ &\leq 2CT \|f\|_{L^q([-T,T];h^1)}. \end{aligned}$$

Similar estimates hold for $\|K^- f\|_{L^q([-T,T];h^1)}$ and $\|R^{\pm} f\|_{L^q([-T,T];h^1)}$. We may extend (6.2) by a density argument to

$$LKf = (I + R)f, \quad f(x, t) \in L^q\big([-T, T]; h^1(\mathbb{R})\big).$$

The operator norm of R in $L^q\big([-T, T]; h^1(\mathbb{R})\big)$ is bounded by $CT < 1$ so, for T small enough, $I + R$ may be inverted to obtain

$$LK(I + R)^{-1} f = f, \quad f(x, t) \in L^q\big([-T, T]; h^1(\mathbb{R})\big),$$

so $u = K(I + R)^{-1} f \in L^q\big([-T, T]; h^1(\mathbb{R})\big)$ solves $Lu = f$ in Ω_T. $\square$

6.2. **Proof of Theorem 6.2.** Let us fix $0 < p < 1$ and write

$$X = \big\{u(t, x) \in L^\infty\big(I, h^p(\mathbb{R})\big), \partial_t u(t, x) \in L^\infty\big(I, h^p(\mathbb{R})\big)\big\}$$

where $I = [-1, 1] \subset \mathbb{R}$. We will also write $\Omega = (-1, 1) \times \mathbb{R}$. We start by stating and proving a lemma.

Lemma 6.1. *The space X endowed with the distance*

$$\|u(t,x)-v(t,x))\|_g$$
$$=\sup_{t\in I}|u(t,x)-v(t,x)\|^p_{h^p(\mathbb{R})}+\sup_{t\in I}\|\partial_t(u(t,x)-v(t,x))\|^p_{h^p(\mathbb{R})},$$

is complete. If the densely defined operator $\partial_t : (X,\|\cdot\|_g)\to(X,\|\cdot\|_g)$ is onto then the a priori estimate

$$\inf_{a\in h^p(\mathbb{R})}\sup_{-1\le t\le 1}\|u(t,x)-a(x)\|^p_{h^p(\mathbb{R})}\le A\sup_{-1\le t\le 1}\|\partial_t u(t,x)\|^p_{h^p(\mathbb{R})} \tag{6.3}$$

must hold for some $A>0$.

Proof: The space $L^\infty([-1,1],h^p(\mathbb{R}))$ with the distance

$$d(u,v)=\|u-v\|=\sup_{-1\le t\le 1}\|u(t,x)-v(t,x)\|^p_{h^p(\mathbb{R})}$$

is an F-space in the sense of Banach [B], i.e., a topological vector space with its topology defined by a complete, translation invariant distance. We denote by $(X,\|\cdot\|_X)$ the subspace X equipped with the induced metric, i.e.,

$$d(u,v)=\|u-v\|_X=\sup_{t\in I}\|u(t,x)-v(t,x)\|^p_{h^p(\mathbb{R})}$$

Note that $(X,\|\cdot\|_g)$ is complete. In fact, if $\{u_n\}$ is a Cauchy sequence in $(X,\|\cdot\|_g)$, it follows from the trivial estimate $\|\cdot\|_X\le\|\cdot\|_g$ and the completeness of $(L^\infty(I,h^p(\mathbb{R})),\|\cdot\|)$ that

$$u_n\to u\in L^\infty(I,h^p(\mathbb{R}))\quad\text{and}$$
$$\partial_t u_n\to v\in L^\infty(I,h^p(\mathbb{R}))$$

with convergence in the $\|\cdot\|$ gauge. Since these sequences also converge in $\mathcal{D}'(\Omega)$ we have that $\partial_t u=v$, showing that $(X,\|\cdot\|_g)$ is complete. Now, the operator $\partial_t : (X,\|\cdot\|_g)\to(X,\|\cdot\|)$ is obviously continuous. If it is also onto, it must be open by the open mapping theorem. Denoting by N the kernel of $\partial_t : (X,\|\cdot\|_g)\to(X,\|\cdot\|)$ we have that

$$Y=X/N\ni\dot u\longmapsto\partial_t u\in X\quad\text{is an isomorphism,}$$

for the quotient metric

$$d_Y(\dot u,\dot v)=\inf_{w\in\dot u,z\in\dot v}\|w-z\|_g.$$

Thus, for some $A > 0$,

$$d_Y(\dot{u}, \dot{v}) \leq A\|\partial_t(u - v)\|_X. \tag{6.4}$$

Since Ω is convex, it is easy to see that $N = h^p(\mathbb{R}_x)$ so (6.4) may be rewritten as (6.3). $\square$

Let us assume now that $\partial_t : X \to X$ is onto. To reach a contradiction we will violate (6.3). Define $f : \mathbb{R} \to \mathbb{R}$, by

$$f(x) = \begin{cases} 0 & \text{if } t \leq 0, \\ t & \text{if } 0 \leq t \leq 1, \\ 1 & \text{if } t \geq 1, \end{cases}$$

and choose $g(x) \in C_c^\infty(\mathbb{R})$, supported in $[0, 1]$, not identically zero and satisfying $\int g(x)\,dx = 0$. Given a positive integer n, define for each integer k, $0 \leq k \leq n - 1$,

$$\begin{aligned} f_k(t) &= \frac{1}{n} f(nt - k), \quad -1 \leq t \leq 1, \quad x \in \mathbb{R}, \\ g_k(x) &= g(nx - k). \end{aligned}$$

Thus,

$$\begin{cases} f_k(1) = 1/n, & 0 \leq k \leq n - 1, \\ f'_k(t) = \chi_{[\frac{k}{n}, \frac{k+1}{n}]}(t), & t \in \mathbb{R}, \end{cases}$$

and

$$\operatorname{supp} g_k \subset \left[\frac{k}{n}, \frac{k+1}{n}\right], \quad \|g_k\|_{h^p}^p \leq \frac{C_p}{n}, \quad 0 \leq k \leq n - 1.$$

For each positive integer n we set

$$u_n(t, x) = \sum_{k=0}^{n-1} f_k(t) g_k(x) + f_k(-t) g_k(-x) \in X.$$

Taking account of (6.3) we may find $a_n \in h^p(\mathbb{R})$ such that

$$\sup_{t \in [-1,1]} \|u_n(t, x) - a_n(x)\|_{h^p(\mathbb{R})}^p \leq 2A \sup_{t \in [-1,1]} \|\partial_t u(t, x)\|_{h^p(\mathbb{R})}^p.$$

In particular,

$$\begin{aligned} \|u_n(-1, x) - a_n(x)\|_{h^p(\mathbb{R})}^p &\leq 2A \sup_{t \in [-1,1]} \|\partial_t u_n(t, x)\|_{h^p(\mathbb{R})}^p, \\ \|u_n(1, x) - a_n(x)\|_{h^p(\mathbb{R})}^p &\leq 2A \sup_{t \in [-1,1]} \|\partial_t u_n(t, x)\|_{h^p(\mathbb{R})}^p. \end{aligned}$$

Therefore, by the triangle inequality,

$$\|u_n(1,x)-u_n(-1,x)\|^p_{h^p(\mathbb{R})} \le 4A \sup_{t\in[-1,1]} \|\partial_t u_n(t,x)\|^p_{h^p(\mathbb{R})}. \tag{6.5}$$

Note that

$$\|u_n(1,x)-u_n(-1,x)\|^p_{h^p(\mathbb{R})} = \frac{1}{n^p}\left\|\sum_{k=0}^{n-1} g_k(x) - \sum_{k=0}^{n-1} g_k(-x)\right\|^p_{h^p}. \tag{6.6}$$

Taking account of the trivial pointwise estimate

$$m_\phi h(x) = \sup_{0<\varepsilon\le 1} |\phi_\varepsilon * h(x)| \ge |h(x)|, \quad h\in L^\infty(\mathbb{R}),$$

we see that the right hand side in (6.6) is bounded below by

$$\frac{1}{n^p}\int \left|\sum_{k=0}^{n-1} g_k(x) - \sum_{k=0}^{n-1} g_k(-x)\right|^p dx,$$

which is greater than of equal to

$$\begin{aligned}\frac{1}{n^p}\int_0^1 \left|\sum_{k=0}^{n-1} g_k(x) - \sum_{k=0}^{n-1} g_k(-x)\right|^p dx &= \frac{1}{n^p}\int_0^1 \left|\sum_{k=0}^{n-1} g_k(x)\right|^p dx\\ &= \frac{1}{n^p}\int_0^1 \sum_{k=0}^{n-1} |g_k(x)|^p d\,x\\ &= \frac{C}{n^p}.\end{aligned}$$

Notice that we have equality in the second displayed equation because the terms in the sum have disjoint supports. Summing up, we have shown that

$$\frac{C}{n^p} \le \|u_n(1,x)-u_n(-1,x)\|^p_{h^p(\mathbb{R})}. \tag{6.7}$$

On the other hand,

$$\begin{aligned}\sup_{t\in I} \chi_{[k/n,(k+1)/n]}(t)\|g_k(x)\|^p_{h^p} &\le C_p/n,\\ \sup_{t\in I} \chi_{[(-k-1)/n,-k/n]}(t)\|g_k(-x)\|^p_{h^p} &\le C_p/n,\end{aligned}$$

and this implies

$$\sup_{t\in[-1,1]} \|\partial_t u_n(t,x)\|^p_{h^p(\mathbb{R})} \le C_p/n \tag{6.8}$$

because the functions $2n$ functions $f_k'(t)$ and $f_k'(-t)$, $0 \le k \le n-1$, have disjoint supports. Thus, (6.5), (6.7) and (6.8) lead to

$$\frac{C}{n^p} \le \frac{4AC_p}{n},$$

which is false for large n because $0 < p < 1$. □

Appendix A. Proof of Theorem 4.2

We will start by stating two well known lemmas; proofs can be found, for instance, in [CM] and [Hou1]. The first one is concerned with L^2 continuity of pseudodifferential operators with symbols in the class $S^0_{0,0}(\mathbb{R}^n)$ with sharp control on the number of derivatives of the symbol.

Lemma A.1. *Let $a(x,\xi)$ be a bounded function defined on $\mathbb{R}^n \times \mathbb{R}^n$ with bounded derivatives $\partial_x^\alpha \partial_\xi^\beta a(x,\xi)$, $|\alpha|, |\beta| \le [n/2]+1$ and assume that $a(x,\xi) = 0$ for $|\xi|$ sufficiently large. There is a positive dimensional constant C such that*

$$\|a(x,D)\|_{\mathcal{L}(L^2)} \le C.$$

The next ingredient is

Lemma A.2. *Let $a(x,D)$ be a pseudodifferential operator with symbol $a(x,\xi)$ such that $a(x,D) \in \mathcal{L}(L^2)$. For $T > 0$, denote by $a^T(x,D)$ the operator with symbol $a^T(x,\xi) = a(Tx, T^{-1}\xi)$. Then $a^T(x,D) \in \mathcal{L}(L^2)$ and*

$$\|a(x,D)\|_{\mathcal{L}(L^2)} = \|a^T(x,D)\|_{\mathcal{L}(L^2)}.$$

We now prove Theorem 4.2.

Proof: Pick $\alpha \in C_c^\infty(\mathbb{R}^n)$ such that $\alpha(\xi) = 1$ for $|\xi| < 1/2$ and $\alpha(\xi) = 0$ for $|\xi| \ge 1$. We may the write $a(x,\xi) \in \mathcal{B}_r S^0_{1,1}$ as

$$a(x,\xi) = \alpha(\xi)\, a(x,\xi) + (1-\alpha(\xi))\, a(x,\xi) = a_0(x,\xi) + a^\flat(x,\xi).$$

The operator $a_0(x,D)$ with symbol $a_0(x,\xi)$ is bounded in $L^2(\mathbb{R}^n)$ by Lemma A.1 and we need only show that $a^\flat(x,D)$ is bounded as well. Hence, we may assume from the start without loss of generality that

$a(x,\xi) \equiv 0$ for $|\xi| \leq 1/2$ and we do so. Choose $\phi \in C_c^\infty(\mathbb{R}^n)$ so that $1/3 \leq |\xi| \leq 1$ for ξ in the support of ϕ and

$$\sum_{j=0}^{\infty} \phi(2^{-j}\xi) = 1 \quad \text{if } |\xi| \geq 1/2.$$

We may write

$$a(x,\xi) = \sum_{j=0}^{\infty} \phi(2^{-j}\xi) a(x,\xi) = \sum_{j=0}^{\infty} a_j(x,\xi).$$

Note that $2^j/3 \leq |\xi| \leq 2^j$ on the support of $a_j(x,\xi)$, so we have

$$|\partial_x^\alpha \partial_\xi^\beta a_j(x,\xi)| \leq C 2^{j(|\alpha|-|\beta|)}, \quad |\alpha|, |\beta| \leq [n/2]+1. \tag{A.1}$$

Next, choose $\psi \in \mathcal{S}(\mathbb{R}^n)$ satisfying $\hat{\psi}(\xi) = 1$ for $|\xi| \leq 1$ and $\hat{\psi}(\xi) = 0$ for $|\xi| \geq 2$ and set

$$p_j(x,\xi) = \int a_j(x-u,\xi)\psi(2^j u) 2^{jn}\, du,$$
$$q_j(x,\xi) = \int \big(a_j(x,\xi) - a_j(x-u,\xi)\big)\psi(2^j u) 2^{jn}\, du$$

so $a_j(x,\xi) = p_j(x,\xi) + q_j(x,\xi)$. Taking Fourier transforms with respect to x we get

$$\hat{p}_j(\eta,\xi) = \hat{a}_j(\eta,\xi)\, \hat{\psi}(\eta/2^j),$$
$$\hat{q}_j(\eta,\xi) = \hat{a}_j(\eta,\xi)\, (1 - \hat{\psi}(\eta/2^j)).$$

Observe that $\hat{q}_j(\eta,\xi) \equiv 0$, since $1 - \hat{\psi}(\eta/2^j) = 0$ if $|\eta| \leq 2^j$ and we know that $|\eta| \leq r|\xi| \leq r2^j < 2^j$ on the support of $\hat{a}_j(\eta,\xi) = \phi(2^{-j}\xi)\hat{a}(\eta,\xi)$. Hence, $a_j(x,D) = p_j(x,D) \doteq P_j$. Set $\tilde{p}_j(x,\xi) = p_j(2^{-j}x, 2^j\xi)$. Then (A.1) yields

$$\left| D_x^\alpha D_\xi^\beta \tilde{p}_j(x,\xi) \right| \leq C_{\alpha,\beta}, \quad |\alpha|, |\beta| \leq [n/2]+1.$$

Thus, $\|P_j\|_{\mathcal{L}(L^2)} = \|\tilde{P}_j\|_{\mathcal{L}(L^2)} \leq C$, by Lemmas A.1 and A.2. Take now an arbitrary $f \in \mathcal{S}(\mathbb{R}^n)$ and let us look at the support of $\widehat{P_j f}$. Notice that

$$\widehat{P_j f}(\eta) = \int \hat{a}_j(\eta-\xi,\xi)\, \hat{\psi}(2^{-j}(\eta-\xi))\, \phi(2^{-j}\xi)\, \hat{f}(\xi)\, d\xi$$

and we have that $\phi(2^{-j}\xi) = 0$ unless $2^j/3 \leq |\xi| \leq 2^j$. Since $|\eta - \xi| \leq r|\xi|$ on the support of $\hat{a}_j(\eta-\xi,\xi)$, we conclude that $\widehat{P_j f}(\eta)$ is supported in the annulus

$$T_j = \{(1-r)2^{j-2} \le |\eta| \le 2^{j+1}\}.$$

If we choose a positive integer ℓ so that $1 - 2^{-\ell} \le r < 1 - 2^{-\ell-1}$ we see that $\widehat{P_j f}$ and $\widehat{P_k f}$ have disjoint supports for $|j-k| \ge \ell + 4$, in particular, $P_j f$ and $P_k f$ are orthogonal in $L^2(\mathbb{R}^n)$ by Parseval's formula. Therefore,

(A.2)

$$\Big\| \sum_{j=0}^{\infty} P_{(\ell+4)j+m} f \Big\|_{L^2}^2 = \sum_{j=0}^{\infty} \| P_{(\ell+4)j+m} f \|_{L^2}^2, \quad \text{for } m = 0, 1, \cdots, \ell+3$$

Let $\chi_j(\xi)$ be the characteristic function of the annulus $\{2^{j-\ell-2} \le |\xi| \le 2^{j+1}\}$ and set $\hat{f}_j = \hat{f}\chi_j$. Thus,

$$f = \sum_{j=0}^{\infty} f_{(\ell+4)j}.$$

It is easy to see that $\widehat{P_j f_k} \equiv 0$ if $|j-k| > \ell + 4$, so

$$\begin{aligned} \Big\| \sum_{j=0}^{\infty} P_{(\ell+4)j} f \Big\|_{L^2}^2 &= \sum_{j=0}^{\infty} \| P_{(\ell+4)j} f_{(\ell+4)j} \|_{L^2}^2 \\ &\le C^2 \sum_{j=0}^{\infty} \| f_{(\ell+4)j} \|_{L^2}^2 \le C^2 \|f\|_{L^2}^2. \end{aligned}$$

This gives an estimate for the left hand side of (A.2) when $m = 0$ and the same estimate holds for $m = 1, 2, \cdots, \ell+3$. Thus,

$$\|a(x,D)\|_{\mathcal{L}(L^2)} \le \sum_{m=0}^{\ell+3} \Big\| \sum_{j=0}^{\infty} P_{(\ell+4)j+m} \Big\|_{\mathcal{L}(L^2)} \le (\ell+4)C. \quad \square$$

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