Euclidean submanifolds with codimension two satisfying the equality in the DDVV inequality.

Marcos Dajczer & Ruy Tojeiro

Abstract

We classify Euclidean submanifolds with codimension two that satisfy the equality in an inequality due to De Smet, Dillen, Verstraelen and Vrancken involving the scalar curvature, the normal scalar curvature and the length of the mean curvature vector.

Let $f\colon M^n \to \mathbb{Q}_c^{n+p}$ be an isometric immersion of a Riemannian manifold of dimension n into a space form of dimension $n+p$ and constant sectional curvature c. Let

$$\rho = \frac{2}{n(n-1)} \sum_{1=i<j}^{n} R(e_i, e_j, e_j, e_i)$$

and

$$\rho^{\perp} = \frac{2}{n(n-1)} \left(\sum_{1=i<j}^{n} \sum_{1=r<s}^{p} \langle R^{\perp}(e_i, e_j)\xi_r, \xi_s \rangle^2 e_j, e_i) \right)^{1/2}$$

be the normalized scalar curvatures of the tangent and normal bundles, respectively, where $\{e_1, \dots, e_n\}$ (resp., $\{\xi_1, \dots, \xi_n\}$) is an orthonormal basis of the tangent (resp., normal) space, and R and $R^{\perp}$ denote the curvature tensors of the tangent and normal bundles, respectively. The following interesting inequality relating ρ, $\rho^{\perp}$ and the length of the mean curvature vector H of f was proved in [**DDVV**] for $p = 2$:

Theorem 1. *If $p = 2$ then*

$$\rho + \rho^{\perp} \leq \|H\|^2 + c. \tag{1}$$

Moreover, equality holds at $x \in M^n$ if and only if there exist orthonormal bases $\{e_1, \dots, e_n\}$ and $\{\eta, \zeta\}$ of the tangent and normal spaces at x such that the shape operators A_η and

A_ζ *have the form*

$$A_\eta = \begin{bmatrix} \lambda & \mu & 0 & \cdots & 0 \\ \mu & \lambda & 0 & \cdots & 0 \\ 0 & 0 & \lambda & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda \end{bmatrix}; \qquad A_\zeta = \begin{bmatrix} \mu & 0 & 0 & \cdots & 0 \\ 0 & -\mu & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix} \tag{2}$$

The inequality (1) is also known to hold for surfaces [**GR**] and submanifolds with flat normal bundle of any codimension [**Ch**] (in particular for hypersurfaces) as well as for various special classes of submanifolds (see [**DFV**] and the references therein), and has been conjectured in [**DDVV**] to hold for any submanifold of a space form. Recently, the conjecture was proved in [**CL**] for any three dimensional submanifolds with arbitrary codimension of any space form.

It was shown in [**DFV**] that, for any isometric immersion $f\colon M^n \to \mathbb{Q}_c^{n+p}$, the inequality (1) holds at a point $x \in M^n$ if and only if the following inequality is satisfied for the traceless parts $B_1, \ldots, B_p$ of the shape operators of f with respect to any orthonormal normal frame at x:

$$\sum_{\alpha,\beta=1}^{p} \|[B_\alpha, B_\beta]\|^2 \leq \left(\sum_{\alpha=1}^{p} \|B_\alpha\|^2 \right)^2. \tag{3}$$

For surfaces $f\colon M^2 \to \mathbb{Q}_c^{2+p}$ of arbitrary codimension p, it was shown in [**GR**] that the equality in (1) holds at $x \in M^2$ if and only if the elipse of curvature of f at x is a circle. For $n \geq 3$ and $p = 2$ it was claimed in [**DFV**] that if the equality in (1) holds then f is either minimal or umbilical. Unfortunately, the proof contained a mistake, and that this can not be the case can already be seen from the fact that the class of submanifolds $f\colon M^n \to \mathbb{Q}_c^{n+p}$ for which the equality holds in (1) is invariant under conformal transformations of the ambient space. In fact, under such a transformation the traceless parts of the shape operators only change by multiplication by a common smooth function on M^n (see Lemma 7 below).

In this paper we give a complete classification of isometric immersions $f\colon M^n \to \mathbb{R}^{n+2}$ for which equality holds eveywhere in the inequality (1). Our classification applies even to the two-dimensional case, namely, we provide an explicit construction of all surfaces $f\colon M^2 \to \mathbb{R}^4$ whose elipse of curvature at any point is a circle.

1 The results

We start with the following preliminary lemma.

Lemma 2. *Let* $g\colon L^2 \to \mathbb{R}^{n+2}$ *and* $h\colon L^2 \to \mathbb{R}^{n+2}$ *be conjugate minimal surfaces,i.e.,* $h_* = g_* \circ J$*, where* J *is the complex structure of* L^2*. Set* $r = \|h\|$*,* $a = (1 - \|\nabla r\|^2)^{1/2}$ *and* $Z = -J\nabla r$*. Then there exists a (unique) smooth unit normal vector field* ξ *to* g *such that*

$$h = -r(g_* Z + a\xi). \tag{4}$$

Moreover, we have

$$\langle B_w Z, X\rangle = a\langle \nabla_X^\perp w, \xi\rangle \ \text{ for all } w \in \Lambda := span\{\xi\}^\perp \subset T_g^\perp L^2, \tag{5}$$

and

$$B_\xi = \frac{1}{a}(Hess\, r - \frac{1}{r}S)J, \tag{6}$$

where $S = I - \langle \nabla r, *\rangle \nabla r$ *and* B_ν *is the shape operator of* g *in the normal direction* ν*.*

Proof: Decompose $h = g_* T + \beta$ into tangent and normal components to g. From $h_* = g_* \circ J$ we obtain

$$\begin{cases} \nabla_X T - B_\beta X = JX, \\ \alpha_g(X,T) + \nabla_X^\perp \beta = 0, \end{cases} \tag{7}$$

for any $X \in TL^2$. It also follows from $h_* = g_* \circ J$ that the tangent components of the position vector h with respect to g and h coincide. Since the latter is $h_*(r\nabla r)$, we obtain

$$g_* T = h_*(r\nabla r) = g_*(rJ\nabla r),$$

hence $T = rJ\nabla r = -rZ$. From $\|h\|^2 = \|T\|^2 + \|\beta\|^2$ we get

$$r^2 = r^2\|\nabla r\|^2 + \|\beta\|^2,$$

which gives $\|\beta\| = ar$. Thus we can write $\beta = -ar\xi$ for a unit normal vector field ξ. Then, the first equation in (7) reduces to (6), whereas the second to (5) and $rB_\xi Z = -\nabla(ar)$. Notice that the latter also follows from (6). ∎

We are now in a position to state and prove our main result.

Theorem 3. *Let* $g\colon L^2 \to \mathbb{R}^{n+2}$*,* $r, a \in C^\infty(L^2)$*,* $\xi \in T_g^\perp L^2$ *and* Λ *be as in Lemma 2. On the unit bundle* Λ_1 *of* Λ *define a map* ϕ *into* $\mathbb{R}^{n+2}$ *by*

$$\phi(y,w) = g(y) - r(y)\eta(y,w), \tag{8}$$

where

$$\eta(y,w) = g_*\nabla r(y) + a(y)w. \tag{9}$$

Then, at regular points, ϕ *parametrizes a submanifold satisfying the equality in the DDVV inequality.*

Conversely, any submanifold satisfying the equality in the DDVV inequality can be constructed in this way, at least locally.

The converse statement means, more precisely, that if $f\colon M^n \to \mathbb{R}^{n+2}$ is an isometric immersion satisfying the equality in the DDVV inequality, then there exist a minimal surface $g\colon L^2 \to \mathbb{R}^{n+2}$ and a difeomorphism $\psi\colon \Lambda_1 \to M^n$ such that $f \circ \psi$ is given by (8), where $r, a \in C^\infty(L^2)$, $\xi \in T_g^\perp L^2$ and Λ as in Lemma 2.

Proof: We first prove the converse. Let $f\colon M^n \to \mathbb{R}^{n+2}$ be an isometric immersion satisfying the equality in the DDVV inequality. Then, by Theorem ? in [**DDVV**], there exist an orthonormal frame $\{\eta, \zeta\}$ of $T_f^\perp M^n$ and an orthonormal tangent frame $\{E_1, \ldots, E_n\}$ with respect to which the shape operators A_η and A_ζ have the form. Moreover, by ?? we have $E_k(\lambda) = 0$ and $\nabla^\perp_{E_k}\eta = 0$ for $k \geq 3$. Therefore $\lambda\eta$ is a *Dupin principal normal* of multiplicity $n-2$. This means that the subspaces

$$E_\eta(x) = \{T \in T_xM\colon \alpha_f(T, X) = \lambda\langle T, X\rangle\eta, \text{ for all } X \in T_xM\},$$

define a smooth distribution E_η of rank $n-2$ satisfying

$$T(\lambda) = 0 \text{ and } \nabla^\perp_T\eta = 0 \text{ for any } T \in E_\eta. \tag{10}$$

In addition, we have that η is *generic*, in the sense that $E_\eta = \ker(A_\eta - \lambda I)$.

Let L^2 be the quotient space of leaves of E_η. Define $g\colon M^n \to \mathbb{R}^{n+2}$ by $g = f + r\eta$, with $r = 1/\lambda$. From (10) we have

$$g_*T = f_*T - rf_*A_\eta T = 0,$$

hence g factors through a map on L^2 that we still denote by g. Also, the function r gives rise to a smooth function on L^2 that is also denoted by r. By Proposition 1 in [**DF**], there exist a smooth unit vector field ξ normal to g and a diffeomorphism $\psi\colon \Lambda_1 \to M^n$, where Λ_1 stands for the unit bundle of the orthogonal complement Λ of ξ in $T_gL^\perp$, such that

$$\eta(y, w) := (\eta \circ \psi)(y, w) = g_*\nabla r(y) + \rho(y)\xi(y) + \Omega(y)w \tag{11}$$

for smooth functions ρ and Ω on L^2 satisfying

$$\|\nabla r\|^2 + \rho^2 + \Omega^2 = 1 \tag{12}$$

and

$$\phi(y, w) := (f \circ \psi)(y, w) = g(y) - r(y)\eta(y, w). \tag{13}$$

We identify the tangent space $T_{(y,w)}\Lambda_1$ with the direct sum $T_yL^2 \oplus \{w\}^\perp$, where $\{w\}^\perp$ denotes the orthogonal complement of span$\{w\}$ in $\Lambda(y)$, and write vectors $Y \in T_{(y,w)}\Lambda_1$ as $Y = (X, V)$ according to this decomposition. We also denote by $\mathcal{V}$ the corresponding *vertical* subbundle of $T\Lambda_1$, that is, the fiber $\mathcal{V}(y, w)$ of $\mathcal{V}$ at $(y, w) \in \Lambda_1$ is $\{w\}^\perp$. Clearly, we have $\mathcal{V} = \psi_*E_\eta$.

Since η is a Dupin principal normal of f, the orthogonal complement $\eta^\perp$ of span$\{\eta\}$ in $T_f^\perp M$ is constant in $\mathbb{R}^{n+2}$ along E_η. Therefore, if ζ is a unit vector field spanning $\eta^\perp$, then the map $\zeta \circ \phi$, which we also denote simply by ζ, is constant along $\mathcal{V}$. Thus we may write

$$\zeta = g_* Z + a\xi \tag{14}$$

for a smooth vector field Z and a smooth function a on L^2 satisfying

$$\|Z\|^2 + a^2 = 1. \tag{15}$$

Since $(\ker(A_\eta - \lambda I)^\perp$ has rank two everywhere, the function a is nowhere vanishing. Otherwise $g_* Z$ would be somewhere tangent to ϕ, which would imply, by taking tangent components in

$$\phi_*(X,0) = g_* X - \langle \nabla r, X\rangle \eta - r\eta_*(X,0), \tag{16}$$

that $Z \in \ker(A_\eta - \lambda I)$, a contradiction.

The orthogonality between η and ζ yields

$$Z(r) + a\rho = 0. \tag{17}$$

From the fact that $\mathcal{V} = \ker A_\zeta$ we obtain

$$0 = \langle \zeta_*(X,0), \phi_*(y,w)(0,V)\rangle \ \text{ for any } \ (y,w) \in \Lambda_1 \ \text{ and any } \ V \in w^\perp.$$

Using that

$$\phi_*(0,V) = -r\Omega V, \tag{18}$$

this gives

$$\langle B_w Z, X\rangle = a\langle \nabla^\perp_X w, \xi\rangle \ \text{ for all } \ w \in \Lambda. \tag{19}$$

Now, taking (19) into account we obtain for any $(X,0) \in T_{(y,w)}\Lambda_1$ that

$$\begin{aligned}
0 &= \langle \zeta(y,w), \phi_*(X,0)\rangle \\
&= X\langle \zeta, g\rangle - \langle \zeta_*(X,0), g - r\eta\rangle \\
&= \langle Z, X\rangle + \langle \tilde{\nabla}_X(g_* Z + a\xi), r\eta\rangle \\
&= \langle Z, X\rangle - r\langle \operatorname{Hess} r(Z) + B_\xi(a\nabla r - \rho Z) + a\nabla\rho, X\rangle,
\end{aligned}$$

hence

$$\operatorname{Hess} r(Z) - \frac{1}{r} Z + B_\xi(a\nabla r - \rho Z) + a\nabla\rho = 0. \tag{20}$$

By the assumption, there exists an orthonormal frame $\{E_1, E_2\}$ of $\mathcal{H} = \mathcal{V}^\perp$ such that

$$\left\{ \begin{array}{l} A_\eta E_1 = \lambda E_1 + \mu E_2 \\ A_\eta E_1 = \lambda E_1 + \mu E_2 \end{array} \right. , \qquad \left\{ \begin{array}{l} A_\zeta E_1 = \mu E_1 \\ A_\zeta E_2 = -\mu E_2 \end{array} \right. \tag{21}$$

Setting $Y_1 = \dfrac{1}{\sqrt{2}}(E_1 + E_2)$ and $Y_2 = \dfrac{1}{\sqrt{2}}(E_1 - E_2)$ we obtain

$$\begin{cases} A_\eta Y_1 = (\lambda + \mu) Y_1 \\ A_\eta Y_2 = (\lambda - \mu) Y_2 \end{cases}, \qquad \begin{cases} A_\zeta E_1 = \mu E_2 \\ A_\zeta E_2 = \mu E_1 \end{cases} \tag{22}$$

Therefore

$$\begin{cases} \eta_* Y_1 = -(\lambda + \mu)\phi_* Y_1 + \langle \eta_* Y_1, \zeta\rangle \zeta \\ \eta_* Y_2 = -(\lambda - \mu)\phi_* Y_2 + \langle \eta_* Y_2, \zeta\rangle \zeta \\ \zeta_* Y_1 = -\mu \phi_* Y_2 + \langle \zeta_* Y_1, \eta\rangle \eta \\ \eta_* Y_2 = -\mu \phi_* Y_1 + \langle \zeta_* Y_2, \eta\rangle \eta \end{cases} \tag{23}$$

and

$$\langle \phi_* Y_i(y, w), \phi_*(0, V)\rangle = 0, \ \ 1 \le i \le 2, \ \ \text{for any} \ \ (y, w) \in \Lambda_1 \ \ \text{and} \ \ V \in \{w\}^\perp. \tag{24}$$

Write $Y_1 = (X_1, V_1)$ and $Y_2 = (X_2, V_2)$ according to the splitting $T_{(y,w)}\Lambda_1 = T_y L^2 \oplus \{w\}^\perp$. Using that

$$\eta_*(0, V) = \Omega V, \quad \phi_*(0, V) = -r\Omega V, \quad \text{and} \ \ \zeta_*(0, V) = 0, \tag{25}$$

we obtain from (24) that

$$(\eta_*(X_i, 0))_{w^\perp} = -\Omega V_i, \quad 1 \le i \le 2. \tag{26}$$

Taking components in $H = TL^2 \oplus \operatorname{span}\{\xi\} \oplus \operatorname{span}\{w\}$ of the equations of (23) gives

$$\begin{cases} -r^2\mu(\eta_*(X_1, 0))_H = (1 + r\mu)(-g_* X_1 + r_1 \eta) + r\langle \eta_*(X_1, 0), \zeta\rangle \zeta \\ r^2\mu(\eta_*(X_2, 0))_H = (1 - r\mu)(-g_* X_2 + r_2 \eta) + r\langle \eta_*(X_2, 0), \zeta\rangle \zeta \\ \zeta_*(X_1, 0) = -\mu g_* X_2 + \mu r(\eta_*(X_2, 0))_H + \mu r_2 \eta) + \langle \zeta_*(X_1, 0), \eta\rangle \eta \\ \zeta_*(X_2, 0) = -\mu g_* X_1 + \mu r(\eta_*(X_1, 0))_H + \mu \langle \nabla r, X_1\rangle \eta) + \langle \zeta_*(X_2, 0), \eta\rangle \eta \end{cases} \tag{27}$$

Now we have

$$\eta_*(X, 0) = g_* Q_w X + \langle T_w, X\rangle \xi + \langle P_w, X\rangle w + (\eta_*(X, 0))_{w^\perp}, \tag{28}$$

with

$$\begin{cases} Q_w = \operatorname{Hess} r - \rho B_\xi - \Omega B_w, \\ T_w = \nabla \rho + B_\xi \nabla r + \dfrac{\Omega}{a} B_w Z, \\ P_w = \nabla \Omega + B_w \nabla r - \dfrac{\rho}{a} B_w Z \end{cases} \tag{29}$$

and

$$\zeta_*(X,0) = g_* DX + \langle K, X\rangle \xi, \tag{30}$$

with

$$\begin{cases} DX = \nabla_X Z - aB_\xi X, \\ K = \nabla a + B_\xi Z \end{cases} \tag{31}$$

Since

$$\begin{aligned} \langle \nabla_X Z, \nabla r\rangle &= XZ(r) - \langle Z, \operatorname{Hess} r(X)\rangle \\ &= -X(a)\rho - aX(\rho) - \langle Z, \operatorname{Hess} r(X)\rangle \\ &= -\langle \rho\nabla a + a\nabla\rho + \operatorname{Hess} r(Z), X\rangle, \end{aligned}$$

where we have used (17) for the second equality, we obtain using (20) that

$$\begin{aligned} \langle DX, \nabla r\rangle &= \langle \nabla_X Z, \nabla r\rangle - a\langle B_\xi \nabla r, X\rangle \\ &= -\langle \rho K + \frac{1}{r} Z, X\rangle. \end{aligned}$$

Therefore,

$$\langle \zeta_*(X,0), \eta\rangle = \langle DX, \nabla r\rangle + \rho\langle K, X\rangle = -\frac{1}{r}\langle Z, X\rangle. \tag{32}$$

Then, denoting $r_i = \langle \nabla r, X_i\rangle$, $1 \le i \le 2$, the w-component of (27) gives

$$\begin{cases} r^2\mu\langle P_w, X_1\rangle = -(1+r\mu)\Omega r_1. \\ r^2\mu\langle P_w, X_2\rangle = (1-r\mu)\Omega r_2. \\ r^2\mu\langle P_w, X_2\rangle = -r\mu\Omega r_2 + \Omega\langle Z, X_1\rangle. \\ r^2\mu\langle P_w, X_1\rangle = -r\mu\Omega r_1 + \Omega\langle Z, X_2\rangle. \end{cases} \tag{33}$$

Replacing the first two equations into the last two yields

$$\begin{cases} r_2 = \langle Z, X_1\rangle. \\ r_1 = -\langle Z, X_2\rangle. \end{cases} \tag{34}$$

Taking the tangent component to g of (27) we obtain

$$\begin{cases} -r^2\mu Q_w(X_1) + (1+r\mu)S(X_1) - \langle Z, X_1\rangle Z = 0. \\ r^2\mu Q_w(X_2) + (1-r\mu)S(X_2) - \langle Z, X_2\rangle Z = 0. \\ rD(X_1) + r\mu S(X_2) - r^2\mu Q_w(X_2) + \langle Z, X_1\rangle \nabla r = 0. \\ rD(X_2) + r\mu S(X_1) - r^2\mu Q_w(X_1) + \langle Z, X_2\rangle \nabla r = 0, \end{cases} \tag{35}$$

where

$$S = I - \langle \nabla r, * \rangle \nabla r. \tag{36}$$

Taking inner products of the first and second equations by X_2 and X_1, respectively, and adding them up we get

$$\langle Z, X_1 \rangle \langle Z, X_2 \rangle = \langle S(X_1), X_2 \rangle = \langle X_1, X_2 \rangle - r_1 r_2. \tag{37}$$

Replacing (34) into (37) implies that $\langle X_1, X_2 \rangle = 0$. Moreover, replacing the first two equations of (35) into the last two gives

$$\begin{cases} rD(X_1) + S(X_2) - \langle Z, X_2 \rangle Z + \langle Z, X_1 \rangle \nabla r = 0, \\ rD(X_2) - S(X_1) + \langle Z, X_1 \rangle Z + \langle Z, X_2 \rangle \nabla r = 0. \end{cases} \tag{38}$$

The ξ-component of (27) yields

$$\begin{cases} -r^2 \mu \langle T_w, X_1 \rangle = \langle (1 + r\mu)\rho \nabla r + aZ, X_1 \rangle, \\ r^2 \mu \langle T_w, X_2 \rangle = \langle (1 - r\mu)\rho \nabla r + aZ, X_2 \rangle, \\ r\langle K, X_1 \rangle = r^2 \mu \langle T_w, X_2 \rangle + r\mu\rho r_2 - \rho \langle Z, X_1 \rangle, \\ r\langle K, X_2 \rangle = r^2 \mu \langle T_w, X_2 \rangle + r\mu\rho r_1 - \rho \langle Z, X_2 \rangle. \end{cases} \tag{39}$$

Replacing the first two equations into the last two gives

$$\begin{cases} r\langle K, X_1 \rangle = \langle \rho \nabla r + aZ, X_2 \rangle - \rho \langle Z, X_1 \rangle, \\ r\langle K, X_2 \rangle = -\langle \rho \nabla r + aZ, X_1 \rangle - \rho \langle Z, X_2 \rangle \end{cases} \tag{40}$$

We now use that

$$\delta_{ij} = \langle \phi_* Y_i, \phi_* Y_j \rangle, \ \ 1 \le i, j \le 2. \tag{41}$$

From (26) we get

$$\langle \phi_*(0, V_1), \phi_*(X_2, 0) \rangle = \langle -r\Omega V_1, -r\eta_*(X_2, 0) \rangle = -r^2 \Omega^2 \langle V_1, V_2 \rangle = \langle \phi_*(0, V_2), \phi_*(X_1, 0) \rangle.$$

On the other hand,

$$\langle \phi_*(0, V_1), \phi_*(0, V_2) \rangle = r^2 \Omega^2 \langle V_1, V_2 \rangle = \langle (\eta_*(X_1, 0))_{w^\perp}, (\eta_*(X_2, 0))_{w^\perp} \rangle,$$

hence (41) reduces to $\delta_{ij} = \langle (\eta_*(X_1, 0))_H, (\eta_*(X_2, 0))_H \rangle$, which gives

$$\begin{aligned} \delta_{ij} = & \langle X_i, X_j \rangle - \langle \nabla r, X_i \rangle \langle \nabla r, X_j \rangle - 2r \langle Q_w X_i, X_j \rangle + r^2 (\langle Q_w X_i, Q_w X_j \rangle \\ & + \langle T_w, X_i \rangle \langle T_w, X_j \rangle + \langle P_w, X_i \rangle \langle P_w, X_j \rangle. \end{aligned} \tag{42}$$

Using (34), we compute from (35) that

$$\begin{cases} r^2\mu\langle Q_w X_1, X_1\rangle = (1+r\mu)(\|X_1\|^2 - r_1^2) - r_2^2, \\ r^2\mu\langle Q_w X_2, X_2\rangle = -(1-r\mu)(\|X_2\|^2 - r_2^2) - r_1^2, \\ r\langle Q_w X_1, X_2\rangle = \langle X_1, X_2\rangle - r_1 r_2^2, \end{cases} \tag{43}$$

and

$$\begin{cases} r^4\mu^2\|Q_w X_1\|^2 = (1+r\mu)^2(\|X_1\|^2 + (\|\nabla r\|^2 - 2)r_1^2) - 2(1+r\mu)r_2^2 \\ \qquad -2(1+r\mu)a\rho r_1 r_2 + (1-a^2)r_2^2, \\ r^4\mu^2\|Q_w X_2\|^2 = (1-r\mu)^2(\|X_2\|^2 + (\|\nabla r\|^2 - 2)r_2^2) - 2(1-r\mu)r_1^2 \\ \qquad +2(1-r\mu)a\rho r_1 r_2 + (1-a^2)r_1^2, \\ -r^4\mu^2\langle Q_w X_1, Q_w X_2\rangle = -(1-r^2\mu^2)(1+\rho^2+\Omega^2)r_1 r_2 + (1+r\mu)(r_1 r_2 + a\rho r_1^2) \\ \qquad -(1-r\mu)(-r_1 r_2 + a\rho r_2^2) - (1-a^2)r_1 r_2. \end{cases} \tag{44}$$

From (34) and (39) we obtain

$$\begin{cases} r^2\mu\langle T_w, X_1\rangle = -(1+r\mu)\rho r_1 - a r_2, \\ r^2\mu\langle T_w, X_2\rangle = (1-r\mu)\rho r_2 - a r_1. \end{cases} \tag{45}$$

Thus

$$\begin{cases} r^4\mu^2\langle T_w, X_1\rangle^2 = (1+r\mu)^2\rho^2 r_1^2 + a^2 r_2^2 + 2(1+r\mu)a\rho r_1 r_2 \\ r^2\langle T_w, X_1\rangle\langle T_w, X_2\rangle = -\frac{1}{r^2\mu^2}((1-r^2\mu^2)\rho^2 r_1 r_2 - (1+r\mu)a\rho r_1^2 \\ \qquad +(1-r\mu)a\rho r_2^2 - a^2 r_1 r_2). \\ r^4\mu^2\langle T_w, X_2\rangle^2 = (1-r\mu)^2\rho^2 r_2^2 + a^2 r_1^2 - 2(1-r\mu)a\rho r_1 r_2 \end{cases} \tag{46}$$

From the first two equations in (33) we get

$$\begin{cases} r^2\langle P_w, X_1\rangle^2 = \frac{(1+r\mu)^2}{r^2\mu^2}\Omega^2 r_1^2. \\ r^2\langle P_w, X_2\rangle^2 = \frac{(1-r\mu)^2}{r^2\mu^2}\Omega^2 r_2^2. \\ r^2\langle P_w, X_1\rangle\langle P_w, X_2\rangle = -\frac{1-r^2\mu^2}{r^2\mu^2}\Omega^2 r_1 r_2. \end{cases} \tag{47}$$

Replacing (43), (44), (46) and (47) into (42) we obtain

$$\|X_1\|^2 = r^2\mu^2 + r_1^2 + r_2^2 = \|X_2\|^2. \tag{48}$$

In particular, this and (34) imply that

$$\langle \nabla r, Z\rangle = 0 \;\text{ and }\; \|Z\| = \|\nabla r\|.$$

We conclude from (17) and the first of the preceding equations that $\rho = 0$. Using this, equations (12) and (15) and the second of the preceding equations we get

$$a^2 = 1 - \|Z\|^2 = 1 - \|\nabla r\|^2 = \Omega^2,$$

hence we may assume that $a = \Omega$. We also get from (48) that

$$\|X_i\|^2 = \frac{r^2\mu^2}{a^2}, \;\; 1 \le i \le 2. \tag{49}$$

Now, we have for S given by (36) that

$$\begin{cases} S(Z) = Z, \\ S(\nabla r) = a^2\nabla r, \end{cases} \tag{50}$$

where we have used that $Z(r) = 0$ for the first equation. Thus, system (35) reduces to

$$rQ_w = S + \frac{a^2}{r\mu}R_w, \tag{51}$$

where R_w denotes the reflection with respect to the line spanned by X_1, together with equations (38), which can be written as

$$rDX + SJX - \langle Z, JX\rangle Z + \langle Z, X\rangle \nabla r = 0,$$

or equivalently,

$$rDX = -JX - \langle \nabla r, X\rangle Z. \tag{52}$$

Taking (34) into account, system (40) is now

$$rB_\xi Z = -\nabla(ar), \tag{53}$$

and the first two equations in (39) reduce to

$$B_\xi \nabla r + B_w Z = -\frac{a}{r^2\mu}R_w Z. \tag{54}$$

On the other hand, system (33) is now

$$rB_w\nabla r + \frac{a}{r\mu}R_w\nabla r = -\nabla(ar). \tag{55}$$

Replacing (53) into (55) gives

$$-B_\xi Z + B_w\nabla r = -\frac{a}{r^2\mu}R_w\nabla r. \tag{56}$$

Using that R_w is symmetric and traceless, it follows from (54) and (56) that

$$\begin{cases} \langle -B_\xi Z + B_w\nabla r, \nabla r\rangle = -\langle B_\xi\nabla r + B_w Z, Z\rangle \\ \langle -B_\xi Z + B_w\nabla r, Z\rangle = \langle B_\xi\nabla r + B_w Z, \nabla r\rangle. \end{cases} \tag{57}$$

The first (resp., second) equation in (57) implies that B_w (resp., B_ξ) has zero trace, therefore g is a minimal surface.

We have from the first equation in (29) and equation (51) that

$$B_w = \frac{1}{a}(\operatorname{Hess} r - \frac{1}{r}S) - \frac{a}{r^2\mu}R_w, \tag{58}$$

and then (54), (56) and (58) yield

$$B_\xi = (B_w + \frac{a}{r^2\mu}R_w)J = \frac{1}{a}(\operatorname{Hess} r - \frac{1}{r}S)J, \tag{59}$$

where J is the complex structure on L^2 defined by $JZ = \nabla r$. From (19) we have

$$\tilde\nabla_X\xi = -g_*B_\xi X + \nabla^\perp_X\xi = -g_*B_\xi X - \frac{1}{a}(\alpha_g(Z,X) - \langle B_\xi Z, X\rangle\xi),$$

hence

$$-ar\tilde\nabla_X\xi + r\langle B_\xi Z, X\rangle\xi = arg_*B_\xi X + r\alpha_g(Z,X).$$

In view of (53), the left-hand-side can be written as $\tilde\nabla_X(-ar\xi)$, whereas for the right-hand-side we have

$$\begin{aligned} arg_*B_\xi X + r\alpha_g(Z,X) &= arg_*B_\xi X + r(\tilde\nabla_X g_*Z - g_*\nabla_X Z) \\ &= arg_*B_\xi X - g_*(r\nabla_X Z) - X(r)g_*Z + \tilde\nabla_X(rg_*Z) \\ &= g_*(arB_\xi X - r\nabla_X Z - X(r)Z) + \tilde\nabla_X(rg_*Z). \end{aligned}$$

Therefore

$$\tilde\nabla_X(-ar\xi - rg_*Z) = g_*(arB_\xi X - r\nabla_X Z - X(r)Z).$$

But

$$arB_\xi X - r\nabla_X Z - X(r)Z = -rDX - X(r)Z = JX,$$

by (52) and the first equation in (31). Thus

$$\tilde{\nabla}_X(-ar\xi - rg_*Z) = g_*JX. \tag{60}$$

This shows that $h = -ar\xi - rg_*Z$ is a conjugate minimal surface to g. Moreover, since

$$\|h\|^2 = r^2(\|Z\|^2 + a^2) = r^2,$$

it follows that r, a and ξ are given as in Lemma 2, and the proof of the converse is completed.

We now prove the direct statement. Let $g\colon L^2 \to \mathbb{R}^{n+2}$, $r, a \in C^\infty(L^2)$, $Z \in TL^2$, $\xi \in T_g^\perp L^2$ and Λ be as in Lemma 2. Define $\phi\colon \Lambda_1 \to \mathbb{R}^{n+2}$ by (8). From

$$\phi_*(X,0) = g_*X - \langle \nabla r, X\rangle \eta - r\eta_*(X,0), \ \text{ and } \ \phi_*(0,V) = -arV$$

it follows that η is a unit normal vector field to ϕ. Moreover, since $\phi + r\eta = g$ does not depend on w, we have that $A_\eta|_\mathcal{V} = r^{-1}I$.

Let ζ be defined by (14). Then ζ has unit length and is orthogonal to η. From (6) we get

$$r\operatorname{Hess} r(Z) - Z + aB_\xi \nabla r = 0. \tag{61}$$

By the computations before (20), it follows using (61) and (5) that ζ is normal to ϕ and satisfies $A_\zeta|_\mathcal{V} = 0$. Therefore, to complete the proof it suffices to show that there exists an orthonormal frame $\{Y_1, Y_2\}$ in $T\Lambda_1$ (with respect to the metric induced by ϕ) satisfying (23) and (24).

First observe that we can find $\mu \in C^\infty(\Lambda_1)$ and, for each $w \in \Lambda_1$, a reflection R_w on TL^2 such that B_w is given by (58). In fact, since B_w and B_ξ are traceless symmetric 2×2 matrices, we have $(B_w + B_\xi J)^2 = \lambda^2 I$ for some smooth function λ. Therefore $B_w + B_\xi J = \lambda R_w$ for some reflection R_w. It now suffices to define μ by $\lambda = -a/(r^2\mu)$.

For each $w \in \Lambda_1$, let $\{\bar{X}_1, \bar{X}_2\}$ be the orthonormal basis of eigenvectors of R_w, with $\bar{X}_1$ corresponding to the eigenvalue $+1$. Define

$$X_i = \frac{r\mu}{a}\bar{X}_i, \ \ 1 \le i \le 2,$$

and set

$$V_i = -\frac{1}{a}(\eta_*(X_i, 0))_{w^\perp}, \ \ 1 \le i \le 2.$$

We claim that $\{Y_1, Y_2\}$ given by $Y_i = (X_i, V_i)$, $1 \le i \le 2$, is the desired orthonormal frame.

It follows from $(\eta_*(X_i, 0))_{w^\perp} = -aV_i$ for $1 \le i \le 2$ that (24) is satisfied. In particular, in order to check that $\{Y_1, Y_2\}$ is an orthonormal frame it suffices to verify (42).

It also follows from $(\eta_*(X_i, 0))_{w^\perp} = -aV_i$, $1 \le i \le 2$, that the $w^\perp$-components of both sides of all equations in (23) coincide. Therefore, it suffices to prove that (27), or equivalently, (33), (35) and (39), holds for X_1 and X_2.

Since we have (34), because X_1, X_2 are orthogonal vectors with the same norm, and so are Z and ∇r, system (33) reduces to its first two equations. These are in turn equivalent to (55), which follows from (58).

Now, since B_w is given by (58), it follows that (51) holds. Moreover, from (59) we get (52), hence (35) is satisfied.

From (59) we obtain (53), and hence (40). Moreover, (59) and (58) imply (54), thus (39) is satisfied.

Finally, we now have (43), (44), (46) and (47), hence (42) follows by using that $\langle X_1, X_2\rangle = 0$ and $\|X_i\| = r\mu/a$ for $1 \leq i \leq 2$.

Corollary 4. *Let $g\colon L^2 \to \mathbb{R}^{n+2}$, $r, a \in C^\infty(L^2)$, $\xi \in T_g^\perp L^2$ and Λ be as in Lemma 2. Choose a smooth unit normal vector field w orthogonal to ξ and define $\phi\colon L^2 \to \mathbb{R}^4$ by*

$$\phi = g - r(g_*\nabla r + aw). \tag{62}$$

Then, at regular points, ϕ parametrizes a surface whose elipse of curvature at any point is a circle.

Conversely, any surface whose elipse of curvature at any point is a circle can be constructed in this way, at least locally.

Among isometric immersions $f\colon M^n \to \mathbb{R}^{n+2}$ for which equality holds eveywhere in the inequality (1), a special class is formed by those obtained by composing a minimal isometric immersion of rank two (for which $\lambda = 0$ in (2)) with an inversion in $\mathbb{R}^{n+2}$. In the nex result we characterize this class in terms of the description in Theorem 3.

Proposition 5. *Let $f\colon M^n \to \mathbb{R}^{n+2}$ be an isometric immersion for which the equality holds eveywhere in the inequality (1). Then f is the composition of a minimal isometric immersion of rank two (for which $\lambda = 0$ in (2)) with an inversion in $\mathbb{R}^{n+2}$ if and only if it is constructed as in Theorem 3 by means of a minimal surface $g\colon L^2 \to \mathbb{R}^{n+2}$ whose elipse of curvature at any point is a circle.*

Proof: Let $g\colon L^2 \to \mathbb{R}^{n+2}$, $r, a \in C^\infty(L^2)$, $\xi \in T_g^\perp L^2$ and Λ be as in Lemma 2. Define $\phi, \eta\colon \Lambda_1 \to \mathbb{R}^{n+2}$ as in Theorem 3. Then ϕ is the composition of a minimal isometric immersion of rank two (for which $\lambda = 0$ in (2)) with an inversion in $\mathbb{R}^{n+2}$ if and only if the images by ϕ of all leaves of $\mathcal{V}$ have a common point $P_0 \in \mathbb{R}^{n+2}$. This is the case if and only if there exists a section $\hat{w}$ of Λ_1 such that

$$\phi(y, \hat{w}(y)) = g(y) - r(y)\eta(y, \hat{w}(y)) = P_0 \text{ for all } y \in L^2. \tag{63}$$

If such a section exists, then we obtain from (63) that

$$\langle g - P_0, g_*X\rangle = \langle rg_*\nabla r, g_*X\rangle \text{ for all } X \in TL^2,$$

which implies that $r = \|g - P_0\|$ up to a constant. Since $r = \|h\|$ and g, h are conjugate minimal surfaces, this can only happen if the associated family to g is trivial, that is, if the elipse of curvature of g at any point is a circle.

Conversely, if the elipse of curvature of g at any point is a circle then the associated family of g is trivial, hence there exists $P_0 \in \mathbb{R}^{n+2}$ such that $h = \tilde{J}(g - P_0)$, where $\tilde{J}$ is a complex structure in $\mathbb{R}^{n+2}$. Define

$$\hat{w} = \frac{1}{ra}(g - rg_*\nabla r - P_0).$$

From $r = \|g\|$ it follows that $\hat{w}$ is normal to g. Then, taking norms in

$$g - P_0 = ra\bar{w} + rg_*\nabla r$$

implies that $\hat{w}$ has unit length. It remais to be shown that $\hat{w}$ is a section of Λ_1, that is, that it is orthogonal to

$$\xi = \frac{1}{ar}(g_*rJ\nabla r - h).$$

Observe that $\hat{w}$ is just the unit vector in the direction of $(g - P_0)^N$, the normal part of the position vector $g - P_0$, whereas ξ is the unit vector in the direction of $-h^N$. Since

$$\langle h^T, (g - P_0)^T\rangle = \langle h_*r\nabla r, g_*r\nabla r\rangle = \langle g_*rJ\nabla r, g_*r\nabla r\rangle = 0,$$

this follows from $\langle h, g - P_0\rangle = 0$ everywhere. ∎

According to Theorem 3, any isometric immersion $f\colon M^n \to \mathbb{R}^{n+2}$ for which equality holds eveywhere in the inequality (1) is determined by a pair (g, h) of conjugate minimal surfaces. In the following result we determine how the pairs (g, h) and $(\tilde{g}, \tilde{h})$ associated to such an isometric immersion and its composition $\tilde{f} = \mathcal{I} \circ f$ with an inversion $\mathcal{I}$ in $\mathbb{R}^{n+2}$ with respect to a sphere with radius R centered at $P_0 \in \mathbb{R}^{n+p}$ are related.

Proposition 6. *We have*

$$(\tilde{g} - P_0, h) = \bar{T} \circ (g - P_0, h),$$

where $T\colon \mathbb{C}^{n+2} \to \mathbb{C}^{n+2}$ *is given by*

$$T(z_1, \ldots, z_n) = \frac{(z_1, \ldots, z_n)}{z_1^2 + \ldots + z_n^2}$$

and $\bar{T}$ *denotes the conjugate map to* T.

For the proof of Proposition 6 we need the following well-known fact.

Lemma 7. *Let $f\colon M^n \to \mathbb{R}^{n+p}$ be an isometric immersion and let $\mathcal{I}$ be an inversion with respect to a sphere with radius R centered at $P_0 \in \mathbb{R}^{n+p}$. Then*

$$\mathcal{P}\xi = \xi - 2\frac{\langle f - P_0, \xi\rangle}{\|f - P_0\|^2}(f - P_0) \tag{64}$$

is a vector bundle isometry between the normal bundles $T_f^\perp M^n$ and $T_{\tilde{f}}^\perp M^n$ of f and $\tilde{f} = \mathcal{I} \circ f$. Moreover, the shape operators A_ξ and $\tilde{A}_{\mathcal{P}\xi}$ are related by

$$\tilde{A}_{\mathcal{P}\xi} = \frac{\|f - P_0\|^2}{R^2}\left(A_\xi + 2\frac{\langle f - P_0, \xi\rangle}{\|f - P_0\|^2} I\right). \tag{65}$$

Proof of Proposition 6: Using Lemma 7 we obtain

$$\tilde{\zeta} = (\bar{\lambda}^2 + \bar{\nu}^2)^{-1/2}(\bar{\nu}\mathcal{P}\eta - \bar{\lambda}\mathcal{P}\zeta)$$

and

$$\tilde{\eta} = (\bar{\lambda}^2 + \bar{\nu}^2)^{-1/2}(\bar{\lambda}\mathcal{P}\eta + \bar{\nu}\mathcal{P}\zeta),$$

where

$$\bar{\nu} = \frac{2\langle \phi - P_0, \zeta\rangle}{R^2}$$

and

$$\bar{\lambda} = \frac{\lambda\|\phi - P_0\|^2}{R^2} + \frac{2\langle \phi - P_0, \eta\rangle}{R^2}.$$

Moreover,

$$\tilde{\lambda} = (\bar{\lambda}^2 + \bar{\nu}^2)^{1/2}.$$

Thus

$$\tilde{h} = -\tilde{r}\tilde{\zeta} = -(\bar{\lambda}^2 + \bar{\nu}^2)^{-1}(\bar{\nu}\mathcal{P}\eta - \bar{\lambda}\mathcal{P}\zeta).$$

Using that $\phi = g - r\eta$, $\lambda = 1/r$ and $h = -r\zeta$ we get

$$R^2\bar{\nu} = -\frac{2}{r}\langle g - P_0, h\rangle$$

and

$$R^2\bar{\lambda} = \frac{\|g - P_0\|^2}{r} - r.$$

On the other hand, from

$$\mathcal{P}\eta = \eta - 2\frac{\langle \phi - P_0, \eta\rangle}{\|\phi - P_0\|^2}(\phi - P_0)$$

and

$$\mathcal{P}\zeta = \eta - 2\frac{\langle \phi - P_0, \zeta\rangle}{\|\phi - P_0\|^2}(\phi - P_0),$$

we obtain

$$
\begin{aligned}
R^2(\bar{\nu}\mathcal{P}\eta - \bar{\lambda}\mathcal{P}\zeta)) &= 2\langle \phi - P_0, \zeta\rangle\eta - 2\langle \phi - P_0, \eta\rangle\zeta - \lambda\|\phi - P_0\|^2\zeta + 2\lambda\langle \phi - P_0, \zeta\rangle(\phi - P_0) \\
&= R^2\bar{\nu}\frac{g - P_0}{r} - R^2\bar{\lambda}\zeta \\
&= \frac{1}{r^2}(-2\langle g - P_0, h\rangle(g - P_0) + (\|g - P_0\|^2 - r^2)h).
\end{aligned}
$$

Therefore

$$
\tilde{h} = R^2\left(\frac{2\langle g - P_0, h\rangle(g - P_0) - (\|g - P_0\|^2 - \|h\|^2)h}{4\langle g - P_0, h\rangle^2 + (\|g - P_0\|^2 - \|h\|^2)^2}\right).
$$

On the other hand, we have

$$
\tilde{g} = \tilde{\phi} + \tilde{r}\tilde{\eta},
$$

with

$$
\tilde{\phi} = P_0 + \frac{R^2}{\|\phi - P_0\|^2}(\phi - P_0)
$$

and

$$
\tilde{r}\tilde{\eta} = \frac{\bar{\lambda}\mathcal{P}\eta + \bar{\nu}\mathcal{P}\zeta}{\bar{\nu}^2 + \bar{\lambda}^2}.
$$

Using that

$$
R^4(\bar{\nu}^2 + \bar{\lambda}^2) = 4\langle \phi - P_0, \zeta\rangle^2 + 4\langle \phi - P_0, \eta\rangle^2 + \frac{\|\phi - P_0\|^4}{r^2} + \frac{4}{r}\langle \phi - P_0, \eta\rangle\|\phi - P_0\|^2
$$

and

$$
\begin{aligned}
R^2(\bar{\lambda}\mathcal{P}\eta + \bar{\nu}\mathcal{P}\zeta) = {} & \frac{\|\phi - P_0\|^2}{r}\eta - \frac{2\langle \phi - P_0, \eta\rangle}{r}(\phi - P_0) + 2\langle \phi - P_0, \eta\rangle\eta \\
& - \frac{4\langle \phi - P_0, \eta\rangle^2}{\|\phi - P_0\|^2}(\phi - P_0) + 2\langle \phi - P_0, \zeta\rangle\zeta - \frac{4\langle \phi - P_0, \zeta\rangle^2}{\|\phi - P_0\|^2}(\phi - P_0).
\end{aligned}
$$

we obtain

$$
\begin{aligned}
\tilde{g} &= \tilde{\phi} + \tilde{r}\tilde{\eta} \\
&= P_0 + R^2\left(\frac{\phi - P_0}{\|\phi - P_0\|^2} + \frac{R^2(\bar{\lambda}\mathcal{P}\eta + \bar{\nu}\mathcal{P}\zeta}{R^4(\bar{\nu}^2 + \bar{\lambda}^2)}\right) \\
&= P_0 + \frac{R^2}{\|\phi - P_0\|^2 R^4(\bar{\nu}^2 + \bar{\lambda}^2)}\left(R^4(\bar{\nu}^2 + \bar{\lambda}^2)(\phi - P_0) + \|\phi - P_0\|^2 R^2(\bar{\lambda}\mathcal{P}\eta + \bar{\nu}\mathcal{P}\zeta)\right) \\
&= P_0 + R^2\frac{(\|g - P_0\|^2 - \|h\|^2)(g - P_0) + 2\langle g - P_0, h\rangle h}{4\langle g - P_0, h\rangle^2 + (\|g - P_0\|^2 - \|h\|^2)^2}.
\end{aligned}
$$

References

[**CL**] CHOI, T., LU, Z.: On the DDVV conjecture and the comass in calibrated geometry. Preprint (avavilable at arXiv:math.DG).

[**Ch**] CHEN, B.Y.: Mean curvature and shape operators of isometric immersions into real-space-forms. *Glasgow Math. J.* **38** (1996), 87-97.

[**DF**] DAJCZER, M., FLORIT, L.: On Chen's Basic Equality. *Illinois J. Math.* **42** (1) (2004), 97-106.

[**DFV**] DILLEN, F., FASTENAKELS, J, VAN DER VEKEN, J.: Remarks on an inequality involving the normal scalar curvature. Preprint (avavilable at arXiv:math.DG/0610721).

[**DDVV**] DE SMET, P.J., DILLEN, F., VERSTRAELEN, L., VRANCKEN, L.: A pointwise inequality in submanifold theory. *Arch. Math. (Brno)* **35** (1999), 115-128.

[**GR**] GUADALUPE, I., RODRIGUEZ, L.: Normal curvature of surfaces in space forms. *Pacific J. Math.* **106** (1983), 95-103.