Layered stable equilibria of a reaction-diffusion equation with nonlinear Neumann boundary condition.

Arnaldo Simal do Nascimento [a,1], Renato José de Moura [b,2]

[a] *Universidade Federal de S. Carlos, D.M. 13565-905 São Carlos, S.P., Brazil*

[b] *Universidade Federal de Minas Gerais, ICEX - D.M. 30123-970 Belo Horizonte, M.G., Brazil*

Abstract

In this work we investigate the existence and asymptotic profile of a family of layered stable stationary solutions to the scalar equation $u_t = \varepsilon^2 \Delta u + f(u)$ in a smooth bounded domain $\Omega \subset \mathbb{R}^3$ under the boundary condition $\varepsilon \partial_\nu u = \delta_\varepsilon g(u)$. It is assumed that Ω has a cross-section which locally minimizes area and $\lim_{\varepsilon \to 0} \varepsilon \ln \delta_\varepsilon = \kappa$, with $0 \leq \kappa < \infty$ and $\delta_\varepsilon > 1$ when $\kappa = 0$.

The functions f and g are of bistable type and do not necessarily have the same zeros what makes the asymptotic geometric profile of the solutions on the boundary to be different from the one in the interior.

Key words: reaction-diffusion equation, internal transition layer, equal-area condition, nonlinear boundary condition.

Email addresses: `arnaldon@dm.ufscar.br` (Arnaldo Simal do Nascimento), `moura@mat.ufmg.br` (Renato José de Moura).

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1 Introduction and statement of the main result

The subject of study in this work is the following boundary and initial value problem

$$\begin{cases} \dfrac{\partial u}{\partial t} = \varepsilon^2 \Delta u + f(u), & (t,x) \in \mathbb{R}^+ \times \Omega \\ \varepsilon \dfrac{\partial u}{\partial \nu} = \delta_\varepsilon \, g(u), & (t,x) \in \mathbb{R}^+ \times \partial\Omega \\ u(0,x) = \psi_0(x), & x \in \overline{\Omega}. \end{cases} \tag{1}$$

where $\Omega \subset \mathbb{R}^3$ is a C^2 bounded domain, ε a small positive parameter, δ_ε a suitable parameter which depends on ε, ν the exterior vector field normal to $\partial\Omega$ and f and g are C^1 functions.

We address the question of existence and asymptotic geometric profile, as $\varepsilon \to 0$, of a family of nonconstant stable stationary solutions to (1), where stability is meant in the usual Liapounov sense. Such solutions will herein be abbreviated patterns for short.

Before stating our main result let us give some background on some works related to (1).

For the case of $g \equiv 0$ and Ω a convex domain it is well-known that (1) possesses no pattern. This also holds when $f \equiv 0$ and Ω is a N-dimensional ball (see [4]).

When Ω is a domain of dumbbell type, $g \equiv 0$ and f is a bistable function, it has long been known that (1) possesses a family of patterns (see [3], for instance).

Still for this type of non-convex domain, Cònsul and Solá-Morales in [5] showed existence of patterns for the case of $f \equiv 0$, $\varepsilon = 1$, δ a positive parameter and g of bistable type. Their method of proof, namely construction of a invariant set for the evolution equation, is suited just for proving existence of patterns but does not yield any information on the asymptotic geometric profile nor on the location of the interface.

In this work, by supposing that f and g are of bistable type with the relative positions of their zeros satisfying a certain order along with the equal-area condition, we prove the existence of a family of patterns which converges to the stable zeros of f in the interior and to the stable zeros of g on the boundary as long as $\lim_{\varepsilon \to 0} \varepsilon \ln \delta_\varepsilon = \kappa$. The interface in Ω is the surface which locally minimizes the area-functional on a dumbbell type domain and its boundary in $\overline{\Omega}$ turns out to be interface in $\partial\Omega$.

Although this has little claim on intuition some difficult technical problems in the proof have to be overcome. The most difficult one is to locally minimize the area-functional (arc-length functional) on Ω (on $\partial\Omega$, respectively) since the competing surfaces (curves) are just rectifiable sets and those with too small area (arc-length) must be ruled out.

Before stating our results in a rigorous manner we describe our hypotheses:

$(\mathbf{f_1})$ $\exists\ \alpha' \leq \alpha < \beta \leq \beta'$ such that $f(l) = 0,\ f'(l) < 0, \forall\, l \in \{\alpha, \beta\}$ and $g(l) = 0,\ g'(l) < 0, \forall\, l \in \{\alpha', \beta'\}$. Moreover $f > 0$ on $(-\infty, \alpha)$, $f < 0$ on (β, ∞), $g > 0$ on $(-\infty, \alpha')$ and $g < 0$ on (β', ∞)

$(\mathbf{f_2})$ $\int_{\alpha'}^{\beta'} g(\xi)d\xi = \int_{\alpha}^{\beta} f(\xi)d\xi = 0$ (the equal-area condition)

$(\mathbf{f_3})$ $\lim_{\varepsilon\to 0} \varepsilon \ln \delta_\varepsilon = \kappa$, where $0 \leq \kappa < \infty$ and $\delta_\varepsilon > 1$ when $\kappa = 0$

$(\mathbf{f_4})$ There are constants $c_1 > 0$, $c_2 > \max\{|\alpha'|, \beta'\}$, r, s, σ_f and σ_g such that

- $\min\{|f(u)|, |g(u)|\} \geq c_1|u|,\ \forall\ |u| > c_2,$
- $|f(u)| \leq r + s|u|^{\sigma_f},\quad 1 \leq \sigma_f \leq 5,$
- $|g(u)| \leq r + s|u|^{\sigma_g},\quad 1 \leq \sigma_g \leq 3.$

Note that $\delta_\varepsilon = \varepsilon^{-n}$ $(n = 1, 2 \ldots)$ satisfies $(\mathbf{f_3})$ with $\kappa = 0$ as well as $\delta_\varepsilon = e^{\kappa/\varepsilon}$ where κ is any positive constant.

Regarding the domain we suppose that

(H) $\exists\, \mathcal{O} \subset \Omega$ such that $\partial\Omega \cap \partial\mathcal{O}$ is a surface of revolution generated by a positive C^2 function $\theta : (-h, h) \longrightarrow \mathbb{R}$, h a small positive real number, where θ has an absolute minimum at 0.

The assumption that a portion of $\partial\Omega$ is a surface of revolution greatly simplifies a future computation. However this symmetry condition could be somewhat relaxed at the cost of some additional work; it would suffice to require that there exists a smooth closed curve $\gamma \subset \partial\mathcal{O} \cap \partial\Omega$ such that γ locally minimizes arc-length.

Let us denote

$$\mathcal{S} \stackrel{\text{def}}{=} \{(x, y, 0) : x^2 + y^2 < \theta^2(0)\} \text{ and } \mathcal{C} = \{(x, y, 0) : x^2 + y^2 \stackrel{\text{def}}{=} \theta^2(0)\},$$

and suppose that $\mathcal{S}$ partitions Ω in two disjoint open sets Ω_α and Ω_β, i.e., $\Omega = \Omega_\alpha \cup \mathcal{S} \cup \Omega_\beta$.

After setting $\mathcal{M}_\alpha = \partial\Omega \cap \partial\Omega_\alpha$ and $\mathcal{M}_\beta = \partial\Omega \cap \partial\Omega_\beta$, we define

$$u_0 \stackrel{\text{def}}{=} \alpha\chi_{\Omega_\alpha} + \beta\chi_{\Omega_\beta} \qquad (2)$$

and

$$v_0 \stackrel{\text{def}}{=} \alpha' \chi_{\mathcal{M}_\alpha} + \beta' \chi_{\mathcal{M}_\beta} \tag{3}$$

where χ_O stands for the characteristic function of the set O.

Also T will denote the trace operator which maps either $H^1(\Omega)$ onto $H^{1/2}(\partial\Omega)$ or $BV(\Omega)$ onto $L^1(\partial\Omega)$, according to the situation.

We rather set

$$F(u) \stackrel{\text{def}}{=} -\int_\alpha^u f(\xi) d\xi \text{ and } G(u) \stackrel{\text{def}}{=} -\int_{\alpha'}^u g(\xi) d\xi$$

Note that by virtue of $(\mathbf{f_1})$, we have $F \geq 0$ $(G \geq 0)$ and F vanishes only at $\{\alpha, \beta\}$ (respectively at $\{\alpha', \beta'\}$).

Let us now state what is the main result of this work.

Theorem 1.1 *In addition to* $(\mathbf{f_1})$-$(\mathbf{f_4})$ *and* $\mathbf{(H)}$ *suppose also that*

$(\mathbf{f_5})$ $\int_{\alpha'}^{\alpha} \sqrt{F} = \int_\beta^{\beta'} \sqrt{F}$.

Then $\exists\, \varepsilon_0 > 0$ *and a sequence* $\{u_{\varepsilon_j}\}_{0<\varepsilon_j \leq \varepsilon_0}$ $(\varepsilon_j \to 0,$ *as* $j \to \infty)$ *of classical stationary solutions to (1) which is stable in* $W^{1,p}(\Omega)$, $p > 3$ *and satisfies*

$$u_{\varepsilon_j} \longrightarrow u_0 \text{ in } L^1(\Omega)$$

and

$$Tu_{\varepsilon_j} \longrightarrow v_0 \text{ in } L^1(\partial\Omega), \text{as}\, j \to \infty.$$

Here u_0 *and* v_0 *are given by* (2) *and* (3), *respectively.*

Remark 1.2 *We will present in the Appendix an argument to show that the new area condition given by* $(\mathbf{f_5})$ *and which relates the zeros of* f *and* g *is actually necessary in our approach.*

As a byproduct of our procedure we prove existence of a family of patterns whose asymptotic behaviour on $\partial\Omega$ as well as on Ω is flat, yet the family of patterns develops boundary layer.

Theorem 1.3 *In addition to* $(\mathbf{f_1})$-$(\mathbf{f_4})$, *suppose that*

$(\mathbf{f_6})$ $\int_{\alpha'}^{\alpha} \sqrt{F} < \int_\beta^{\beta'} \sqrt{F}$.

Then $\exists\, \varepsilon_0 > 0$ *and a sequence* $\{u_{\varepsilon_j}\}_{0<\varepsilon_j \leq \varepsilon_0}$ *of classical stable (in* $W^{1,p}(\Omega)$, $p > 3$*) stationary solutions to (1) satisfying*

$$u_{\varepsilon_j} \longrightarrow \alpha \text{ in } L^1(\Omega)$$

and

$$Tu_{\varepsilon_j} \longrightarrow \alpha' \text{ in } L^1(\partial\Omega), \text{ as } j \to \infty.$$

By reversing the inequality in ($\mathbf{f_6}$), a similar conclusion holds with β' and β in place of α' and α, respectively.

Our approach is variational and uses a theorem by De Giorgi which, under suitable hypotheses, guarantees that if the Γ-limit of the family of the energy functionals associated with our problem has an isolated local minimum u_0 say (in the L^1-topology) then this family itself has a sequence of minima which converge to u_0.

Let us now justify our hypotheses:

($\mathbf{f_1}$) is used (along with ($\mathbf{f_2}$)) to guarantee that in the computation of the Γ-limit, the corresponding potentials for f and g are of the type double-well with equal depth. The functions f and g could have been allowed to have more zeros at the additional cost of a truncation argument.

($\mathbf{f_2}$) is the well-known area-condition and it has been proved in [2] that it actually is a necessary condition for the existence of a family of stationary solutions to (1) which develops internal and superficial transition layers, as is the case here.

($\mathbf{f_3}$) is a technical condition which appears in the computation of the Γ-limit. It also reflects the different diffusibility scales in Ω and $\partial\Omega$, a case also contemplated by the present ansatz.

As for ($\mathbf{f_4}$), it is a growth condition used only to assure that the energy functional is well-defined and satisfies a compactness condition.

2 Local minimizers via De Giorgi's result

For the sake of brevity in notation, we rather define

$$\mathbb{L}^1 \stackrel{\text{def}}{=} L^1(\Omega) \times L^1(\partial\Omega),$$

which endowed with the norm $\|(u,v)\|_{\mathbb{L}^1} \stackrel{\text{def}}{=} \|u\|_{L^1(\Omega)} + \|v\|_{L^1(\partial\Omega)}$ is a Banach space.

The reader is referred to [9] for a comprehensive text on Γ-convergence. In our setting the definition is the following.

Definition 2.1 *A sequence* $\{\mathcal{E}_{\varepsilon_j}\}_{\varepsilon_j>0}$ *of real-extended functionals defined in* $\mathbb{L}^1$ *is said to* Γ*- converge, as* $\varepsilon_j \to 0$*, to the functional* $\mathcal{E}_0$ *if:*

– *For each* $(u,v) \in \mathbb{L}^1$ *and for any sequence* $\{(u_{\varepsilon_j}, v_{\varepsilon_j})\}$ *in* $\mathbb{L}^1$ *such that* $(u_{\varepsilon_j}, v_{\varepsilon_j}) \to (u,v)$ *in* $\mathbb{L}^1$*, as* $\varepsilon_j \to 0$*, it holds that*

$$\mathcal{E}_0(u,v) \leq \liminf_{\varepsilon_j \to 0} \mathcal{E}_{\varepsilon_j}(u_{\varepsilon_j}, v_{\varepsilon_j}).$$

– *For each* $(u,v) \in \mathbb{L}^1$ *there is a sequence* $\{(u_{\varepsilon_j}, v_{\varepsilon_j})\}$ *in* $\mathbb{L}^1$ *such that* $(u_{\varepsilon_j}, v_{\varepsilon_j}) \to (u,v)$ *in* $\mathbb{L}^1$*, as* $\varepsilon_j \to 0$*, and*

$$\mathcal{E}_0(u,v) \geq \limsup_{\varepsilon_j \to 0} \mathcal{E}_{\varepsilon_j}(u_{\varepsilon_j}, v_{\varepsilon_j}).$$

Definition 2.2 *It is said that* $(u_0, v_0) \in \mathbb{L}^1$ *is a* $\mathbb{L}^1$*-local minimizer of a functional* $\mathcal{E}_0$ *if there is* $\rho > 0$ *such that*

$$\mathcal{E}_0(u_0,v_0) \leq \mathcal{E}_0(u,v) \quad \textit{whenever} \quad 0 < \|(u,v) - (u_0,v_0)\|_{\mathbb{L}^1} < \rho\ .$$

Moreover if $\mathcal{E}_0(u_0,v_0) < \mathcal{E}_0(u,v)$ *for* $0 < \|(u,v) - (u_0,v_0)\|_{\mathbb{L}^1} < \rho$*, then* (u_0,v_0) *is called an isolated* $\mathbb{L}^1$*-local minimizer of* $\mathcal{E}_0$*.*

The following theorem is a variational version of a rather general result due to De Giorgi [6].

Its proof, when the energy functional has a contribution from the boundary, will be given in the Appendix.

Theorem 2.3 *Given a sequence of real-extended functional* $\{\mathcal{E}_{\varepsilon_j}\}_{\varepsilon_j>0}$ *in* $\mathbb{L}^1$*, suppose that the following hypotheses hold:*

(**2.3.i**) $\mathcal{E}_{\varepsilon_j}$ Γ*-converges to* $\mathcal{E}_0$ *in* $\mathbb{L}^1$*, as* $\varepsilon_j \to 0$*.*

(**2.3.ii**) *There exists an isolated* $\mathbb{L}^1$*-local minimizer* (u_0,v_0) *of* $\mathcal{E}_0$*.*

(**2.3.iii**) *If* $\mathcal{E}_{\varepsilon_j}(u_{\varepsilon_j}, v_{\varepsilon_j}) \leq$ *constant, then* $\{(u_{\varepsilon_j}, v_{\varepsilon_j})\}$ *has a subsequence which converges in* $\mathbb{L}^1$*.*

(**2.3.iv**) *For each* $\delta > 0$ *and* $\varepsilon > 0$*, there exists* $(u_\varepsilon, v_\varepsilon) \in B_\delta(u_0,v_0) \stackrel{\text{def}}{=} \Big\{(u,v) : \|(u,v) - (u_0,v_0)\|_{\mathbb{L}^1} \leq \delta\Big\}$ *satisfying*

$$\mathcal{E}_\varepsilon(u_\varepsilon, v_\varepsilon) = \inf\left\{\mathcal{E}_\varepsilon(u,v) : (u,v) \in B_\delta\right\}.$$

Then there exists a sequence $\{(u_{\varepsilon_j}, v_{\varepsilon_j})\}$ *in* $\mathbb{L}^1$*, with* $\varepsilon_j \to 0$ *as* $j \to \infty$*, such that*

- $(u_{\varepsilon_j}, v_{\varepsilon_j})$ *is a* $\mathbb{L}^1$*-local minimizer of* $\mathcal{E}_{\varepsilon_j}$ *and*
- $||(u_{\varepsilon_j}, v_{\varepsilon_j}) - (u_0, v_0)||_{\mathbb{L}^1} \longrightarrow 0$*, as* $j \to \infty$.

The next sections are devoted to set the appropriated scenario in which the hypotheses of Theorem 2.3 are verified by the family of energy functionals corresponding to our problem.

Then Theorem 2.3 will be used to prove Theorem 1.1 and Theorem 1.3.

3 The Γ-limit of the energy functional

In this section we verify hypothesis (**2.3.i**) of Theorem 2.3, for a sequence of energy functionals whose critical points are the stationary solutions to (1), i.e.,

$$\begin{cases} \varepsilon^2 \Delta u + f(u) = 0, \ x \in \Omega \\ \varepsilon \dfrac{\partial u}{\partial \nu} = \delta_\varepsilon g(u), x \in \partial\Omega \end{cases} \tag{4}$$

The family of energy functional $E_\varepsilon : L^1(\Omega) \longmapsto \mathbb{R} \cup \{\infty\}$ is defined by

$$E_\varepsilon(u) = \begin{cases} \frac{\varepsilon}{2} \int_\Omega |\nabla u(x)|^2 \, dx + \varepsilon^{-1} \int_\Omega F(u) \, dx + \delta_\varepsilon \int_{\partial\Omega} G(Tu) \, d\mathcal{H}^2, \\ \text{if } u \in H^1(\Omega) \\ +\infty, \text{if } u \in L^1(\Omega) \backslash H^1(\Omega) \end{cases}$$

where $\mathcal{H}^2$ stands for the 2-dimensional Hausdorff measure.

In the sequel $BV(X, \{a, b\})$ will denote the space of functions of bounded variation in X which takes values a and b only. If $u \in BV(\Omega, \{\alpha, \beta\})$ then $S_u \stackrel{\text{def}}{=} \partial^* \{x \in \Omega : u(x) = \beta\} \cap \Omega$, where ∂^* stands for the reduced boundary, is a rectifiable set. The definition of S_v for $v \in BV(\partial\Omega, \{\alpha', \beta'\})$ is analogous.

Note that the total variation $\int_\Omega |Du|$ of $u \in BV(\Omega, \{\alpha, \beta\})$ is given by

$$\int_\Omega |Du| = (\beta - \alpha)\mathcal{H}^2(S_u).$$

For $u \in BV(\Omega, \{\alpha, \beta\})$ and $v \in BV(\partial\Omega, \{\alpha', \beta'\})$ let us define

$$\Upsilon(u, v) \stackrel{\text{def}}{=} \sigma \mathcal{H}^2(S_u) + \int_{\partial\Omega} |h(Tu) - h(v)| d\mathcal{H}^2 + c\mathcal{H}^1(S_v),$$

where $c = \frac{(\beta-\alpha)^2}{\pi}\kappa$ (κ as in $(\mathbf{f_3})$), $h(w) = 2^{3/2}\int_0^w \sqrt{F(\zeta)}\,d\zeta$ and $\sigma = |h(\beta) - h(\alpha)|$.

Theorem 3.1 ([1]) *Suppose* $(\mathbf{f_1})-(\mathbf{f_4})$ *hold with* $0 < \kappa < \infty$. *Then the* Γ*-limit on* $L^1(\Omega)$ *of the functionals* E_ε, *is given by*

$$\overline{\Upsilon}(u) = \begin{cases} \inf\{\Upsilon(u,v) : v \in BV(\partial\Omega, \{\alpha', \beta'\})\} \text{ if } u \in BV(\Omega, \{\alpha, \beta\}) \\ \infty, \ \text{otherwise in } L^1(\Omega). \end{cases}$$

The above computation of the Γ-limit in $L^1(\Omega)$ is not suitable for our purposes since we want to obtain information of the asymptotic profile of the solutions on $\partial\Omega$ as well. Therefore the topology in which the limit problem is going to be framed must change and we compute the Γ-limit in $\mathbb{L}^1$ as follows.

We define $\mathcal{E}_\varepsilon, \mathcal{E}_0 : \mathbb{L}^1 \to \mathbb{R} \cup \{\infty\}$ by

$$\mathcal{E}_\varepsilon(u,v) = \begin{cases} E_\varepsilon(u), \ \text{if } u \in H^1(\Omega) \text{ and } v = Tu \\ \infty, \ \text{otherwise in } \mathbb{L}^1. \end{cases} \tag{5}$$

and

$$\mathcal{E}_0(u,v) = \begin{cases} \sigma\mathcal{H}^2(S_u) + \int_{\partial\Omega} |h(Tu) - h(v)| d\mathcal{H}^2 + c\mathcal{H}^1(S_v), \\ \text{if } (u,v) \in BV(\Omega, \{\alpha, \beta\}) \times BV(\partial\Omega, \{\alpha', \beta'\}) \\ \infty, \ \text{otherwise in } \mathbb{L}^1. \end{cases} \tag{6}$$

Using Theorem 3.1, we can thus compute the $\Gamma-$limit for the penalized problem as follows.

Lemma 3.2 *If* $\kappa > 0$ *then* $\mathcal{E}_\varepsilon$ Γ*-converge to* $\mathcal{E}_0$ *in* $\mathbb{L}^1$.

Proof. In order to verify the first condition of Definition 2.1 we take $(u,v) \in \mathbb{L}^1$ and a sequence $\{(u_{\varepsilon_j}, v_{\varepsilon_j})\}$ in $\mathbb{L}^1$ such that $(u_{\varepsilon_j}, v_{\varepsilon_j}) \to (u,v)$ in $\mathbb{L}^1$, as $\varepsilon_j \to 0$.

The cases $u_{\varepsilon_j} \in L^1(\Omega) \setminus H^1(\Omega)$ and $Tu_{\varepsilon_j} \not\equiv v_{\varepsilon_j}$ follow by the penalization hypothesis.

On the hand if $u_\varepsilon \in H^1(\Omega)$ and $Tu_{\varepsilon_j} \equiv v_{\varepsilon_j}$, then it follows from Theorem 2.6 in [1] that $\mathcal{E}_0(u,v) \le \liminf_{\varepsilon_j \to 0} \mathcal{E}_{\varepsilon_j}(u_{\varepsilon_j}, v_{\varepsilon_j})$.

It remains to analyze the case

$$(u,v) \notin BV(\Omega, \{\alpha, \beta\}) \times BV(\partial\Omega, \{\alpha', \beta'\}).$$

If $\liminf_{\varepsilon_j \to 0} \mathcal{E}_{\varepsilon_j}(u_{\varepsilon_j}, v_{\varepsilon_j}) = \infty$ then there is nothing to prove. So let us suppose that there is a subsequence $\{u_{\varepsilon_j}\} \in H^1(\Omega)$ such that

$$\lim_{j \to \infty} \mathcal{E}_{\varepsilon_j}(u_{\varepsilon_j}, v_{\varepsilon_j}) < \infty.$$

Evoking again Theorem 2.6 (i) in [1] we conclude that $\{(u_{\varepsilon_j}, Tu_{\varepsilon_j})\}$ is relatively compact in $\mathbb{L}^1$ and every cluster point belongs to $BV(\Omega, \{\alpha, \beta\}) \times BV(\partial\Omega, \{\alpha', \beta'\})$. But this contradicts our hypothesis.

As for the second requirement in the definition of Γ-convergence it follows again from Theorem 2.6 (iii) in [1] along with the penalization hypothesis. ■

Lemma 3.3 *If* $\kappa = 0$ *then* $\mathcal{E}_\varepsilon$ *Γ-converge to* $\overline{\mathcal{E}}_0$ *in* $\mathbb{L}^1$ *where*

$$\overline{\mathcal{E}}_0(u, v) = \begin{cases} \sigma \mathcal{H}^2(S_u) + \int_{\partial\Omega} |h(Tu) - h(v)| d\mathcal{H}^2, \\ \textit{if } (u, v) \in BV(\Omega, \{\alpha, \beta\}) \times BV(\partial\Omega, \{\alpha', \beta'\}) \\ \infty, \quad \textit{otherwise in } \mathbb{L}^1 \end{cases} \tag{7}$$

Although in the proof of Theorem 3.1 in [1] the authors only care for the case $\lim_{\varepsilon \to 0} \varepsilon \ln \delta_\varepsilon = \kappa > 0$, a careful reading of the proof shows that the same holds true when $\kappa = 0$, except that now the third term on the right-hand side of $\mathcal{E}_0$, namely $\mathcal{H}^1(S_v)$, is dropped.

The computation of the Γ-limit above holds regardless of hypothesis **(H)** which will only be needed in the next section.

4 Existence of an isolated minimizer for the Γ-limit

It is worthwhile to mention that the problem of finding local minimizers of our original functional was reduced to finding a local isolated minimizer of the Γ-limits $\mathcal{E}_0$ and $\overline{\mathcal{E}}_0$ (see (6) and (7)), which is a more tractable geometric problem though not a simple one.

It would be simpler, as will be seen in the proof of the next lemma, if when minimizing the area- and the arc-length-functionals we could restrict the class of competing sets to those rectifiable sets whose orthogonal projection would cover all of the interface.

Lemma 4.1 *If* (**H**) *holds then* (u_0, v_0) (*given by* (2) *and* (3)) *is a* $\mathbb{L}^1$*-local isolated minimizer of* $\mathcal{E}_0$ *if* $0 < \kappa < \infty$ *and a* $\mathbb{L}^1$*-local isolated minimizer of* $\overline{\mathcal{E}}_0$ *if* $\kappa = 0$.

Proof It suffices to prove that $\exists\ \delta > 0$ such that for any $(u, v) \in BV(\Omega, \{\alpha, \beta\}) \times BV(\partial\Omega, \{\alpha', \beta'\})$, with $0 < \|(u, v) - (u_0, v_0)\|_{\mathbb{L}^1} < \delta$, it holds that

(4.1.i) $\mathcal{H}^2(S_{u_0}) = \mathcal{H}^2(\mathcal{S}) < \mathcal{H}^2(S_u)$,

(4.1.ii) $\mathcal{H}^1(S_{v_0}) = \mathcal{H}^1(\mathcal{C}) < \mathcal{H}^1(S_v)$ and

(4.1.iii) $\int_{\partial\Omega} |h(Tu_0) - h(v_0)| d\mathcal{H}^2 < \int_{\partial\Omega} |h(Tu) - h(v)| d\mathcal{H}^2$.

Proof of (4.1.i). We prove that $\mathcal{H}^2(\mathcal{S}) < \mathcal{H}^2(S_u)$ for any $u \in BV(\mathcal{O}, \{\alpha, \beta\})$ with $0 < \|u - u_0\|_{L^1(\mathcal{O})} < \delta$, for suitable $\delta > 0$, where $\mathcal{O}$ is given in (**H**).

From (**H**), for some $h > 0$ we may write

$$\partial\Omega \cap \partial\mathcal{O} = \{(\theta(z)\cos t, \theta(z)\sin t, z), 0 \leq t < 2\pi, -h < z < h\}. \tag{8}$$

In order to explore the local geometry of the domain, we make a change of variables $\Lambda : \mathcal{O} \to K$ which takes $\mathcal{O}$ into a right circular cylinder and is defined by

$$\Lambda(x, y, z) = \left(\left(\frac{\theta(0)}{\theta(z)}\right) x, \left(\frac{\theta(0)}{\theta(z)}\right) y, z\right).$$

Λ is a diffeomorphism and taking into account that 0 is absolute minimum of θ, we obtain

$$\left|J\Lambda^{-1}(x, y, z)\right| = \left(\frac{\theta(z)}{\theta(0)}\right)^2 > 1, \tag{9}$$

$\forall\ (x, y, z) \in K,\ z \neq 0$.

We may suppose without loss that $\mathcal{O} = \mathcal{O}_\alpha \cup \mathcal{S} \cup \mathcal{O}_\beta$ where

$$\mathcal{O}_\alpha = \{(x, y, z) \in \Omega : 0 < z < h\}$$

and

$$\mathcal{O}_\beta = \{(x, y, z) \in \Omega : -h < z < 0\}.$$

Note that $\Lambda(\mathcal{S}) = \mathcal{S}$ and as such

$$\mathcal{H}^2(\mathcal{S}) = \mathcal{H}^2(S_{\overline{u}_0}), \tag{10}$$

where $\overline{u}_0 = u_0 \circ \Lambda^{-1} = \alpha\chi_{K_\alpha} + \beta\chi_{K_\beta}$, $K_\alpha \stackrel{\text{def}}{=} \Lambda(\mathcal{O}_\alpha)$ and $K_\beta \stackrel{\text{def}}{=} \Lambda(\mathcal{O}_\beta)$.

Define $\overline{u} = u \circ \Lambda^{-1} \in BV(K, \{\alpha, \beta\})$ and denote

$$\begin{aligned} K_\alpha^{\overline{u}} &= \{x \in K : \overline{u}(x) = \alpha\} \\ K_\beta^{\overline{u}} &= \{x \in K : \overline{u}(x) = \beta\}. \end{aligned} \tag{11}$$

In the sequel $P : \mathbb{R}^3 \to \mathbb{R}^2$ stands for the usual orthogonal projection and $\mathbb{R}^2$ is identified with $\mathbb{R}^2 \times \{0\}$. Hence it follows that either

$(a) P(S_{\overline{u}}) = \mathcal{S}$ or

(b) $P(S_{\overline{u}}) \subsetneq \mathcal{S}$.

If (a) holds then $\mathcal{H}^2(S_{\overline{u}} \setminus \mathcal{S}) > 0$, because $0 < \|\overline{u} - \overline{u}_0\|_{L^1(K)}$. Therefore there exists a set $W_{\overline{u}} \subset S_{\overline{u}}$ such that $\overline{W_{\overline{u}}} \cap \mathcal{S} = \emptyset$ and $\mathcal{H}^2(W_{\overline{u}}) > 0$. This fact along with (10) and Corollary 1, p. 76, of [8] yield

$$\begin{aligned} \mathcal{H}^2(S_u) &= \int_{S_{\overline{u}} \setminus W_{\overline{u}}} \left(\frac{\theta(z)}{\theta(0)}\right)^2 d\mathcal{H}^2 + \int_{W_{\overline{u}}} \left(\frac{\theta(z)}{\theta(0)}\right)^2 d\mathcal{H}^2 \\ &> \int_{S_{\overline{u}} \setminus W_{\overline{u}}} d\mathcal{H}^2 + \int_{W_{\overline{u}}} d\mathcal{H}^2 \\ &= \mathcal{H}^2(S_{\overline{u}}) \geq \mathcal{H}^2(P(S_{\overline{u}})) = \mathcal{H}^2(\mathcal{S}), \end{aligned} \tag{12}$$

and this case is proved.

If (b) holds then we set for $t \in [0, h)$

$$l_t \stackrel{\text{def}}{=} \mathcal{S} \times \{t\} \text{ and } l_{-t} \stackrel{\text{def}}{=} \mathcal{S} \times \{-t\}$$

and either:

(b_1) $l_t \cap S_{\overline{u}} \neq \emptyset$ for any $t \in (0, h)$ or $l_{-t} \cap S_{\overline{u}} \neq \emptyset$ for any $t \in (0, h)$;

(b_2) $\exists\, t_0 \in (0, h)$ such that $l_t \cap S_{\overline{u}} = \emptyset$ and $l_{-t} \cap S_{\overline{u}} = \emptyset$, $\forall\, t \in (t_0, h)$.

Suppose that (b_2) holds. In this case one easily check that

$$\mathcal{H}^2(S_{\overline{u}}) \geq 2\mathcal{H}^2(P(S_{\overline{u}})). \tag{13}$$

Since $\|\overline{u} - \overline{u}_0\|_{L^1(K)} \leq \|u - u_0\|_{L^1(\mathcal{O})}$, in this case we may take δ, the bound for $\|u - u_0\|_{L^1(\mathcal{O})}$, to be

$$\delta = \frac{(\beta - \alpha)}{4} \mathcal{H}^2(\mathcal{S})\, h \tag{14}$$

thus implying

$$0<\int_K |\overline{u}-\overline{u}_0|\,dx=(\beta-\alpha)\left\{|K_\alpha\cap K_\beta^{\overline{u}}|+|K_\beta\cap K_\alpha^{\overline{u}}|\right\}$$
$$<\frac{(\beta-\alpha)}{4}\mathcal{H}^2(\mathcal{S})\,h=\delta,$$

where $K_\alpha^{\overline{u}}$ and $K_\beta^{\overline{u}}$ are defined in (11). Therefore

$$\max\left\{|K_\alpha\cap K_\beta^{\overline{u}}|,|K_\beta\cap K_\alpha^{\overline{u}}|\right\}<\frac{1}{4}\mathcal{H}^2(\mathcal{S})\,h. \tag{15}$$

One easily verifies that $K_l=(K_l\cap K_\beta^{\overline{u}})\cup(K_l\cap K_\alpha^{\overline{u}})\cup(\partial K_\beta^{\overline{u}}\cap K_l)$, since $|\partial K_\beta^{\overline{u}}\cap K_l|=0$ $(l=\alpha,\beta)$.

Since $|K_\alpha|=|K_\beta|=\mathcal{H}^2(\mathcal{S})\,h$, (15) and the last remark yield

$$|K_\alpha\cap K_\alpha^{\overline{u}}|>\frac{3}{4}\mathcal{H}^2(\mathcal{S})h \;\text{ and }\; |K_\beta\cap K_\beta^{\overline{u}}|>\frac{3}{4}\mathcal{H}^2(\mathcal{S})h.$$

Define

$$\begin{aligned}\Sigma_\alpha &\stackrel{\text{def}}{=}\left\{(x,y,z)\in K:(x,y,0)\in P(K_\alpha\cap K_\alpha^{\overline{u}}),0<z<h\right\};\\ \Sigma_\beta &\stackrel{\text{def}}{=}\left\{(x,y,z)\in K:(x,y,0)\in P(K_\beta\cap K_\beta^{\overline{u}}),-h<z<0\right\}.\end{aligned} \tag{16}$$

Thus $K_\alpha\cap K_\alpha^{\overline{u}}\subseteq\Sigma_\alpha$, $K_\beta\cap K_\beta^{\overline{u}}\subseteq\Sigma_\beta$ and

$$\mathcal{H}^2(P(K_l\cap K_l^{\overline{u}}))h=|\Sigma_l|\geq|K_l\cap K_l^{\overline{u}}|>\frac{3}{4}\mathcal{H}^2(\mathcal{S})h\quad(l=\alpha,\beta),$$

from where we conclude that

$$\mathcal{H}^2(P(K_l\cap K_l^{\overline{u}}))>\frac{3}{4}\mathcal{H}^2(\mathcal{S})\quad(l=\alpha,\beta).$$

Altogether these facts produce

$$\mathcal{H}^2(P(K_\alpha\cap K_\alpha^{\overline{u}})\cap P(K_\beta\cap K_\beta^{\overline{u}}))>\frac{1}{2}\mathcal{H}^2(\mathcal{S}).$$

Since $P(S_{\overline{u}})\supseteq P(K_\alpha\cap K_\alpha^{\overline{u}})\cap P(K_\beta\cap K_\beta^{\overline{u}})$ using (13) we conclude that

$$\mathcal{H}^2(S_{\overline{u}})\geq 2\mathcal{H}^2(P(S_{\overline{u}}))>\mathcal{H}^2(\mathcal{S})$$

and the proof of this case is complete.

Now if (b_1) holds, we may suppose without loss that $l_t\cap S_{\overline{u}}\neq\emptyset$, for any $t\in(0,h)$. By defining

$$\mathcal{A}_s\stackrel{\text{def}}{=}\{(x,y,0)\in\mathcal{S}:-\theta(0)<x\leq\theta(0)-s\}.$$

we obtain

$$2s\sqrt{2\theta(0)s - s^2} > \mathcal{H}^2(\mathcal{S}) - \mathcal{H}^2(\mathcal{A}_s) \tag{17}$$

on the account that $2\sqrt{2\theta(0)s - s^2}$ and s are, respectively, the length and the height of a rectangle containing $\mathcal{S}\backslash\mathcal{A}_s$.

In this case by taking δ, the bound for $\|u - u_0\|_{L^1(\mathcal{O})}$, to be

$$\delta = \frac{(\beta - \alpha)}{32}\sqrt{2\theta(0)h - h^2}\, h^2$$

the same proof for (15) yields

$$\max\left\{|K_\alpha \cap K_\beta^{\overline{u}}|, |K_\beta \cap K_\alpha^{\overline{u}}|\right\} < \frac{1}{32}\sqrt{2\theta(0)h - h^2}\; h^2.$$

Thus $|K_l \cap K_l^{\overline{u}}| > \mathcal{H}^2(\mathcal{S})h - \frac{1}{32}\sqrt{2\theta(0)h - h^2}\; h^2 \quad (l = \alpha, \beta)$. Using again the sets Σ_α and Σ_β (see 16) we conclude that

$$\mathcal{H}^2(P(K_l \cap K_l^{\overline{u}})) > \mathcal{H}^2(\mathcal{S}) - \frac{1}{32}\sqrt{2\theta(0)h - h^2}\, h \quad (l = \alpha, \beta)$$

and since $P(S_{\overline{u}}) \supseteq P(K_\alpha \cap K_\alpha^{\overline{u}}) \cap P(K_\beta \cap K_\beta^{\overline{u}})$ we have

$$\begin{aligned}\mathcal{H}^2(P(S_{\overline{u}})) &\geq \mathcal{H}^2(P(K_\alpha \cap K_\alpha^{\overline{u}}) \cap P(K_\beta \cap K_\beta^{\overline{u}})) \\ &> \mathcal{H}^2(\mathcal{S}) - \frac{1}{16}\sqrt{2\theta(0)h - h^2}\; h.\end{aligned}$$

Thus we infer that $\exists\, s_1 \in (0, h/4)$ such that

$$\mathcal{H}^2(P(S_{\overline{u}})) = \mathcal{H}^2(\mathcal{A}_{s_1}). \tag{18}$$

Now by defining the sets $L_t \stackrel{\text{def}}{=} \mathcal{S} \times [-t, t]$, $t \in [h/2, h)$,

$$\begin{aligned}\widetilde{\Sigma}_\alpha^t &\stackrel{\text{def}}{=} \left\{(x, y, z) \in K : (x, y, 0) \in P(K_\alpha \cap K_\alpha^{\overline{u}} \cap L_t), 0 < z < t\right\} \text{ and} \\ \widetilde{\Sigma}_\beta^{-t} &\stackrel{\text{def}}{=} \left\{(x, y, z) \in K : (x, y, 0) \in P(K_\beta \cap K_\beta^{\overline{u}} \cap L_t), -t < z < 0\right\}\end{aligned}$$

we find $P(S_{\overline{u}} \cap L_t) \supseteq P(K_\alpha \cap K_\alpha^{\overline{u}} \cap L_t) \cap P(K_\beta \cap K_\beta^{\overline{u}} \cap L_t)$ and using the foregoing arguments the following inequality is established

$$\mathcal{H}^2(P(S_{\overline{u}} \cap L_t)) > \mathcal{H}^2(\mathcal{S}) - \frac{h^2}{16t}\sqrt{2\theta(0)h - h^2}.$$

In particular for $t = h/2$ we conclude that there exist $s_2 \in [s_1, h/4)$ such that

$$\mathcal{H}^2(P(S_{\overline{u}} \cap L_{h/2})) = \mathcal{H}^2(\mathcal{A}_{s_2}) \leq \mathcal{H}^2(\mathcal{A}_{s_1}), \tag{19}$$

with s_1 as in (18). Thus, for each $t \in (h/2, h)$, $\exists\ s = s(t) \in [s_1, s_2]$ satisfying

$$\mathcal{H}^2(\mathcal{A}_{s_2}) \leq \mathcal{H}^2(P(S_{\overline{u}} \cap L_t)) = \mathcal{H}^2(\mathcal{A}_s) \leq \mathcal{H}^2(\mathcal{A}_{s_1}). \tag{20}$$

Since $\overline{u} \in BV(K, \{\alpha, \beta\})$ satisfies (b_1) for each $t \in [h/2, h)$ the set $l_t \cap (K_\alpha \cap K_\alpha^{\overline{u}}) \subset \mathbb{R}^2$ is rectifiable and

$$\mathrm{Per}_{l_t}(K_\alpha \cap K_\alpha^{\overline{u}}) = \mathcal{H}^1(l_t \cap S_{\overline{u}}) > 0,$$

where $\mathrm{Per}_X A$ stands for the perimeter of the set A in X.

Since $P(l_t \cap S_{\overline{u}}) \subset P(S_{\overline{u}} \cap L_t)$, from the fact that $\mathcal{S}$ is a disc and the definition of $\mathcal{A}_s$ one readily verifies that

$$\mathcal{H}^1(l_t \cap S_{\overline{u}}) \geq 2\sqrt{2\theta(0)s - s^2} \geq 2\sqrt{2\theta(0)s_1 - s_1^2}, \tag{21}$$

for each $t \in [h/2, h)$, with s as in (20).

Given that $S_{\overline{u}} = (S_{\overline{u}} \cap L_{h/2}) \cup (S_{\overline{u}} \cap \{K \setminus L_{h/2}\})$ and that the application $t \mapsto \mathrm{Per}_{l_t}(K_\alpha \cap K_\alpha^{\overline{u}})$ is integrable (see [7], for instance), the co-area formula and (21) yield

$$\begin{aligned}
\mathcal{H}^2(S_{\overline{u}} \cap \{K \setminus L_{h/2}\}) &= \int_{h-s_1}^{h} \mathrm{Per}_{l_t}(K_\alpha \cap K_\alpha^{\overline{u}})dt + \int_{h/2}^{h-s_1} \mathrm{Per}_{l_t}(K_\alpha \cap K_\alpha^{\overline{u}})dt \\
&\geq 2s_1\sqrt{2\theta(0)s_1 - s_1^2} + \int_{s_2}^{s_1} 2\sqrt{2\theta(0)s - s^2}\, ds \\
&= 2s_1\sqrt{2\theta(0)s_1 - s_1^2} + \mathcal{H}^2(\mathcal{A}_{s_1}) - \mathcal{H}^2(\mathcal{A}_{s_2}).
\end{aligned}$$

This fact along with (17), (19) and Corollary 1, p. 76, of [8] imply

$$\begin{aligned}
\mathcal{H}^2(S_{\overline{u}}) &\geq \mathcal{H}^2(S_{\overline{u}} \cap \{K \setminus L_{h/2}\}) + \mathcal{H}^2(P(S_{\overline{u}} \cap L_{h/2})) \\
&\geq 2s_1\sqrt{2\theta(0)s_1 - s_1^2} + \mathcal{H}^2(\mathcal{A}_{s_1}) \\
&> \mathcal{H}^2(\mathcal{S})
\end{aligned}$$

and the proof for this case is established.

Proof of (4.1.ii). Since the proof is basically the same as the previous one we just mention the modifications needed.

The transformation Λ now takes the set $\partial\mathcal{O} \cap \partial\Omega$ into

$$\begin{aligned}
\widetilde{\partial K} &\stackrel{\text{def}}{=} \Lambda(\partial\mathcal{O} \cap \partial\Omega) \\
&= \{(\theta(0)\cos t, \theta(0)\mathrm{sen} t, z),\ t \in [0, 2\pi),\ -h < z < h\}.
\end{aligned}$$

Moreover, for each $\overline{v} \in BV(\widetilde{\partial K}, \{\alpha', \beta'\})$, such that $\mathcal{H}^1(S_{\overline{v}}) > 0$, either

(a) $P(S_{\overline{v}}) = \mathcal{C}$

or

(b) $P(S_{\overline{v}}) \subset \mathcal{C}$.

If $\overline{v} = v \circ \Lambda^{-1}$ satisfies (a), the very same computation used to prove (12), can be utilized to prove that $\mathcal{H}^1(S_v) > \mathcal{H}^1(\mathcal{C})$.

Now if $\overline{v} = v \circ \Lambda^{-1}$ satisfies (b), a few and natural modifications (regarding the sets $K_\alpha^{\overline{v}}$ and $K_\beta^{\overline{v}}$, l_t and l_{-t} now defined on $\widetilde{\partial K}$) allow us to conclude that one of the following cases occur:

(b_1) $l_t \cap S_{\overline{v}} \neq \emptyset$ for any $t \in (0, h)$ or $l_{-t} \cap S_{\overline{v}} \neq \emptyset$ for any $t \in (0, h)$;

(b_2) $\exists\, t_0 \in (0, h)$ such that $l_t \cap S_{\overline{v}} = \emptyset$ and $l_{-t} \cap S_{\overline{v}} = \emptyset$, $\forall\, t \in (t_0, h)$.

If (b_2) holds, then by choosing δ, the bound for $\|u - u_0\|_{L^1(\mathcal{O})}$, to be $\delta = \dfrac{(\beta' - \alpha')}{4}\mathcal{H}^1(\mathcal{C})h$, one shows (using the very same arguments from $(4.1.i)$) that

$$\mathcal{H}^1(S_{\overline{v}}) \geq 2\mathcal{H}^1(P(S_{\overline{v}})) > \mathcal{H}^1(\mathcal{C}).$$

Now if (b_1) holds, in a similar fashion we define the sets $L_{h/2}$ and $\widetilde{\Sigma}_\alpha^{h/2}$, $\widetilde{\Sigma}_\beta^{-h/2}$ in $\widetilde{\partial K}$. In this case we may take $\delta = \dfrac{(\beta' - \alpha')}{8}h^2$ and from the fact that $S_{\overline{v}} = (S_{\overline{v}} \cap L_{h/2}) \cup (S_{\overline{v}} \cap \{\widetilde{\partial K} \setminus L_{h/2}\})$ we obtain

$$\mathcal{H}^1(S_{\overline{v}} \cap L_{h/2}) \geq \mathcal{H}^1(P(S_{\overline{v}} \cap L_{h/2})) > \mathcal{H}^1(\mathcal{C}) - h/2 \quad \text{and}$$
$$\mathcal{H}^1((S_{\overline{v}} \cap \{\widetilde{\partial K} \setminus L_{h/2}\}) \geq h/2,$$

which implies $\mathcal{H}^1(S_{\overline{v}}) > \mathcal{H}^1(\mathcal{C})$. In this way the proof of $(4.1.ii)$ can be completed.

Proof of $(4.1.iii)$. By $(\mathbf{f_5})$ we have

$$\int_{\mathcal{M}_l} |h(Tu_0) - h(v_0)| d\mathcal{H}^2 = |h(\alpha) - h(\alpha')|\, \mathcal{H}^2(\mathcal{M}_l) \quad (l \in \{\alpha, \beta\}).$$

For (u, v) as above; $i, l \in \{\alpha, \beta\}$ and $j \in \{\alpha', \beta'\}$, we define

$$\mathcal{M}_{ij}^l \stackrel{\text{def}}{=} \{x \in \mathcal{M}_l : Tu(x) = i, v(x) = j\}.$$

Then $\mathcal{M}_l = \mathcal{M}^l_{\alpha\alpha'} \cup \mathcal{M}^l_{\alpha\beta'} \cup \mathcal{M}^l_{\beta\alpha'} \cup \mathcal{M}^l_{\beta\beta'}$ and

$$\int_{\mathcal{M}_l} |h(Tu) - h(v)| d\mathcal{H}^2 = \sum_{\substack{i \in \{\alpha,\beta\} \\ j \in \{\alpha',\beta'\}}} |h(i) - h(j)| \mathcal{H}^2(\mathcal{M}^l_{ij}).$$

From ($\mathbf{f_5}$) and the definition of h, one easily sees that

$$|h(\beta) - h(\alpha')| > |h(\beta) - h(\beta')| = |h(\alpha) - h(\alpha')|.$$

Likewise $|h(\alpha) - h(\beta')| > |h(\alpha) - h(\alpha')|$ and thus for $l \in \{\alpha, \beta\}$,

$$\int_{\mathcal{M}_l} |h(Tu) - h(v)| d\mathcal{H}^2 > |h(\alpha') - h(\alpha)| \mathcal{H}^2(\mathcal{M}_l).$$

Therefore

$$\int_{\partial\Omega} |h(Tu_0) - h(v_0)| d\mathcal{H}^2 < \int_{\partial\Omega} |h(Tu) - h(v)| d\mathcal{H}^2,$$

and the proof of (4.1.*iii*) is also complete.

At last the proof of Lemma 4.1 is established by choosing δ, the bound for $\|u - u_0\|_{L^1(\mathcal{O})}$, as the minimum of the bounds picked above and noting that

$$\|(u,v) - (u_0,v_0)\|_{\mathbb{L}^1} = \|u - u_0\|_{L^1(\Omega)} + \|v - v_0\|_{L^1(\partial\Omega)}.$$

In this way if $(u,v) \in \mathbb{L}^1$ satisfies $0 < \|(u,v) - (u_0,v_0)\|_{\mathbb{L}^1} < \delta$ then $\mathcal{E}_0(u_0,v_0) < \mathcal{E}_0(u,v)$ if $0 < \kappa < 0$, and $\overline{\mathcal{E}_0}(u_0,v_0) < \overline{\mathcal{E}}_0(u,v)$ if $\kappa = 0$. ■

5 The remaining hypotheses of De Giorgi's Theorem

In this section we verify hypotheses (2.3.iii) and (2.3.iv) of Theorem 2.3.

The compactness result (2.3.iii) has been proved in [1] (see Theorem 2.6 (i)) and we now prove (2.3.iv) using the usual direct method of Calculus of Variations.

Let $\mathcal{E}_\varepsilon$ defined by (5). From ($\mathbf{f_4}$), $\exists C > 0$, $p \geq 2$ and $d \in \mathbb{R}$ such that $F(t) \geq C|t|^p - d$.

Let us fix $\delta > 0$ and $\varepsilon > 0$. Since $\mathcal{E}_\varepsilon \geq 0$, there is a constant $M \geq 0$ such that

$$M = \inf\{\mathcal{E}_\varepsilon(u,v) : (u,v) \in B_\delta\}.$$

Let $\{(u_i, v_i)\}_{i\in N}$ be a minimizing sequence. From the above remark on the growth of F it follows that (u_i) is a bounded sequence in $H^1(\Omega)$ and as such there exists $u_\varepsilon \in H^1(\Omega)$ and a subsequence of (u_i) (still denoted by (u_i)) such that

$$u_i \rightharpoonup u_\varepsilon \text{ in } H^1(\Omega) \text{ and } \quad u_i \to u_\varepsilon \text{ in } L^2(\Omega).$$

Moreover by the properties of the trace operator,

$$Tu_i \rightharpoonup Tu_\varepsilon \text{ in } H^{1/2}(\partial\Omega) \text{ and } \quad Tu_i \to Tu_\varepsilon = v_\varepsilon \text{ in } L^2(\partial\Omega).$$

We take another subsequence of (u_i), still denoted by (u_i), so that $u_i(x) \to u_\varepsilon(x)$ a.e. in Ω and $Tu_i(y) \to v_\varepsilon(y)$ a.e. in $\partial\Omega$. Since $F(u_i(x)) \to F(u_\varepsilon(x))$ a.e. in Ω and $G(Tu_i(y)) \to G(v_\varepsilon(y))$ a.e. in $\partial\Omega$, we now resort to Fatou's Lemma and results from semi-continuity to conclude that

$$\mathcal{E}_\varepsilon(u_\varepsilon, v_\varepsilon) \le \liminf_{i\to\infty} \mathcal{E}_\varepsilon(u_i, v_i) = M.$$

Since B_δ is closed in $\mathbb{L}^1$ and $(u_i, Tu_i) \to (u_\varepsilon, v_\varepsilon)$ in $\mathbb{L}^1$, we conclude that $(u_\varepsilon, v_\varepsilon) \in B_\delta$. ■

6 Proofs of the main results

Once Theorem 2.3 is proved our main results will follow from standard procedures.

Proof of Theorem 1.1

As mentioned we have verified in the previous sections all the hypotheses of Theorem 2.3 for the family of functionals $\mathcal{E}_\varepsilon$.

Then there exists an $\varepsilon_0 > 0$ and a sequence $\{(u_{\varepsilon_j}, v_{\varepsilon_j})\}_{0<\varepsilon_j<\varepsilon_0}$ in $\mathbb{L}^1$ ($\varepsilon_j \to 0$ as $j \to \infty$) such that each $(u_{\varepsilon_j}, v_{\varepsilon_j})$ is a $\mathbb{L}^1$-local minimizer of $\mathcal{E}_{\varepsilon_j}$.

Moreover $||(u_{\varepsilon_j}, v_{\varepsilon_j}) - (u_0, v_0)||_{\mathbb{L}^1} \longrightarrow 0$, as $\varepsilon_j \to 0$.

From the penalization imposed on $\mathcal{E}_\varepsilon$, we have $u_{\varepsilon_j} \in H^1(\Omega)$ and $v_{\varepsilon_j} = Tu_{\varepsilon_j}$. Since $H^1(\Omega)$ and $H^{1/2}(\partial\Omega)$ are continuously imbedded in $L^1(\Omega)$ and $L^1(\partial\Omega)$, respectively, there are constants C_1 and C_2 satisfying

$$\begin{aligned} &\|u\|_{L^1(\Omega)} \le C_1\|u\|_{H^1(\Omega)}, \ \forall u \in H^1(\Omega) \text{ and} \\ &\|v\|_{L^1(\partial\Omega)} \le C_2\|v\|_{H^{1/2}(\partial\Omega)}, \ \forall v \in H^{1/2}(\partial\Omega). \end{aligned} \tag{22}$$

Also by continuity of the trace operator $T : H^1(\Omega) \to H^{1/2}(\partial\Omega)$, there exists a constant C_3 such that

$$\|Tu\|_{H^{1/2}(\partial\Omega)} \leq C_3 \|u\|_{H^1(\Omega)}, \ \ \forall\, u \in H^1(\Omega). \tag{23}$$

Let $\delta' > 0$ be such that

$$\mathcal{E}_{\varepsilon_j}(u_{\varepsilon_j}, v_{\varepsilon_j}) \leq \mathcal{E}_{\varepsilon_j}(u, v) \tag{24}$$

whenever $(u, v) \in \mathbb{L}^1$ satisfies $\|(u, v) - (u_{\varepsilon_j}, v_{\varepsilon_j})\|_{\mathbb{L}^1} < \delta'$.

Also let $u \in H^1(\Omega)$ be such that

$$\|u - u_{\varepsilon_j}\|_{H^1(\Omega)} < \delta, \text{ where } \quad \delta = \frac{\delta'}{C_1 + (C_2 C_3)}\,. \tag{25}$$

Hence using (22) and (23), one easily obtains

$$\|(u, Tu) - (u_{\varepsilon_j}, Tu_{\varepsilon_j})\|_{\mathbb{L}^1} \leq (C_1 + C_2 C_3)\|u - u_{\varepsilon_j}\|_{H^1(\Omega)} < \delta'.$$

Therefore, in view of (24), $E_{\varepsilon_j}(u_{\varepsilon_j}) \leq E_{\varepsilon_j}(u)$.

Summing up, for $u \in H^1(\Omega)$ satisfying (25), we have $E_{\varepsilon_j}(u_{\varepsilon_j}) \leq E_{\varepsilon_j}(u)$.

But critical points of E_{ε_j} are weak solutions of (4) and now, as usual, we conclude that u_{ε_j} is a classical solution by resorting to bootstrap arguments.

Since $W^{1,p}(\Omega)$, for $p > 3$, is continuously imbedded in $H^1(\Omega)$, we infer that u_{ε_j} is also a local minimum of E_{ε_j} in $W^{1,p}(\Omega)$. With this information and using the variational characterization of the eigenvalues of the corresponding linearized problem at u_{ε_j}, we evoke the results established in [5] to conclude that in fact u_{ε_j} is a stable (in the sense of Liapounov) stationary solution to (1). ∎

Proof of Theorem 1.3

A careful reading of the proof of Theorem 1.1 reveals that the replacement of $(\mathbf{f_5})$ with $(\mathbf{f_6})$ only requires a new proof (actually simpler) of Lemma 4.1 in order to establish Theorem 1.3.

It suffices to prove that $(u_0, v_0) = (\alpha, \alpha')$ is an isolated $\mathbb{L}^1$-local minimizer of $\mathcal{E}_0$ if $0 < \kappa < 0$, and of $\overline{\mathcal{E}}_0$ in case of $\kappa = 0$.

Noting that $\mathcal{H}^2(S_{u_0}) = \mathcal{H}^1(S_{v_0}) = 0$, Lemma 4.1 will be proved by showing that $(\mathbf{f_6})$ implies the strict inequality (4.1.*iii*).

We compute

$$\mathcal{E}_0(u_0,v_0)=\int_{\partial\Omega}|h(\alpha)-h(\alpha')|d\mathcal{H}^2=|h(\alpha)-h(\alpha')|\mathcal{H}^2(\partial\Omega), \tag{26}$$

and take $(u,v)\in BV(\Omega,\{\alpha,\beta\})\times BV(\partial\Omega,\{\alpha',\beta'\})$ satisfying $0<\|(u,v)-(u_0,v_0)\|_{\mathbb{L}^1(\Omega)}<\delta$.

Then defining

$$\mathcal{M}_{ij}\overset{\text{def}}{=}\{x\in\partial\Omega: Tu(x)=i, v(x)=j\}\ \ (i\in\{\alpha,\beta\}; j\in\{\alpha',\beta'\})$$

we obtain

$$\int_{\partial\Omega}|h(Tu)-h(v)|d\mathcal{H}^2=\sum_{\substack{i=\alpha,\beta\\ j=\alpha',\beta'}}|h(i)-h(j)|\mathcal{H}^2(\mathcal{M}_{ij}).$$

Since $|h(\beta)-h(\alpha')|>|h(\alpha)-h(\alpha')|$ and $|h(\alpha)-h(\beta')|>|h(\beta)-h(\beta')|$, $(\mathbf{f_6})$ implies

$$\begin{aligned}\mathcal{E}_0(u,v)&=\sigma\mathcal{H}^2(S_u)+\int_{\partial\Omega}|h(Tu)-h(v)|d\mathcal{H}^2+c\mathcal{H}^1(S_v)\\ &\geq\int_{\partial\Omega}|h(Tu)-h(v)|d\mathcal{H}^2=\sum_{\substack{i=\alpha,\beta\\ j=\alpha',\beta'}}|h(i)-h(j)|\mathcal{H}^2(\mathcal{M}_{ij})\\ &>\sum_{\substack{i=\alpha,\beta\\ j=\alpha',\beta'}}|h(\alpha)-h(\alpha')|\mathcal{H}^2(\mathcal{M}_{ij})=\mathcal{E}_0(u_0,v_0).\end{aligned}$$

Thus (u_0,v_0) is an isolated $\mathbb{L}^1$-local minimizer of $\mathcal{E}_0$ (likewise for $\overline{\mathcal{E}}_0$).

The rest of the proof now follows exactly as in the previous case. ■

7 Appendix

The necessity of $(\mathbf{f_5})$ in Theorem 1.1

We claim that $(\mathbf{f_5})$ is a necessary hypothesis for (**2.3.ii**) of Theorem 2.3 to be satisfied.

Indeed if $(\mathbf{f_5})$ does not hold we may suppose, for instance, that

$$\int_{\alpha'}^{\alpha}\sqrt{F}>\int_{\beta}^{\beta'}\sqrt{F}. \tag{27}$$

Next we take the function θ in **(H)** to be $\theta(z) = 1 + z^2$. For each $t > 0$ define

$$H_t = \{(x, y, z) \in \overline{\mathcal{O}} : z = t\}$$

and choose $u \in BV(\Omega, \{\alpha, \beta\})$ and $v \in BV(\partial\Omega, \{\alpha', \beta'\})$ such that

$S_u = H_t \cap \mathcal{O}, \{x \in \mathcal{O} : u(x) = \beta\} \supset \mathcal{O}_\beta, S_v = H_t \cap \partial\mathcal{O}$, $\{y \in \partial\Omega : Tu(y) = l\} = \{y \in \partial\Omega : v(y) = l'\}(l = \alpha, \beta)$.

Then

$$\begin{aligned}\Phi(u, v) &= \sigma\mathcal{H}^2(S_u) + \int_{\partial\Omega} |h(Tu) - h(v)|\, d\mathcal{H}^2 + c\mathcal{H}^1(S_v)\\ &= \sigma\pi\theta^2(t) + |h(\alpha') - h(\alpha)||\mathcal{M}^\alpha_{\alpha\alpha'}|\\ &\quad + |h(\beta') - h(\beta)||\mathcal{M}^\alpha_{\beta\beta'}| + |h(\beta') - h(\beta)||\mathcal{M}^\beta_{\beta\beta'}| + c2\pi\theta(t)\end{aligned}$$

Since $\Phi(u_0, v_0) = \sigma\pi + |h(\alpha') - h(\alpha)||\mathcal{M}^\alpha| + |h(\beta') - h(\beta)||\mathcal{M}^\beta| + c2\pi$, it follows that $\Phi(u, v) \leq \Phi(u_0, v_0)$ if and only if

$$\sigma\pi(\theta^2(t) - 1) + c2\pi(\theta(t) - 1) \leq (|h(\alpha') - h(\alpha)| - |h(\beta') - h(\beta)|)|\mathcal{M}^\alpha_{\beta\beta'}|$$

for in this case $\mathcal{M}^\alpha = \mathcal{M}^\alpha_{\beta\beta'} \cup \mathcal{M}^\alpha_{\alpha\alpha'}$.

Note that by assumption

$$|\mathcal{M}^\alpha_{\beta\beta'}| = 2\pi \int_0^t (1 + z^2)\sqrt{1 + z^2}\, dz,$$

and then above inequality reads

$$\frac{\sigma\pi(\theta^2(t) - 1) + c2\pi(\theta(t) - 1)}{2\pi \int_0^t (1 + z^2)\sqrt{1 + z^2}\, dz}$$

$$\leq |h(\alpha') - h(\alpha)| - |h(\beta') - h(\beta)| = 2^{2/3}\{\int_{\alpha'}^{\alpha} \sqrt{F} - \int_{\beta}^{\beta'} \sqrt{F}\}.$$

Note that the left-hand side of the above inequality goes to zero as $t \to 0$ whereas the right-hand side does not depend on t and is positive by virtue of (27). We conclude that (u_0, v_0) is not a local minimum of the Γ-limit functional Φ.

Proof of Theorem 2.3

Underlying our procedure is De Giorgi's Theorem ([6]) whose proof in our case for the reader's convenience is presented below.

By (2.3.ii) there exists an isolated minimiser (u_0, v_0) of $\mathcal{E}_0$ in $B_\delta(u_0, v_0)$, for some $\delta > 0$.

Given that $\mathcal{E}_{\varepsilon_j}$ Γ-converge to $\mathcal{E}_0$, there exists a sequence $\{(a_{\varepsilon_j}, b_{\varepsilon_j})\} \subset \mathbb{L}^1$ such that $(a_{\varepsilon_j}, b_{\varepsilon_j}) \to (u_0, v_0)$ in $\mathbb{L}^1$ and

$$\lim_{j\to\infty} \mathcal{E}_{\varepsilon_j}(a_{\varepsilon_j}, b_{\varepsilon_j}) = \mathcal{E}_0(u_0, v_0). \tag{28}$$

By its turn, (2.3.iv) guarantees the existence of a sequence $\{(u_{\varepsilon_j}, v_{\varepsilon_j})\} \subset \mathbb{L}^1$ of minima of $\mathcal{E}_{\varepsilon_j}$ in $B_\delta(u_0, v_0)$.

For j large enough, $(a_{\varepsilon_j}, b_{\varepsilon_j}) \in B_\delta(u_0, v_0)$. Hence since $(u_{\varepsilon_j}, v_{\varepsilon_j})$ is a minimum in $B_\delta(u_0, v_0)$, it follows that

$$\mathcal{E}_{\varepsilon_j}(u_{\varepsilon_j}, v_{\varepsilon_j}) \le \mathcal{E}_{\varepsilon_j}(a_{\varepsilon_j}, b_{\varepsilon_j}), \tag{29}$$

thus implying that $\{\mathcal{E}_{\varepsilon_j}(u_{\varepsilon_j}, v_{\varepsilon_j})\}$ is a bounded sequence.

On the other hand, it follows from (29) and (28) that for any subsequence $\{\varepsilon_{j_k}\}$ it holds

$$\begin{aligned}\liminf_{k\to\infty} \mathcal{E}_{\varepsilon_{j_k}}(u_{\varepsilon_{j_k}}, v_{\varepsilon_{j_k}}) &\le \liminf_{k\to\infty} \mathcal{E}_{\varepsilon_{j_k}}(a_{\varepsilon_{j_k}}, b_{\varepsilon_{j_k}}) = \lim_{k\to\infty} \mathcal{E}_{\varepsilon_{j_k}}(a_{\varepsilon_{j_k}}, b_{\varepsilon_{j_k}}) \\ &= \mathcal{E}_0(u_0, v_0).\end{aligned} \tag{30}$$

We claim that $(u_{\varepsilon_j}, v_{\varepsilon_j})$ lies in the interior of $B_\delta(u_0, v_0)$. In other words, $(u_{\varepsilon_j}, v_{\varepsilon_j})$ is a local minimum of $\mathcal{E}_{\varepsilon_j}$.

Suppose by contradiction that there exists a subsequence (keeping the same notation) $\{(u_{\varepsilon_j}, v_{\varepsilon_j})\} \subset \mathbb{L}^1$ such that

$$\|(u_{\varepsilon_j}, v_{\varepsilon_j}) - (u_0, v_0)\|_{\mathbb{L}^1} = \delta. \tag{31}$$

By (2.3.iii) there is another subsequence $\{(u_{\varepsilon_{j_k}}, v_{\varepsilon_{j_k}})\}$ and $(\overline{u}, \overline{v}) \in \mathbb{L}^1$ satisfying $(u_{\varepsilon_{j_k}}, v_{\varepsilon_{j_k}}) \overset{k\to\infty}{\longrightarrow} (\overline{u}, \overline{v})$ in $\mathbb{L}^1$.

Therefore from (31) we infer that $\|(\overline{u}, \overline{v}) - (u_0, v_0)\|_{\mathbb{L}^1} = \delta$.

Once again from (2.3.i) and (30) we conclude that

$$\mathcal{E}_0(\overline{u}, \overline{v}) \le \liminf_{k\to\infty} \mathcal{E}_{\varepsilon_{j_k}}(u_{\varepsilon_{j_k}}, v_{\varepsilon_{j_k}}) = \mathcal{E}_0(u_0, v_0),$$

which contradicts the fact that (u_0, v_0) is a isolated minimum, thus proving our claim.

We assert that $\|(u_{\varepsilon_j}, v_{\varepsilon_j}) - (u_0, v_0)\|_{\mathbb{L}^1} \to 0$.

To that end, we suppose by contradiction that $\exists\,\gamma > 0$ and a subsequence $\{(u_{\varepsilon_{j_k}}, v_{\varepsilon_{j_k}})\} \subset \{(u_{\varepsilon_j}, v_{\varepsilon_j})\}$ such that

$$0 < \gamma \leq \|(u_{\varepsilon_{j_k}}, v_{\varepsilon_{j_k}}) - (u_0, v_0)\|_{\mathbb{L}^1} \leq \delta.$$

Since the sequence $\{\mathcal{E}_{\varepsilon_{j_k}}(u_{\varepsilon_{j_k}}, v_{\varepsilon_{j_k}})\}$ is bounded, by (2.3.iii) we find another subsequence (keeping the same notation) $\{(u_{\varepsilon_{j_k}}, v_{\varepsilon_{j_k}})\} \subset \mathbb{L}^1$ and $(\overline{u}, \overline{v}) \in \mathbb{L}^1$ such that $(u_{\varepsilon_{j_k}}, v_{\varepsilon_{j_k}}) \to (\overline{u}, \overline{v})$ in $\mathbb{L}^1$, and as such

$$0 < \gamma \leq \|(\overline{u}, \overline{v}) - (u_0, v_0)\|_{\mathbb{L}^1} \leq \delta.$$

This implies that $(\overline{u}, \overline{v}) \neq (u_0, v_0)$. Now (2.3.i) and (30) yield

$$\mathcal{E}_0(\overline{u}, \overline{v}) \leq \liminf_{k\to\infty} \mathcal{E}_{\varepsilon_{j_k}}(u_{\varepsilon_{j_k}}, v_{\varepsilon_{j_k}}) = \mathcal{E}_0(u_0, v_0),$$

thus contradicting the fact that (u_0, v_0) is an isolated minimum and this completes the proof. ■

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