

A NONLINEAR PARABOLIC APPROXIMATION OF THE EULER EQUATIONS FOR ISOTHERMAL GAS FLOWS

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ABSTRACT. A notion of entropy quasisolution is introduced for the Euler equations of isothermal gas flows. Such a solution is obtained by means of nonlinear parabolic approximation with a small parameter ε . Compensated compactness argument is applied to justify the passage to limit as $\varepsilon \rightarrow 0$. It is verified that smooth entropy quasisolution is necessarily a classic solution. An example of entropy solution with a shock front is constructed to reveal that it is not an entropy quasisolution. The study is motivated by the explosion physics experiments in which the mass conservation law may be violated at a shock front passing through the gas.

1. INTRODUCTION

It is well known that hyperbolic equations serve well as the first level models to describe waves and shocks since they admit solutions with discontinuities. But they alone are not enough. To characterize types and properties of shock waves, one should exploit models of the next level allowing for a number of different small physical effects such as viscosity, diffusion, dispersion etc.

In Lagrangian variables, the one-dimensional isothermal flows of a gas with plane waves are described by the system

$$(1.1) \quad u_t = -p_x, \quad v_t = u_x, \quad p = k^2/v, \quad (t, x) \in \Pi = (0, T) \times \mathbb{R},$$

where u is the velocity, v is the specific volume, p is the pressure, and k is a constant. The existence of weak solutions (containing jump discontinuities) for the Cauchy problem associated with (1.1) was first established by Nishida [19]. The solutions obtained by Nishida have bounded variation and remain bounded away from the vacuum. For background on the BV theory we refer to [5, 14].

The existence of entropy L^∞ -solutions was established by Huang, Wang [10] and LeFloch, Shelukhin [15] in the Eulerian formulation. Given initial data $u_0, v_0 \in L^\infty(\mathbb{R})$,

$$(1.2) \quad u|_{t=0} = u_0(x), \quad v|_{t=0} = v_0(x),$$

we approximate the Cauchy problem (1.1), (1.2), with the parabolic system

$$(1.3) \quad u_t^\varepsilon = -p(v^\varepsilon)_x + \varepsilon u_{xx}^\varepsilon, \quad v_t^\varepsilon = u_x^\varepsilon + \varepsilon v_{xx}^\varepsilon - \varepsilon (v_x^\varepsilon)^2/v^\varepsilon,$$

equipped with the initial conditions

$$(1.4) \quad u^\varepsilon|_{t=0} = u_0^\varepsilon, \quad v^\varepsilon|_{t=0} = v_0^\varepsilon.$$

We consider the limit problem as $\varepsilon \rightarrow 0$ and verify that the limit functions

$$u = \lim_{\varepsilon \rightarrow 0} u^\varepsilon, \quad v = \lim_{\varepsilon \rightarrow 0} v^\varepsilon$$

solve the Cauchy problem (1.1), (1.2) in the following sense.

Definition. Functions $u, v \in L^\infty(\Pi)$, $v > 0$, are called an entropy quasisolution to the Cauchy problem (1.1)-(1.2) if

$$(1.5) \quad \int_0^\infty \int_{\mathbb{R}} [\eta(u, v) - \eta(u_0, v_0)] \varphi_t + q(u, v) \varphi_x \, dx dt \geq 0$$

for each nonnegative function $\varphi(t, x) \in \mathcal{D}(\mathbb{R}^2)$ and all entropy pairs $(\eta, q) \in C^1(\mathbb{R}^2)$ solving the system

$$(1.6) \quad \eta_v = -q_u, \quad k^2 \eta_u = -v^2 q_v$$

with the condition

$$(1.7) \quad B_U[\xi, \zeta] \equiv \eta_{uu} \xi^2 + (\eta_{vv} + \eta_v/v) \zeta^2 + 2\eta_{uv} \xi \zeta \geq 0, \quad \forall (\xi, \zeta) \in \mathbb{R}^2.$$

Note, that the definition on entropy solution [10, 15] coincides with the above if we replace (1.7) with the convexity condition on the function $\eta(u, v)$.

The equation

$$(1.8) \quad \eta_{vv} = -p'(v) \eta_{uu},$$

which is obtained by elimination of q from (1.6), is called the entropy-wave equation.

The principal peculiarity of notion of entropy quasisolution is as follows. The momentum equation $u_t = -p_x$ holds in the distributional sense and ensues from (1.5). This is a consequence of the fact that both the pair $\eta = u$, $q = p$ and the pair $\eta = -u$, $q = -p$ satisfy (1.6). However, whereas the pair $\eta = v$, $q = -u$ satisfies (1.6), the pair $\eta = -v$, $q = u$ fails to do so. Therefore, the entropy condition guarantees only validity of the relations

$$u_t = -p(v)_x, \quad v_t \leq u_x \quad \text{in } \mathcal{D}'(\Pi).$$

Below, to justify the notion of entropy quasisolution, we show that a smooth entropy quasisolution solves equations (1.1) in the classical sense. Thus, an entropy quasisolution describes a process in which the law of mass conservation of gas may be violated at a discontinuity point. Such a phenomenon appears in explosion physics. For example, when a shock wave passes through gaseous carbon, solid carbon particles, diamonds, precipitate from the gas [9, 27] and hydrogen may acquire metallic properties in the front of the shock wave [28].

Our main result is the following.

Theorem 1.1. *Given initial data $u_0, v_0 \in L^\infty(\mathbb{R})$, there is an entropy quasisolution (u, v) to the Cauchy problem (1.1)-(1.2) such that $\inf v > 0$ and $\sup v < \infty$.*

The scheme of proof is as follows. First, we prove that the viscous approximations $(u^\varepsilon, v^\varepsilon)$ enjoy the estimates

$$|u^\varepsilon| \leq c_0, \quad c_0^{-1} \leq v^\varepsilon \leq c_0,$$

with a constant c_0 independent of ε . These estimates enable us to assume [1, 24] that there is a sequence $(u^\varepsilon, v^\varepsilon)$ such that

$$(1.9) \quad u^\varepsilon \rightharpoonup u, \quad v^\varepsilon \rightharpoonup v \quad \text{*weakly in } L_{loc}^\infty(\Pi),$$

and there is a probability Young measure $\nu_{t,x} \geq 0$ on the plane \mathbb{R}^2 , defined for a. e. $(t, x) \in \Pi$, such that

$$\text{spt } \nu_{t,x} \subset K = \{(u, v) \in \mathbb{R}^2 : |u| \leq c_0, \quad c_0^{-1} \leq v \leq c_0\}$$

and

$$(1.10) \quad \lim_{\varepsilon \rightarrow 0} f(u_\varepsilon, v_\varepsilon) = \int f(u, v) d\nu_{t,x}(u, v) \equiv \langle \nu_{t,x}, f(u, v) \rangle \equiv \langle f \rangle$$

in $\mathcal{D}'(\mathbb{R}^2)$ for every continuous function $f(u, v)$. Next, we are to check a.e. in Π the Tartar-Murat commutation relation [26] for the measure $\nu_{t,x}$ and two quasientropy pairs (η_i, q_i) :

$$(1.11) \quad \langle \eta_1 q_2 - \eta_2 q_1 \rangle = \langle \eta_1 \rangle \langle q_2 \rangle - \langle \eta_2 \rangle \langle q_1 \rangle.$$

We prove (1.11) by the compensated compactness method and the div-curl lemma [17].

Then we are to prove that the measure $\nu_{t,x}$ is a Dirac δ -measure on the u, v -plane. It means in particular that there exist functions u and v such that $u^\varepsilon \rightarrow u$ and $v^\varepsilon \rightarrow v$ a.e. in Π . Afterward we easily check that (u, v) is an entropy quasisolution.

The need of a large family of entropies for the Young measure reduction to point mass measures was demonstrated by DiPerna for the isentropic gas dynamics equations with the pressure law $p = \rho^\gamma$, $\gamma > 1$. When $\gamma = \frac{2n+3}{2n+1}$, with n being integer, DiPerna used weak entropies which are progressive waves given by Lax. The method of Tartar and DiPerna was then extended by Serre [23] to strictly hyperbolic systems of two conservation laws, by Chen, et al. [2, 6] to fluid equations with $\gamma \in (1, 5/3]$ and by Lions, Perthame, Souganidis, and Tadmor [16, 12] to the full range $\gamma > 1$. The theory was also extended to real fluid equations by Chen and LeFloch [3, 13, 4]. We also mention the extensive work by Perthame and Tzavaras on the kinetic formulation of systems of two conservation laws; see [21, 22].

To generate entropies, we calculate all the Lie groups associated with the entropy equation (1.8) for the function η . By using one of them we construct several families of entropies which serve as the Lax progressive waves in the paper of DiPerna [7].

In the final section we give an example of an entropy solution which is not an entropy quasisolution. The stability of a sequence of bounded entropy quasisolutions was proved earlier in [25].

2. PARABOLIC APPROXIMATION

In this section we establish the existence of smooth solutions to problem (1.3)-(1.4), assuming that the smooth initial data u_0^ε and v_0^ε are chosen in such a way that $u_0^\varepsilon \rightarrow u_0$, $v_0^\varepsilon \rightarrow v_0$ in $L^\infty(\mathbb{R})$ as $\varepsilon \rightarrow 0$. First, we derive a priori estimates for the solution $(u^\varepsilon, v^\varepsilon)$. For simplicity, we omit the superindex ε .

Lemma 2.1. *There are constants c_0 and c_1 such that*

$$(2.1) \quad |u| \leq c_0, \quad c_0^{-1} \leq v \leq c_0, \quad \int_0^T \int_{-R}^R (\varepsilon u_x^2 + v_x^2) dx dt \leq c_1(T, R),$$

uniformly in ε .

Proof. Passing to the Riemann invariant variables

$$w = u + k \ln v, \quad z = u - k \ln v,$$

we can rewrite system (1.3) as

$$(2.2) \quad w_t - kv^{-1}w_x = \varepsilon w_{xx}, \quad z_t + kv^{-1}z_x = \varepsilon z_{xx}.$$

By the maximum principle, the first two estimates in (2.1) are valid.

It follows from (1.3) that

$$(2.3) \quad \frac{\partial}{\partial t} (u^2/2 + k^2(v - \ln v - 1)) + \varepsilon u_x^2 + \varepsilon k^2 v_x^2/v = \frac{\partial}{\partial x} J$$

where

$$J = k^2 u - pu + \varepsilon u u_x + k^2 \varepsilon (v - \ln v - 1).$$

Given $R > 0$, let $\psi : \mathbb{R}_+ \rightarrow \mathbb{R}$ be a non-increasing function of class C^2 such that $\psi(x) = 1$ for $x \in [0, R]$, $\psi(x) = e^{-x}$ for $x \geq 2R$, and $\psi(x)$ is a non-negative polynomial for $R \leq x \leq 2R$. Denote $\Psi(x) = \psi(|x|)$ for $x \in \mathbb{R}$. Clearly,

$$(2.4) \quad |\Psi'(x)| \leq c_2 \Psi(x)/R, \quad |\Psi''(x)| \leq c_2 \Psi(x)/R^2$$

for some constant $c_2 > 0$. We multiply (2.3) by $\Psi(x)$ and integrate in x to obtain the equality

$$(2.5) \quad \frac{\partial}{\partial t} \int \Psi \left(\frac{u^2}{2} + k^2(v - \ln v - 1) \right) dx + \int \Psi \left(\varepsilon u_x^2 + \frac{\varepsilon k^2 v_x^2}{v} \right) dx = - \int J \Psi_x dx.$$

By the Young inequality, with $\delta > 0$,

$$|J| \leq \frac{u^2}{2} \left(k^2 + \frac{1}{2} + \frac{\varepsilon}{\delta} \right) + \delta \varepsilon u_x^2/2 + (k^2 + p^2)/2 + k^2 \varepsilon (v - \ln v - 1).$$

With a proper choice of δ , where is a constant c_3 such that

$$\begin{aligned} \int |J \Psi_x| dx &\leq \frac{1}{2} \int \Psi (\varepsilon u_x^2 + \varepsilon k^2 v_x^2/v) dx + \\ &c_3 \int \Psi (u^2/2 + k^2(v - \ln v - 1)) dx. \end{aligned}$$

Now, the third estimate in (2.1) holds by the Gronwall inequality.

We rewrite equations (1.3) as a quasi-linear parabolic system:

$$(2.6) \quad u_t + a_1(v, v_x) = \varepsilon u_{xx}, \quad v_t + a_2(v, u_x, v_x) = \varepsilon v_{xx},$$

where we have set

$$a_1 := -k^2 v^{-2} v_x, \quad a_2 := \varepsilon v_x^2/v - u_x.$$

With the estimates of Lemma 2.1 at hand, it is a standard matter to derive estimates in Hölder's norms, depending on ε , by standard techniques of the theory quasi-linear parabolic equations [11]. We will only sketch the derivation. Let $\zeta(x, t)$ be a smooth function such that $\zeta \neq 0$ only if $x \in \omega$, where ω is an interval $[x_0 - \sigma, x_0 + \sigma]$.

Let $\mathbf{1}_A(x)$ stand for the characteristic function of a set $A \subset \mathbb{R}$. We multiply the first equation in (1.3) by

$$\zeta^2 \max\{u - n, 0\} \equiv \zeta^2 u^{(n)}$$

and integrate with respect to x :

$$\frac{d}{dt} \int \zeta^2 |u^{(n)}|^2 dx + \varepsilon \int \zeta^2 |u_x^{(n)}|^2 dx \leq \gamma \int (\zeta_x^2 + \zeta |\zeta_t|) |u^{(n)}|^2 + \zeta^2 \mathbf{1}_{u \geq n} dx.$$

Similarly, for the variable v one gets

$$\frac{d}{dt} \int \zeta^2 |v^{(n)}|^2 dx + \varepsilon \int \zeta^2 |v_x^{(n)}|^2 dx \leq \gamma \int (\zeta_x^2 + \zeta |\zeta_t|) |v^{(n)}|^2 + \zeta^2 \mathbf{1}_{v \geq n} dx.$$

These inequalities imply that u and v belong to a class $\mathcal{B}_2(Q, M, \gamma, r, \delta, n)$ [11] (Chapter II, §7, formula (7.5)), for some parameters Q, M, γ, r, δ , and n . Then it follows that the estimate

$$\|u, v\|_{H^{\alpha, \alpha/2}(\omega \times [0, T])} \leq c$$

holds for some $\alpha \in (0, 1)$.

In the same manner, one can estimate the Hölder norm of the derivatives $u_x, u_{xx}, u_t, v_x, v_{xx}$, and v_t , in the same way as done in [8] for a general class of parabolic systems.

We now arrive at the main existence result, concerning the viscous approximation (1.3).

Lemma 2.2. (Existence of smooth solution of the regularized system.) *Let $u_0^\varepsilon, v_0^\varepsilon \in L^\infty \cap H_{loc}^{2+\beta}$, $0 < \beta < 1$. Then the Cauchy problem (1.3)-(1.4) has a unique solution such that*

$$u, v \in L^\infty(\Pi) \cap H_{loc}^{2+\beta, 1+\beta/2}(\Pi).$$

Now, we study compactness of the viscous solutions $(u^\varepsilon, v^\varepsilon)$ as $\varepsilon \rightarrow 0$.

Lemma 2.3. *Given an entropy-entropy flux pair $\eta(u, v), q(u, v)$, the sequence*

$$\theta^\varepsilon \equiv \frac{\partial \eta^\varepsilon}{\partial t} + \frac{\partial q^\varepsilon}{\partial x}$$

is compact in $W_{loc}^{-1,2}(\Pi)$, where $\eta^\varepsilon = \eta(u^\varepsilon, v^\varepsilon)$, $q^\varepsilon = q(u^\varepsilon, v^\varepsilon)$.

Proof. We use the following lemma due to Murat's lemma [18].

Let $Q \subset \mathbb{R}^2$ be a bounded domain, $Q \in C^{1,1}$. Let A be a compact set in $W^{-1,2}(Q)$, B be a bounded set in the space of bounded Radon measures $M(Q)$, and C be a bounded set in $W^{-1,p}(Q)$ for some $p \in (2, \infty]$. Further, let $D \subset \mathcal{D}'(Q)$ be such that

$$D \subset (A + B) \cap C.$$

Then there exists E , a compact set in $W^{-1,2}(Q)$ such that $D \subset E$.

By definition, the functions $\eta(u, v)$ and $q(u, v)$ solve the system

$$q_u = -\eta_v, \quad k^2 \eta_u = -v^2 q_v.$$

Hence, calculations show that

$$(2.7) \quad \frac{\partial \eta^\varepsilon}{\partial t} + \frac{\partial q^\varepsilon}{\partial x} = \varepsilon \eta_{xx}^\varepsilon - \varepsilon \{ \eta_{uu}^\varepsilon (u_x^\varepsilon)^2 + (\eta_{vv}^\varepsilon + \eta_v^\varepsilon / v^\varepsilon) (v_x^\varepsilon)^2 + 2\eta_{uv}^\varepsilon u_x^\varepsilon v_x^\varepsilon \}_1.$$

We check the conditions of Murat's lemma. By Lemma 2.1, the sequence θ^ε is bounded in $W_{loc}^{-1,\infty}(\Pi)$. Hence, it is enough to show that $\varepsilon \eta_x^\varepsilon \rightarrow 0$ in $L_{loc}^2(\Pi)$ and the residual sequence $\theta^\varepsilon - \varepsilon \eta_{xx}^\varepsilon$ is bounded in $L_{loc}^1(\Pi)$.

We have

$$\varepsilon \eta_x^\varepsilon = \varepsilon (\eta_u^\varepsilon u_x^\varepsilon + \eta_v^\varepsilon v_x^\varepsilon).$$

Thus, by estimates (2.1), $\varepsilon \eta_x^\varepsilon \rightarrow 0$ in $L_{loc}^2(\Pi)$.

Consider the sequence $\theta^\varepsilon - \varepsilon \eta_{xx}^\varepsilon$. We have $\theta^\varepsilon - \varepsilon \eta_{xx}^\varepsilon = -\varepsilon \{ \dots \}_1$, and the sequence $\varepsilon \{ \dots \}_1$ is bounded in L_{loc}^1 due to estimate (2.1)₃. The lemma is proved.

Given two entropy pairs $(\eta_i(u, v), q_i(u, v))$, $(i = 1, 2)$, from Lemma 2.3, the functions

$$\frac{\partial \eta_i^\varepsilon}{\partial t} + \frac{\partial q_i^\varepsilon}{\partial x}$$

are compact in $W_{loc}^{-1,2}(\Pi)$. Hence, by the div-curl lemma [26], Tartar's commutation relation

$$(2.8) \quad \langle \eta_1 q_2 - \eta_2 q_1 \rangle = \langle \eta_1 \rangle \langle q_2 \rangle - \langle \eta_2 \rangle \langle q_1 \rangle$$

is valid for a Young measure $\nu_{t,x}$ associated with a sequence from (1.9).

For reader's convenience, we remind that the div-curl lemma states the following.

Let $Q \subset \mathbb{R}^2$ be a bounded domain, $Q \in C^{1,1}$. Let

$$w_1^n \rightharpoonup w, \quad w_2^n \rightharpoonup w_2, \quad v_1^n \rightharpoonup v_1, \quad v_2^n \rightharpoonup v_2,$$

weakly in $L^2(Q)$, as $n \rightarrow \infty$. With $\text{curl}(w_1, w_2)$ denoting $\partial w_2 / \partial x_1 - \partial w_1 / \partial x_2$, suppose that the sequences $\text{div}(v_1^n, v_2^n)$ and $\text{curl}(w_1^n, w_2^n)$ lie in a compact subset E of $W^{-1,2}(Q)$. Then, for a subsequence,

$$v_1^n w_1^n + v_2^n w_2^n \rightarrow v_1 w_1 + v_2 w_2 \quad \text{in } \mathcal{D}'(Q) \quad \text{as } n \rightarrow \infty.$$

3. FAMILY OF ENTROPIES

Here, we construct entropies which mit the definition of entropy quasisolution. By the method of separation of variables, we find that the entropy wave equation (1.8) enjoys a particular solution

$$(3.1) \quad \eta = v^\beta e^{\alpha u} \quad \text{provided} \quad \beta(\beta - 1) = k^2 \alpha^2.$$

Starting from this solution, we construct several families of new solutions by applying the group symmetry analysis [20]. Using the Riemann invariants

$$w = u + k \log(Mv), \quad z = u - k \log(Mv), \quad M = \text{const} > 0,$$

we can write down the entropy wave equation as the Euler-Poisson-Darboux equation

$$(3.2) \quad F(\eta_w, \eta_z, \eta_{wz}) \equiv \eta_{wz} - a(\eta_z - \eta_w) = 0, \quad a = (4k)^{-1}.$$

Assume that a one-parameter group admitted by (3.2) is given by the infinitesimal operator

$$X = \xi(w, z, \eta) \frac{\partial}{\partial w} + \tau(w, z, \eta) \frac{\partial}{\partial z} + \varphi(w, z, \eta) \frac{\partial}{\partial \eta}.$$

Calculations of the first and second prolongation of this operator yields

$$X^1 = X + \zeta^{\eta w} \frac{\partial}{\partial \eta_w} + \zeta^{\eta z} \frac{\partial}{\partial \eta_z},$$

$$X^2 = X^1 + \zeta^{\eta ww} \frac{\partial}{\partial \eta_{ww}} + \zeta^{\eta wz} \frac{\partial}{\partial \eta_{wz}} + \zeta^{\eta zz} \frac{\partial}{\partial \eta_{zz}},$$

where

$$\zeta^{\eta w} = D_w \varphi - \eta_w D_w \xi - \eta_z D_w \tau, \quad D_w = \frac{\partial}{\partial w} + \eta_w \frac{\partial}{\partial \eta},$$

$$\zeta^{\eta z} = D_z \varphi - \eta_w D_z \xi - \eta_z D_z \tau, \quad D_z = \frac{\partial}{\partial z} + \eta_z \frac{\partial}{\partial \eta},$$

$$\begin{aligned} \zeta^{\eta wz} &= D_z \varphi_w + \eta_w D_z \varphi_\eta + \varphi_\eta \eta_{wz} - \eta_{ww} D_z \xi - \eta_{wz} (D_w \xi + D_z \tau) \\ &\quad - \eta_w (D_z \xi_w + \eta_w D_z \xi_\eta + \xi_\eta \eta_{wz}) - \eta_{zz} D_w \tau - \eta_z (D_z \tau_w + \eta_w D_z \tau_\eta + \eta_{wz} \tau_\eta). \end{aligned}$$

Note that we need not calculate the coefficients $\zeta^{\eta ww}$ and $\zeta^{\eta zz}$. Application of the operator X^2 to F and analysis of this application on the manifold $F = 0$ enable

us to conclude that equation (3.2) admits four one-dimensional groups G_i and one infinite-dimensional group G_5 with the infinitesimal operators

$$\frac{\partial}{\partial w}, \quad \frac{\partial}{\partial z}, \quad \eta \frac{\partial}{\partial \eta}, \quad w \frac{\partial}{\partial w} - z \frac{\partial}{\partial z} + a(w+z)\eta \frac{\partial}{\partial \eta}, \quad \beta(w, z) \frac{\partial}{\partial \eta},$$

where $\beta(w, z)$ is a solution to (3.2). The fact that equation (3.2) admits the group G_i means that if $\eta(w, z)$ is a solution then for arbitrary $c, \xi \in \mathbb{R}$ the following functions are solutions as well:

$$\eta(w+c, z), \quad \eta(w, z+c), \quad c\eta(w, z), \\ \eta(e^{-\xi}w, e^{\xi}z) \exp(aw(1-e^{-\xi}) - az(1-e^{\xi})), \quad \eta(w, z) + \beta(w, z).$$

Observe that, once this assertion is obtained, its validity can be easily verified directly without group analysis.

Each solution (3.1) generates a family of new solutions to the entropy-wave equation by means of the group G_4 :

$$(3.3) \quad \eta(u, v, \xi) = v^B e^{Au}, \\ B = \frac{e^{\xi}}{2} \left(\beta - \frac{1}{2} - \alpha k \right) + \frac{e^{-\xi}}{2} \left(\beta - \frac{1}{2} + \alpha k \right) + \frac{1}{2}, \\ A = -\frac{e^{\xi}}{2k} \left(\beta - \frac{1}{2} - \alpha k \right) + \frac{e^{-\xi}}{2k} \left(\beta - \frac{1}{2} + \alpha k \right).$$

We select two families of solutions

$$\eta_i(u, v, \xi) = v^{B_i(\xi)} e^{A_i(\xi)u} = M^{-B_i(\xi)} e^{wn_i(\xi) + zm_i(\xi)},$$

corresponding to pairs (α_i, β_i) such that $\alpha_1 < 0$, $\beta_1 < 0$, $\alpha_2 > 0$, and $\beta_2 > 1$, where

$$n_i = \frac{e^{-\xi}}{2k} \left(\beta_i - \frac{1}{2} + \alpha_i k \right) + \frac{1}{4k}, \quad m_i = -\frac{e^{\xi}}{2k} \left(\beta_i - \frac{1}{2} - \alpha_i k \right) - \frac{1}{4k}.$$

Observe that

$$(3.4) \quad n_1 \rightarrow -\infty, \quad m_1 \rightarrow -\frac{1}{4k} \quad \text{as } \xi \rightarrow -\infty,$$

$$(3.5) \quad n_2 \rightarrow \infty, \quad m_2 \rightarrow -\frac{1}{4k} \quad \text{as } \xi \rightarrow -\infty.$$

We bind (α_1, β_1) and (α_2, β_2) by the equality

$$(3.6) \quad (\beta_2 - 1/2 + \alpha_2 k) = -(\beta_1 - 1/2 + \alpha_1 k).$$

Such matched pairs can be chosen in many ways. For example, if

$$\alpha_2 = \frac{1}{k} (> 0), \quad \beta_2 = \frac{1 + \sqrt{5}}{2} (> 1),$$

the pair (α_2, β_2) satisfies $(3.1)_2$. Let λ be a positive root of the equation

$$2\lambda + 2\sqrt{\lambda(1+\lambda)} = 1 + \sqrt{5}.$$

Then the pair

$$\alpha_1 = -\sqrt{\frac{\lambda(1+\lambda)}{k^2}} (< 0), \quad \beta_1 = -\lambda (< 0)$$

satisfies $(3.1)_2$ and (3.6).

Alongside the matched families $\eta_1(\xi)$ and $\eta_2(\xi)$, of importance in the sequel will be the two more matched families

$$(3.7) \quad \eta_i(u, v, \zeta) = v^{B_i(\zeta)} e^{A_i(\zeta)u} = M^{-B_i(\zeta)} e^{wn_i(\zeta)+zm_i(\zeta)}, \quad i \in \{1, 2\},$$

$$n_i = \frac{A_i}{2} + \frac{B_i}{2k}, \quad m_i = \frac{A_i}{2} - \frac{B_i}{2k},$$

corresponding to pairs (α_i, β_i) , such that

$$\alpha_1 < 0, \quad \beta_1 < 0, \quad \alpha_2 > 0, \quad \beta_2 > 1, \quad \beta_i(\beta_i - 1) = k^2 \alpha_i^2.$$

The matching condition for the families $\eta_i(\zeta)$ is

$$(3.8) \quad (\beta_2 - 1/2 - \alpha_2 k) = -(\beta_1 - 1/2 - \alpha_1 k).$$

It is easy to see that such matching is possible. The parameters $n_i(\zeta)$ and $m_i(\zeta)$ are given by the equalities

$$n_i(\zeta) = \frac{e^{-\zeta}}{2k} \left(\beta_i - \frac{1}{2} + \alpha_i k \right) + \frac{1}{4k}, \quad m_i(\zeta) = -\frac{e^\zeta}{2k} \left(\beta_i - \frac{1}{2} - \alpha_i k \right) - \frac{1}{4k}.$$

Therefore,

$$(3.9) \quad m_1(\zeta) + m_2(\zeta) = -\frac{1}{2k}, \quad n_i(\zeta) \rightarrow \frac{1}{4k} \quad \text{as } \zeta \rightarrow \infty,$$

$$(3.10) \quad m_1(\zeta) \rightarrow \infty, \quad m_2 \rightarrow -\infty \quad \text{as } \zeta \rightarrow \infty.$$

Note, the entropies constructed above satisfy condition (1.7).

4. REDUCTION OF YOUNG MEASURES TO DIRAC MEASURES

With estimates (2.1) at hand, the correspondence between the variables (u, v) and (w, z) is one-to-one. Therefore, it suffices to prove the equality $\nu_{t,x} = \delta$ considering measures on the w, z -plane. Let $E = [w_1, w_2] \times [z_1, z_2]$ be the minimal rectangle containing the support of the measure $\nu_{t,x}$. In view of (2.1), we can choose the constant M in the definition of the Riemann invariants so as to have $w_1 > 0$ and $z_2 < 0$.

In what follows, an important role is played by the measures $\mu_{t,x}^i$ defined by the equality

$$\langle \mu_{t,x}^i, h(u, v) \rangle = \lim_{\xi \rightarrow -\infty} \frac{\langle \nu_{t,x}, h\eta_i(\xi) \rangle}{\langle \nu_{t,x}, \eta_i(\xi) \rangle}, \quad i \in \{1, 2\}.$$

Such measures were first introduced by DiPerna [7].

Lemma 4.1. *If $w_1 < w_2$ then*

$$(4.1) \quad \text{spt } \mu_{t,x}^i \subset \{w = w_i\}.$$

Proof. Suppose that $h(w, z) \in C^1(E)$ and $h = 0$ for $w = w_2$. Then the inequality

$$|h| \leq r(w_2 - w), \quad \forall (w, z) \in E.$$

holds for a sufficiently large $r > 0$. Omitting the index (t, x) in the notation of measures, we write the obvious inequality

$$|\langle \mu^2, h \rangle| \leq \lim_{\xi \rightarrow -\infty} \frac{\int_E r(w_2 - w) e^{(w-w_2)n_2+zm_2} d\nu}{\int_E e^{(w-w_2)n_2+zm_2} d\nu} \equiv \lim_{\xi \rightarrow -\infty} J_2(\xi).$$

We estimate $J_2(\xi)$ by making use the following partitions of E for the numerator and denominator:

$$E = [w_1, w_2 - \delta] \times [z_1, z_2] \cup [w_2 - \delta, w_2] \times [z_1, z_2],$$

$$E = [w_1, w_2 - \delta/2] \times [z_1, z_2] \cup [w_2 - \delta/2, w_2] \times [z_1, z_2].$$

We have

$$J_2 \leq \delta r + \frac{r(w_2 - w_1)e^{-\delta n_2} \int_{z_1}^{z_2} \int_{w_1}^{w_2 - \delta} e^{zm_2} d\nu}{e^{-\delta n_2/2} \int_{z_1}^{z_2} \int_{w_2 - \delta/2}^{w_2} e^{zm_2} d\nu}$$

$$\leq \delta r + \frac{ce^{-\delta n_2/2} \int_{z_1}^{z_2} \int_{w_2 - \delta/2}^{w_2} e^{zm_2} d\nu}{\int_{z_1}^{z_2} \int_{w_2 - \delta/2}^{w_2} e^{zm_2} d\nu}.$$

Since the set E is chosen from the minimality condition, we obtain, passing to the limit as $\xi \rightarrow -\infty$, that

$$|\langle \mu^2, h \rangle| \leq \delta r.$$

Since the embedding $C^1(E) \subset C(E)$ is dense, inclusion (4.1) for the measure μ^2 is proved.

Consider the measure μ^1 . Suppose that $h(w, z) \in C^1(E)$ and $h = 0$ for $w = w_1$. We can assume that

$$|h| \leq r(w - w_1), \quad \forall (w, z) \in E.$$

The following estimate is valid:

$$|\langle \mu^1, h \rangle| \leq \lim_{\xi \rightarrow -\infty} \frac{\int_E r(w - w_1)e^{(w-w_1)n_1+zm_1} d\nu}{\int_E e^{(w-w_1)n_1+zm_1} d\nu} \equiv \lim_{\xi \rightarrow -\infty} J_1(\xi).$$

We estimate $J_1(\xi)$ by making use the partition

$$E = [w_1, w_1 + \delta] \times [z_1, z_2] \cup [w_1 + \delta, w_2] \times [z_1, z_2],$$

for the numerator and the partition

$$E = [w_1, w_1 + \delta/2] \times [z_1, z_2] \cup [w_1 + \delta/2, w_2] \times [z_1, z_2].$$

for the denominator. We have

$$J_1 \leq \delta r + \frac{ce^{\delta n_1(\xi)/2} \int_{z_1}^{z_2} \int_{w_1 + \delta}^{w_2} e^{zm_1(\xi)} d\nu}{\int_{z_1}^{z_2} \int_{w_1}^{w_1 + \delta/2} e^{zm_1(\xi)} d\nu}.$$

Hence, the sought assertion for the measure μ^1 follows.

Lemma 4.2. *The following equality holds:*

$$\langle \mu^1, 1/v \rangle = \langle \mu^2, 1/v \rangle.$$

Proof. One can verify easily that entropy flux corresponding to the entropy η_i is given by the formula

$$q = q_i = -\frac{k^2 A_i \eta_i}{(B_i - 1)v}.$$

Let us write down (2.8) for the pairs $(\eta_1(\xi), q_1(\xi))$ and $(\eta_2(\xi), q_2(\xi))$:

$$(4.2) \quad \frac{\langle \nu, q_2 \rangle}{\langle \nu, \eta_2 \rangle} - \frac{\langle \nu, q_1 \rangle}{\langle \nu, \eta_1 \rangle} = \frac{\langle \nu, \eta_1 q_2 - \eta_2 q_1 \rangle}{\langle \nu, \eta_1 \rangle \langle \nu, \eta_2 \rangle} \equiv J.$$

Note that J admits the representation

$$J = k^2 \frac{j_1}{j_2}, \quad j_1 = \left(\frac{A_1}{B_1 - 1} - \frac{A_2}{B_2 - 1} \right) \int_E e^{wP(\xi) + zQ(\xi)} d\nu,$$

$$j_2 = \int_E e^{wn_2(\xi) + zm_2(\xi)} d\nu \int_E e^{wn_1(\xi) + zm_1(\xi)} d\nu,$$

$$P = n_1 + n_2 - 1/(2k), \quad Q = m_1 + m_2 + 1/(2k).$$

Due to the matching condition (3.6) and the limit relations (3.4) and (3.5),

$$P = 0, \quad Q \rightarrow 0, \quad \frac{A_1}{B_1 - 1} - \frac{A_2}{B_2 - 1} \rightarrow 0 \quad \text{as } \xi \rightarrow -\infty.$$

Therefore, $j_1 \rightarrow 0$ as $\xi \rightarrow -\infty$. We estimate j_2 by making use the limit relations (3.4) and (3.5) and positivity of the variable w . We have

$$\begin{aligned} j_2 &\geq \int_{z_1}^{z_2} \int_{w_1}^{w_1 + \varepsilon} e^{wn_1 + zm_1} d\nu \int_{z_1}^{z_2} \int_{w_2 - \varepsilon}^{w_2} e^{wn_2 + zm_2} d\nu \\ &\geq e^{w_1 n_1 + w_2 n_2 + \varepsilon(n_1 - n_2)} \int_{z_1}^{z_2} \int_{w_1}^{w_1 + \varepsilon} e^{zm_1} d\nu \int_{z_1}^{z_2} \int_{w_2 - \varepsilon}^{w_2} e^{zm_2} d\nu \equiv e^{\Lambda(\xi)} c(\varepsilon, \xi). \end{aligned}$$

We take $2\varepsilon < w_2 - w_1$. Then by (3.6)

$$\Lambda = \frac{w_1 + w_2}{4k} + \frac{e^{-\xi}}{k} \left(\beta_2 - \frac{1}{2} + \alpha_2 k \right) \left(\frac{w_2 - w_1}{2} - \varepsilon \right) \rightarrow \infty$$

as $\xi \rightarrow -\infty$. On the other hand, the limit of $c(\varepsilon, \xi)$, as $\xi \rightarrow -\infty$, is finite, since the rectangle E satisfies the minimality condition and the limits of m_i , as $\xi \rightarrow -\infty$, are finite. Hence, $\lim j_1/j_2 = 0$ as $\xi \rightarrow -\infty$.

Since

$$(4.3) \quad \lim_{\xi \rightarrow -\infty} \frac{k^2 A_i}{B_i - 1} = k,$$

passage to the limit in (4.2) leads to the assertion of the lemma.

Lemma 4.3. *The following equality holds for every quasientropy pair (η, q) :*

$$(4.4) \quad \langle \mu^1, q + \eta k/v \rangle = \langle \mu^2, q + \eta k/v \rangle.$$

Proof. We write (2.8) for the pairs (η, q) and $(\eta_1(\xi), q_1(\xi))$:

$$\langle \nu, q \rangle - \frac{\langle \nu, \eta \rangle \langle \nu, q_1(\xi) \rangle}{\langle \nu, \eta_1(\xi) \rangle} = \frac{\langle \nu, q\eta_1(\xi) - \eta q_1(\xi) \rangle}{\langle \nu, \eta_1(\xi) \rangle}.$$

Passing to the limit as $\xi \rightarrow -\infty$ and making use of (4.3), we obtain

$$\langle \nu, q \rangle - \langle \nu, \eta \rangle \langle \mu^1, -k/v \rangle = \langle \mu^1, q + \eta k/v \rangle.$$

Similarly, we prove the equality

$$\langle \nu, q \rangle - \langle \nu, \eta \rangle \langle \mu^2, -k/v \rangle = \langle \mu^2, q + \eta k/v \rangle.$$

Now, validity of (4.4) ensues from Lemma 4.2.

Lemma 4.4. *The equality $w_1 = w_2$ is valid.*

Proof. Assume $w_1 < w_2$. We write down (4.4) for the pair $(\eta_2(\xi), q_2(\xi))$:

$$\langle \mu^1, \eta_2(\xi)k/v \rangle = \langle \mu^2, \eta_2(\xi)k/v \rangle.$$

Since the measure μ^i is supported in the set $w = w_i$, this equality amounts to the following:

$$(4.5) \quad e^{(w_2-w_1)(n_2(\xi)-\frac{1}{2k})} = \frac{\int_{z_1}^{z_2} e^{z(m_2(\xi)+\frac{1}{2k})} d\mu^1}{\int_{z_1}^{z_2} e^{z(m_2(\xi)+\frac{1}{2k})} d\mu^2}.$$

The measures μ^i are probability measures and the limit $\lim m_2(\xi)$ as $\xi \rightarrow -\infty$ is finite. Therefore, the right-hand side of (4.5) has a finite limit as $\xi \rightarrow -\infty$. The left-hand side tends to $+\infty$, since $n_2 \rightarrow +\infty$ as $\xi \rightarrow -\infty$. This contradiction proves the lemma.

Lemma 4.5. *The equality $z_1 = z_2$ is valid.*

Proof. Since the arguments are the same as in the case of $w_1 = w_2$, we only sketch the proof. Here we use the family $\eta_i(\zeta)$ of (3.7) matched by equality (3.8). Assume that $z_1 < z_2$ and introduce the probability measures

$$\langle \pi_{t,x}^1, h(u, v) \rangle = \lim_{\zeta \rightarrow \infty} \frac{\langle \nu_{t,x}, h\eta_2(\zeta) \rangle}{\langle \nu_{t,x}, \eta_2(\zeta) \rangle}, \quad \langle \pi_{t,x}^2, h(u, v) \rangle = \lim_{\zeta \rightarrow \infty} \frac{\langle \nu_{t,x}, h\eta_1(\zeta) \rangle}{\langle \nu_{t,x}, \eta_1(\zeta) \rangle}.$$

The first step in the proof of the lemma is to verify the inclusion

$$(4.6) \quad \text{spt } \pi^i \subset \{z = z_i\}.$$

Derivation of this inclusion is the same as in Lemma 4.1. For example, for the measure π^1 we have the estimate

$$|\langle \pi^1, h \rangle| \leq \delta c + \frac{c e^{\delta m_2(\zeta)} \int_{w_1}^{w_2} \int_{z_1}^{z_2} e^{wn_2(\zeta)} d\nu}{e^{m_2(\zeta)\delta/2} \int_{w_1}^{w_2} \int_{z_1}^{z_1+\delta/2} e^{wn_2(\zeta)} d\nu}.$$

for every smooth function $h(w, z)$ vanishing on the line $z = z_1$ and for every δ . Therefore, $\text{spt } \pi^1 \subset \{z = z_1\}$, since the domain E satisfies the minimality condition and the limit relations (3.9) and (3.10) are valid.

The second step is to verify the equality

$$(4.7) \quad \langle \pi^1, 1/v \rangle = \langle \pi^2, 1/v \rangle.$$

Its derivation is based on equality (4.2) in which we use the families $(\eta_i(\zeta), q_i(\zeta))$ matched by equality (3.8). By (3.9), the function $Q(\zeta)$ in (4.2) vanishes. Moreover, in (4.2)

$$P \rightarrow 0, \quad \frac{A_i}{B_i - 1} \rightarrow -\frac{1}{k} \quad \text{as } \zeta \rightarrow \infty.$$

Therefore, the numerator $j_1(\zeta)$ vanishes as $\zeta \rightarrow \infty$.

Since the variable z takes negative values while the parameters $n_i(\zeta)$ and $m_i(\zeta)$ have the asymptotic behavior indicated in (3.9), the denominator j_2 admits the following estimate from below:

$$j_2(\zeta) \geq \int_{z_1}^{z_1+\varepsilon} \int_{w_1}^{w_2} e^{wn_2+zm_2} d\nu \int_{z_2-\varepsilon}^{z_2} \int_{w_1}^{w_2} e^{wn_1+zm_1} d\nu \geq e^\gamma \int_{z_1}^{z_1+\varepsilon} \int_{w_1}^{w_2} e^{wn_2} \int_{z_2-\varepsilon}^{z_2} \int_{w_1}^{w_2} e^{wn_1}, \quad \gamma = -\frac{z_1+z_2}{4k} + \frac{e^\zeta}{k} \left(\beta_2 - \frac{1}{2} - \alpha_2 k \right) \left(\frac{z_2-z_1}{2} - \varepsilon \right).$$

Now, we can validate equalities (4.7) by making use the limit relation

$$q_i/\eta_i \rightarrow k/v \quad \text{as } \zeta \rightarrow \infty.$$

The third step is to prove the equality

$$(4.8) \quad \langle \pi^1, q - \eta k/v \rangle = \langle \pi^2, q - \eta k/v \rangle.$$

We can derive it by the scheme of Lemma 4.3 if we replace the functions $\eta_i(\xi)$, $q_i(\xi)$ with $\eta_i(\zeta)$, $q_i(\zeta)$ and use equality (4.7).

The last step repeats the arguments of Lemma 4.4. We write down (4.8) for the pairs $\eta_2(\zeta)$, $q_2(\zeta)$ with (4.6) taken into account. We then obtain

$$e^{(z_1-z_2)(m_2(\zeta)+\frac{1}{2k})} = \frac{\int_{w_1}^{w_2} e^{w(n_2(\zeta)-\frac{1}{2k})} d\pi_2}{\int_{w_1}^{w_2} e^{w(n_2(\zeta)-\frac{1}{2k})} d\pi_1}.$$

Since this equality is violated in the limit as $\zeta \rightarrow \infty$, Lemma 4.5 is proved.

By lemmas 4.4, 4.5, and (1.10), there exist two functions $u_*(t, x)$, $v_*(t, x)$, and a sequence $\varepsilon \rightarrow 0$ such that

$$(4.9) \quad |u_*(t, x)| \leq c_0, \quad c_0^{-1} \leq v_*(t, x) \leq c_0, \quad \nu_{t,x} = \delta_{(u_*(t,x), v_*(t,x))},$$

and

$$(4.10) \quad \int_{\Pi} f(u^\varepsilon(t, x), v^\varepsilon(t, x)) \varphi(t, x) dx dt \rightarrow \int_{\Pi} f(u_*(t, x), v_*(t, x)) \varphi(t, x) dx dt$$

for any $f \in C(K)$ and for every $\varphi \in \mathcal{D}(\mathbb{R}^2)$.

5. PASSAGE TO THE LIMIT

Taking $f(u, v) = u$ and $f(u, v) = v$ in (4.10), we conclude from (1.9) that $u_*(t, x) = u(t, x)$ and $v_*(t, x) = v(t, x)$ where

$$(5.1) \quad u^\varepsilon \rightharpoonup u, \quad v^\varepsilon \rightharpoonup v \quad \text{* -weakly in } L_{loc}^\infty(\Pi) \quad \text{and weakly in } L_{loc}^2(\Pi).$$

Taking $f(u, v) = u^2$ and $f(u, v) = v^2$ in (4.10), we derive that

$$(5.2) \quad (u^\varepsilon)^2 \rightarrow u^2, \quad (v^\varepsilon)^2 \rightarrow v^2 \quad \text{weakly in } L_{loc}^2(\Pi).$$

It follows easily from (6.1) and (6.2) that

$$u^\varepsilon \rightarrow u, \quad v^\varepsilon \rightarrow v \quad \text{strongly in } L_{loc}^2(\Pi) \quad \text{and a. e. in } \Pi.$$

Let us show that (u, v) is an entropy quasisolution of problem (1.1) and (1.2). To this end we multiply equality (2.7) by $\varphi \in \mathcal{D}(\mathbb{R}^2)$, integrate over Π , and let ε

go to zero. If functions $\eta(u, v)$, $q(u, v)$ obey the restrictions (1.6) and (1.7), one obtains

$$\begin{aligned} & \int (\eta(u^\varepsilon, v^\varepsilon) - \eta(u_0^\varepsilon, v_0^\varepsilon)) \varphi_t + q(u^\varepsilon, v^\varepsilon) \varphi_x \, dxdt \rightarrow \\ & \int (\eta(u, v) - \eta(u_0, v_0)) \varphi_t + q(u, v) \varphi_x \, dxdt, \\ & \varepsilon \int \eta(u^\varepsilon, v^\varepsilon) \varphi_{xx} \, dxdt \rightarrow 0 \end{aligned}$$

for any $\varphi \in \mathcal{D}(\mathbb{R}^2)$. Hence, inequality (1.5) holds and Theorem 1.1 is proved.

Taking $(\eta, q) = (u, p(v))$ and $(\eta, q) = (-u, -p(v))$ in (1.5), we conclude that

$$u_t = -p(v)_x \quad \text{in } \mathcal{D}'.$$

When $(\eta, q) = (v, -u)$, equality (2.7) implies that

$$\int_{\Pi} (v_t^\varepsilon - u_x^\varepsilon - \varepsilon v_{xx}^\varepsilon + \varepsilon (v_x^\varepsilon)^2 / v^\varepsilon) \varphi(t, x) \, dxdt = 0$$

for every $\varphi \in \mathcal{D}(\Pi)$. Clearly, the limit

$$\lim_{\varepsilon \rightarrow 0} \varepsilon (v_x^\varepsilon)^2 / v^\varepsilon \equiv \overline{\varepsilon (v_x^\varepsilon)^2 / v^\varepsilon}$$

exists in $\mathcal{D}'(\Pi)$ and $\overline{\varepsilon (v_x^\varepsilon)^2 / v^\varepsilon}$ is a nonnegative measure on Π . Thus,

$$v_t = u_x - \overline{\varepsilon (v_x^\varepsilon)^2 / v^\varepsilon} \quad \text{in } \mathcal{D}'.$$

6. REGULAR ENTROPY QUASISOLUTIONS

Here, we verify that any smooth entropy quasisolution (u, v) solves equations (1.1) in the classical sense, i. e. we show that the measure $\overline{\varepsilon (v_x^\varepsilon)^2 / v^\varepsilon}$ vanishes at the points (t, x) where the functions $u(t, x)$ and $v(t, x)$ are smooth. Let (u, v) be an entropy quasisolution such that $u, v \in C^1(\overline{G})$, where G is some domain in Π .

Denote

$$M = 2 \sup_G v > 0, \quad U(t, x) = u(t, x/M), \quad V(t, x) = v(t, x/M)/M.$$

The invariance of equations (3.2) under translations along the variables w and z enables us to conclude that the entropy-wave equation (1.8) is invariant with respect to scaling the variable v . Also, we can immediately verify that the function $\eta(u, cv)$ is a solution to (1.8) for every constant c if $\eta(u, v)$ is a solution. Then we easily verify that the following invariance property of system (1.6): if (η, q) is an entropy pair, i.e. if it satisfies (1.6) and (1.7), then

$$\eta_c(u, v) = \eta(u, cv), \quad q_c(u, v) = cq(u, cv)$$

is an entropy pair as well.

We check that (U, V) is an entropy quasisolution. Let $\varphi(t, x)$ be an arbitrary function in $\mathcal{D}(\Pi)$. Then for $c = 1/M$ we have

$$\begin{aligned} J & \equiv \int_{\Pi} \eta(U, V) \varphi_t + q(U, V) \varphi_x \, dxdt = \\ & \int_{\Pi} \eta_c(u(t, cx), v(t, cx)) \varphi_t(t, x) + M q_c(u(t, cx), v(t, cx)) \varphi_x(t, x) \, dxdt. \end{aligned}$$

Executing the change of variables $y = cx$ and denoting $\Phi(t, y) = \varphi(t, My)$, we obtain

$$J = M \int_{\Pi} \eta_c(u(t, y), v(t, y)) \Phi_t(t, y) + q_c(u(t, y), v(t, y)) \Phi_y(t, y) dy dt \geq 0.$$

Consequently, (U, V) is an entropy quasisolution which is smooth in the domain

$$G_M = \{(t, x) : (t, x/M) \in G\}.$$

Hence, the equalities

$$(6.1) \quad U_t = -p(V)_x, \quad V_t - U_x = -\overline{\varepsilon(v_x^\varepsilon)^2/v^\varepsilon} \equiv -\mu(t, x),$$

hold in G_M , where μ is a nonnegative continuous function in G_M . Multiplying (6.1)₂ by $k^2(1 - 1/V)$ and multiplying (6.1)₁ by U , we arrive at the equality

$$(6.2) \quad \Psi(U, V)_t + H(U, V)_x = -k^2\mu(1 - 1/V),$$

where

$$\Psi(U, V) = U^2/2 + k^2(V - \ln V - 1), \quad H(U, V) = k^2U(1/V - 1).$$

It is easy to verify that (Ψ, H) is an entropy pair. Therefore,

$$\Psi(U, V)_t + H(U, V)_x \leq 0 \quad \mathcal{D}'(\Pi).$$

Hence, the equality

$$(6.3) \quad \Psi(U, V)_t + H(U, V)_x = \nu(t, x),$$

is valid in G_M , with $0 \leq \nu \in C(\overline{G_M})$. Comparing (6.2) and (6.3), we arrive at the contradictory equality $k^2\mu(1 - 1/V) = \nu$, since the inequality $\sup_{G_M} V \leq 1/2$ holds in G_M . Thus,

$$u_t = -p(v)_x, \quad v_t = u_x \quad \text{for } (t, x) \in G.$$

It means that the measure $\overline{\varepsilon(v_x^\varepsilon)^2/v^\varepsilon}$ concentrates on shocks of the functions $u(t, x)$ and $v(t, x)$.

7. EXAMPLE

Here, we verify that an entropy solution (in the common sense) is not necessary an entropy quasisolution. For the gas dynamics equations (1.1), we consider the Riemann problem with the initial data

$$(u, v)|_{t=0} = \begin{cases} (u_l, v_l), & x < 0 \\ (u_r, v_r), & x > 0. \end{cases}$$

In what follows we assume that $v_l > v_r$ and $u_l = u_r = 0$. Let us describe the corresponding entropy solution $U = (u, v)$. In the sector $-\infty < x/t < s$ the solution is constant:

$$u = u_l, \quad v = v_l,$$

where s is the velocity of the 1-shock wave propagating to the left. The state (\hat{u}, \hat{v}) behind this shock can be found from the Hugoniot equalities

$$(7.1) \quad s(\hat{v} - v_l) + (\hat{u} - u_l) = 0, \quad s(\hat{u} - u_l) = (\hat{p} - p_l).$$

In the sector $s < x/t < \lambda_2(\hat{U}) = k/\hat{v}$ the solution is constant

$$u = \hat{u}, \quad v = \hat{v}.$$

In the sector $\lambda_2(\hat{U}) < x/t < \lambda_2(U_r) = k/v_r$, the solution is a 2-rarefaction wave such that

$$u + k \ln v = \text{const} = u_r + k \ln v_r, \quad \lambda_2(U) \equiv k/v = x/t.$$

Finally, in the sector $\lambda_2(U_r) = k/v_r < x/t < \infty$, we have

$$u = u_r, \quad v = v_r.$$

Let us find s, \hat{u} and \hat{v} . By continuity of the rarefaction wave solution,

$$(7.2) \quad \hat{u} + k \ln \hat{v} = u_r + k \ln v_r$$

Thus the three unknowns s, \hat{u} and \hat{v} satisfy the three equations (7.1)-(7.2).

We find that

$$(7.3) \quad s = -\frac{k \ln(v_r/\hat{v})}{\hat{v} - v_l}, \quad \hat{u} = k \ln \frac{v_r}{\hat{v}}, \quad v_l > \hat{v} > v_r$$

To determine \hat{v} , one should solve the following equation for $\sigma = \sqrt{\hat{v}/v_l}$:

$$(7.4) \quad 1/\sigma - 2 \ln \sigma - \sigma = \ln(v_l/v_r).$$

This equation has a unique solution such that $0 < \sigma < 1$.

We verify that the entropy solution constructed above is not an entropy quasisolution.

Lemma 7.1. *There are initial data v_l and v_r , $v_l > v_r$, such that the entropy solution constructed above is not an entropy quasisolution.*

Proof. We recall that the functions

$$\eta = v^B e^{Au}, \quad q = -\frac{k^2 A v^{B-1} e^{Au}}{B-1}$$

satisfy (1.6) and (1.7) provided

$$k^2 A^2 = B(B-1).$$

It is enough to show that the inequality

$$(7.5) \quad s(\hat{\eta} - \eta_l) - \hat{q} + q_l \geq 0$$

does not hold for some choice of B, v_l , and v_r , where

$$\hat{\eta} = \eta(\hat{v}, \hat{u}), \quad \hat{q} = q(\hat{v}, \hat{u}), \quad \eta_l = \eta(v_l, u_l), \quad q_l = q(v_l, u_l).$$

It follows from (7.3) that the left-hand side of (7.5) is equal to

$$F(\sigma, B) \equiv k \left(\frac{1}{\sigma} + \sqrt{\frac{B}{B-1}} - \left(\sigma + \sqrt{\frac{B}{B-1}} \right) \sigma^{2B-2} e^{(1/\sigma - \sigma)\sqrt{B(B-1)}} \right).$$

Setting $\sigma = 1/2$ and denoting $\delta = \sqrt{B/(B-1)}$, $0 < \delta < 1$, we obtain $F = kF_1$ with

$$F_1(1/2, \delta) = 2 + \delta - (2 + 4\delta)(e^{3\delta/2}/4)^{1/(1-\delta^2)}.$$

One can verify easily that $F_1 \rightarrow -\infty$ as $\delta \rightarrow 1$. Lemma 6.1 is proved.

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