ON NONSYMMETRIC THEOREMS FOR (H, G)-COINCIDENCES

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Abstract. Let X be a compact Hausdorff space, $\varphi : X \to S^n$ a continuous map into the n-sphere S^n that induces a nonzero homomorphism $\varphi^* : H^n(S^n; \mathbb{Z}_p) \to H^n(X; \mathbb{Z}_p)$, Y a k-dimensional CW-complex and $f : X \to Y$ a continuous map. Let G a finite group which acts freely on S^n. Suppose that $H \subset G$ is a normal cyclic subgroup of a prime order. In this paper, we define and we estimate the cohomological dimension of the set $A_\varphi(f, H, G)$ of (H, G)-coincidence points of f relative to φ.

Key words: Borsuk-Ulam theorem, $\mathbb{Z}_p$-index, (H, G)-coincidence, free actions

1. Introduction

K.D. Joshi [10] has proved a nonsymmetric generalization of the Borsuk-Ulam theorem [1], in which the n-sphere S^n is replaced by a certain compact subset X of the $(n + 1)$-dimensional Euclidean space $\mathbb{R}^{n+1}$. In this context, a pair of points $x, y \in X$ are said to be antipodal if $y = -\lambda x$, for some $\lambda > 0$. The Joshi's theorem shows that for every continuous map $f : X \to \mathbb{R}^n$ there exist antipodal points $x, y \in X$ such that $f(x) = f(y)$.

K. Borsuk has suggested to define antipodal points in an arbitrary space in the following way: $x_1, x_2 \in X$ are said to be antipodal points relative to an essential map [1] $\varphi : X \to S^n$ if $\varphi(x_1) = -\varphi(x_2)$. Using the Borsuk's suggestion, Spież [11] has proved that if X is a compact Hausdorff space and if $\varphi : X \to S^n$ is an essential map, then for every continuous map $f : X \to \mathbb{R}^k$, the covering dimension of the set

$$A_\varphi(f) = \{x \in X;\ \exists\, y \in X, \text{ such that } \varphi(x) = -\varphi(y) \text{ and } f(x) = f(y)\}$$

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[1] A map $\varphi : X \to S^n$ is said to be an essential map if φ induces nonzero homomorphism $\varphi^* : H^n(S^n; \mathbb{Z}_p) \to H^n(X; \mathbb{Z}_p)$

is not less than $n-k$, obtaining thus a generalization of the Joshi's theorem.

M. Izydorek [7] has extended the proposition of Borsuk for a cyclic group G of order prime which acts freely on a n-dimensional sphere S^n and has proved the following generalization of the Spież's theorem: if X is a compact Hausdorff space and if $\varphi : X \to S^n$ is an essential map, then for every continuous map $f : X \to \mathbb{R}^k$, the covering dimension of the set

$$\begin{aligned} A_\varphi(f) = \{x \in X;\ \exists\, x_2, \ldots, x_p \mid \quad & \varphi \quad (x) = g^{-1}\varphi(x_2) = \cdots = g^{1-p}\varphi(x_p) \\ \text{and} \quad & f \quad (x) = f(x_2) = \cdots = f(x_p)\} \end{aligned}$$

is not less than $n-(p-1)k$, where g is a fixed generator of G. Moreover, if $\mathbb{R}^k$ is replace by a generalized k- dimensional manifold M^k over $\mathbb{Z}_p$, then an analogous theorem has been proved (see [7, Theorem 4]).

Gonçalves, Jaworowski and Pergher [3] have defined (H,G)-concidence for a continuous map f from a n-sphere S^n into a k-dimensional CW-complex Y, where G is a finite group which acts freely on S^n and have proved that if H is a nontrivial normal cyclic subgroup of a prime order, then

$$\text{cohom.dim } A(f,H,G) \geq n - |G|k,$$

where $A(f,H,G)$ is the set of (H,G)-coincidence points of f and cohom.dim denotes the cohomological dimension. The other papers closely related to [3] are [4, 5, 6], [8, 9] and [13].

The purpose of this paper is to define the set $A_\varphi(f,H,G)$ of (H,G)-coincidence points of a continuous map $f : X \to Y$ relative to an essential map $\varphi : X \to S^n$, where X is a compact Hausdorff space, Y is a topological space, G is a finite group which acts freely on the n-dimensional sphere S^n and H is a subgroup of G. Using this definition, under certain conditions, we estimate the cohomological dimension of the set $A_\varphi(f,H,G)$. Specifically, we will prove the following nonsymmetric version of the main Theorem of [3],

Theorem 1.1. *Let X be a compact Hausdorff space, Y a k-dimensional CW-complex and $\varphi : X \to S^n$ an essential map. Given a finite group G which acts freely on S^n and H a normal cyclic subgroup of prime order, then for every continuous map $f : X \to Y$ such that $f^* : H^i(Y;\mathbb{Z}_p) \to H^i(X;\mathbb{Z}_p)$ is trivial for $i \geq 1$,* cohom.dim $A_\varphi(f,H,G) \geq n - |G|k$.

For the proof of Theorem 1.1, it was fundamental to prove the following version of the main Theorem of [3],

Theorem 1.2. *Let X be a paracompact Hausdorff space, Y a k-dimensional CW-complex, G a finite group which acts freely on X and $H \subset G$ a normal cyclic subgroup of prime order. Let $f : X \to Y$ be a continuous map such that $f^* : H^i(Y; \mathbb{Z}_p) \to H^i(X; \mathbb{Z}_p)$ is trivial, for $i \geq 1$. Suppose that the $\mathbb{Z}_p$-index of X is greater than or equal to n, then the $\mathbb{Z}_p$-index of the set $A(f, H, G)$ is greater than or equal to $n - |G|k$. Consequently,* cohom.dim $A(f, H, G) \geq n - |G|k$.

2. Preliminaries

Throughout this paper the symbols H_* and H^* will denote Čech homology and cohomology groups with coefficients in $\mathbb{Z}_p$, unless otherwise indicated. If G is a group which acts on a topological space X, we will denote by X^* the orbit space X/G.

We start by introducing some basic notions and definitions as follows.

2.1. (H, G)**-coincidence.** Suppose that X,Y are topological spaces, G is a group acting freely on X and $f : X \to Y$ is a map. If H is a subgroup of G, then H acts on the right on each orbit Gx of G as follows: if $y \in Gx$ and $y = gx$, $g \in G$, then $hy = ghx$ (such action may depend on the choice of the reference point x). Following [4, 6, 9] the concept of G-coincidence is generalized as follows: a point $x \in X$ is said to be a (H, G)-coincidence point of f if f sends every orbit of the action of H on the G-orbit of x to a single point (see [5]). We will denote by $A(f, H, G)$ the set of all (H, G)-coincidence points of f. If H is the trivial subgroup, then every point of X is a (H, G)-coincidence. If $H = G$, this is the usual definition of coincidence. If $G = Z_p$ with p prime, then a nontrivial (H, G)-coincidence point is a G-coincidence point.

2.2. **The space X_φ and the set $A_\varphi(f, H, G)$.** Let us consider X a compact Hausdorff space and an essential map $\varphi : X \longrightarrow S^n$. Suppose G be a finite group of order r which acts freely on S^n and H be a subgroup of order p of G. Let $G = \{g_1, g_2, \ldots, g_r\}$ be a fixed enumeration of elements of G, where g_1 is the identity of G. A nonempty space X_φ can be associated with the essential map $\varphi : X \to S^n$ as follows:

$$\begin{aligned} X_\varphi &= \{(x_1, x_2, \ldots, x_r) \in X^r \mid \varphi(x_1) = (g_2)^{-1}\varphi(x_2) = \cdots = (g_r)^{-1}\varphi(x_r)\} \\ &= \{(x_1, x_2, \ldots, x_r) \in X^r \mid g_i\varphi(x_1) = \varphi(x_i), i = 1, \ldots, r\}, \end{aligned}$$

where X^r denotes the r-fold cartesian product of X. The set X_φ is a closed subset of X^r and so it is compact. We define a G-action on X_φ as follows: for each $g_i \in G$ and for each $(x_1, \dots, x_r) \in X_\varphi$,

$$g_i(x_1, \dots, x_r) = (x_{\sigma_{g_i}(1)}, \dots, x_{\sigma_{g_i}(r)}), \tag{2.1}$$

where the permutation σ_{g_i} is defined by $\sigma_{g_i}(k) = j$, se $g_k g_i = g_j$. We observe that if $x = (x_1, \dots, x_r) \in X_\varphi$ then $x_i \neq x_j$, for any $i \neq j$ and therefore G acts freely on X_φ.

Now, let us consider a continuous map $f : X \to Y$, where Y is a topological space and $\tilde{f} : X_\varphi \to Y$ given by $\tilde{f}(x_1, \dots, x_r) = f(x_1)$. Let $y = (x_1, \dots, x_r) \in A(\tilde{f}, H, G)$ and consider the orbit $Gy = \{g_1 y, g_2 y, \dots, g_r y\}$. Note that
(i) From (2.1), we have that for each i, the 1-th coordinate of $g_i y$ is x_i.
(ii) The action of H on Gy determines a partition of the orbit Gy in $s = (r/p)$ disjoint suborbits, and we can be rewrite

$$Gy = \{g_{1_1} y, \dots, g_{1_p} y; \dots; g_{j_1} y, \dots, g_{j_p} y; \dots; g_{s_1} y, \dots, g_{s_p} y\},$$

where $\{1, 2, \dots, r\} \longleftrightarrow \{1_1, \dots, 1_p; \dots; j_1, \dots, j_p; \dots; s_1, \dots, s_p\}$ is a bijection.

Since y is a (H, G) - coincidence point of $\tilde{f}$, it follows from (i) and (ii) that,

$$f(x_{1_1}) = \cdots = f(x_{1_p}); \dots; f(x_{j_1}) = \cdots = f(x_{j_p}); \dots; f(x_{s_1}) = \cdots = f(x_{s_p}).$$

In these conditions, we have the following

Definition 2.1. The set $A_\varphi(f, H, G)$ of (H, G)-coincidence points of f relative to φ is defined by

$$\begin{aligned} A_\varphi(f, H, G) = A(\tilde{f}, H, G) \;&=\; \{(x_1, \dots, x_r) \in X^r \mid g_i \varphi(x_1) = \varphi(x_i), i = 1, \dots, r \\ &\qquad \text{and } \; f(x_{j_1}) = \cdots = f(x_{j_p}), j = 1, \dots, s\}. \end{aligned}$$

Remark 2.2. Let us observe that if $G = H = \mathbb{Z}_p$,

$$\begin{aligned} A_\varphi(f, H, G) = A_\varphi(f) \;&=\; \{(x_1, \dots, x_p) \in X^p \mid g_i \varphi(x_1) = \varphi(x_i), i = 1, \dots, p \\ &\qquad \text{and } \; f(x_1) = \cdots = f(x_p)\}. \end{aligned}$$

2.3. **The $\mathbb{Z}_p$-index.** Suppose that the cyclic group $G = \mathbb{Z}_p$ of order prime p acts freely on a Hausdorff and paracompact space X. Then $X \to X^*$ is a principal $\mathbb{Z}_p$-bundle and one can take $h : X^* \to B\mathbb{Z}_p$ a classifying map for the G-bundle $X \to X^*$.

Remark 2.3. It is well known that if $\hat{h}$ is another classifying map for the principal $\mathbb{Z}_p$-bundle $X \to X^*$, then there is a homotopy between h and $\hat{h}$.

We will be considering the following definition for the $\mathbb{Z}_p$-index of X (see [7]).

Definition 2.4. We say that the $\mathbb{Z}_p$-index of X is greater than or equal to k if the homomorphism $h^* : H^k(B\mathbb{Z}_p) \to H^k(X^*)$ is nontrivial.
We say that the $\mathbb{Z}_p$-index of X is equal to k if it is greater than or equal to k and moreover $h^* : H^i(B\mathbb{Z}_p) \to H^i(X^*)$ is zero, for any $i \geq k+1$.

Remark 2.5. A model for $B\mathbb{Z}_2$ the classifying space for $\mathbb{Z}_2$ is the infinite real projective space P^∞. Then $H^*(B\mathbb{Z}_2) \cong H^*(P^\infty)$ is isomorphic to $\mathbb{Z}_2[a]$, where $a \in H^1(P^\infty)$ is the generator. The generator of $H^i(B\mathbb{Z}_2)$ is a^i for any $i \geq 0$. If $p > 2$ a model for $B\mathbb{Z}_p$ the classifying space for $\mathbb{Z}_p$ is the infinite lens space $L_p^\infty = S^\infty/\mathbb{Z}_p$. Thus, $H^i(B\mathbb{Z}_p) = H^i(L_p^\infty) \cong \mathbb{Z}_p$ for any $i \geq 0$ and given any nonzero element $a \in H^1(L_p^\infty)$, one has that $b = \beta(a)$ is a nonzero element of $H^2(L_p^\infty)$, where $\beta : H^1(L_p^\infty) \to H^2(L_p^\infty)$ is the Bockstein homomorphism. More generally, a generator $\mu \in H^i(B\mathbb{Z}_p)$ is given by

$$\mu = \begin{cases} a \smile b^{(i-1)/2}, & \text{if} \quad i \quad \text{is odd} \\ b^{i/2}, & \text{if} \quad i \quad \text{is even.} \end{cases}$$

2.4. **The Smith special cohomology groups with coefficients in $\mathbb{Z}_p$.** In this work, we will be considering the definition of the Smith special cohomology groups with coefficients in $\mathbb{Z}_p$ in the sense of [2]. Smith homology and cohomology were originally defined in [12] and in a series of subsequent papers. A systematic exposition of the Smith theory can be found in [2]. Let X be a topological space; given a finite group G of prime order p which acts freely on X, let g be a fixed generator of G and put

$$\sigma = 1 + g + g^2 + \cdots + g^{p-1} \quad \text{and} \quad \tau = 1 - g,$$

in the group ring $\mathbb{Z}_p(G)$. We have that $\sigma = \tau^{p-1}$. If $\rho = \tau^i$, we put $\bar{\rho} = \tau^{p-i}$, then $\tau = \bar{\sigma}$ and $\sigma = \bar{\tau}$. There exists an exact sequence with coefficients in $\mathbb{Z}_p$ [2, p.125],

$$\longrightarrow H_\rho^n(X) \xrightarrow{\rho^*} H^n(X) \xrightarrow{\mathcal{T}} H^n(X^*) \xrightarrow{\delta} H_\rho^{n+1}(X) \xrightarrow{\rho^*}$$

called Smith exact sequence, where $H_\rho^*(X)$ denotes the Smith special cohomology groups and $\mathcal{T}$ is the transfer homomorphism.

Remark 2.6. The Smith cohomology groups are natural with respect to $\mathbb{Z}_p$-equivariant maps, that is, if $f : X \to Y$ is a $\mathbb{Z}_p$-equivariant map then f induces homomorphism ${f_\rho}^* : H^*_\rho(Y) \to H^*_\rho(X)$ which commutes with the homomorphisms in the Smith sequence.

3. The $\mathbb{Z}_p$-index of X_φ

Let us consider the free G-space X_φ as defined in Section 2.2. If $H \subset G$ is a cyclic subgroup of prime order p, then X_φ is a free $H \cong \mathbb{Z}_p$-space. In these conditions, as in [7, Theorem 3], we have the following

Theorem 3.1. *Let X be a compact Hausdorff space and $\varphi : X \longrightarrow S^n$ an essential map. Then, the $\mathbb{Z}_p$-index of X_φ is equal to n.*

Proof. For $i = 1, 2, \dots, r$, let us consider the maps $\varphi_i : X \to S^n$ given by $\varphi_i(x) = (g_i)^{-1}\varphi(x)$, where $g_i \in G$. Then, we can define the map

$$\psi = \varphi_1 \times \cdots \times \varphi_r : X^r \to [S^n]^r,$$

where $[S^n]^r$ denotes the r-fold cartesian product of r copies of S^n, such that $X_\varphi = \psi^{-1}\Delta[S^n]^r$, where $\Delta[S^n]^r$ is the diagonal in $[S^n]^r$. In these conditions, we prove the following

Lemma 3.2. *The homomorphism $\psi^* : H^*([S^n]^r) \to H^*(X^r)$ induced by $\psi : X^r \to [S^n]^r$ is a monomorphism in each dimension.*

Proof. Let m be an integer and for each $t = 1, 2, \dots, r$ consider $H^m([S^n]^t)$. If m is not divisible by n then $H^m([S^n]^t) = 0$ and the result follows. Suppose that m is divisible by n; one then has that there exists $\alpha = 0, 1, 2, \dots,$ such that $m = \alpha n$. Let us consider the following commutative diagram

(3.1)

$$\begin{array}{ccc}
H^{\alpha n}([S^n]^t) \otimes H^0(S^n) \oplus H^{(\alpha-1)n}([S^n]^t) \otimes H^n(S^n) & \xrightarrow{\times} & H^{\alpha n}([S^n]^{t+1}) \\
\downarrow {\scriptstyle (\varphi_1 \times \cdots \times \varphi_t)^* \otimes \varphi_{t+1}^*} & & \downarrow {\scriptstyle (\varphi_1 \times \cdots \times \varphi_{t+1})^*} \\
H^{\alpha n}(X^t) \otimes H^0(X) \oplus H^{(\alpha-1)n}(X^t) \otimes H^n(X) & \xrightarrow{\times} & H^{\alpha n}(X^{t+1})
\end{array}$$

Applying the Künneth's formula, we have that the upper row of the above diagram is an isomorphism and the lower row is a monomorphism, for each $\alpha = 0, 1, 2, \dots$ and $t = 1, \dots, r$.

The proof will be done by induction on t. We assume inductively that for some $t = 1, 2, \dots, r-1$ and for each $\alpha = 0, 1, 2, \dots$ the homomorphism $(\varphi_1 \times \cdots \times \varphi_t)^* : H^{\alpha n}([S^n]^t) \to H^{\alpha n}(X^t)$ is a monomorphism and we will show that $(\varphi_1 \times \cdots \times \varphi_t)^* \otimes \varphi_{t+1}^*$ is a monomorphism. The result will follow from the commutativity of the diagram 3.1. By induction hypothesis it suffices to show that φ_{t+1}^* is a monomorphism. For this, observe that $\varphi_i^* = \varphi_j^*$, for any $1 \le i, j \le r$. If φ_{t+1}^* is not a monomorphism, then φ_i^* is not a monomorphism, for any $1 \le i \le t$, which completes the proof. $\square$

The next step is to use Lemma 3.2 to show that the homomorphism induced by $\psi|_{X_\varphi} : X_\varphi \to \Delta[S^n]^r$ is a monomorphism. Let us consider the following commutative diagram

$$\begin{array}{ccccc} H^n(X^r, X_\varphi) & \xrightarrow{j^*} & H^n(X^r) & \xrightarrow{k^*} & H^n(X_\varphi) \\ \uparrow{\scriptstyle \psi^*} & & \uparrow{\scriptstyle \psi^*} & & \uparrow{\scriptstyle (\psi|_{X_\varphi})^*} \\ H^n([S^n]^r, \Delta[S^n]^r) & \longrightarrow & H^n([S^n]^r) & \xrightarrow{i^*} & H^n(\Delta[S^n]^r) \end{array} \tag{3.2}$$

whose rows are exact. If γ is a generator of $H^n(S^n) \cong \mathbb{Z}_p$ let us denote by α_i the element $q_i^*(\gamma) \in H^n([S^n]^r)$, where $q_i : [S^n]^r \to S^n$ is the natural projection on the i-th coordinate for $i = 1, 2, \dots, r$. Let us observe that $q_i \circ i = q_j \circ i$ for any $1 \le i, j \le r$, where $i : \Delta[S^n]^r \hookrightarrow [S^n]^r$ is the natural inclusion. In this way, one has that

$$i^*(\alpha_i) = i^* \circ q_i^*(\gamma) = (q_i \circ i)^*(\gamma) = (q_j \circ i)^*(\gamma) = i^*(\alpha_j). \tag{3.3}$$

Lemma 3.3. $(\psi|_{X_\varphi})^* : H^n(\Delta[S^n]^r) \to H^n(X_\varphi)$ *is a monomorphism.*

Proof. Since $\Delta[S^n]^r$ is homeomorphic to S^n, it follows from (3.3) that $i^*(\alpha_i)$ is a nonzero element in $H^n(\Delta[S^n]^r)$ for any $i = 1, \dots, r$. Thus, it suffices to show that $(\psi|_{X_\varphi})^*(i^*(\alpha_1)) \neq 0$. Let us assume that this does not happen. From the diagram (3.2) we have that

$$k^* \circ \psi^*(\alpha_1) = (\psi|_{X_\varphi})^* \circ i^*(\alpha_1) = 0,$$

which implies that $\psi^*(\alpha_1) \in \mathrm{Ker}(k^*) = \mathrm{Im}(j^*)$ and there exists an element $U \in H^n(X^r, X_\varphi)$ such that

$$j^*(U) = \psi^*(\alpha_1) \neq 0, \tag{3.4}$$

since by Lemma 3.2 ψ^* is a monomorphism and $\alpha_1 = q_1^*(\gamma) \in H^n([S^n]^r)$ is a nonzero element. Let us consider the following commutative diagram,

(3.5)

$$\begin{array}{ccccc}
H^{(r-1)n}(X^r, X^r - X_\varphi) & \xrightarrow{j^*} & H^{(r-1)n}(X^r) & \xrightarrow{k^*} & H^{(r-1)n}(X^r - X_\varphi) \\
\uparrow \psi^* & & \uparrow \psi^* & & \uparrow \psi^* \\
H^{(r-1)n}([S^n]^r, [S^n]^r - \Delta[S^n]^r) & \xrightarrow{j_1^*} & H^{(r-1)n}([S^n]^r) & \xrightarrow{k_1^*} & H^{(r-1)n}([S^n]^r - \Delta[S^n]^r) \\
\uparrow D^{-1} & & \uparrow D^{-1} & & \\
H_n(\Delta[S^n]^r) & \xrightarrow{i_*} & H_n([S^n]^r) & &
\end{array}$$

where the first and the second rows are exact, D is the Alexander-Spanier Duality which is an isomorphism and all others maps are induced by appropriate inclusions.

Let us denote by $a_1, a_2, \dots, a_r$ the elements of $H_n([S^n]^r)$ which are conjugated to $\alpha_1, \alpha_2, \dots, \alpha_r$ in $H^n([S^n]^r)$. More precisely,

$$\langle \alpha_j, a_i \rangle = \begin{cases} 1, & \text{if} \quad i = j \\ 0, & \text{if} \quad i \neq j \end{cases}$$

where the map $\langle\ ,\ \rangle : H^n([S^n]^r) \times H_n([S^n]^r) \to \mathbb{Z}_p$ denotes the Kronecker product. Let us denote by $c \in H_n(\Delta[S^n]^r)$ the conjugated element to $i^*(\alpha_1) = \cdots = i^*(\alpha_r)$ in $H^n(\Delta[S^n]^r)$. Then for any $j = 1, \dots, r$ one has $\langle i^*(\alpha_j), c\rangle = 1$. Furthermore, it follows from properties of the Kronecker product that for any $j = 1, \dots, r$

$$\langle i^*(\alpha_j), c\rangle = \langle \alpha_j, i_*(c)\rangle = 1,$$

which implies that

$$i_*(c) = \sum_{i=1}^{r} a_i. \tag{3.6}$$

Let us consider for each $i = 1, \dots, r$ the elements

$$\beta_i = \alpha_1 \smile \cdots \smile \alpha_{i-1} \smile \hat{\alpha}_i \smile \alpha_{i+1} \smile \cdots \smile \alpha_r \in H^{(n-1)r}([S^n]^r),$$

where the symbol $\hat{\alpha}_i$ means that the element α_i is omitted.

To simplify notation, here we will also denote by $[S^n]^r$ the generator of $H_{nr}([S^n]^r)$, which is called *the fundamental class of* $[S^n]^r$. We will show that

$$D^{-1}(a_i) = (-1)^{i+1}\beta_i, \text{ that is, } (-1)^{i+1}\beta_i \frown [S^n]^r = a_i.$$

In fact, it follows from properties of the cup and cap products with respect to the Kronecker product and by definition of β_i that

$$\begin{aligned}\langle \alpha_i, (-1)^{i+1}\beta_i \frown [S^n]^r\rangle &= \langle \alpha_i \smile (-1)^{i+1}\beta_i, [S^n]^r\rangle \\ &= \langle (-1)^{i+1}(-1)^{n(n(i-1))}\alpha_1 \smile \cdots \smile \alpha_r, [S^n]^r\rangle \\ &= \langle \alpha_1 \smile \cdots \smile \alpha_r, [S^n]^r\rangle = 1\end{aligned}$$

observing that $\alpha_1 \smile \cdots \smile \alpha_r$ is the generator of $H^{nr}([S^n]^r)$. Thus,

$$D^{-1}\left(\sum_{i=1}^{r} a_i\right) = \sum_{i=1}^{r} D^{-1}(a_i) = \sum_{i=1}^{r}(-1)^{i+1}\beta_i. \tag{3.7}$$

It follows from (3.6), (3.7) and from commutativity of diagram(3.5) that

$$j_1^* \circ D^{-1}(c) = D^{-1} \circ i_*(c) = \sum_{i=1}^{r}(-1)^{i+1}\beta_i,$$

that is,

$$\sum_{i=1}^{r}(-1)^{i+1}\beta_i \in \mathrm{Im}(j_1^*).$$

Since the second row of diagram (3.5) is exact, one has that

$$k_1^*\left(\sum_{i=1}^{r}(-1)^{i+1}\beta_i\right) = 0.$$

By using again the commutativity of diagram (3.5)

$$k^* \circ \psi^*\left(\sum_{i=1}^{r}(-1)^{i+1}\beta_i\right) = \psi^* \circ k_1^*\left(\sum_{i=1}^{r}(-1)^{i+1}\beta_i\right) = 0,$$

which implies that

$$\psi^*\left(\sum_{i=1}^{r}(-1)^{i+1}\beta_i\right) \in \ker(k^*) = \mathrm{Im}(j^*).$$

Thus, there exists an element $V \in H^{(r-1)n}(X^r, X^r - X_\varphi)$ such that

$$j^*(V) = \psi^*\left(\sum_{i=1}^{p}(-1)^{i+1}\beta_i\right) \neq 0, \tag{3.8}$$

since ψ^* is a monomorphism. Using the naturality of the cup product in the following diagram

$$\begin{array}{ccc} H^n(X^r)\otimes H^{(r-1)n}(X^r) & \xrightarrow{\smile} & H^{rn}(X^r) \\ \uparrow \psi^* & & \uparrow \psi^* \\ H^n([S^n]^r)\otimes H^{(r-1)n}([S^n]^r) & \xrightarrow{\smile} & H^{rn}([S^n]^r) \end{array}$$

and observing that

$$\sum_{i=2}^{r} \alpha_1 \smile (-1)^{i+1}\beta_i = 0$$

one has that

$$\begin{aligned} \psi^*(\alpha_1)\smile\psi^*\left(\sum_{i=1}^{r}(-1)^{i+1}\beta_i\right) &= \psi^*\left(\alpha_1\smile\sum_{i=1}^{r}(-1)^{i+1}\beta_i\right) \\ &= \psi^*\left(\alpha_1\smile\beta_1+\sum_{i=2}^{r}\alpha_1\smile(-1)^{i+1}\beta_i\right) \\ &= \psi^*(\alpha_1\smile\beta_1) \\ &= \psi^*\Big(\alpha_1\smile\cdots\smile\alpha_r\Big)\neq 0, \end{aligned} \tag{3.9}$$

since $\alpha_1 \smile \cdots \smile \alpha_r$ is the generator of $H^{rn}([S^n]^r)$ and ψ^* is a monomorphism.

On the other hand, from naturality of the cup product in the diagram

$$\begin{array}{ccc} H^n(X^r, X_\varphi)\otimes H^{(r-1)n}(X^r, X^r - X_\varphi) & \xrightarrow{\smile} & H^{rn}(X^r, X^r) \\ j^*\downarrow & & \downarrow j^* \\ H^n(X^r)\otimes H^{(r-1)n}(X^r) & \xrightarrow{\smile} & H^{rn}(X^r) \end{array}$$

and from equations (3.4) and (3.8) we conclude that

$$\begin{aligned} \psi^*(\alpha_1\smile\cdots\smile\alpha_r) &= \psi^*(\alpha_1)\smile\psi^*\left(\sum_{i=1}^{p}(-1)^{i+1}\beta_i\right) \\ &= j^*(U)\smile j^*(V) = j^*(U\smile V) = j^*(0) = 0, \end{aligned}$$

which contradicts (3.9). This completes the proof. □

Now, let us consider the map

$$\theta = q_1\circ\psi|_{X_\varphi} : X_\varphi \to S^n,$$

where $q_1 : \Delta[S^n]^r \to S^n$ is the natural projection on the 1-th coordinate, which is an homeomorphism. Since by Lemma 3.3, $(\psi|_{X_\varphi})^*$ is a monomorphism, one then has that $\theta^* : H^n(S^n) \to H^n(X_\varphi)$ is a monomorphism. Note that, if $(x_1, \dots, x_r) \in X_\varphi$, we have that for each i, $g_i\theta(x_1, \dots, x_r) = g_i\varphi(x_1) = \varphi(x_i) = \theta g_i(x_1, \dots, x_r)$, thus θ is a G- equivariant map, and consequently, θ is a H- equivariant map, where $H \subset G$ is a cyclic subgroup of prime order. Thus, in particular for $\rho = \sigma$, we can consider the homomorphism induced by θ, $\theta_\sigma^* : H_\sigma^n(S^n) \to H_\sigma^n(X_\varphi)$, where H_σ^n denotes the n-dimensional Smith special cohomology group with coefficients in $\mathbb{Z}_p$ in the sense of Section 2.4.

By remarks in [2, Results following 5.2] whose dual holds in cohomology, $i^* : H^n(S^n) \to H_\sigma^n(S^n)$ is an isomorphism, and since $\theta^* : H^n(S^n) \to H^n(X_\varphi)$ is a monomorphism it follows that θ_σ^* is a monomorphism. To conclude that the $\mathbb{Z}_p$-index of X_φ is equal to n it suffices to verify that the map between the orbit spaces $\bar\theta : X_\varphi/H \to S^n/H$ induces a monomorphism in cohomology. From results in Section 2.4, we have that $H_\sigma^n(S^n) \cong H^n(S^n/H)$ and $H_\sigma^n(X_\varphi) \cong H^n(X_\varphi/H)$, and considering the commutative diagram

$$\begin{array}{ccc} H_\sigma^n(S^n) & \xrightarrow{\theta_\sigma^*} & H_\sigma^n(X_\varphi) \\ \uparrow\cong & & \cong\uparrow \\ H^n(S^n/H) & \xrightarrow{\bar\theta^*} & H^n(X_\varphi/H) \end{array}$$

it follows that $\bar\theta^* : H^n(S^n/H) \to H^n(X_\varphi/H)$ is a monomorphism. Therefore, the $\mathbb{Z}_p$-index of X_φ is equal to n. □

4. Proof of Theorems 1.1 and 1.2

Proof of theorem 1.2. By following the similar steps of [3], we first prove Theorem 1.2 in the case that $G = H = Z_p$, where $p \geq 2$. We need to show that the $\mathbb{Z}_p$-index of the set

$$A_f = \{x \in X; f(x) = f(gx) = \cdots, f(g^{p-1}x)\} \tag{4.1}$$

is greater than or equal to $n - pk$. For this, let us consider $F : X \to Y^p$ given by $F(x) = (f(x), f(gx), \dots, f(g^{p-1}x))$, where $Y^p = Y \times \cdots \times Y$ denotes the p-fold cartesian product of Y and g is a fixed generator of $\mathbb{Z}_p$. In these conditions, we prove the following

Lemma 4.1. *The homomorphism $F^*: H^q(Y^p) \to H^q(X)$ induced by the map $F: X \to Y^p$ is zero for any $q \geq 1$.*

Proof. We have that $F = (f_0 \times f_1 \times \cdots \times f_{p-1}) \circ d$, where $d: X \to X^p$ is the diagonal map and $f_i(x) = f(g^i x)$, for any $x \in X$ and $i = 0, \cdots p-1$. In this way, it suffices to show that $(f_0 \times f_1 \times \cdots \times f_{p-1})^*: H^q(Y^p) \to H^q(X^p)$ is trivial for any $q \geq 1$. Let us consider the following commutative diagram

$$\begin{array}{ccc} \bigoplus_{i+j=q} H^i(Y^t) \otimes H^j(Y) & \xrightarrow{\cong} & H^q(Y^{t+1}) \\ \downarrow{\scriptstyle (f_0\times\cdots\times f_t)^*\otimes f_{t+1}^*} & & \downarrow{\scriptstyle (f_0\times\cdots\times f_{t+1})^*} \\ \bigoplus_{i+j=q} H^i(X^t) \otimes H^j(X) & \xrightarrow{\times} & H^q(X^{t+1}) \end{array} \tag{4.2}$$

Since Y is a CW-complex, applying the Künneth's formula we have that the upper row of diagram (4.2) is an isomorphism for any $q = 1, 2, \cdots$ and $t = 1, \cdots, p-1$.

The proof will be done by induction on t. Suppose inductively that $(f_0 \times \cdots \times f_t)^*: H^i(Y^t) \to H^i(X^t)$ is zero for some $t = 1, 2, \cdots, p-1$ and for each $i = 1, 2, \cdots$. By hypothesis, f induces the zero homomorphism in each dimension; in particular f_{t+1}^* is zero and thus $(f_0 \times \cdots \times f_t)^* \otimes f_{t+1}^*$ is trivial. It follows from commutativity of the diagram (4.2) that $(f_0 \times \cdots \times f_{t+1})^*$ is zero, which completes the proof. □

We can define a $\mathbb{Z}_p$-action on Y^p as follows: for each $(y_1, \cdots, y_p) \in Y^p$ $g(y_1, \cdots, y_{p-1}, y_p) = (y_p, y_1, \cdots, y_{p-1})$. Since p is a prime, this action is free on $Y^p - \Delta$, where Δ is the diagonal in Y^p. Let us observe that $A_f = F^{-1}(\Delta)$, thus F determines a $\mathbb{Z}_p$-equivariant map $F_0: X - A_f \to Y^p - \Delta$, which induces a map between the orbit spaces $\bar{F}_0: [X - A_f]^* \to [Y^p - \Delta]^*$. In these conditions, we prove that

Lemma 4.2. *The map $\bar{F}_0^*: H^{pk}([Y^p - \Delta]^*) \to H^{pk}([X - A_f]^*)$ is zero.*

Proof. Let us consider the map of pairs $(F, F_0): (X, X - A_f) \to (Y, Y - \Delta)$. One then has the following commutative diagram

$$\begin{array}{ccccccccc} \longrightarrow & H^{pk}(X) & \xrightarrow{i^*} & H^{pk}(X - A_f) & \longrightarrow & H^{pk+1}(X, X - A_f) & \longrightarrow \\ & \uparrow{\scriptstyle F^*} & & \uparrow{\scriptstyle F_0^*} & & \uparrow{\scriptstyle (F,F_0)^*} & \\ \longrightarrow & H^{pk}(Y^p) & \xrightarrow{j^*} & H^{pk}(Y^p - \Delta) & \longrightarrow & H^{pk+1}(Y^p, Y^p - \Delta) & \longrightarrow \end{array}$$

where the homomorphisms i^* and j^* are induced by appropriate inclusions. Since $\dim(Y^p)$ is less than or equal to pk we have that $H^{pk+1}(Y^p, Y^p - \Delta)$ is trivial and thus j^* is surjective. On the other hand $F_0 : (X - A_f) \to (Y^p - \Delta)$ is a $\mathbb{Z}_p$-equivariant map and it follows from Remark 2.6 that the diagram

$$\begin{array}{ccccc} H^{pk}(X - A_f) & \xrightarrow{\mathcal{T}} & H^{pk}([X - A_f]^*) & \longrightarrow & H^{pk+1}_\rho(X - A_f) \\ \uparrow {\scriptstyle F_0{}^*} & & \uparrow {\scriptstyle \bar{F}_0^*} & & \uparrow \\ H^{pk}(Y^p - \Delta) & \xrightarrow{\mathcal{T}} & H^{pk}([Y^p - \Delta]^*) & \longrightarrow & H^{pk+1}_\rho(Y^p - \Delta) \end{array}$$

between the Smith sequences of $X - A_f$ and $Y^p - \Delta$ is commutative and since $H^{pk+1}_\rho(Y^p - \Delta)$ is zero, $\mathcal{T}$ is surjective.

Putting together these diagrams, one obtains a new commutative diagram

$$\begin{array}{ccccc} H^{pk}(X) & \xrightarrow{i^*} & H^{pk}(X - A_f) & \xrightarrow{\mathcal{T}} & H^{pk}([X - A_f]^*) \\ \uparrow {\scriptstyle F^*} & & \uparrow {\scriptstyle F_0^*} & & \uparrow {\scriptstyle \bar{F}_0^*} \\ H^{pk}(Y^p) & \xrightarrow{j^*} & H^{pk}(Y^p - \Delta) & \xrightarrow{\mathcal{T}} & H^{pk}([Y^p - \Delta]^*) \end{array} \tag{4.3}$$

where the horizontal sequences are not necessarily exacts, but the composition $\mathcal{T} \circ j^*$ is surjective. Therefore, as F^* is zero by Lemma 4.1, it follows from commutativity of the diagram (4.3) that $\bar{F}_0^*$ is zero. □

Let $h : X^* \to B\mathbb{Z}_p$ be a classifying map for the principal $\mathbb{Z}_p$-bundle $X \to X^*$. Then the compositions

$$h \circ i_1 : A_f^* \to B\mathbb{Z}_p \text{ and } h \circ i_2 : [X - A_f]^* \to B\mathbb{Z}_p$$

are classifying maps for the following principal $\mathbb{Z}_p$-bundles $A_f \to A_f^*$ and $X - A_f \to [X - A_f]^*$ respectively, where the maps $i_1 : A_f^* \to X^*$ and $i_2 : [X - A_f]^* \to X^*$ are induced by the inclusions between the orbit spaces.

Let us consider $G : Y^p - \Delta \to B\mathbb{Z}_p$ a classifying map for the principal $\mathbb{Z}_p$-bundle $Y^p - \Delta \to [Y^p - \Delta]^*$. Since $F_0 : X - A_f \to Y^p - \Delta$ is a $\mathbb{Z}_p$-equivariant map, one the has that

$$G \circ \bar{F}_0 : [X - A_f]^* \to B\mathbb{Z}_p$$

also classifies the principal $\mathbb{Z}_p$-bundle $X - A_f \to [X - A_f]^*$. In this way,

$$i_2^* \circ h^* = \bar{F}_0^* \circ G^* : H^*(B\mathbb{Z}_p) \to H^*([X - A_f]^*). \tag{4.4}$$

To conclude that the $\mathbb{Z}_p$-index of A_f is greater than or equal to $n-pk$, it suffices to show that $i_1^* \circ h^*(\mu) \neq 0$, where μ is the generator of $H^{n-pk}(B\mathbb{Z}_p)$.

We first consider the case when k is odd. Let us observe that n must be necessarily odd, since $p > 2$ is a prime. Then, $n - pk$ is even and it follows from Remark 2.5 that $\mu = b^{(n-pk)/2} \in H^{n-pk}(B\mathbb{Z}_p)$. Suppose that $i_1^* \circ h^*(\mu) = 0$. From continuity of the cohomology, there exists a neighborhood V of A_f in X which is invariant by the action of $\mathbb{Z}_p$ and such that $i_1^* \circ h^*(\mu) = 0$ in $H^{n-pk}(V^*)$. From the exact cohomology sequence of the pair (X^*, V^*) one has

$$h^*(\mu) \in \operatorname{Im}\,[H^{n-pk}(X^*, V^*) \to H^{n-pk}(X^*)]. \tag{4.5}$$

Since pk is odd from Remark 2.5 $\eta = a \smile b^{(pk-1)/2}$ is a generator of $H^{pk}(B\mathbb{Z}_p)$. It follows from Lemma 4.2 and (4.4) that

$$i_2^* \circ h^*(\eta) = \bar{F}_0^* \circ G^*(\eta) = 0 \in H^{pk}([X - A_f]^*)$$

and from the exact cohomology sequence of the pair $(X^*, [X - A_f]^*)$ one has

$$h^*(\eta) \in \operatorname{Im}\,[H^{pk}(X^*, [X - A_f]^*) \to H^{pk}(X^*)]. \tag{4.6}$$

Thus from (4.5), (4.6) and by the naturality of the cup product we have

$$h^*(\eta \smile \mu) = h^*(\eta) \smile h^*(\mu) \in \operatorname{Im}\,[H^n(X^*, [X - A_f]^* \cup V^*) \to H^n(X^*)].$$

Let us note that the element

$$\eta \smile \mu = a \smile b^{(pk-1)/2} \smile b^{(n-pk)/2} = a \smile b^{(n-1)/2}$$

is a generator of $H^n(B\mathbb{Z}_p)$. Furthermore,

$$H^n(X^*, [X - A_f]^* \cup V^*) = H^n(X^*, X^*) = 0$$

and then $h^*(\eta \smile \mu) = 0 \in H^n(X^*)$, that is, $h^* : H^n(B\mathbb{Z}_p) \to H^n(X^*)$ is trivial which contradicts the hypothesis that the $\mathbb{Z}_p$-index of X is greater than or equal to n.

If k is even, then $n - pk$ is odd and pk is even. In this case, the proof is analogous to the previous case, considering now the generators

$$\mu = a \smile b^{(n-pk-1)/2} \in H^{n-pk}(B\mathbb{Z}_p) \text{ and } \eta = b^{(pk)/2} \in H^{pk}(B\mathbb{Z}_p).$$

Let us examine the case where $G = \mathbb{Z}_2$. Here, n can be any positive integer and the generator of $H^{n-2k}(B\mathbb{Z}_2)$ is $\mu = a^{n-2k}$. To show that the $\mathbb{Z}_2$-index of A_f is greater than or equal to $n - 2k$, it suffices to prove that $i_1^* \circ h^*(\mu) \neq 0$. Let us assume that $i_1^* \circ h^*(\mu) = 0$. Then there exists a neighborhood V of A_f in

X which is invariant with respect to the action and such that $i_1^* \circ h^*(\mu) = 0$ in $H^{n-2k}(V^*)$. From exact cohomology sequence of the pair (X^*, V^*) one has that

$$h^*(\mu) \in \text{Im}\,[H^{n-2k}(X^*, V^*) \to H^{n-2k}(X^*)]. \tag{4.7}$$

On the other hand, $\eta = a^{2k}$ is the generator of $H^{2k}(B\mathbb{Z}_2)$ and it follows from Lemma 4.2 and (4.4) that

$$i_2^* \circ h^*(\eta) = \bar{F}_0^* \circ G^*(\eta) = 0 \in H^{2k}([X - A_f]^*).$$

Moreover from exact cohomology sequence of $(X^*, [X - A_f]^*)$ one has that

$$h^*(\eta) \in \text{Im}\,[H^{2k}(X^*, [X - A_f]^*) \to H^{2k}(X^*)]. \tag{4.8}$$

Thus, from (4.7), (4.8) and by the naturality of the cup product we have

$$h^*(\eta \smile \mu) = h^*(\eta) \smile h^*(\mu) \in \text{Im}\,[H^n(X^*, [X - A_f]^* \cup V^*) \to H^n(X^*)].$$

Let us observe that $\eta \smile \mu = a^{2k} \smile a^{n-2k} = a^n$ is the generator of $H^n(B\mathbb{Z}_2)$. Furthermore, $H^n(X^*, [X - A_f]^* \cup V^*) = H^n(X^*, X^*)$ is trivial and then $h^*(\eta \smile \mu) = 0 \in H^n(X^*)$ which contradicts the hypothesis that the $\mathbb{Z}_2$-index of X is greater than or equal to n. This concludes the proof of Theorem 1.2 in the case $G = H = \mathbb{Z}_p$.

For the general case, suppose that G is a finite group which acts freely on X and let $H \subset G$ be a normal cyclic subgroup of prime order p. We denote by $s = |G|/p$, the number of the left cosets of G/H and let $a_1, \ldots, a_s$ be a set of representatives of the cosets. We define the map $F : X \to Y^s$ by

$$F(x) = (f(a_1 x), \ldots, f(a_i x), \ldots, f(a_s x)).$$

We need to show that

$$A(f, H, G) = A_F = \{x \in X; F(x) = F(hx), \forall h \in H\}. \tag{4.9}$$

Let x be a point in the set $A(f, H, G)$, then f collapses each orbit determined by the action of H on $a_i x$ to a single point, for each $i = 1, \ldots, s$. If $h \in H$

$$F(hx) = (f(ha_1 x), \ldots, f(ha_i x), \ldots, f(ha_s x)).$$

Since H is a normal subgroup of G, $ha_i x = a_i \hat{h} x$. Furthermore, $a_i x$ and $a_i \hat{h} x = ha_i x$ belongs to the same H-orbit and $f(a_i x) = f(ha_i x)$, for each $i = 1, \ldots, s$ which implies that $F(x) = F(hx)$. Therefore $x \in A_F$. The proof of the another inclusion is entirely analogous.

To conclude, let us observe that $H \cong \mathbb{Z}_p$ acts freely on X by restriction and by hypothesis the $\mathbb{Z}_p$-index of X is greater than or equal to n. By using Lemma 4.1 for the map $F : X \to Y^s$ defined in (4.9) one has that $F^* : H^q(Y^s) \to H^q(X)$ is trivial for any $q \geq 1$. Since dimension of Y^s is ks and Theorem 1.2 is true for $G = \mathbb{Z}_p$ we can conclude that the $\mathbb{Z}_p$-index of $A_F = A(f, H, G)$ is greater than or equal $n - p(ks) = n - pk(|G|/p) = n - |G|k$ and this completes the proof. □

Proof of Theorem 1.1. Let $\tilde{f} : X_\varphi \to Y$ given by $\tilde{f}(x_1, \ldots, x_r) = f(x_1)$, that is $\tilde{f} = f \circ \pi_1$ where π_1 is the natural projection on the 1-th coordinate. By hypothesis f induces the zero homomorphism in each dimension, then we have that $\tilde{f}^* : H^i(Y) \to H^i(X_\varphi)$ is trivial for any $i \geq 1$. Moreover, the $\mathbb{Z}_p$-index of X_φ is equal to n by Theorem 3.1. In this way, X_φ and $\tilde{f}$ satisfy the hypotheses of Theorem 1.2 which implies that the $\mathbb{Z}_p$-index of the set $A(\tilde{f}, H, G)$ is greater than or equal to $n - |G|k$. By Definition 2.1 $A_\varphi(f, H, G) = A(\tilde{f}, H, G)$, and then cohom.dim $A_\varphi(f, H, G) \geq n - |G|k$. □

Remark 4.3. In the particular case that $G = H = \mathbb{Z}_p$ with p prime, Volovikov in [13, Theorem 3.2] proved a version of Theorem1.1.

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