ON MULTIPLE SOLUTIONS FOR A SINGULAR QUASILINEAR ELLIPTIC SYSTEM INVOLVING CRITICAL HARDY-SOBOLEV EXPONENTS

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Abstract. This paper is concerned with the existence of nontrivial solutions for a class of degenerate quasilinear elliptic systems involving critical Hardy-Sobolev type exponents. The lack of compactness is overcame by using the Brezis-Nirenberg approach, and the multiplicity result is obtained by combining a version of the Ekeland's variational principle due to Mizoguchi with the Ambrosetti-Rabinowitz mountain pass theorem.

1. Introduction

In this paper, we will combine a version of the well-known mountain pass theorem due to Ambrosetti and Rabinowitz [5] with the Ekeland's variational principle [14] to establish conditions for the existence of nontrivial solutions of the quasilinear elliptic system with nonlinear perturbation involving critical Hardy-Sobolev exponents of the type

(1)
$$\begin{cases} -L_{a,p}u &=& \theta|x|^{-c_1p^*}h\,|u|^{\theta-2}|v|^{\delta}u+\mu\alpha|x|^{-c_1p^*}|u|^{\alpha-2}|v|^{\gamma}u & \text{in} \quad \mathbb{R}^N, \\ -L_{b,q}v &=& \delta|x|^{-c_2q^*}h|u|^{\theta}|v|^{\delta-2}v+\mu\gamma|x|^{-c_2q^*}|u|^{\alpha}|v|^{\gamma-2}v & \text{in} \quad \mathbb{R}^N, \\ u,v &\geq& 0 \text{ and } u,v\not\equiv 0, & \text{in} \quad \mathbb{R}^N, \end{cases}$$

where $L_{e,r}w \equiv div(|x|^{-er}|\nabla w|^{r-2}\nabla w)$ and the exponents verifies

$$\begin{gathered} 1<p,q<N,\ -\infty<a<(N-p)/p,\ -\infty<b<(N-q)/q, \\ a\leq c_1<a+1,\ b\leq c_2<b+1,\ d_1=1+a-c_1,\ d_2=1+b-c_2, \\ p^*=Np/(N-d_1p),\ q^*=Nq/(N-d_2q),\ c_1p^*=c_2q^*, \\ \alpha,\gamma,\theta,\delta>1 \text{ and } \mu>0, \end{gathered} \tag{H_{exp}}$$

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with

$$\frac{\theta}{p} + \frac{\delta}{q} < 1 \text{ and } \frac{\alpha}{p^*} + \frac{\gamma}{q^*} = 1, \quad (p, q\text{−sublinear/critical}) \tag{2}$$

and

$$h \in L^{k'}(\mathbb{R}^N, |x|^{-c_1 p^*}),\ h \geq 0, \tag{3}$$

where $p_1 \in (1, p)$, $q_1 \in (1, q)$, and $k, k' > 1$ are taken such that $kp_1 = p^*$, $kq_1 = q^*$ and $\theta/p_1 + \delta/q_1 = 1/k' + 1/k = 1$.

The study of the elliptic systems has been motivated by a great number of applications, for instance, in fluid mechanics, in newtonian fluids, in flow through porous media, reaction-diffusion problems, nonlinear elasticity, petroleum extraction, astronomy, glaciology, etc, see [15] and references cited there.

Technically, the variational systems behave in a certain sense like in the scalar case. However, the study of systems has additional difficulty, mainly, coming from the mutual action of the variables u and v, for instance, the possibility of the existence of semisolutions, in other words, the existence of weak solutions of the type $(u, 0)$ and $(0, v)$, see [21, 29]. Besides this fact, in our work, there exists another difficulty due to the lack of compactness of the embedding $W_0^{1,p}(\Omega, |x|^{-ap}) \hookrightarrow L^{p^*}(\Omega, |x|^{-c_1 p^*})$, where $\Omega \subset \mathbb{R}^N$ is a smooth domain. In the next section, the precise definitions of these spaces as well as some properties of them will be established.

A pioneer work on problems involving critical Sobolev exponents is the celebrated paper of Brezis and Nirenberg [9], in which the authors proved, among others, that problem

$$\begin{cases} -\Delta u = u^{2^*-1} + \mu u^{r-1} \text{ in } \Omega, \\ u > 0 \text{ in } \Omega \text{ and } u = 0 \text{ on } \partial\Omega \end{cases}$$

possesses a solution for each $\mu > 0$, where Ω is a bounded smooth domain of $\mathbb{R}^N$ $(N \geq 3)$, $2 < r < 2^*$. Here $2^* = 2N/(N-2)$ is the critical Sobolev exponent of the embedding $H_1^0(\Omega)$ into $L^{2^*}(\Omega)$. From this paper, many authors have widely studied this type of the problems with much greater generalities. See e.g. [3, 17, 24]. The above problem involving the p-laplacian operator, that is, the operator given by $\Delta_p u \equiv div(|\nabla u|^{p-2}\nabla u)$, $1 < p < N$, we would like to cite, e.g., [7, 12, 16, 19, 27, 33] and see also references therein. We recall that in this case the critical Sobolev exponent is $p^* = pN/(N-p)$. Now, elliptic equations involving more general operator, such as the degenerate quasilinear operator given by $L_{a,p}u \equiv div(|x|^{-ap}|\nabla u|^{p-2}\nabla u)$, we refer the papers by [11, 13, 30, 31, 32] and references cited there. Here the best Sobolev (Hardy) exponent, of the embedding

$W_0^{1,p}(\Omega,|x|^{-ap}) \hookrightarrow L^{p^*}(\Omega,|x|^{-c_1p^*})$, is defined by $p^* = Np/(N-(1+a-c_1)p)$. The lack of compactness of the above mentioned embedding is overcame by analyzing the critical level $(1/N)S^{N/p}$, where S is the best Sobolev constant, which is given by

$$S = \inf_{u\in W_0^{1,p}(\Omega)\setminus\{0\}} \left\{ \frac{\int_\Omega |\nabla u|^p dx}{\left(\int_\Omega |u|^{p^*} dx\right)^{\frac{p}{p^*}}} \right\}.$$

While, for more general operator the critical level is motivated by the best Hardy-Sobolev constant characterized by

$$C^*_{a,p} = C^*_{a,p}(\Omega) = \inf_{u\in W_0^{1,p}(\Omega,|x|^{-ap})\setminus\{0\}} \left\{ \frac{\int_\Omega |x|^{-ap}|\nabla u|^p dx}{\left(\int_\Omega |x|^{-c_1p^*}|u|^{p^*} dx\right)^{\frac{p}{p^*}}} \right\}.$$

When we are working with a system involving Δ_p and Δ_q operators is hard to find a well appropriated critical level, mainly, when $p \neq q$. Actually, Adriouch and Hamidi in [1] pointed out that this is a open question. But, recently, Silva and Xavier in [28] were able to treat, in a certain context, the case $p \neq q$. For the particular case $p = q$, Morais and Souto in [26] defined the following critical level number S_H/p, where

$$S_H = \inf_{W\setminus\{0\}} \left\{ \frac{\int_\Omega (|\nabla u|^p + |\nabla v|^p dx}{\left(\int_\Omega H(u,v)dx\right)^{\frac{p}{p^*}}} \right\},$$

$W = W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega)$ and H is homogeneous nonlinearity of degrees p^*.

Our first result deal with the case $p \neq q$.

Theorem 1.1. *Suppose that* (H_{exp}), (2), *and* (3), *hold true. Assume also that* $\max\{p,q\} < \min\{p^*,q^*\}$. *Then, for each* $\mu > 0$, *there exists* $\lambda_0 = \lambda_0(\mu) > 0$ *such that system* (1) *possesses a weak solution, where each component is nontrivial and nonnegative, for each* $h \in L^{k'}(\mathbb{R}^N,|x|^{-c_1p^*})$ *with* $0 < ||h||_{L^{k'}(\mathbb{R}^N,|x|^{-c_1p^*})} < \lambda_0$.

Before enunciating the second result, we will introduce the following definition:

Definition 1. Consider $1 < p < N$, $-\infty < a < (N-p)/p$, $a \le c_1 < a+1$, $d_1 = 1+a-c_1$, $p^* = Np/(N-d_1p)$, k, k', $p_1 = q_1 > 1$ as in (3). Fixing c_0 and R_0 positive constants, we define the set $\mathbb{E}$ as the subset of $L^{k'}(\mathbb{R}^N,|x|^{-c_1p^*})$ given by nonnegative functions $h \in L^{k'}(\mathbb{R}^N,|x|^{-c_1p^*})$ satisfying

$$\left(1 + R^{\frac{d_1p(N-p-ap)}{(p-1)(N-d_1p)}}\right)^{\frac{-(N-d_1p)p_1}{d_1p}} R^{N-c_1p^*} \inf_{B(0,2R)} h \ge c_0, \text{ for some } R \in (0,R_0).$$

Also, for $\lambda > 0$, consider $\mathbb{E}_\lambda$ the subset of $\mathbb{E}$ given by

$$\mathbb{E}_\lambda = \left\{ h \in \mathbb{E} : ||h||_{L^{k'}(\mathbb{R}^N,|x|^{-c_1p^*})} < \lambda \right\}.$$

Theorem 1.2. *In addition to* (H_{exp}), (2), *and* (3), *assume that* $p = q$, $a = b \geq 0$, $p_1 = q_1$. *Then, for each* $\mu > 0$, *there exists* $\lambda_0 = \lambda_0(\mu) > 0$ *such that system* (1) *possesses at least two weak solutions, where each component is nontrivial and nonnegative, for each* $h \in \mathbb{E}_\lambda$, *provided that* $0 < \lambda < \lambda_0$.

2. Preliminaries

We will set some spaces and their norms. Consider Ω a smooth open domain, not necessarily bounded, of $\mathbb{R}^N$ with $0 \in \Omega$. If $\alpha \in \mathbb{R}$ and $l \geq 1$, let $L^l(\Omega, |x|^\alpha)$ be the subspace of $L^l(\Omega)$ of the Lebesgue measurable functions $u : \mathbb{R}^N \to \mathbb{R}$ satisfying

$$||u||_{L^l(\Omega,|x|^\alpha)} := \left(\int_\Omega |x|^\alpha |u|^l dx\right)^{\frac{1}{l}} < \infty.$$

If $1 < p < N$ and $-\infty < a < (N-p)/p$, we define $W^{1,p}(\Omega, |x|^{-ap})$ (resp. $W_0^{1,p}(\Omega, |x|^{-ap})$) as being the completion of $C^\infty(\Omega)$ (resp. $C_0^\infty(\Omega)$) with respect to the norm $|| \cdot ||$ defined by

$$||u|| := \left(\int_\Omega |x|^{-ap} |\nabla u|^p dx\right)^{\frac{1}{p}}.$$

First of all, from the Caffarelli, Kohn and Nirenberg inequality (see [10]) and the standard approximation argument, it is easy to see that there exists $C = C_{a,c_1} > 0$ such that

$$\left(\int_\Omega |x|^{-c_1 p^*} |u|^{p^*} dx\right)^{\frac{p}{p^*}} \leq C \left(\int_\Omega |x|^{-ap} |\nabla u|^p dx\right), \forall\, u \in W_0^{1,p}(\Omega, |x|^{-ap}).$$

The proof of the next lemma is completely similar to the proof of [4, Theorem 5] (see [26, Lemma 3] for $p \neq 2$).

Lemma 2.1. *Suppose that* Ω *is a smooth open domain, not necessarily bounded, of* $\mathbb{R}^N$ *with* $0 \in \Omega$, $1 < p < N$, $-\infty < a < (N-p)/p$, $a \leq c_1 < a+1$, $d_1 = 1+a-c_1$, $p^* = Np/(N-d_1 p)$, *and* $\alpha + \gamma = p^*$, *then*

$$\tilde{S} := \inf_{(u,v)\in \tilde{W}} \left\{ \frac{\int_\Omega |x|^{-ap} \left(|\nabla u|^p + |\nabla v|^p\right) dx}{\left(\int_\Omega |x|^{-c_1 p^*} |u|^\alpha |v|^\gamma dx\right)^{\frac{p}{p^*}}} \right\},$$

where

$$\tilde{W} = \left\{ (u,v) \in \left(W_0^{1,p}(\Omega, |x|^{-ap})\right)^2 : |u||v| \not\equiv 0 \right\},$$

satisfies

$$\tilde{S} = \left[(\alpha/\gamma)^{\gamma/p^*} + (\alpha/\gamma)^{-\alpha/p^*}\right] C^*_{a,p}.$$

Furthermore, if $C^*_{a,p}$ is achieved by $w_0 \in W^{1,p}_0(\Omega, |x|^{-ap})$, then $\tilde{S}$ is achieved by (sw_0, tw_0), for all $s, t > 0$ satisfying $s/t = (\alpha/\gamma)^{1/p}$.

We define the space

$$W^{1,p}_{a,c_1}(\Omega) = \left\{ u \in L^{p^*}(\Omega, |x|^{-c_1 p^*}) : |\nabla u| \in L^p(\Omega, |x|^{-ap}) \right\},$$

equipped with the norm

$$||u||_{W^{1,p}_{a,c_1}(\Omega)} = ||u||_{L^{p^*}(\Omega,|x|^{-c_1p^*})} + ||\nabla u||_{L^p(\Omega,|x|^{-ap})}.$$

We consider the best Hardy-Sobolev constant given by

$$\tilde{S}_{a,p} = \inf_{W^{1,p}_{a,c_1}(\mathbb{R}^N)\backslash\{0\}} \left\{ \frac{\int_{\mathbb{R}^N} |x|^{-ap}|\nabla u|^p dx}{\left(\int_{\mathbb{R}^N} |x|^{-c_1p^*}|u|^{p^*}dx\right)^{\frac{p}{p^*}}} \right\}.$$

Also, we define

$$R^{1,p}_{a,c_1}(\Omega) = \left\{ u \in W^{1,p}_{a,c_1}(\Omega) : u(x) = u(|x|) \right\},$$

endowed with the norm

$$||u||_{R^{1,p}_{a,c_1}(\Omega)} = ||u||_{W^{1,p}_{a,c_1}(\Omega)}.$$

Actually, Horiuchi in [22] proved that, if $a \geq 0$,

$$\tilde{S}_{a,p,R} = \inf_{R^{1,p}_{a,c_1}(\mathbb{R}^N)\backslash\{0\}} \left\{ \frac{\int_{\mathbb{R}^N} |x|^{-ap}|\nabla u|^p dx}{\left(\int_{\mathbb{R}^N} |x|^{-c_1p^*}|u|^{p^*}dx\right)^{\frac{p}{p^*}}} \right\} = \tilde{S}_{a,p}. \tag{4}$$

and it is achieved by functions of the form

$$y_\epsilon(x) := k_{a,p}(\epsilon) U_{a,p,\epsilon}(x), \forall\, \epsilon > 0,$$

where

$$k_{a,p}(\epsilon) = \tilde{c}\epsilon^{(N-d_1p)/d_1p^2} \text{ and } U_{a,p,\epsilon}(x) = \left(\epsilon + |x|^{\frac{d_1p(N-p-ap)}{(p-1)(N-d_1p)}}\right)^{-\left(\frac{N-d_1p}{d_1p}\right)}.$$

Moreover, y_ϵ satisfies

$$\int_{\mathbb{R}^N} |x|^{-ap}|\nabla y_\epsilon|^p dx = \int_{\mathbb{R}^N} |x|^{-c_1p^*}|y_\epsilon|^{p^*}dx. \tag{5}$$

See also Clément, Figueiredo and Mitidieri [13, Proposition 1.4].

The proof of the next lemma can be proved arguing as in [9], more exactly, see [31, Lemma 5.1].

Lemma 2.2. *Consider Ω a smooth open domain, not necessarily bounded, of $\mathbb{R}^N$, $0 \in \Omega$, $1 < p < N$, $-\infty < a < (N-p)/p$, $a \le c_1 < a+1$, $d_1 = 1 + a - c_1$, $p^* = Np/(N - d_1 p)$, and $l, k, k' > 1$, where k and k' are conjugated exponents with $kl = p^*$. Let R_0, c_0 be positive constants as in definition 1 and $\psi \in C_0^\infty(B(0, 3R_0))$ with $\psi \ge 0$ for a.e. in $B(0, 3R_0)$ and $\psi \equiv 1$ for a.e. in $B(0, 2R_0)$, then the function given by*

$$u_\epsilon(x) = \frac{\psi(x) U_{a,p,\epsilon}(x)}{||\psi U_{a,p,\epsilon}||_{L^{p^*}(\Omega, |x|^{-c_1 p^*})}}$$

satisfies

$$||u_\epsilon||^{p^*}_{L^{p^*}(\Omega,|x|^{-c_1 p^*})} = 1,\ ||\nabla u_\epsilon||^p_{L^p(\Omega,|x|^{-ap})} \le \tilde{S}_{a,p,R} + O(\epsilon^{(N-d_1 p)/d_1 p}), \tag{6}$$

and

$$||h^{1/l}\, u_\epsilon||^l_{L^l(\Omega,|x|^{-c_1 p^*})} \ge \begin{cases} O(\epsilon^{(N-d_1 p)l/d_1 p^2}) \text{ if } l < \frac{(N-c_1 p^*)(p-1)}{N-p-ap}, \\ O(\epsilon^{(N-d_1 p)l/d_1 p^2} |ln(\epsilon)|) \text{ if } l = \frac{(N-c_1 p^*)(p-1)}{N-p-ap}, \\ O(\epsilon^{\frac{(N-d_1 p)(p-1)[(N-c_1 p^*)p - (N-p-ap)l]}{d_1 p^2 (N-p-ap)}}) \\ \text{if } l > \frac{(N-c_1 p^*)(p-1)}{N-p-ap}, \end{cases} \tag{7}$$

for all $h \in L^{k'}(\Omega, |x|^{-c_1 p^*})$ with $\inf_{B(0,2R)} h > 0$ for some $0 < R < R_0$ and $h \ge 0$. Moreover, the inequality (7) is uniform in $h \in L^{k'}(\Omega, |x|^{-c_1 p^*})$, $h \ge 0$ for a.e. in Ω, satisfying

$$\frac{R^{N-c_1 p^*}}{(1 + R^{d_1 p(N-a-ap)/(p-1)(N-d_1 p)})^{(N-d_1 p)l/d_1 p}} \inf_{B(0,2R)} h \ge c_0, \tag{8}$$

for some $0 < R < R_0$.

Definition 2. Let us consider $\{(u_n, v_n)\}$ a sequence in $W_0^{1,p}(\mathbb{R}^N, |x|^{-ap}) \times W_0^{1,q}(\mathbb{R}^N, |x|^{-bq})$. We say that the sequence $\{(u_n, v_n)\}$ is a *Palais Smale* sequence for the operator I at the level c (or simply, $(PS)_c$-sequence) if

$$I(u_n, v_n) \longrightarrow c \text{ and } I'(u_n, v_n) \longrightarrow 0, \text{ as } n \to \infty.$$

Our approach will be variational techniques, that is, we have to find the critical points of the Euler-Lagrange functional $I : W_0^{1,p}(\mathbb{R}^N, |x|^{-ap}) \times W_0^{1,q}(\mathbb{R}^N, |x|^{-bq}) \to \mathbb{R}$ given by

$$\begin{aligned} I(u,v) &= \frac{1}{p}\int_{\mathbb{R}^N} |x|^{-ap} |\nabla u|^p dx + \frac{1}{q}\int_{\mathbb{R}^N} |x|^{-bq} |\nabla v|^q dx \\ &\quad - \int_{\mathbb{R}^N} |x|^{-c_1 p^*} h\, u_+^\theta v_+^\delta dx - \mu \int_{\mathbb{R}^N} |x|^{-c_1 p^*} u_+^\alpha v_+^\gamma dx; \end{aligned}$$

which is well defined and is of class C^1, with Gâteaux derivative given by

$$\begin{aligned}\langle I(u,v),(w,z)\rangle &= \int_{\mathbb{R}^N}|x|^{-ap}|\nabla u|^{p-2}\nabla u\nabla w dx+\int_{\mathbb{R}^N}|x|^{-bq}|\nabla v|^{q-2}\nabla v\nabla z dx\\ &\quad -\theta\int_{\mathbb{R}^N}|x|^{-c_1p^*}h\, u_+^{\theta-1}v_+^{\delta}w dx-\delta\int_{\mathbb{R}^N}|x|^{-c_1p^*}h\, u_+^{\theta}v_+^{\delta-1}z dx\\ &\quad -\mu\alpha\int_{\mathbb{R}^N}|x|^{-c_1p^*}u_+^{\alpha-1}v_+^{\gamma}w dx-\mu\gamma\int_{\mathbb{R}^N}|x|^{-c_1p^*}u_+^{\alpha}v_+^{\gamma-1}z dx,\end{aligned}$$

where $u_\pm = \max\{0,\pm u\} \in W_0^{1,p}(\mathbb{R}^N,|x|^{-ap})$ (Similarly $v_\pm = \max\{0,\pm v\} \in W_0^{1,q}(\mathbb{R}^N,|x|^{-bq})$) (see [2]).

Lemma 2.3. *Let be $\lambda_0>0$ and assume that (H_{exp}) and (3) hold. If $\{(u_n,v_n)\}$ is a Palais Smale sequence in $W_0^{1,p}(\mathbb{R}^N,|x|^{-ap})\times W_0^{1,q}(\mathbb{R}^N,|x|^{-bq})$ for the operator I at the level c, then $\{(u_{n_+},v_{n_+})\}$ is a bounded Palais Smale sequence for the operator I at the level c, uniformly in $h\in\mathbb{E}_\lambda$ and $\mu>0$ with $0<\lambda<\lambda_0$.*

Proof. By definition of $(PS)_c-$sequence we obtain

$$\begin{aligned}c+||(u_n,v_n)||+O_n(1) &\geq I(u_n,v_n)-\left\langle I'(u_n,v_n),(u_n/p^*,v_n/q^*)\right\rangle\\ &\geq (\tfrac{1}{p}-\tfrac{1}{p^*})||u_n||^p+(\tfrac{1}{q}-\tfrac{1}{q^*})||v_n||^q+(\tfrac{\theta}{q^*}+\tfrac{\delta}{q^*}-1)\\ &\quad\times(\tfrac{\theta C^{p_1/p}}{p_1}+\tfrac{\delta C^{q_1/q}}{q_1})(||u_n||^{p_1}+||v_n||^{q_1})\lambda_0.\end{aligned}$$

Therefore, as $p_1\in(1,p)$ and $q_1\in(1,q)$, it follows that $\{(u_n,v_n)\}$ is bounded in $W_0^{1,p}(\mathbb{R}^N,|x|^{-ap})\times W_0^{1,q}(\mathbb{R}^N,|x|^{-bq})$, uniformly in $h\in\mathbb{E}_\lambda$ and $\mu>0$ with $0<\lambda<\lambda_0$. In particular, we have that $\{(u_{n_-},v_{n_-})\}$ and $\{(u_{n_+},v_{n_+})\}$ are bounded sequences in $W_0^{1,p}(\mathbb{R}^N,|x|^{-ap})\times W_0^{1,q}(\mathbb{R}^N,|x|^{-bq})$, then

$$-||u_{n_-}||^p=\left\langle I'(u_n,v_n),(u_{n_-},0)\right\rangle\longrightarrow 0 \text{ as } n\to\infty \tag{9}$$

and similarly

$$-||v_{n_-}||^q=\left\langle I'(u_n,v_n),(0,v_{n_-})\right\rangle\longrightarrow 0 \text{ as } n\to\infty. \tag{10}$$

Moreover, we get

$$\begin{aligned}I(u_{n_+},v_{n_+}) &= I(u_n,v_n)+||u_{n_-}||^p+||v_{n_-}||^q\\ &= I(u_n,v_n)+O_n(1),\end{aligned}$$

where $O_n(1)\to 0$ as $n\to\infty$. Hence, from (9) and (10), we obtain

$$I(u_{n_+},v_{n_+})\longrightarrow c \text{ as } n\to\infty.$$

Similarly, if $(w,z) \in W_0^{1,p}(\mathbb{R}^N, |x|^{-ap}) \times W_0^{1,q}(\mathbb{R}^N, |x|^{-bq})$, we prove that

$$\left\langle I^{'}(u_{n+}, v_{n+}), (w,z)\right\rangle = \left\langle I^{'}(u_n, v_n), (w,z)\right\rangle + O_n(1),$$

hence $I^{'}(u_{n_+}, u_{n_+}) \to 0$ as $n \to \infty$. □

Lemma 2.4. *Suppose that* (H_{exp}) *and* (3) *hold. Then, for every bounded* $(PS)_c$*−sequence* $\{(u_n, v_n)\}$ *with* $u_n, v_n \geq 0$, *for a.e. in* $\mathbb{R}^N$, *there exists a weak solution* $(u,v) \in W_0^{1,p}(\mathbb{R}^N, |x|^{-ap}) \times W_0^{1,q}(\mathbb{R}^N, |x|^{-bq})$ *of system* (1), *where* $u, v \geq 0$ *for a.e. in* Ω.

Proof. By boundedness of $\{(u_n, v_n)\}$ there exists $(u,v) \in W_0^{1,p}(\mathbb{R}^N, |x|^{-ap}) \times W_0^{1,q}(\mathbb{R}^N, |x|^{-bq})$ such that $u_n \rightharpoonup u$ weakly in $W_0^{1,p}(\mathbb{R}^N, |x|^{-ap})$ and $v_n \rightharpoonup v$ weakly in $W_0^{1,q}(\mathbb{R}^N, |x|^{-bq})$, as $n \to \infty$.

First of all, by the diagonal argument, up to a subsequence if necessary, we can admit that $u_n(x) \to u(x)$ and $v_n(x) \to v(x)$, for a.e. in $\mathbb{R}^N$, as $n \to \infty$. Also, arguing as in [20] (see also, [6, 18]) we can prove that $\nabla u_n(x) \to \nabla u(x)$ and $\nabla v_n(x) \to \nabla v(x)$ for a.e. $x \in \mathbb{R}^N$, as $n \to \infty$.

Again, since $\{(u_n, v_n)\}$ is bounded $W_0^{1,p}(\mathbb{R}^N, |x|^{-ap}) \times W_0^{1,q}(\mathbb{R}^N, |x|^{-bq})$, we obtain that $\{u_n^{(\theta-1)k}\}$ is bounded in $L^{\frac{p^*}{(\theta-1)k}}(\mathbb{R}^N, |x|^{-c_1 p^*})$ and $\{v_n^{\delta k}\}$ is bounded in $L^{\frac{q^*}{\delta k}}(\mathbb{R}^N, |x|^{-c_2 q^*})$, and from (3) we get $(\theta-1)k/p^* + \delta k/q^* + k/p^* = 1$; consequently, by Hölder's inequality we have that $\{u_n^{\theta-1} v_n^{\delta} w\}$ is bounded in $L^k(\mathbb{R}^N, |x|^{-c_1 p^*})$, for each $w \in W_0^{1,p}(\mathbb{R}^N, |x|^{-ap})$. Similarly, we get that $\{u_n^{\theta} v_n^{\delta-1} z\}$ is bounded in $L^k(\mathbb{R}^N, |x|^{-c_2 q^*})$, for each $z \in W_0^{1,q}(\mathbb{R}^N, |x|^{-bq})$. Also, it is easy check that $\{|\nabla u_n|^{p-2}\nabla u_n\}$ is bounded in $\left(L^{\frac{p}{p-1}}(\mathbb{R}^N, |x|^{-ap})\right)^N$ and $\{|\nabla v_n|^{q-2}\nabla v_n\}$ is bounded in $\left(L^{\frac{q}{q-1}}(\mathbb{R}^N, |x|^{-bq})\right)^N$. Moreover, as $\alpha/p^* + \gamma/q^* = 1$ we get

$$\tfrac{\alpha-1}{p^*-1} + \tfrac{\gamma p^*}{q^*(p^*-1)} = \tfrac{\gamma-1}{q^*-1} + \tfrac{\alpha q^*}{p^*(q^*-1)} = 1 \text{ and } \tfrac{p^*-1}{\alpha-1}, \tfrac{q^*-1}{\gamma-1} > 1,$$

and using Hölder's inequality again, we achieve that $\{{u_n}^{\alpha-1}{v_n}^{\gamma}\}$ is bounded in $L^{\frac{p^*}{p^*-1}}(\mathbb{R}^N, |x|^{-c_1 p^*})$ and $\{{u_n}^{\alpha}{v_n}^{\gamma-1}\}$ is bounded in $L^{\frac{q^*}{q^*-1}}(\mathbb{R}^N, |x|^{-c_2 q^*})$.

Therefore, from [23, Lemma 4.8] we obtain the following weakly convergences

$$\nabla u_n \rightharpoonup \nabla u \text{ weakly in } (L^{\frac{p}{p-1}}(\mathbb{R}^N, |x|^{-ap}))^N \text{ as } n \to \infty,$$

$$\nabla v_n \rightharpoonup \nabla v \text{ weakly in } (L^{\frac{q}{q-1}}(\mathbb{R}^N, |x|^{-bq}))^N \text{ as } n \to \infty,$$

$$u_n^{\theta-1} v_n^{\delta} w \rightharpoonup u^{\theta-1} v^{\delta} w \text{ weakly in } L^k(\mathbb{R}^N, |x|^{-c_1 p^*}) \text{ as } n \to \infty,$$

$$u_n^{\theta} v_n^{\delta-1} z \rightharpoonup u^{\theta} v^{\delta-1} z \text{ weakly in } L^k(\mathbb{R}^N, |x|^{-c_1 p^*}) \text{ as } n \to \infty,$$

$$u_n^{\alpha-1} v_n^{\gamma} \rightharpoonup u^{\alpha-1} v^{\gamma} \text{ weakly in } L^{\frac{p^*}{p^*-1}}(\mathbb{R}^N, |x|^{-c_1 p^*}) \text{ as } n \to \infty,$$

$$u_n^{\alpha} v_n^{\gamma-1} \rightharpoonup u^{\alpha} v^{\gamma-1} \text{ weakly in } L^{\frac{q^*}{q^*-1}}(\mathbb{R}^N, |x|^{-c_1 p^*}) \text{ as } n \to \infty,$$

where $(w,z) \in W_0^{1,p}(\mathbb{R}^N, |x|^{-ap}) \times W_0^{1,q}(\mathbb{R}^N, |x|^{-bq})$.

Hence, by the definition of $(PS)_c-$ sequence we conclude

$$\langle I'(u,v),(w,z)\rangle = \lim_{n\to\infty} \langle I'(u_n,v_n),(w,z)\rangle = 0 \text{ as } n \to \infty,$$

for all $(w,z) \in W_0^{1,p}(\mathbb{R}^N, |x|^{-ap}) \times W_0^{1,q}(\mathbb{R}^N, |x|^{-bq})$, that is, (u,v) is a weak solution of system (1). □

Lemma 2.5. *Let (u,v) be the weak solution of system* (1) *above obtained, which is a weak limit of a $(PS)_c$-sequence. Then, each component of (u,v) is nontrivial provided that one of the conditions below hold*
i) $c < 0$.
ii) $p = q$, $a = b$, $p^* = q^*$, *and* $0 < c < (\frac{1}{p} - \frac{1}{p^*})(\mu p^*)^{\frac{-p}{p^*-p}} \tilde{S}^{\frac{p^*}{p^*-p}}$.

Proof. Let $\{(u_n,v_n)\} \subset W_0^{1,p}(\mathbb{R}^N, |x|^{-ap}) \times W_0^{1,q}(\mathbb{R}^N, |x|^{-bq})$ be a bounded $(PS)_c-$sequence, with $u_n, v_n \geq 0$ for a.e. in $\mathbb{R}^N$, such that $u_n \rightharpoonup u$ weakly in $W_0^{1,p}(\mathbb{R}^N, |x|^{-ap})$ and $v_n \rightharpoonup v$ weakly in $W_0^{1,q}(\mathbb{R}^N, |x|^{-bq})$, as $n \to \infty$.

Suppose by contradiction that $u(x) = 0$ for a.e. $x \in \mathbb{R}^N$. As in the proof of Lemma 2.4 we get

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} |x|^{-c_1 p^*} h\, u_n^{\theta} v_n^{\delta} dx = 0.$$

Then, we obtain

$$0 = \lim_{n\to\infty} \left\langle I'(u_n,v_n),(u_n,0)\right\rangle = \lim_{n\to\infty} \left(||u_n||^p - \mu\alpha \int_{\mathbb{R}^N} |x|^{-c_1 p^*} u_n^{\alpha} v_n^{\gamma} dx \right)$$

and

$$0 = \lim_{n\to\infty} \left\langle I'(u_n,v_n),(0,v_n)\right\rangle = \lim_{n\to\infty} \left(||v_n||^q - \mu\gamma \int_{\mathbb{R}^N} |x|^{-c_1 p^*} u_n^{\alpha} v_n^{\gamma} dx \right).$$

We can take $l \geq 0$ such that

$$l = \lim_{n\to\infty} \frac{||u_n||^p}{\alpha} = \lim_{n\to\infty} \frac{||v_n||^q}{\gamma} = \mu \lim_{n\to\infty} \int_{\mathbb{R}^N} |x|^{-c_1 p^*} u_n^{\alpha} v_n^{\gamma} dx. \tag{11}$$

Also, we get

$$c = \lim_{n\to\infty} I(u_n,v_n) = \left(\frac{\alpha}{p} + \frac{\gamma}{q} - 1\right) l \geq 0, \tag{12}$$

so we have (i).

Now, supposing $a = b$, $p = q$, and $p^* = q^*$, then by Lemma 2.1 we see that the constant $\tilde{S}$ is well defined and $\tilde{S} > 0$. If $l = 0$, then $c = 0$, which is an absurd. Thus, we can suppose that $l > 0$, and by the definition of $\tilde{S}$ we achieve

$$\left(\int_{\mathbb{R}^N} |x|^{-c_1 p^*} u_n^{\alpha} v_n^{\gamma} dx\right)^{\frac{p}{p^*}} \tilde{S} \le ||u_n||^p + ||v_n||^p, \forall n.$$

Therefore, by taking the limit in the above inequality and using (11), we have

$$(\tfrac{l}{\mu})^{\frac{p}{p^*}} \tilde{S} \le (\alpha + \gamma) l = p^* l,$$

then

$$l \ge (\mu)^{\frac{-p}{p^*-p}} (p^*)^{\frac{-p^*}{p^*-p}} \tilde{S}^{\frac{p^*}{p^*-p}}. \tag{13}$$

Substituting the equation (13) in (12) we obtain

$$c \ge (\tfrac{1}{p} - \tfrac{1}{p^*})(p^* \mu)^{\frac{-p}{p^*-p}} \tilde{S}^{\frac{p^*}{p^*-p}},$$

hence we conclude (ii). □

Lemma 2.6. *In addition to* (H_{exp}) *and* (3), *assume that* $p = q$, $0 \le a = b < (N-p)/p$, *and* $p^* = q^*$. *Then, all the Palais Smale sequences* $\{(u_n, v_n)\} \subset \left(W_0^{1,p}(\mathbb{R}^N, |x|^{-ap})\right)^2$ *for the operator* I *at the level* c, *with* $u_n, v_n \ge 0$, *is precompact provided that*

$$c < (\tfrac{1}{p} - \tfrac{1}{p^*})(\mu p^*)^{\frac{-p}{p^*-p}} \tilde{S}^{\frac{p^*}{p^*-p}} - K(h), \tag{14}$$

where

$$K(h) = \left[1 - \left(\tfrac{\theta+\delta}{p^*}\right)\right] \lim_{n\to\infty} \int_{\mathbb{R}^N} |x|^{-c_1 p^*} h\, u_n^{\theta} v_n^{\delta} dx.$$

Proof. We have by Lemma 2.3 that $\{(u_n, v_n)\}$ is a bounded sequence in $\left(W_0^{1,p}(\mathbb{R}^N, |x|^{-ap})\right)^2$; consequently, there exists $(u, v) \in \left(W_0^{1,p}(\mathbb{R}^N, |x|^{-ap})\right)^2$ such that $u_n \rightharpoonup u$ and $v_n \rightharpoonup v$ weakly in $W_0^{1,p}(\mathbb{R}^N, |x|^{-ap})$, as $n \to \infty$. By the diagonal argument, up to a subsequence if necessary, we can admit that $u_n(x) \to u(x)$ and $v_n(x) \to v(x)$, for a.e. in $\mathbb{R}^N$, as $n \to \infty$. Also, as in Lemma 2.4 we can suppose that $\nabla u_n(x) \to \nabla u(x)$ and $\nabla v_n(x) \to \nabla v(x)$, for a.e. $x \in \mathbb{R}^N$, as $n \to \infty$.

Define $\tilde{u}_n = u_n - u$ and $\tilde{v}_n = v_n - v$. Then, by the Young's inequaltity, we infer that $\{u_n^{\theta} v_n^{\delta}\}$ is a bounded sequence in $L^k(\mathbb{R}^N, |x|^{-c_1 p^*})$, therefore, by [23, Lemma 4.8] we obtain $u_n^{\theta} v_n^{\delta} \rightharpoonup u^{\theta} v^{\delta}$ weakly in $L^k(\mathbb{R}^N, |x|^{-c_1 p^*})$ as $n \to \infty$. In particular, we have

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} |x|^{-c_1 p^*} h\, u_n^{\theta} v_n^{\delta} dx = \int_{\mathbb{R}^N} |x|^{-c_1 p^*} h\, u^{\theta} v^{\delta} dx, \tag{15}$$

for all function $h \in L^{k'}(\mathbb{R}^N, |x|^{-c_1 p^*})$ with $h \geq 0$.

Now, by a Brezis and Lieb result [8, Theorem 1], we have

i.: $||u_n||^p = ||\tilde{u}_n||^p + ||u||^p + O_n(1)$, as $n \to \infty$.

ii.: $||v_n||^p = ||\tilde{v}_n||^p + ||v||^p + O_n(1)$, as $n \to \infty$.

iii.:

$$\int_{\mathbb{R}^N} |x|^{-c_1 p^*} |u_n|^\alpha |v_n|^\gamma dx - \int_{\mathbb{R}^N} |x|^{-c_1 p^*} |\tilde{u}_n|^\alpha |\tilde{v}_n|^\gamma dx$$
$$= \int_{\mathbb{R}^N} |x|^{-c_1 p^*} |u|^\alpha |v|^\gamma dx + O_n(1), \text{ as } n \to \infty.$$

We recall that the proof of identity **iii.** follows arguing as in [26, Lemma 8].

In virtue of Lemma 2.4 we have that (u, v) is a weak solution of system (1), that is, $\langle I'(u, v), (w, z)\rangle = 0$ for all $(w, z) \in (W_0^{1,p}(\mathbb{R}^N, |x|^{-ap}))^2$. By using (15) and **i**. $-$ **iii**., we get

$$\begin{aligned} &||\tilde{u}_n||^p - \mu\alpha \int_{\mathbb{R}^N} |x|^{-c_1 p^*} |\tilde{u}_n|^\alpha |\tilde{v}_n|^\gamma dx \\ &= ||u_n||^p - ||u||^p - \mu\alpha \left[\int_{\mathbb{R}^N} |x|^{-c_1 p^*} u_n^\alpha v_n^\gamma dx - \int_{\mathbb{R}^N} |x|^{-c_1 p^*} u^\alpha v^\gamma dx \right] + O_n(1) \\ &= \left\langle I'(u_n, v_n), (u_n, 0) \right\rangle - \left\langle I'(u, v), (u, 0) \right\rangle + O_n(1) \\ &= O_n(1), \text{ as } n \to \infty. \end{aligned}$$

Analogously, we obtain

$$||\tilde{v}_n||^p - \mu\gamma \int_{\mathbb{R}^N} |x|^{-c_1 p^*} |\tilde{u}_n|^\alpha |\tilde{v}_n|^\gamma dx = O_n(1).$$

Thus, we can take $l \geq 0$ such that

$$l = \lim_{n\to\infty} \frac{||\tilde{u}_n||^p}{\alpha} = \lim_{n\to\infty} \frac{||\tilde{v}_n||^p}{\gamma} = \mu \lim_{n\to\infty} \int_{\mathbb{R}^N} |x|^{-c_1 p^*} |\tilde{u}_n|^\alpha |\tilde{v}_n|^\gamma dx.$$

If $l = 0$ the result is proved. Suppose by contradiction that $l > 0$. By the definition of $(PS)_c-$sequence we get

$$\begin{aligned} c + O_n(1) &= I(u_n, v_n) - \tfrac{1}{p^*} \langle I'(u_n, v_n), (u_n, v_n)\rangle \\ &= (\tfrac{1}{p} - \tfrac{1}{p^*})(||\tilde{u}_n||^p + ||\tilde{v}_n||^p) + (\tfrac{1}{p} - \tfrac{1}{p^*})(||u||^p + ||v||^p) \\ &\quad + (\tfrac{\theta+\delta}{p^*} - 1) \int_{\mathbb{R}^N} |x|^{-c_1 p^*} h\, u_n^\theta v_n^\delta dx + O_n(1) \\ &\geq (\tfrac{1}{p} - \tfrac{1}{p^*}) p^* l + (\tfrac{\theta+\delta}{p^*} - 1) \int_{\mathbb{R}^N} |x|^{-c_1 p^*} h\, u_n^\theta v_n^\delta dx + O_n(1). \end{aligned} \tag{16}$$

By using the definition of $\tilde{S}$ we achieve

$$\left(\int_{\mathbb{R}^N}|x|^{-c_1p^*}u_n^\alpha v_n^\gamma dx\right)^{\frac{p}{p^*}}\tilde{S}\leq ||u_n||^p+||v_n||^p,\ \forall n.$$

Hence, taking the limit in the above inequality we get

$$\left(\frac{l}{\mu}\right)^{\frac{p}{p^*}}\tilde{S}\leq(\alpha+\gamma)l=p^*l$$

then

$$l\geq(\mu)^{\frac{-p}{p^*-p}}(p^*)^{\frac{-p^*}{p^*-p}}\tilde{S}^{\frac{p^*}{p^*-p}}. \tag{17}$$

Substituting (17) in (16) and taking the limit, we obtain

$$c\geq(\tfrac{1}{p}-\tfrac{1}{p^*})(\mu p^*)^{\frac{-p}{p^*-p}}\tilde{S}^{\frac{p^*}{p^*-p}}-K(h),$$

which contradicts the inequality (14). □

3. Proof of Theorem 1.1

Consider $h\in L^{k'}(\mathbb{R}^N,|x|^{-c_1p^*})\setminus\{0\}$ with $h\geq 0$ for a.e. in $\mathbb{R}^N$ and $(u,v)\in W_0^{1,p}(\mathbb{R}^N,|x|^{-ap})\times W_0^{1,q}(\mathbb{R}^N,|x|^{-bq})$ with $||(u,v)||\leq 1$, then we obtain

(18)

$$\begin{aligned}I(u,v)\geq&\min\{\tfrac{1}{p},\tfrac{1}{q}\}\left(||u||^{\max\{p,q\}}+||v||^{\max\{p,q\}}\right)\\&-\left(\tfrac{\theta}{p_1}C^{p_1/p}+\tfrac{\delta}{q_1}C^{q_1/q}\right)||h||_{L^{k'}(\Omega,|x|^{-c_1p^*})}\left(||u||^{\min\{p_1,q_1\}}+||v||^{\min\{p_1,q_1\}}\right)\\&-\mu\left(\tfrac{\alpha}{p^*}C^{p^*/p}+\tfrac{\gamma}{q^*}C^{q^*/q}\right)\left(||u||^{\min\{p^*,q^*\}}+||v||^{\min\{p^*,q^*\}}\right)\\\geq&||(u,v)||^{\max\{p,q\}}\Big[2^{1-\max\{p,q\}}\min\{\tfrac{1}{p},\tfrac{1}{q}\}\\&-k_1(h)||(u,v)||^{\min\{p_1,q_1\}-\max\{p,q\}}-k_2(\mu)||(u,v)||^{\min\{p^*,q^*\}-\max\{p,q\}},\end{aligned}$$

where

$$k_1(h)=2\left(\tfrac{\theta}{p_1}C^{p_1/p}+\tfrac{\delta}{q_1}C^{q_1/q}\right)||h||_{L^{k'}(\mathbb{R}^N,|x|^{-c_1p^*})}$$

and

$$k_2(\mu)=2\mu\left(\frac{\alpha C^{p^*/p}}{p^*}+\frac{\gamma C^{q^*/q}}{q^*}\right).$$

Let $f_{h,\mu}:(0,+\infty)\longrightarrow\mathbb{R}$ be given by

$$f_{h,\mu}(s)=k_1(h)s^{\min\{p_1,q_1\}-\max\{p,q\}}+k_2(\mu)s^{\min\{p^*,q^*\}-\max\{p,q\}},$$

where $\min\{p_1,q_1\}<\max\{p,q\}<\min\{p^*,q^*\}$. Therefore, the minimum point of $f_{h,\mu}$ is

$$s_{h,\mu}=\left(\frac{(\max\{p,q\}-\min\{p_1,q_1\})k_1(h)}{(\min\{p^*,q^*\}-\max\{p,q\})k_2(\mu)}\right)^{\frac{1}{\min\{p^*,q^*\}-\min\{p_1,q_1\}}},$$

for all $||h||_{L^{k'}(\mathbb{R}^N,|x|^{-c_1p^*})},\mu>0$.

Then, for each $\mu > 0$, we can choose $\lambda_0 = \lambda_0(\mu) > 0$ satisfying for all $h \in L^{k'}(\mathbb{R}^N, |x|^{-c_1 p^*})$ with $0 < ||h||_{L^{k'}(\mathbb{R}^N,|x|^{-c_1 p^*})} < \lambda_0$ the conditions: $0 < s_{h,\mu} < 1$ and $2^{1-\max\{p,q\}} \min\{1/p, 1/q\} - f_{h,\mu}(s_{h,\mu}) > 0$. Consequently, by (18) we have

$$I(u,v) \geq (s_{h,\mu})^{\max\{p,q\}}(2^{1-\max\{p,q\}} \min\{\tfrac{1}{p}, \tfrac{1}{q}\} - f_{h,\mu}(s_{h,\mu})) > 0,$$

for all $(u,v) \in W_0^{1,p}(\mathbb{R}^N, |x|^{-ap}) \times W_0^{1,q}(\mathbb{R}^N, |x|^{-bq})$ with $||(u,v)|| = s_{h,\mu}$, where $0 < ||h||_{L^{k'}(\mathbb{R}^N,|x|^{-c_1 p^*})} < \lambda_0$.

Now, we will adapt an argument used by Mizoguchi in [25, Theorem 1] (see also [3]). The operator $I|_{\overline{B(0,s_{h,\mu})}} : \overline{B(0, s_{h,\mu})} \longrightarrow \mathbb{R}$ is bounded from below and continuous. Evidently

$$\inf_{\partial \overline{B(0,s_{h,\mu})}} I > 0.$$

Let us take $(u_0, v_0) \in W_0^{1,p}(\mathbb{R}^N, |x|^{-ap}) \times W_0^{1,q}(\mathbb{R}^N, |x|^{-bq})$ with $u_0, v_0 \geq 0$ for a.e. in $\mathbb{R}^N$ and $h.u_0.v_0 \not\equiv 0$, then we get $t_0 \in (0,1)$ such that

$$I(t_0^{1/p}u_0, t_0^{1/q}v_0) \leq \left(\tfrac{1}{p}||u_0||^p + \tfrac{1}{q}||v_0||^q\right) t - t^{\frac{\theta}{p}+\frac{\delta}{q}} \int_{\mathbb{R}^N} |x|^{-c_1 p^*} h\, u_0^{\theta} v_0^{\delta} dx < 0$$

and $(t_0^{1/p}u_0, t_0^{1/q}v_0) \in B(0, s_{h,\mu})$. Therefore, we have

$$\inf_{\overline{B(0,s_{h,\mu})}} I \leq \inf_{B(0,s_{h,\mu})} I < 0 < \inf_{\partial \overline{B(0,s_{h,\mu})}} I.$$

Now, for each ξ satisfying

$$0 < \xi < \inf_{\partial \overline{B(0,s_{h,\mu})}} I - \inf_{B(0,s_{h,\mu})} I, \tag{19}$$

follow from definition of the infimun of I in $\overline{B(0, s_{h,\mu})}$ that there exists $(w_{0,\xi}, z_{0,\xi}) \in \overline{B(0, s_{h,\mu})}$ such that

$$I(w_{0,\xi}, z_{0,\xi}) \leq \inf_{\overline{B(0,s_{h,\mu})}} I + \xi. \tag{20}$$

Thus, the Ekeland's variational principle implies that there exists $(u_\xi, v_\xi) \in \overline{B(0, s_{h,\mu})}$ satisfying

$$I(u_\xi, v_\xi) \leq I(w_{0,\xi}, z_{0,\xi}) \tag{21}$$

and

$$I(u_\xi, v_\xi) < I(w,z) + \xi||(w,z) - (u_\xi, v_\xi)||, \ (w,z) \neq (u_\xi, v_\xi). \tag{22}$$

In particular, by combining (19), (20), and (21), we obtain

$$I(u_\xi, v_\xi) < \inf_{\partial \overline{B(0,s_{h,\mu})}} I,$$

then $(u_\xi, v_\xi) \in B(0, s_{h,\mu})$.

Let us define the operator $T : \overline{B(0, s_{h,\mu})} \longrightarrow \mathbb{R}$ given by

$$T(w, z) = I(w, z) + \xi||(w, z) - (u_\xi, v_\xi)||,$$

so by (22) follows that (u_ξ, v_ξ) is a strict local minimum of T, therefore

$$\frac{T((u_\xi, v_\xi) + \eta(w, z)) - T(u_\xi, v_\xi)}{\eta} \geq 0$$

for $\eta > 0$ small and (w, z) belongs to unity sphere of $W_0^{1,p}(\mathbb{R}^N, |x|^{-ap}) \times W_0^{1,q}(\mathbb{R}^N, |x|^{-bq})$. Thus, we obtain

$$\frac{I((u_\xi, v_\xi) + \eta(w, z)) - I(u_\xi, v_\xi)}{\eta} + \xi||(w, z)|| \geq 0,$$

and passing to the limit as $\eta \to 0$ it follows that

$$\langle I^{'}(u_\xi, v_\xi), (w, z)\rangle \geq -\xi||(w, z)||,$$

therefore

$$||I^{'}(u_\xi, v_\xi)|| \leq \xi.$$

Hence, we can construct $\{(u_n, v_n)\} \subset W_0^{1,p}(\mathbb{R}^N, |x|^{-ap}) \times W_0^{1,q}(\mathbb{R}^N, |x|^{-bq})$ a $(PS)_c-$sequence with $c := \inf_{B(0,s_{h,\mu})} I$, which, by virtue of Lemma 2.3, can be considered bounded and $u_n, v_n \geq 0$ for a.e. in $\mathbb{R}^N$.

Consequently, by Lemma 2.4 we have that system (1) possesses a weak solution $(u, v) \in W_0^{1,p}(\mathbb{R}^N, |x|^{-ap}) \times W_0^{1,q}(\mathbb{R}^N, |x|^{-bq})$ with $u, v \geq 0$ for a.e. in $\mathbb{R}^N$. Finally, as $\inf_{B(0,s_{h,\mu})} I < 0$, by Lemma 2.5 we have that u, v are nontrivial. □

4. Proof of Theorem 1.2

Analogously to the proof of Theorem 1.1, for each $\mu > 0$, we can find $\lambda_0 > 0$, a $(PS)_c-$sequence $\{(u_n, v_n)\} \subset \left(W_0^{1,p}(\mathbb{R}^N, |x|^{-ap})\right)^2$ with $u_n, v_n \geq 0$ for a.e. in $\mathbb{R}^N$, and $\sigma, \rho \in (0, 1)$ satisfying

$$I(w, z) \geq \sigma \quad \text{if} \quad ||(w, z)|| = \rho, \tag{23}$$

and

$$I(u_n, v_n) \longrightarrow \inf_{B(0,\rho)} I < 0 \text{ and } I^{'}(u_n, v_n) \longrightarrow 0, \text{ as } n \to \infty,$$

for each $h \in \mathbb{E}_\lambda$, provided that $0 < \lambda < \lambda_0$.

From the boundedness of $\{(u_n, v_n)\}$, uniformly in $h \in \mathbb{E}_\lambda$ and $\mu > 0$ with $0 < \lambda < \lambda_0$, we obtain $K(h) \leq M||h||_{L^{k'}(\mathbb{R}^N,|x|^{-c_1 p^*})} \to 0$ as $||h||_{L^{k'}(\mathbb{R}^N,|x|^{-c_1 p^*})} \to 0$, for some $M > 0$. Therefore, changing λ_0 by other smaller, if necessary, we get

$$\inf_{\overline{B(0,\rho)}} I < 0 \leq (\tfrac{1}{p} - \tfrac{1}{p^*})(p^*\mu)^{\frac{-p}{p^*-p}} \tilde{S}^{\frac{p^*}{p^*-p}} - K(h),$$

for all $h \in \mathbb{E}_\lambda$, since that $0 < \lambda < \lambda_0$. Consequently, by Lemma 2.6 there exists a subsequence of $\{(u_n, v_n)\}$, that we will be denoted by $\{(u_n, v_n)\}$, such that $u_n \to u$ and $v_n \to v$ strongly in $W_0^{1,p}(\mathbb{R}^N, |x|^{-ap})$, as $n \to \infty$. Hence, we conclude

$$I(u_n, v_n) \longrightarrow I(u, v) < 0 \text{ and } I^{'}(u_n, v_n) \longrightarrow I^{'}(u, v) \equiv 0, \text{ as } n \to \infty,$$

that is, (u, v) is a weak solution of system (1), where u, v are nonnegatives and $I(u, v) < 0$. Moreover, from Lemma 2.5 we obtain that u, v are nontrivial.

In the next steep, we will prove the existence of the second weak solution.

Consider $(u_0, v_0) \in \left(W_0^{1,p}(\mathbb{R}^N, |x|^{-ap})\right)^2$ with $u_0.v_0 \not\equiv 0$ and $u_0, v_0 \geq 0$ for a.e. in $\mathbb{R}^N$, then we obtain

(24)

$$I(t^{1/p}u_0, t^{1/p}v_0) \leq \tfrac{1}{p}\left(||u_0||^p + ||v_0||^p\right)t - \mu t^{\frac{\alpha+\gamma}{p}} \int_{\mathbb{R}^N} |x|^{-c_1 p^*} u_0^\alpha v_0^\gamma dx \longrightarrow -\infty,$$

as $t \to \infty$.

Thus, by (23) and (24) we have the geometric condition of mountain pass theorem, provided that $h \in \mathbb{E}_\lambda$ with $0 < \lambda < \lambda_0$.

Lemma 4.1. *Let us consider* $s_0 = s_1/(s_1^\alpha t_1^\gamma)^{\frac{1}{p^*}}$ *and* $t_0 = t_1/(s_1^\alpha t_1^\gamma)^{\frac{1}{p^*}}$*, where* $s_1, t_1 > 0$ *and* $s_1/t_1 = (\alpha/\gamma)^{1/p}$*, and* u_ϵ *the function defined in Lemma* 2.2*. Then, there exists* $\epsilon, \eta > 0$ *such that*

$$\sup_{t\geq 0} I(t(s_0u_\epsilon), t(t_0u_\epsilon)) \leq \eta < \left(\frac{1}{p} - \frac{1}{p^*}\right)(\mu p^*)^{\frac{-p}{p^*-p}} \tilde{S}^{\frac{p^*}{p^*-p}},$$

uniformly in $h \in \mathbb{E}_\lambda$ *with* $||h||_{L^{k'}(\mathbb{R}^N, |x|^{-c_1 p^*})} > 0$.

Proof. Due to the geometric conditions of the mountain pass theorem, for each $\epsilon > 0$ there exists $t_\epsilon > 0$ such that

$$0 < \sigma \leq \sup_{t\geq 0} I(t(s_0u_\epsilon), t(t_0u_\epsilon)) = I(t_\epsilon(s_0u_\epsilon), t_\epsilon(t_0u_\epsilon)).$$

Moreover, if we suppose by contradiction that there exists a subsequence $\{t_{\epsilon_n}\}$ such that $t_{\epsilon_n} \to 0$ as $n \to \infty$, we obtain, by using Lemma 2.2, that

$$\begin{aligned} 0 &< \sigma \leq \sup\nolimits_{t\geq 0} I(t(s_0u_{\epsilon_n}), t(t_0u_{\epsilon_n})) \\ &\leq \tfrac{t_{\epsilon_n}^p}{p}(s_0^p + t_0^p)(\tilde{S}_{a,p,R} + O(\epsilon_n^{(N-d_1p)/d_1p})) \longrightarrow 0, \end{aligned}$$

as $n \to \infty$, which is an absurd. Then, we have $l > 0$ with $t_\epsilon \geq l$, for all $\epsilon > 0$. Consequently, by using Lemma 2.2 and putting $c_0 = l^{p_1} s_0^\theta t_0^\delta$, we get

(25)

$$\sup_{t\geq 0} I(t(s_0u_\epsilon), t(t_0u_\epsilon)) \leq \left(\frac{s_1^p + t_1^p}{(s_1^\alpha t_1^\gamma)^{\frac{p}{p^*}}}||u_\epsilon||^p\right)\frac{t_\epsilon^p}{p} - \lambda c_0 \int_{\mathbb{R}^N} |x|^{-c_1 p^*} u_\epsilon^{p_1} dx - \mu t_\epsilon^{p^*}.$$

Notice that

$$t_{1_\epsilon} = (\mu p^*)^{\frac{-1}{p^*-p}} \left(\frac{s_1^p + t_1^p}{(s_1^\alpha t_1^\gamma)^{p/p^*}} \right)^{\frac{1}{p^*-p}} ||u_\epsilon||^{\frac{p}{p^*-p}} \tag{26}$$

is the unique maximum point of $f_\epsilon : (0,\infty) \to \mathbb{R}$, given by

$$f_\epsilon(t) = \frac{(s_1^p + t_1^p)\ t^p}{(s_1^\alpha t_1^\gamma)^{p/p^*} p} ||u_\epsilon||^p - \mu t^{p^*}.$$

It is easy to see that the following inequality holds,

$$(A+B)^k \leq A^k + k(A+B)^{k-1}B, \tag{27}$$

for all $A, B \geq 0$ and $k \geq 1$ (See [24]). Observe that

$$\left[\frac{s_1^p + t_1^p}{(s_1^\alpha t_1^\gamma)^{p/p^*}} \right] = \left[(\alpha/\gamma)^{\gamma/p^*} + (\alpha/\gamma)^{-\alpha/p^*} \right]. \tag{28}$$

By using the Caffarelli-Kohn-Nirenberg's inequality it follows that $W_0^{1,p}(\mathbb{R}^N, |x|^{-ap}) \subset W_{a,c_1}^{1,p}(\mathbb{R}^N)$, then

$$\tilde{S}_{a,p} \leq C_{a,p}^*. \tag{29}$$

Substituting (26) in (25), from (27), (28), and using Lemma 2.2(6), we obtain

$$\begin{aligned} \sup_{t\geq 0} I(t(s_0 u_\epsilon), t(t_0 u_\epsilon)) &\leq (\tfrac{1}{p} - \tfrac{1}{p^*})(\mu p^*)^{\frac{-p}{p^*-p}} \Big\{ [(\tfrac{\alpha}{\gamma})^{\frac{\gamma}{p^*}} + (\tfrac{\alpha}{\gamma})^{\frac{-\alpha}{p^*}}] \tilde{S}_{a,p,R} \\ &\quad + O(\epsilon^{\frac{N-d_1 p}{d_1 p}}) \Big\}^{\frac{p^*}{p^*-p}} - c_0 \int_{\mathbb{R}^N} |x|^{-c_1 p^*} h\, u_\epsilon^{p_1}\, dx \end{aligned}$$

Combining (4) with (29) we get

$$\begin{aligned} \sup_{t\geq 0} I(t(s_0 u_\epsilon), t(t_0 u_\epsilon)) &\leq (\tfrac{1}{p} - \tfrac{1}{p^*})(\mu p^*)^{\frac{-p}{p^*-p}} \Big\{ [(\tfrac{\alpha}{\gamma})^{\frac{\gamma}{p^*}} + (\tfrac{\alpha}{\gamma})^{\frac{-\alpha}{p^*}}] C_{a,p}^* \Big\}^{\frac{p^*}{p^*-p}} \\ &\quad + O(\epsilon^{\frac{N-d_1 p}{d_1 p}}) - c_0 \int_{\mathbb{R}^N} |x|^{-c_1 p^*} h\, u_\epsilon^{p_1}\, dx \end{aligned}$$

Now, from Lemma 2.1, we have

$$\begin{aligned} \sup_{t\geq 0} I(t(s_0 u_\epsilon), t(t_0 u_\epsilon)) &\leq (\tfrac{1}{p} - \tfrac{1}{p^*})(\mu p^*)^{\frac{-p}{p^*-p}} \tilde{S}^{\frac{p^*}{p^*-p}} + O(\epsilon^{\frac{N-d_1 p}{d_1 p}}) \\ &\quad - c_0 \int_{\mathbb{R}^N} |x|^{-c_1 p^*} h\, u_\epsilon^{p_1}\, dx. \end{aligned} \tag{30}$$

Suppose that $p_1 < (N - c_1 p^*)(p-1)/(N - p - ap)$ and observe that

$$\frac{(N - d_1 p) p_1}{d_1 p^2} < \frac{N - d_1 p}{d_1 p}, \tag{31}$$

then, by Lemma 2.2 and (30), we get

$$\begin{aligned}\sup_{t\geq 0} I(t(s_0u_\epsilon), t(t_0u_\epsilon)) &\leq (\tfrac{1}{p}-\tfrac{1}{p^*})(\mu p^*)^{\frac{-p}{p^*-p}}\tilde{S}^{\frac{p^*}{p^*-p}} + O(\epsilon^{(N-d_1p)/d_1p}) \\ &\quad -O(\epsilon^{(N-d_1p)p_1/d_1p^2}) \\ &\leq \eta < (\tfrac{1}{p}-\tfrac{1}{p^*})(\mu p^*)^{\frac{-p}{p^*-p}}\tilde{S}^{\frac{p^*}{p^*-p}},\end{aligned}$$

uniformly in $h \in \mathbb{E}_\lambda$, for some $\eta > 0$ and $\epsilon > 0$ small enough.

If $p_1 = (N - c_1p^*)(p-1)/(N-p-ap)$, by (31), (30), and Lemma 2.2, we can choose $\epsilon > 0$ small enough such that

$$\begin{aligned}\sup_{t\geq 0} I(t(s_0u_\epsilon), t(t_0u_\epsilon)) &\leq (\tfrac{1}{p}-\tfrac{1}{p^*})(\mu p^*)^{\frac{-p}{p^*-p}}\tilde{S}^{\frac{p^*}{p^*-p}} + O(\epsilon^{(N-d_1p)/d_1p}) \\ &\quad -O(\epsilon^{(N-d_1p)p_1/d_1p^2}|\ln(\epsilon)|) \\ &\leq \eta < (\tfrac{1}{p}-\tfrac{1}{p^*})(\mu p^*)^{\frac{-p}{p^*-p}}\tilde{S}^{\frac{p^*}{p^*-p}},\end{aligned}$$

uniformly in $h \in \mathbb{E}_\lambda$ and for some $\eta > 0$.

Assume that $p_1 > (N - c_1p^*)(p-1)/(N-p-ap)$, then we have

$$\tfrac{(N-c_1p^*)(p-1)(N-d_1p)}{d_1p(N-p-ap)} < \tfrac{(N-d_1p)p_1}{d_1p},$$

so, from (31), (30), and Lemma 2.2, there exists $\epsilon > 0$ small enough satisfying

$$\begin{aligned}\sup_{t\geq 0} I(t(s_0u_\epsilon), t(t_0u_\epsilon)) &\leq (\tfrac{1}{p}-\tfrac{1}{p^*})(\mu p^*)^{\frac{-p}{p^*-p}}\tilde{S}^{\frac{p^*}{p^*-p}} + O(\epsilon^{(N-d_1p)/d_1p}) \\ &\quad -O(\epsilon^{\frac{(N-d_1p)(p-1)(N-c_1p^*)}{d_1p(N-p-ap)} - \frac{(N-d_1p)p_1}{d_1p} + \frac{(N-d_1p)p_1}{d_1p^2}}) \\ &\leq \eta < (\tfrac{1}{p}-\tfrac{1}{p^*})(\mu p^*)^{\frac{-p}{p^*-p}}\tilde{S}^{\frac{p^*}{p^*-p}},\end{aligned}$$

uniformly in $h \in \mathbb{E}_\lambda$ and some $\eta > 0$. Thus, we conclude the proof of Lemma 4.1. □

Fix $\epsilon > 0$ as in the above claim.

By equation (24), we obtain $\tilde{t} > 0$ such that

$$I(\tilde{t}(s_0u_\epsilon), \tilde{t}(t_0u_\epsilon)) < 0,$$

uniformly in $h \in \mathbb{E}_\lambda$, for each $0 < \lambda < \lambda_0$.

Applying the mountain pass theorem [5] we get a $(PS)_c$-sequence $\{(w_n, z_n)\}$ in $(W_0^{1,p}(\mathbb{R}^N, |x|^{-ap}))^2$, where

$$0 < \sigma \leq c = \inf_{h\in\Gamma}\max_{t\in[0,1]} I(h(t)) \leq \sup_{0\leq t\leq \tilde{t}} I(t(s_0u_\epsilon), t(t_0u_\epsilon)) \tag{32}$$

and

$$\Gamma = \left\{h \in C([0,1], W_0^{1,p}(\mathbb{R}^N, |x|^{-ap}) \times W_0^{1,q}(\mathbb{R}^N, |x|^{-bq})) : h(0) = 0, h(1) = e\right\}, \tag{33}$$

with $e := (\tilde{t}(s_0 u_\epsilon), \tilde{t}(t_0 u_\epsilon))$.

Thus, from Lemma 2.3 we can assume that $\{(w_n, z_n)\}$ is a bounded sequence and $w_n, z_n \geq 0$ for a.e. in Ω, uniformly in $h \in \mathbb{E}_\lambda$ and $\mu > 0$ with $0 < \lambda < \lambda_0$. Also, changing $\lambda_0 > 0$ by other smaller, if necessary, from Lemma 4.1 we obtain

$$0 < c \leq \eta < (\tfrac{1}{p} - \tfrac{1}{p^*})(\mu p^*)^{\frac{-p}{p^*-p}} \tilde{S}^{\frac{p^*}{p^*-p}} - K(h),$$

uniformly in $h \in \mathbb{E}_\lambda$, if $0 < \lambda < \lambda_0$. Then, by Lemma 2.6 we obtain $(w, z) \in (W_0^{1,p}(\mathbb{R}^N, |x|^{-ap}))^2$ and a subsequence of $\{(w_n, z_n)\}$, that we will denote by $\{(w_n, z_n)\}$, satisfying $w_n \to w$ and $z_n \to z$ strongly in $W_0^{1,p}(\mathbb{R}^N, |x|^{-ap})$, as $n \to \infty$. Hence, we have

$$I(w_n, z_n) \longrightarrow I(w, z) = c \text{ and } I^{'}(w_n, z_n) \longrightarrow I^{'}(w, z) \equiv 0, \text{ as } n \to \infty,$$

that is, (w, z) is a weak solution of system (1) with $w, z \geq 0$ for a.e. in $\mathbb{R}^N$. Moreover, from Lemma 2.5 we obtain that w, z are nontrivial.

Evidently $(u, v) \neq (w, z)$, because $I(u, v) < 0 < I(w, z)$. □

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5. Appendix

Proof of Lemma 2.2. The proof of (6) can be found in [31, Lemma 5.1] (see also [13]). We are going to prove only the inequality (7).

First of all notice that by equation (5) we obtain

$$||\nabla y_\epsilon||^p_{L^p(\mathbb{R}^N,|x|^{-ap})} = (\tilde{S}_{a,p,R})^{p^*/(p^*-p)} = (k_{a,p}(\epsilon))^p ||\nabla U_{a,p,\epsilon}||^p_{L^p(\mathbb{R}^N,|x|^{-ap})}$$

and

$$||y_\epsilon||^{p^*}_{L^{p^*}(\mathbb{R}^N,|x|^{-c_1 p^*})} = (\tilde{S}_{a,p,R})^{p^*/(p^*-p)} = (k_{a,p}(\epsilon))^{p^*} ||U_{a,p,\epsilon}||^{p^*}_{L^{p^*}(\mathbb{R}^N,|x|^{-ap})}.$$

Therefore, we get

$$\begin{aligned} \int_\Omega |x|^{-c_1p^*}|\psi U_{a,p,\epsilon}|^{p^*}dx &= O(1) + \int_{|x|<R_0} |x|^{-c_1p^*}|U_{a,p,\epsilon}|^{p^*}dx \\ &= O(1) + \int_{\mathbb{R}^N} |x|^{-c_1p^*}|U_{a,p,\epsilon}|^{p^*}dx \\ &= O(1) + (\tilde{S}_{a,p,R})^{\frac{p^*}{p^*-p}}(k_{a,p}(\epsilon))^{-p^*}. \end{aligned} \tag{34}$$

We will prove that $||h^{1/l}U_{a,p,\epsilon}||^l_{L^l(\Omega,|x|^{-c_1p^*})}$ verifies the inequalities (7). Considering the change of variables by the polar coordinates we obtain

(35)

$$\begin{aligned} &\int_\Omega |x|^{-c_1p^*} h\,|\psi U_{a,p,\epsilon}|^l dx \\ &\quad \geq \int_{B(0,2R)} |x|^{-c_1p^*} h|\psi U_{a,p,\epsilon}|^l dx \\ &\quad \geq \inf_{B(0,2R)} h \left(\int_{B(0,2R)\setminus B(0,R)} |x|^{-c_1p^*}|U_{a,p,\epsilon}|^l dx + \int_{B(0,R)} |x|^{-c_1p^*}|U_{a,p,\epsilon}|^l dx \right) \\ &\quad \geq \omega_N \inf_{B(0,2R)} h \left(\int_R^{2R} r^{-c_1p^*+N-1}|U_{a,p,\epsilon}|^l dr + \int_0^R r^{-c_1p^*+N-1}|U_{a,p,\epsilon}|^l dr \right) \end{aligned}$$

Now, we will calculate each one of the integrals of the above sum. Let us consider $\alpha = \frac{d_1p(N-p-ap)}{(p-1)(N-d_1p)}$, then

(36)

$$\begin{aligned} \int_R^{2R} r^{-c_1p^*+N-1}|U_{a,p,\epsilon}|^l dr &\geq \int_R^{2R} \frac{r^{-c_1p^*+N-1}}{(1+r^\alpha)^{(N-d_1p)l/d_1p}} dr \\ &= \int_R^{2R} \frac{r^{-c_1p^*+N-1-[\alpha(N-d_1p)l/d_1p]}}{(r^{-\alpha}+1)^{(N-d_1p)l/d_1p}} dr \\ &\geq \frac{1}{(R^{-\alpha}+1)^{\frac{(N-d_1p)l}{d_1p}}} \int_R^{2R} r^{-c_1p^*+N-1-\alpha\frac{(N-d_1p)l}{d_1p}} dr, \end{aligned}$$

and making the change of variables $s = R^{-1}\epsilon^{-1/\alpha}r$, in the second integral of the sum in (35), we get

$$\int_0^R r^{-c_1p^*+N-1}|U_{a,p,\epsilon}|^l dr = (R^\alpha\epsilon)^{\frac{-(N-d_1p)l}{d_1p}+\frac{(N-c_1p^*)(p-1)(N-d_1p)}{d_1p(N-p-ap)}} \times\int_0^{\epsilon^{-1/\alpha}} \frac{s^{-c_1p^*+N-1}}{(R^{-\alpha}+s^\alpha)^{(N-d_1p)l/d_1p}}ds \tag{37}$$

If $l < (N-c_1p^*)(p-1)/(N-p-ap)$, we have that

$$-\tfrac{(N-d_1p)l}{d_1p}+\tfrac{(N-c_1p^*)(p-1)(N-d_1p)}{d_1p(N-p-ap)} > 0 \text{ and } N-c_1p^*-\alpha\tfrac{(N-d_1p)l}{d_1p} > 0.$$

therefore, by (35) and (36) we obtain

$$\begin{aligned}\int_\Omega |x|^{-c_1p^*}h\,|\psi U_{a,p,\epsilon}|^l dx &\geq \frac{\omega_N(\inf_{B(0,2R)}h)}{(R^{-\alpha}+1)^{(N-d_1p)l/d_1p}}\int_R^{2R} r^{-c_1p^*+N-1-\alpha\frac{(N-d_1p)l}{d_1p}}dr\\ &\geq \left(\frac{w_nR^{N-c_1p^*}\inf_{B(0,2R)}h}{(1+R^\alpha)^{(N-d_1p)l/d_1p}}\right)\frac{\left(2^{-c_1p^*+N-\alpha\frac{(N-d_1p)l}{d_1p}}-1\right)}{\left(-c_1p^*+N-\alpha\frac{(N-d_1p)l}{d_1p}\right)}\\ &= O(R,h).\end{aligned}$$

Then, since $h \geq 0$ for a.e. in Ω and $\inf_{B(0,2R)}h > 0$, from (34) we get

$$\begin{aligned}\int_\Omega |x|^{-c_1p^*}h\,|u_\epsilon|^l dx &= \frac{||h^{1/l}\psi U_{a,p,\epsilon}||^l_{L^l(\Omega,|x|^{-c_1p^*})}}{||\psi U_{a,p,\epsilon}||^l_{L^{p^*}(\Omega,|x|^{-c_1p^*})}}\\ &\geq \frac{O(R,h)}{\left(O(1)+(S_{a,p,R})^{p^*/(p^*-p)}(k_{a,p}(\epsilon))^{-p^*}\right)^{l/p^*}}\\ &\geq \frac{O(R,h)(k_{a,p}(\epsilon))^l}{\left(O(1)+(S_{a,p,R})^{p^*/(p^*-p)}\right)^{l/p^*}}\\ &= O(R,h)O(\epsilon^{l(N-d_1p)/d_1p^2}).\end{aligned}$$

Moreover, we have for h satisfying (8) that

$$O(R,h) \geq \omega_N c_0\left[\frac{\left(2^{-c_1p^*+N-[\alpha(N-d_1p)l/d_1p]}-1\right)}{-c_1p^*+N-[\alpha(N-d_1p)l/d_1p]}\right] = \tilde{c}_0 > 0,$$

therefore

$$\int_\Omega |x|^{-c_1p^*}h\,|u_\epsilon|^l dx \geq O(\epsilon^{l(N-d_1p)/d_1p^2}),$$

uniformly in $h \in L^{k'}(\Omega,|x|^{-c_1p^*})$ satisfying (8).

Supposing $l = (N-c_1p^*)(p-1)/(N-p-ap)$, we see that

$$-\tfrac{(N-d_1p)l}{d_1p}+\tfrac{(p-1)(N-c_1p^*)(N-d_1p)}{d_1p(N-p-ap)} = 0 \text{ and } N-c_1p^*-\alpha\tfrac{(N-d_1p)l}{d_1p} = 0,$$

consequently, by (35) and (37), we obtain

$$\begin{aligned}
\int_\Omega |x|^{-c_1p^*} h\,|\psi U_{a,p,\epsilon}|^l dx &\geq \omega_N\big(\inf_{B(0,2R)} h\big)\int_0^{\epsilon^{-1/\alpha}} \frac{s^{-c_1p^*+N-1}}{(R^{-\alpha}+s^{\alpha})^{(N-d_1p)l/d_1p}} ds \\
&\geq \omega_N\big(\inf_{B(0,2R)} h\big)\int_1^{\epsilon^{-1/\alpha}} \frac{s^{-c_1p^*+N-1-[\alpha(N-d_1p)l/d_1p]}}{((Rs)^{-\alpha}+1)^{(N-d_1p)l/d_1p}} ds \\
&\geq \frac{\omega_N\big(\inf_{B(0,2R)} h\big)}{(R^{-\alpha}+1)^{(N-d_1p)l/d_1p}}\int_1^{\epsilon^{-1/\alpha}} s^{-1} ds \\
&= \omega_N \frac{R^{[\alpha(N-d_1p)l/d_1p]}\big(\inf_{B(0,2R)} h\big)}{(1+R^{\alpha})^{(N-d_1p)l/d_1p}} \ln(\epsilon^{-1/\alpha}) \\
&= \frac{\omega_N}{\alpha}\frac{R^{N-c_1p^*}\big(\inf_{B(0,2R)} h\big)}{(1+R^{\alpha})^{(N-d_1p)l/d_1p}} |\ln(\epsilon)|.
\end{aligned}$$

Thus, as $h \geq 0$ a.e. in Ω and $\inf_{B(0,2R)} h > 0$, from (34) we get we have

$$\begin{aligned}
\int_\Omega |x|^{-c_1p^*} h\,|u_\epsilon|^l dx &\geq \frac{\left(\frac{\omega_N R^{N-c_1p^*}\inf_{B(0,2R)} h}{\alpha(1+R^{\alpha})^{(N-d_1p)l/d_1p}}\right)|\ln(\epsilon)|}{\big(O(1)+(S_{a,p,R})^{p^*/(p^*-p)}(k_{a,p}(\epsilon))^{-p^*}\big)^{l/p^*}} \\
&\geq \frac{O(1)\left(\frac{R^{N-c_1p^*}\inf_{B(0,2R)} h}{(1+R^{\alpha})^{(N-d_1p)l/d_1p}}\right)\epsilon^{(N-d_1p)l/d_1p^2}|\ln(\epsilon)|}{\big(O(1)+(S_{a,p,R})^{p^*/(p^*-p)}\big)^{l/p^*}} \\
&= O(h,R)\epsilon^{(N-d_1p)l/d_1p^2}|\ln(\epsilon)|,
\end{aligned}$$

and if h satisfies (8), it follows that $O(h,R) \geq \tilde{c}_0 > 0$, then

$$\int_\Omega |x|^{-c_1p^*} h\,|u_\epsilon|^l dx = O(\epsilon^{(N-d_1p)l/d_1p^2}|\ln(\epsilon)|),$$

uniformly in h satisfying (8).

Assuming $l > (N-c_1p^*)(p-1)/(N-p-ap)$, we get

$$\tfrac{-(N-d_1p)l}{d_1p} + \tfrac{(N-c_1p^*)(p-1)(N-d_1p)}{d_1p(N-p-ap)} < 0 \text{ and } N-c_1p^* - \alpha\tfrac{(N-d_1p)l}{d_1p} < 0.$$

By using (35) and (36), we obtain

$$
\begin{aligned}
\int_\Omega |x|^{-c_1 p^*} h|\psi U_\epsilon|^l dx &\geq \omega_N(\inf_{B(0,R)} h)(R^\alpha \epsilon)^{-(N-d_1p)l/d_1p+\frac{(N-c_1p^*)(p-1)(N-d_1p)}{d_1p(N-p-ap)}} \\
&\quad \times \int_{1/2}^1 \frac{s^{-c_1p^*+N-1}}{(R^{-\alpha}+s^\alpha)^{(N-d_1p)l/d_1p}} ds \\
&\geq \left(\frac{\omega_N R^{N-c_1p^*}(\inf_{B(0,R)} h)}{(1+R^\alpha)^{(N-d_1p)l/d_1p}}\right) \\
&\quad \times \epsilon^{\frac{-(N-d_1p)l}{d_1p}+\frac{(N-c_1p^*)(p-1)(N-d_1p)}{d_1p(N-p-ap)}} \int_{1/2}^1 s^{-c_1p^*+N-1} ds \\
&\geq \left(\frac{O(1)R^{N-c_1p^*}(\inf_{B(0,R)} h)}{(1+R^\alpha)^{(N-d_1p)l/d_1p}}\right)\epsilon^{\frac{-(N-d_1p)l}{d_1p}+\frac{(N-c_1p^*)(p-1)(N-d_1p)}{d_1p(N-p-ap)}} \\
&\geq O(h,R)\epsilon^{\frac{-(N-d_1p)l}{d_1p}+\frac{(N-c_1p^*)(p-1)(N-d_1p)}{d_1p(N-p-ap)}},
\end{aligned}
$$

therefore, we conclude

$$
\begin{aligned}
\int_\Omega |x|^{-c_1p^*} h\,|u_\epsilon|^l dx &\geq \frac{O(h,R)\,\epsilon^{\frac{-(N-d_1p)l}{d_1p}+\frac{(N-c_1p^*)(p-1)(N-d_1p)}{d_1p(N-p-ap)}}}{\left(O(1)+(S_{a,p,R})^{p^*/(p^*-p)}(k_{a,p}(\epsilon))^{-p^*}\right)^{l/p^*}} \\
&\geq \frac{O(h,R)\,\epsilon^{\frac{-(N-d_1p)l}{d_1p}+\frac{(N-c_1p^*)(p-1)(N-d_1p)}{d_1p(N-p-ap)}}\,(k_{a,p}(\epsilon))^l}{\left(O(1)+(S_{a,p,R})^{p^*/(p^*-p)}\right)^{l/p^*}} \\
&= O(h,R)O\left(\epsilon^{\frac{(N-c_1p^*)(p-1)(N-d_1p)}{d_1p(N-p-ap)}-\frac{(N-d_1p)(p-1)l}{d_1p^2}}\right),
\end{aligned}
$$

and if h satisfies (8), it follows that $O(h,R)\geq \tilde{c}_0>0$, hence we get

$$
\int_\Omega |x|^{-c_1p^*} h\,|u_\epsilon|^l dx \quad \geq \quad O\left(\epsilon^{\frac{(N-d_1p)(p-1)(N-c_1p^*)}{d_1p(N-p-ap)}-\frac{(N-d_1p)(p-1)l}{d_1p^2}}\right),
$$

uniformly in h satisfying (8). □

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