

# $Z_2^k$ -ACTIONS FIXING $K_d P^{2^s} \cup K_d P^{\text{even}}$

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ABSTRACT. The main result of this paper is the determination, up to equivariant cobordism, of all manifolds with  $Z_2^k$ -action whose fixed point set is  $F = K_d P^{2^s} \cup K_d P^n$ , where  $n \geq 2^{s+1}$  is even and  $s \geq 1$ . Here,  $K_d P^n$  is the real ( $d = 1$ ), complex ( $d = 2$ ) or quaternionic ( $d = 4$ )  $n$ -dimensional projective space, with real dimension  $dn$ . This extends a previous and recent result of the authors, concerning the case  $d = 1$  and  $s = 1$ . We also obtain this equivariant cobordism classification for  $d = 2$  and  $4$  in the cases  $F = \{point\} \cup K_d P^n$ , where  $n \geq 2$  is even, and  $F = K_d P^m \cup K_d P^n$ , where  $m$  is odd and  $n \geq 0$  is even; for  $d = 1$  and  $k = 1$ , these results are due to D. C. Royster.

## 1. Introduction

If  $M$  is a smooth, closed manifold and  $T : M \rightarrow M$  is a smooth involution defined on  $M$ , then it is well known that the fixed point set of  $T$ ,  $F$ , is a finite and disjoint union of closed submanifolds of  $M$ . In this setting, for a given  $F$ , a natural question is the classification, up to equivariant cobordism, of the pairs  $(M, T)$  for which the fixed point set is  $F$ . For related results, see for example Royster [4], Hou and Torrence [5; 6], Pergher [13], Stong [19; 20], Conner and Floyd [9, Theorem 27.6], Kosniowski and Stong [3, page 309], Lu [22; 23], and Pergher, Ramos and Oliveira [12].

For  $F = RP^n$  the real and  $n$ -dimensional projective space, the classification was established in [9] and [20]. D. C. Royster [4] then studied this problem with  $F$  the disjoint union of two real projective spaces,  $F = RP^m \cup RP^n$ . He established the results via a case-by-case method depending on the parity of  $m$

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and  $n$ , with special arguments when one of the components is  $RP^0 = \{point\}$ , but his methods were not sufficient to handle the case when  $m$  and  $n$  are even and positive. The results when one of the components is  $RP^0 = \{point\}$  can be so described: consider, in general, the involution  $(RP^{m+n+1}, T_{m,n})$ , for any  $m$  and  $n$ , defined in homogeneous coordinates by

$$T_{m,n}[x_0, x_1, \dots, x_{m+n+1}] = [-x_0, -x_1, \dots, -x_m, x_{m+1}, \dots, x_{m+n+1}] .$$

The fixed point set of  $T_{m,n}$  is  $RP^m \cup RP^n$ . From  $T_{m,n}$ , it may be possible to obtain other involutions fixing  $RP^m \cup RP^n$ : in general, for a given involution  $(W, T)$  with fixed point set  $F$  and  $W$  a boundary, the involution  $\Gamma(W, T) = (\frac{S^1 \times W}{-Id \times T}, \tau)$  is equivariantly cobordant to an involution fixing  $F$ ; here,  $S^1$  is the 1-sphere,  $Id$  is the identity map and  $\tau$  is the involution induced by  $c \times Id$ , where  $c$  is complex conjugation (see Conner and Floyd [9]). If  $\frac{S^1 \times W}{-Id \times T}$  is a boundary, we can repeat the process taking  $\Gamma^2(W, T)$ , and so on. If  $F$  is nonbounding, this process finishes, that is, there exists a smallest natural number  $r \geq 1$  for which the underlying manifold of  $\Gamma^r(W, T)$  is nonbounding; this follows from the 5/2-theorem of J. Boardman in [8] and its strengthened version in [3]. In particular, if  $m$  and  $n$  are even and  $m < n$ ,  $RP^m \cup RP^n$  does not bound and  $RP^{m+n+1}$  bounds, so this number  $r$  makes sense for  $(RP^{m+n+1}, T_{m,n})$  and we denote  $r$  by  $h_{m,n}$ . In [4], Royster proved the following theorem:

**Theorem.** *Let  $(M, T)$  be an involution fixing  $\{point\} \cup RP^n$ , where  $n \geq 1$ . Then, if  $n$  is odd,  $(M, T)$  is equivariantly cobordant to  $(RP^{n+1}, T_{0,n})$ , and if  $n$  is even,  $(M, T)$  is equivariantly cobordant to  $\Gamma^j(RP^{n+1}, T_{0,n})$  for some  $0 \leq j \leq h_{0,n}$ .*

Later, in [16], R. E. Stong and P. Pergher determined the value of  $h_{0,n}$ , thus answering the question posed by Royster in [4; page 271]: writing  $n = 2^p q$  with  $p \geq 1$  and  $q \geq 1$  odd, they showed that  $h_{0,n} = 2$  if  $p = 1$  and  $h_{0,n} = 2^p - 1$  if  $p > 1$ . In the recent paper [12], Pergher, Ramos and Oliveira generalized this result of Stong and Pergher, calculating the general value of  $h_{m,n}$ ; specifically, they showed that, for  $m, n$  even, with  $0 \leq m < n$  and  $n - m = 2^p q$  with  $p \geq 1$  and  $q \geq 1$  odd,  $h_{m,n} = 2$  if  $p = 1$  and  $h_{m,n} = 2^p - 1$  if  $p > 1$ .

Denote by  $K_d P^n$  the real ( $d = 1$ ), complex ( $d = 2$ ) and quaternionic ( $d = 4$ )  $n$ -dimensional projective space, with real dimension  $dn$ . The first objective of this paper is to obtain complex and quaternionic versions of the results above; specifically, we establish the classification in question for  $F = K_d P^m \cup K_d P^n$ , where  $d = 2$  or  $4$ ,  $m \geq 0$  is even and  $n \geq 1$  is odd, and for  $F = \{point\} \cup K_d P^n$ , where  $d = 2$  or  $4$  and  $n \geq 2$  is even. The precise statement of these results will be given in Sections 3 and 4. For  $d = 2$  or  $4$ , the case  $(m, n) = (odd, odd)$  was treated in [7], and the case  $F = K_d P^m$  with  $m$  even was treated in [21]. The arguments for  $d = 2$  and  $4$  are similar to those used by Royster for  $d = 1$ , with few and minor technical differences (for example, Royster used the fact that  $K_1 P^{2n}$  is indecomposable, while  $K_2 P^{2n}$  and  $K_4 P^{2n}$  are not).

We have the complex and quaternionic version of the involution  $(RP^{m+n+1}, T_{m,n})$ , which we similarly denote by  $(K_d P^{m+n+1}, T_{m,n})$ , and of the number  $h_{m,n}$ , denoted by  $h_{m,n}^d$  (that is,  $h_{m,n} = h_{m,n}^1$ ). We also calculate the value of  $h_{m,n}^d$  for every  $m, n$  even,  $0 \leq m < n$ , showing that  $h_{m,n}^d = dh_{m,n}^1 = dh_{m,n}$ ; this will be made in Section 5, and makes numerically precise the classifications obtained in this paper.

Concerning the question left open by Royster, with  $m$  and  $n$  even and positive,  $(m, n) = (n, n)$  was the first case solved: Kosniowski and Stong had shown in [3] that in this case  $(M, T)$  is an equivariant boundary when  $dim(M) \geq 2n$ , and Hou and Torrence showed in [5] that the same fact is true when  $n \leq dim(M) < 2n$ . The complex and quaternionic version of this particular case was treated in [21]. In [12], we solved the case  $(2, n)$ , where  $n \geq 4$ . The second and main objective of this paper is to solve the case  $(2^s, n)$ , where  $s \geq 1$  and  $n \geq 2^{s+1}$ , which extends the previous case ( $s = 1$ ). This classification, given in Section 6, includes the complex and quaternionic cases. Specifically, we will prove the following:

**Theorem.** *Let  $(M, T)$  be an involution fixing  $K_d P^{2^s} \cup K_d P^n$ , where  $M$  is connected,  $s \geq 1$  and  $n \geq 2^{s+1}$  is even. Then, if  $n > 4$ ,  $(M, T)$  is equivariantly cobordant to  $\Gamma^j(K_d P^{n+2^s+1}, T_{2^s, n})$  for some  $0 \leq j \leq h_{2^s, n}^d$ ; if  $n = 4$  (and thus*

$s = 1$ ),  $(M, T)$  is either equivariantly cobordant to  $\Gamma^j(K_d P^7, T_{2,4})$  for some  $0 \leq j \leq h_{2,4}^d$ , or equivariantly cobordant to  $\Gamma^{2d}(K_d P^3, T_{0,2}) \cup (K_d P^5, T_{0,4})$ .

Finally, as in [12], the classifications referring to the cases  $F = \{point\} \cup K_d P^n$  (every  $n \geq 1$ ) and  $F = K_d P^{2^s} \cup K_d P^n$  ( $s \geq 1, n \geq 2^{s+1}$  even) are automatically extended for  $Z_2^k$ -actions. These extensions follow, respectively, from the combination of the corresponding  $k = 1$  results of this paper and of [21], with results of [14], [15] and [17]. The details concerning these extensions will be given in Section 7.

**Dedication.** This paper is dedicated to Professor Robert E. Stong, who left us recently. Some techniques used here were learned from him.

## 2. Preliminaries

In this section we give the general background information needed to handle the proposed questions. Let  $(M, T)$  be an involution pair like those of Section 1, with fixed point set  $F$ , and let  $\eta \rightarrow F$  be the normal bundle of  $F$  in  $M$ . We say in this case that  $\eta \rightarrow F$  is a *fixed data*. Consider  $RP(\eta) \rightarrow F$  the real projective space bundle associated to  $\eta$ , and denote by  $\lambda_\eta \rightarrow RP(\eta)$  the line bundle of the double cover  $S(\eta) \rightarrow RP(\eta)$ ,  $S(\eta)$  the sphere bundle of  $\eta$ . Then  $\lambda_\eta \rightarrow RP(\eta)$  bounds as an element of the cobordism group  $\mathcal{N}_*(BO(1))$ , that is, as a closed manifold with line bundle. More generally, an arbitrary vector bundle  $\eta \rightarrow F$  is a fixed data if and only if  $\lambda_\eta \rightarrow RP(\eta)$  bounds in the above sense; also, two involution pairs  $(M, T), (M', T')$  are equivariantly cobordant if and only if their fixed data are cobordant as bundles, and if  $\eta \rightarrow F$  and  $\eta' \rightarrow F'$  are cobordant then  $\lambda_\eta \rightarrow RP(\eta)$  and  $\lambda_{\eta'} \rightarrow RP(\eta')$  are cobordant, which implies that if  $\eta \rightarrow F$  and  $\eta' \rightarrow F'$  are cobordant then  $\eta \rightarrow F$  is a fixed data if and only if  $\eta' \rightarrow F'$  is a fixed data. These facts are due to Conner and Floyd [9]. Conner and Floyd also provided an algebraic scheme to determine the cobordism class of an  $r$ -dimensional vector bundle  $\eta^r$  over a closed  $n$ -dimensional manifold  $F^n$ , given by the set of *Stiefel-Whitney numbers* (or *characteristic numbers*) of this bundle: write  $W(F^n) = 1 + w_1(F^n) + \dots + w_n(F^n)$

and  $W(\eta^r) = 1 + v_1(\eta^r) + \dots + v_r(\eta^r)$  for the total Stiefel-Whitney classes of  $F^n$  and  $\eta^r$ . A general characteristic number of  $\eta^r$  is a modulo 2 number obtained by evaluating an  $n$ -dimensional  $Z_2$ -cohomology class of the form  $P(w_1(F^n), \dots, w_n(F^n), v_1(\eta^r), \dots, v_r(\eta^r)) \in H^n(F^n, Z_2)$ , given by a homogeneous polynomial over  $Z_2$  of dimension  $n$  in the classes  $w_i(F^n)$ ,  $v_j(\eta^r)$ , on the fundamental homology class  $[F^n] \in H_n(F^n, Z_2)$ . These numbers form a complete set of invariants of the cobordism class of  $\eta^r$ . In particular, if  $\eta \rightarrow F$  is a fixed data, then any class  $P(w_1(RP(\eta)), \dots, w_{n+r-1}(RP(\eta)), c) \in H^{n+r-1}(RP(\eta), Z_2)$ , given by a polynomial of dimension  $n+r-1$  in the classes  $w_i(RP(\eta))$  and  $c = w_1(\lambda_\eta)$ , gives the zero characteristic number  $P(w_1(RP(\eta)), \dots, w_{n+r-1}(RP(\eta)), c)[RP(\eta)]$ ; in this case,  $P(w_1(RP(\eta)), \dots, w_{n+r-1}(RP(\eta)), c)[RP(\eta)]$  splits into a modulo 2 sum of factors corresponding to the connected components of  $F$ . To handle these numbers, for a component  $\eta^r \rightarrow F^n$  of  $\eta \rightarrow F$ , it is needed to know the  $Z_2$ -cohomology  $H^*(RP(\eta^r), Z_2)$  and the Stiefel-Whitney class of  $RP(\eta^r)$ : according to the Leray-Hirsch Theorem [2, page 129],  $H^*(RP(\eta^r), Z_2)$  is a free graded  $H^*(F^n, Z_2)$ -module with basis  $1, c, c^2, \dots, c^{r-1}$ , and from [1, page 517] the total Stiefel-Whitney class of  $RP(\eta^r)$  is

$$W(RP(\eta^r)) = (1 + w_1(F^n) + \dots + w_n(F^n))((1+c)^r + (1+c)^{r-1}v_1(\eta^r) + \dots + v_r(\eta^r)),$$

where here we are suppressing bundle maps. In fact,  $(1+c)^r + (1+c)^{r-1}v_1(\eta^r) + \dots + v_r(\eta^r)$  is the Stiefel-Whitney class of the bundle of tangent vectors parallel to the fibre, which is  $(r-1)$ -dimensional; then  $c^r + c^{r-1}v_1(\eta^r) + \dots + v_r(\eta^r) = 0$ , and this relation determines the ring structure of  $H^*(RP(\eta^r), Z_2)$ . This leads to the useful *Conner formula* (see [11; Lemma 3.1]): if  $\overline{W}(\eta^r) = 1 + \overline{w}_1(RP(\eta^r)) + \dots + \overline{w}_r(RP(\eta^r))$  is the dual Stiefel-Whitney class defined by  $W(\eta^r)\overline{W}(\eta^r) = 1$  and  $\alpha \in H^*(F^n, Z_2)$ , then  $c^j\alpha[RP(\eta^r)] = \overline{w}_{j-r+1}(\eta^r)\alpha[F^n]$  when  $j \geq r-1$ .

For a vector bundle  $\eta \rightarrow F$  and a natural number  $p \geq 0$ , write  $p \rightarrow F$  for the trivial  $p$ -dimensional bundle over  $F$  and  $p\eta \rightarrow F$  for the Whitney sum of  $p$  copies of  $\eta$ , with the agreement that  $0\eta \rightarrow F$  means the zero bundle over  $F$ . Denote by  $\lambda_{n,d} \rightarrow K_dP^n$  the canonical (real, complex or quaternionic) line bundle over  $K_dP^n$ , with real dimension  $d$ . From the structure of the Grothendieck ring of

orthogonal bundles over projectives spaces, one has that any bundle  $\eta \rightarrow K_d P^n$  is stably equivalent to  $p\lambda_{n,d} \rightarrow K_d P^n$  for some  $p \geq 0$ , which implies that  $W(\eta) = (1 + \alpha)^p$ , where  $\alpha \in H^d(K_d P^n, Z_2)$  is the generator;  $p$  is unique modulo  $2^s$ , where  $s$  is the smallest number with  $n < 2^s$ . For the model involutions  $\Gamma^j(K_d P^{m+n+1}, T_{m,n})$ ,  $0 \leq j \leq h_{m,n}^d$ , the fixed components  $K_d P^m$  and  $K_d P^n$  have normal bundles  $(n+1)\lambda_{m,d} \oplus j \rightarrow K_d P^m$  and  $(m+1)\lambda_{n,d} \oplus j \rightarrow K_d P^n$ , with  $W((n+1)\lambda_{m,d} \oplus j) = (1 + \alpha)^{n+1}$ ,  $W((m+1)\lambda_{n,d} \oplus j) = (1 + \beta)^{m+1}$ , where  $\alpha \in H^d(K_d P^m, Z_2)$ ,  $\beta \in H^d(K_d P^n, Z_2)$  are the generators. Now for a given involution  $(M, T)$  with fixed data  $\eta \rightarrow F = (\eta|_{K_d P^m} \rightarrow K_d P^m) \cup (\eta|_{K_d P^n} \rightarrow K_d P^n)$ , one has  $W(\eta|_{K_d P^m}) = (1 + \alpha)^p$ ,  $W(\eta|_{K_d P^n}) = (1 + \alpha)^q$  for some  $p, q \geq 0$ . Thus, to obtain the desired classifications, the general strategy consists in showing that  $p = n + 1$  and  $q = m + 1$ , by comparing the 2-adic expansions of these numbers. This comparison is performed with systems of equations in the variables  $p, q$  and  $\dim(M)$ , where the equations are of the form  $P(w_1(RP(\eta)), \dots, w_j(RP(\eta)), \dots, c)[RP(\eta)] = 0$  for certain polynomials in the characteristic classes. The key point is the choice of suitable polynomials. In this setting, it will be useful to consider certain variants of the Stiefel-Whitney class  $W(RP(\eta))$ , given by the classes

$$W[r] = \frac{W(RP(\eta))}{(1 + c)^r} = 1 + W[r]_1 + W[r]_2 + \dots,$$

where  $r$  is any integer. Each homogeneous part  $W[r]_j$  of  $W[r]$  is a polynomial in the classes  $w_i(RP(\eta))$  and  $c$ , and thus can be used to yield characteristic numbers. These classes were introduced by Stong and Pergher in [16], and their usefulness lies in the fact that  $W[r]_j$  may be simpler than  $w_j(RP(\eta))$  for adequate values of  $r$ . Other key tool in this setting are the Steenrod operations  $Sq^j$ : by the Wu formula,  $Sq^j$  evaluated on a characteristic class gives a polynomial in the characteristic classes, and by the Cartan formula this extends to evaluations on polynomials in the characteristic classes. By applying Steenrod operations on low dimensional monomials in the characteristic classes, it is possible to show that certain simple and high dimensional cohomology classes are

in fact polynomials in the characteristic classes. This was extensively used in [12].

### 3. The case $F = \{point\} \cup K_d P^n$ with $n \geq 2$ even

The classification in this case is given by the following

**Theorem.** *Let  $(M, T)$  be an involution fixing  $\{point\} \cup K_d P^n$  with  $n \geq 2$  even. Then  $(M, T)$  is equivariantly cobordant to  $\Gamma^i(K_d P^{n+1}, T_{0,n})$  for some  $0 \leq i \leq h_{0,n}^d$ .*

**Proof.** Denote by  $\eta \rightarrow K_d P^n$  the normal bundle of  $K_d P^n$  in  $M$ , and set  $r = \dim(M)$ ,  $k = \dim(\eta) = r - nd$ . One has  $W(\eta) = (1 + \alpha)^p$  for some  $p \geq 0$ , where  $\alpha \in H^d(K_d P^n, Z_2)$  is the generator. If  $2^u$  is the greatest power of 2 of the 2-adic expansion of  $n$ , we can assume  $p \leq 2^{u+1}$ . Here and throughout the paper we use several times the fact that a binomial coefficient  $\binom{a}{b}$  is nonzero modulo 2 if and only if the 2-adic expansion of  $b$  is a subset of the 2-adic expansion of  $a$ . Setting  $p' = 2^{u+1} - p$ , the dual Stiefel-Whitney class of  $\eta$  is  $\overline{W}(\eta) = (1 + \alpha)^{p'}$ . Over the point the normal bundle is the trivial  $r$ -dimensional bundle, and thus the real projective space bundle is  $RP^{r-1}$ , with line bundle the canonical line bundle over  $RP^{r-1}$ . Write  $\beta \in H^1(RP^{r-1}, Z_2)$  for the generator and set  $w_1(\lambda_\eta) = c$ . The Stiefel-Whitney classes then are  $W(RP^{r-1}) = (1 + \beta)^r$  and

$$W(RP(\eta)) = (1 + \alpha)^{n+1} \left( \sum_i (1 + c)^{k-id} \binom{p}{i} \alpha^i \right).$$

One has  $w_d(RP^{r-1}) = \binom{r}{d} \beta^d$  and  $w_d(RP(\eta)) = (p+1)\alpha + \binom{k}{d} c^d$ . Since  $\binom{r}{d} = \binom{k}{d}$ ,  $w_d(RP(\eta)) + \binom{k}{d} c^d = (p+1)\alpha$  and  $w_d(RP^{r-1}) + \binom{k}{d} \beta^d = 0$  are corresponding classes, and as remarked in Section 2

$$0 = (w_d(RP^{r-1}) + \binom{k}{d} \beta^d)^n \beta^{k-1} [RP^{r-1}] = (p+1)\alpha^n c^{k-1} [RP(\eta)] = p+1.$$

Hence  $p$  is odd. Since  $p + p' = 2^{u+1}$ , we get in particular that  $\binom{p}{2^t} + \binom{p'}{2^t} = 1$  for every  $1 \leq t \leq u$ . By Conner's formula (which will be used without mention and several times in characteristic number calculations),

$$c^{r-1} [RP(\eta)] = \binom{p'}{n} = \beta^{r-1} [RP^{r-1}] = 1,$$

which means that  $2^u$  belongs to the 2-adic expansion of  $p'$  and thus does not belong to the 2-adic expansion of  $p$ . Since  $p \leq 2^{u+1}$  and  $p$  is odd, this gives that  $p < 2^u \leq n$ . In this way,  $w_{pd}(\eta) = \binom{p}{p} \alpha^p = \alpha^p \neq 0$ . Thus  $k \geq pd$  and consequently it makes sense to consider the vector bundle  $p\lambda_{n,d} \oplus (k - pd) \rightarrow K_d P^n$ , where, as in Section 2,  $\lambda_{n,d}$  denotes the canonical line bundle over  $K_d P^n$ , with real dimension  $d$ . This bundle has dimension  $k$  and Stiefel-Whitney class  $W = (1 + \alpha)^p$ , hence it is cobordant to  $\eta$ . As explained in Section 2, our aim is to show that  $p = 1$ , which will give the desired classification. One has that  $(p\lambda_{n,d} \oplus (k - pd) \rightarrow K_d P^n) \cup (r \rightarrow \{\text{point}\})$  is a fixed data; by removing sections, if necessary, we conclude that also  $(p\lambda_{n,d} \rightarrow K_d P^n) \cup (pd + nd \rightarrow \{\text{point}\})$  is a fixed data (see [9; Theorem 26.4]). In this case, the involved projective space bundles are  $RP^{nd+pd-1}$  and  $RP(p\lambda_{n,d})$ . The fact that  $\binom{p'}{n} = 1$  implies that  $p$  and  $n$  have disjoint 2-adic expansions, and thus  $w_{pd}(RP^{nd+pd-1}) = \binom{nd+pd}{pd} \beta^{pd} = \beta^{pd}$ . To calculate the corresponding class  $w_{pd}(RP(p\lambda_{n,d}))$ , first note that the previous fact also implies that  $\binom{n+1}{i} \binom{p}{i} = 0$  for each  $i > 1$ . It follows that

$$w_{pd}(RP(p\lambda_{n,d})) = \sum_i \binom{n+1}{i} \binom{p}{p-i} \alpha^i (c^d + \alpha)^{p-i} = \sum_i \binom{n+1}{i} \binom{p}{i} \alpha^i (c^d + \alpha)^{p-i} \\ = (c^d + \alpha)^p + \alpha (c^d + \alpha)^{p-1} = c^d (c^d + \alpha)^{p-1}.$$

Now,  $p < n$  implies that  $2pd \leq nd + pd - 1$ , and so  $nd - pd - 1 \geq 0$ . Then it makes sense to consider the characteristic number

$$w_{pd}(RP(p\lambda_{n,d}))^2 c^{nd-pd-1} [RP(p\lambda_{n,d})] = (c^d + \alpha)^{2p-2} c^{nd-pd-1+2d} [RP(p\lambda_{n,d})].$$

Note that  $(c^d + \alpha)^p = 0$ : in fact, one has the relation

$$0 = c^{dp} + \sum_{i=1}^{dp} w_i(p\lambda_{n,d}) c^{dp-i} = \sum_{i=0}^p \binom{p}{i} \alpha^i c^{dp-id}$$

and the last term is equal to  $(c^d + \alpha)^p$ .

By contradiction, if  $p > 1$  the above number is zero because  $(c^d + \alpha)^{2p-2} = (c^d + \alpha)^p (c^d + \alpha)^{p-2}$ . On the other hand, the corresponding characteristic number over the point is

$$\{w_{pd}(RP^{nd+pd-1})\}^2 \beta^{nd-pd-1} [RP^{nd+pd-1}] = \beta^{nd+pd-1} [RP(nd + pd - 1)] = 1,$$

which gives the contradiction.  $\square$



#### 4. The case $m$ odd and $n$ even

To study this case, we first establish some notations and facts. Consider  $m \geq 1$  odd,  $n \geq 0$  even. For an involution  $(M, T)$  fixing  $K_d P^m \cup K_d P^n$ , denote by  $\xi \rightarrow K_d P^m$  and  $\eta \rightarrow K_d P^n$  the normal bundles, and set  $r = \dim(M)$ ,  $l = \dim(\xi) = r - md$ ,  $k = \dim(\eta) = r - nd$ . If  $\beta \in H^d(K_d P^m, \mathbb{Z}_2)$  and  $\alpha \in H^d(K_d P^n, \mathbb{Z}_2)$  are the generators, one has  $W(\xi) = (1 + \beta)^q$ ,  $W(\eta) = (1 + \alpha)^p$  for some  $q, p \geq 0$ . Besides  $(K_d P^{m+n+1}, T_{m,n})$ , we present two more models of involutions fixing  $K_d P^m \cup K_d P^n$ . From [9], one knows that if  $q$  is even and  $d = 1$  then  $\xi \rightarrow K_d P^m$  bounds; the same arguments of [9] work to show that this fact is also true for  $d = 2$  and  $4$ . On  $K_d P^n \times K_d P^n$  one has the *twist* involution  $(x, y) \rightarrow (y, x)$ , that fixes a copy of  $K_d P^n$  with normal bundle  $\tau \rightarrow K_d P^n$ , where  $\tau$  is the tangent bundle. If  $md \leq 2nd$ , consider  $\xi \rightarrow K_d P^m$  any bundle with  $\dim(\xi) = 2nd - md$  and with  $q$  even (for example, the  $(2nd - md)$ -dimensional trivial bundle). Then  $(\xi \rightarrow K_d P^m) \cup (\tau \rightarrow K_d P^n)$  is the fixed data of an involution cobordant to  $(K_d P^n \times K_d P^n, \text{twist})$ , which provides a new model as desired. To give the last model, take  $m > n$  satisfying  $(m+1)d \leq (n+1)d + h_{0,n}^d$ . Then the involution  $(K_d P^{m+1}, T_{0,m}) \cup \Gamma^{(m-n)d}(K_d P^{n+1}, T_{0,n})$  is cobordant to an involution with fixed data  $(\xi \rightarrow K_d P^m) \cup (\eta \rightarrow K_d P^n)$ , where  $W(\xi) = 1 + \beta$  and  $W(\eta) = 1 + \alpha$ , which gives the desired model.

**Theorem.** *All involution  $(M, T)$  fixing  $K_d P^m \cup K_d P^n$  with  $m \geq 1$  odd and  $n \geq 0$  even is equivariantly cobordant to one of the three models described above.*

**Proof.** Set  $w_1(\lambda_\eta) = c$ ,  $w_1(\lambda_\xi) = e$ . Then

$$\begin{aligned} W(RP(\eta)) &= (1 + \alpha)^{n+1} \left( \sum_i (1 + c)^{k-id} \binom{p}{i} \alpha^i \right), \\ W(RP(\xi)) &= (1 + \beta)^{m+1} \left( \sum_i (1 + e)^{l-id} \binom{q}{i} \beta^i \right). \end{aligned}$$

We split the proof in several cases.

i)  $n = 0$ , that is,  $K_d P^m \cup K_d P^n = K_d P^m \cup \{\text{point}\}$ : in this case,  $W(RP(\eta)) = W(RP^{r-1}) = (1+c)^r$ ,  $w_d(RP^{r-1}) = \binom{r}{d} c^d$ ,  $w_d(RP(\xi)) = (m+1)\beta + \binom{l}{d} e^d + q\beta = \binom{l}{d} e^d + q\beta$ . If  $q$  is even,  $\xi$  bounds and then it can be equivariantly removed, thus giving a contradiction because an involution cannot have precisely one fixed point [9; 25.1]. Then  $q$  is odd and  $w_d(\xi) = q\beta = \beta \neq 0$ , which means

that  $l \geq d$ ; also,  $w_d(RP(\xi)) = \binom{l}{d}e^d + \beta$ . Because  $m$  is odd,  $\binom{r}{d} + \binom{l}{d} = 1$  and  $w_d(RP^{r-1}) = \binom{r}{d}c^d = \binom{l}{d}c^d + c^d$ . Thus,  $w_d(RP(\xi)) + \binom{l}{d}e^d = \beta$  and  $w_d(RP^{r-1}) + \binom{l}{d}c^d = c^d$  are corresponding classes; if  $l > d$ , one then has

$$0 = \beta^{m+1}e^{l-1-d}[RP(\xi)] = c^{dm+d}c^{l-1-d}[RP^{r-1}] = c^{r-1}[RP^{r-1}] = 1.$$

Hence  $l = d$  and  $W(\xi) = 1 + \beta$ , which means that  $(M, T)$  is cobordant to  $(K_dP^{m+1}, T_{m,0})$ . This ends this case, and so in the next cases we suppose  $n \geq 2$ . In these next cases, to calculate characteristic numbers we use many times the polynomial of degree  $d$  written formally as  $\tilde{w}_d(RP(\ )) = w_d(RP(\ )) + \binom{k}{d}c^d$ . Note that  $l + md = k + nd$ , and hence  $\binom{k}{d} + \binom{l}{d} = 1$ . We have  $w_d(RP(\eta)) = (p+1)\alpha + \binom{k}{d}c^d$  and  $w_d(RP(\xi)) = q\beta + \binom{l}{d}e^d = \beta + \binom{k}{d}e^d + e^d$ . Thus  $\tilde{w}_d(RP(\eta)) = (p+1)\alpha$  and  $\tilde{w}_d(RP(\xi)) = \beta + e^d$ .

ii)  $q$  even: as we remarked above, in this case  $\xi$  bounds and then  $(M, T)$  is cobordant to an involution with fixed point set  $F = K_dP^n$ . From [20] and [5],  $(M, T)$  is cobordant to  $(K_dP^n \times K_dP^n, twist)$ . This ends this case, and so in the next cases we suppose  $q$  odd.

iii)  $p$  odd: in this case,  $\tilde{w}_d(RP(\eta)) = 0$ , and thus  $\tilde{w}_d(RP(\eta)) + c^d = c^d$  and  $\tilde{w}_d(RP(\xi)) + e^d = \beta$  are corresponding classes. Since  $q$  is odd,  $w_d(\xi) \neq 0$  and so  $l \geq d$ ; if  $l > d$ ,  $l - 1 - d \geq 0$  and

$$\begin{aligned} 0 &= \beta^{m+1}e^{l-1-d}[RP(\xi)] = c^{dm+d}c^{l-1-d}[RP(\eta)] \\ &= c^{dm}c^{l-1}[RP(\eta)] = \beta^m e^{l-1}[RP(\xi)] = 1. \end{aligned}$$

We conclude that  $l = d$  and  $W(\xi) = 1 + \beta$ . Therefore the involution  $(M, T) \cup (K_dP^{m+1}, T_{m,0})$  is cobordant to an involution with fixed data  $((md + d) \rightarrow \{point\}) \cup (\eta \rightarrow K_dP^n)$ . From the result of Section 3,  $md + d \leq nd + d + h_{0,n}^d$  and this last involution is cobordant to  $\Gamma^{md-nd}(K_dP^{n+1}, T_{0,n})$ . It follows that  $(M, T)$  is cobordant to  $(K_dP^{m+1}, T_{0,m}) \cup \Gamma^{md-nd}(K_dP^{n+1}, T_{0,n})$ , which finishes this case.

iv)  $p$  even: in this case, we will show that  $(M, T)$  is cobordant to  $(K_dP^{m+n+1}, T_{m,n})$ . To do this, it suffices to show that  $p = m + 1$ ,  $q = n + 1$  and  $r = (m + n + 1)d$ . The first step is to show that  $r \geq (m + n + 1)d$ . We have  $\tilde{w}_d(RP(\eta)) = \alpha$  and  $\tilde{w}_d(RP(\xi)) = \beta + e^d$ . Take  $u$  sufficiently large so that  $2^u > \max\{m, n, p, q\}$  and set  $p' = 2^u - p$ ,  $q' = 2^u - q$ . Then  $\overline{W}(\eta) = (1 + \alpha)^{p'}$

and  $\overline{W}(\xi) = (1 + \beta)^{q'}$ . Note that  $\binom{q}{2^i} + \binom{q'}{2^i} = 1 = \binom{p-1}{2^i} + \binom{p'+1}{2^i}$  for each  $1 \leq i < u$ . First suppose  $m > n$ . Then the polynomial  $\tilde{w}_d^n (\tilde{w}_d + c^d)^{m-n} c^{l-1}$  gives rise to the following characteristic number equation:

$$1 = \alpha^n (\alpha + c^d)^{m-n} c^{l-1} [RP(\eta)] = (\beta + e^d)^n \beta^{m-n} e^{l-1} [RP(\xi)] = \binom{q'+n}{n}.$$

Hence  $q'$  and  $n$  have disjoint 2-adic expansions, which implies that the 2-adic expansion of  $n$  is a subset of the 2-adic expansion of  $q$ . Since also  $q$  is odd and  $n$  is even, we then have  $\binom{q}{n+1} = 1$  and thus  $w_{(n+1)d}(\xi) = \binom{q}{n+1} \beta^{n+1} = \beta^{n+1} \neq 0$ . In this way,  $l = r - md \geq (n+1)d$  and  $r \geq (m+n+1)d$ . Now suppose  $n > m$ . First using the polynomial  $(\tilde{w}_d + c^d)^m \tilde{w}_d^{n-m} c^{k-1}$ , we get

$$1 = \beta^m (\beta + e^d)^{n-m} e^{k-1} [RP(\xi)] = (\alpha + c^d)^m \alpha^{n-m} c^{k-1} [RP(\eta)] = \binom{p'+m}{m}.$$

Hence  $p'$  and  $m$  have disjoint 2-adic expansions and so  $\binom{p-1}{m} = 1$ . Next, the polynomial  $(\tilde{w}_d + c^d)^{m+1} \tilde{w}_d^{n-(m+1)} c^{k-1}$  gives rise to the equation

$$0 = \beta^{m+1} (\beta + e^d)^{n-(m+1)} e^{k-1} [RP(\xi)] = (\alpha + c^d)^{m+1} \alpha^{n-(m+1)} c^{k-1} [RP(\eta)] = \binom{p'+m+1}{m+1}.$$

Since  $p'$  and  $m+1$  are even, we also have  $\binom{(p'+1)+(m+1)}{m+1} = 0$ . Then there exists at least one power of 2, say  $2^v$ , that belongs to the 2-adic expansions of  $m+1$  and  $p'+1$ . Therefore  $2^v$  does not belong to the 2-adic expansion of  $p-1$  and  $\binom{p-1}{m+1} = 0$ . It follows that  $\binom{p}{m+1} = \binom{p-1}{m} + \binom{p-1}{m+1} = 1 + 0 = 1$  and thus  $w_{(m+1)d}(\eta) = \binom{p}{m+1} \alpha^{m+1} = \alpha^{m+1} \neq 0$ . In this way,  $k = r - nd \geq (m+1)d$  and  $r \geq (m+n+1)d$ , thus ending the first step of the proof. The next step is to show that  $p = m+1$  and  $q = n+1$ . Note that the previous step makes possible to consider polynomials involving  $c^{r-1-jd}$  with  $j \leq m+n$ . For each  $i \leq n$ , we have

$$\begin{aligned} 1 &= (\beta + e^d)^{n-i} \beta^m e^{r-1-(m+n-i)d} [RP(\xi)] = \tilde{w}_d^{n-i} (\tilde{w}_d + e^d)^m e^{r-1-(m+n-i)d} [RP(\xi)] \\ &= \tilde{w}_d^{n-i} (\tilde{w}_d + c^d)^m c^{r-1-(m+n-i)d} [RP(\eta)] = \alpha^{n-i} (\alpha + c^d)^m c^{r-1-(m+n-i)d} [RP(\eta)] \\ &= \binom{p'+m}{i}. \end{aligned}$$

Thus  $(1 + \alpha)^{p'+m} = (1 + \alpha)^{-1} = (1 + \alpha)^{p'} (1 + \alpha)^m$  and so  $W(\eta) = \frac{1}{(1+\alpha)^{p'}} = (1 + \alpha)^{m+1}$ .

Similarly, for each  $i \leq m$  we have

$$\begin{aligned} 1 &= \alpha^n (\alpha + c^d)^{m-i} c^{r-1-(m+n-i)d} [RP(\eta)] = \tilde{w}_d^n (\tilde{w}_d + c^d)^{m-i} c^{r-1-(m+n-i)d} [RP(\eta)] \\ &= \tilde{w}_d^n (\tilde{w}_d + e^d)^{m-i} e^{r-1-(m+n-i)d} [RP(\xi)] = (\beta + e^d)^n \beta^{m-i} e^{r-1-(m+n-i)d} [RP(\xi)] \end{aligned}$$

$$= \binom{q'+n}{i}.$$

Thus  $(1 + \beta)^{q'+n} = (1 + \beta)^{-1} = (1 + \beta)^q (1 + \beta)^n$  and so  $W(\xi) = \frac{1}{(1+\beta)^q} = (1 + \beta)^{n+1}$ . This means that  $p = m + 1$  and  $q = n + 1$ , and hence it remains to show that  $r = (m + n + 1)d$ . By contradiction, suppose  $r > (m + n + 1)d$ . Without loss, we can suppose in this case that  $(\xi \rightarrow K_d P(m)) \cup (\eta \rightarrow K_d P^n) = ((n+1)\lambda_{m,d} \oplus (r-d(m+n+1)) \rightarrow K_d P(m)) \cup ((m+1)\lambda_{n,d} \oplus (r-d(m+n+1)) \rightarrow K_d P^n)$ . By removing sections, if necessary, we then have that  $((n+1)\lambda_{m,d} \oplus 1 \rightarrow K_d P(m)) \cup ((m+1)\lambda_{n,d} \oplus 1 \rightarrow K_d P^n)$  is the fixed data of an involution  $(V, S)$ . Then  $\Gamma(K_d P^{m+n+1}, T_{m,n}) \cup (V, S)$  is cobordant to an involution with fixed data  $1 \rightarrow K_d P^{m+n+1}$ . The contradiction follows from the fact that an involution with codimension one fixed point set bounds and  $K_d P^{m+n+1}$  does not bound.

## 5. Calculation of $h_{m,n}^d$

In this section we show that  $h_{m,n}^d = dh_{m,n} = dh_{m,n}^1$ ; in other words, we prove the following

**Theorem.** *For  $m, n$  even,  $0 \leq m < n$ , write  $n - m = 2^p q$  with  $p \geq 1$  and  $q \geq 1$  odd. Then  $h_{m,n}^d = \begin{cases} 2d & \text{if } p = 1, \\ (2^p - 1)d & \text{if } p > 1. \end{cases}$*

For a bundle  $\eta \rightarrow X$  and a natural number  $k > 0$ , write  $\eta^k \rightarrow X^k$  for the cartesian product of  $k$  copies of  $\eta$ . It is known (and in fact it follows from a routine argument involving the comparison between the corresponding characteristic numbers) the fact that  $\lambda_{n,2} \rightarrow K_2 P^n$  is cobordant to  $\lambda_{n,1}^2 \rightarrow (K_1 P^n)^2$  and  $\lambda_{n,4} \rightarrow K_4 P^n$  is cobordant to  $\lambda_{n,2}^2 \rightarrow (K_2 P^n)^2$ ; in particular,  $\lambda_{n,4} \rightarrow K_4 P^n$  is cobordant to  $\lambda_{n,1}^4 \rightarrow (K_1 P^n)^4$ . We recall that if  $(M, T)$  has fixed data  $\eta \rightarrow F$ , then  $\Gamma(M, T)$  has fixed data  $(\eta \oplus 1 \rightarrow F) \cup (1 \rightarrow M)$ . Therefore, from Sections 1 and 2, it suffices to prove the following two assertions:

A) Set  $\ell = h_{m,n}^1$ . Then  $((n+1)\lambda_{m,d} \oplus (d\ell) \rightarrow K_d P^m) \cup ((m+1)\lambda_{n,d} \oplus (d\ell) \rightarrow K_d P^n)$  is the fixed data of an involution.

B)  $((n+1)\lambda_{m,d} \oplus (d\ell+1) \rightarrow K_d P^m) \cup ((m+1)\lambda_{n,d} \oplus (d\ell+1) \rightarrow K_d P^n)$  is not the fixed data of an involution; equivalently, there exists at least one characteristic number of the usual line bundle over the projective space bundle corresponding to the component  $K_d P^m$  that is different from the corresponding number coming from the component  $K_d P^n$ .

Assertion A) follows from the combination of the  $d=1$  case and the fact that if  $(M, T)$  has fixed data  $\eta \cup \xi$ , then  $(M \times M, T \times T)$  is equivariantly cobordant to an involution with fixed data  $\eta^2 \cup \xi^2$ . To prove assertion B), we maintain the notations of the previous section for the characteristic classes of the involved projective space bundles; to simplify these notations, set  $\theta_m = (n+1)\lambda_{m,d} \oplus (d\ell+1) \rightarrow K_d P^m$ ,  $\theta_n = (m+1)\lambda_{n,d} \oplus (d\ell+1) \rightarrow K_d P^n$ . One has

$$\begin{aligned} W(RP(\theta_m)) &= (1+\beta)^{m+1} \left( \sum_i (1+e)^{(n+1)d+d\ell+1-i} w_i(\theta_m) \right) \\ &= (1+\beta)^{m+1} (1+e)^{d\ell+1} \left( \sum_i (1+e^d)^{n+1-i} \binom{n+1}{i} \beta^i \right) \\ &= (1+\beta)^{m+1} (1+e)^{d\ell+1} (1+e^d+\beta)^{n+1}; \end{aligned}$$

$$\begin{aligned} W(RP(\theta_n)) &= (1+\alpha)^{n+1} \left( \sum_i (1+c)^{(m+1)d+d\ell+1-i} w_i(\theta_n) \right) \\ &= (1+\alpha)^{n+1} (1+c)^{d\ell+1} \left( \sum_i (1+c^d)^{m+1-i} \binom{m+1}{i} \alpha^i \right) \\ &= (1+\alpha)^{n+1} (1+c)^{d\ell+1} (1+c^d+\alpha)^{m+1}. \end{aligned}$$

Consider the class  $\widehat{W}(RP(\ )) = \frac{W(RP(\ ))}{(1+c)^{d\ell+1}} = 1 + \widehat{w}_1(RP(\ )) + \widehat{w}_2(RP(\ )) + \dots$

We have:

$$\begin{aligned} \widehat{W}(RP(\theta_m)) &= (1+\beta)^{m+1} (1+e^d+\beta)^{n+1}, \\ \widehat{W}(RP(\theta_n)) &= (1+\alpha)^{n+1} (1+c^d+\alpha)^{m+1}, \text{ and} \\ \widehat{w}_{(m+n)d}(RP(\theta_m)) &= \beta^m (e^d+\beta)^n + \binom{m+1}{2} \beta^{m-1} (e^d+\beta)^{n+1}, \\ \widehat{w}_{(m+n)d}(RP(\theta_n)) &= \alpha^n (c^d+\alpha)^m + \binom{n+1}{2} \alpha^{n-1} (c^d+\alpha)^{m+1}, \\ \widehat{w}_{(\ell+1)d}(RP(\theta_m)) &= \binom{n+1}{\ell+1} (e^d+\beta)^{\ell+1} + \binom{n+1}{\ell} \beta (e^d+\beta)^\ell + \\ &+ \text{terms involving } \beta^2 \text{ and} \\ \widehat{w}_{(\ell+1)d}(RP(\theta_n)) &= \binom{m+1}{\ell+1} (c^d+\alpha)^{\ell+1} + \binom{m+1}{\ell} \alpha (c^d+\alpha)^\ell + \\ &+ \text{terms involving } \alpha^2. \end{aligned}$$

Since  $\beta^i = 0$  for every  $i > m$  and  $\alpha^i = 0$  for every  $i > n$ , we then have

$$\begin{aligned} \widehat{w}_{(m+n)d} \widehat{w}_{(\ell+1)d}(RP(\theta_m)) &= \left( \binom{n+1}{\ell+1} + \binom{m+1}{2} \binom{n+1}{\ell} \right) \beta^m e^{d(n+\ell+1)} + \\ &+ \binom{m+1}{2} \binom{n+1}{\ell+1} \beta^{m-1} e^{d(n+\ell+2)} + \binom{m+1}{2} \binom{n+1}{\ell+1} \binom{n+\ell+2}{1} \beta^m e^{d(n+\ell+1)}, \\ \widehat{w}_{(m+n)d} \widehat{w}_{(\ell+1)d}(RP(\theta_n)) &= \left( \binom{m+1}{\ell+1} + \binom{n+1}{2} \binom{m+1}{\ell} \right) \alpha^n c^{d(m+\ell+1)} + \end{aligned}$$

$$+ \binom{n+1}{2} \binom{m+1}{\ell+1} \alpha^{n-1} c^{d(m+\ell+2)} + \binom{n+1}{2} \binom{m+1}{\ell+1} \binom{m+\ell+2}{1} \alpha^n c^{d(m+\ell+1)}.$$

We also have

$$\beta^{m-1} e^{d(n+\ell+2)} [RP(\theta_m)] = \beta^{m-1} \bar{w}_1(\theta_m) [K_d P(m)] = \beta^m [K_d P(m)] = 1,$$

and similarly  $\alpha^{n-1} c^{d(m+\ell+2)} [RP(\theta_n)] = 1$ .

Thus we get

$$\begin{aligned} \widehat{w}_{(m+n)d} \widehat{w}_{(\ell+1)d} [RP(\theta_m)] &= \\ &= \binom{n+1}{\ell+1} + \binom{m+1}{2} \binom{n+1}{\ell} + \binom{m+1}{2} \binom{n+1}{\ell+1} + \binom{m+1}{2} \binom{n+1}{\ell+1} \binom{n+\ell+2}{1}, \\ \widehat{w}_{(m+n)d} \widehat{w}_{(\ell+1)d} [RP(\theta_n)] &= \\ &= \binom{m+1}{\ell+1} + \binom{n+1}{2} \binom{m+1}{\ell} + \binom{n+1}{2} \binom{m+1}{\ell+1} + \binom{n+1}{2} \binom{m+1}{\ell+1} \binom{m+\ell+2}{1}. \end{aligned}$$

If  $p = 1$ , then  $\ell = 2$  and  $\binom{m}{2} \neq \binom{n}{2}$ ; in this way,

$$\widehat{w}_{(m+n)d} \widehat{w}_{(\ell+1)d} [RP(\theta_m)] = \binom{n+1}{3} \neq \binom{m+1}{3} = \widehat{w}_{(m+n)d} \widehat{w}_{(\ell+1)d} [\mathbb{R}P(\theta_n)].$$

If  $p > 1$ , then  $\ell = 2^p - 1$ . Since  $n - m = 2^p$ ,  $p > 1$  and  $q$  is odd,

$$\binom{m+1}{2^p-1} = \binom{n+1}{2^p-1} \quad , \quad \binom{m+1}{2} \binom{n+1}{2}$$

and

$$\binom{m+1}{2^p} \neq \binom{n+1}{2^p}.$$

Therefore  $\widehat{w}_{(m+n)d} \widehat{w}_{(\ell+1)d} [RP(\theta_m)] \neq \widehat{w}_{(m+n)d} \widehat{w}_{(\ell+1)d} [RP(\theta_n)]$ , and the calculation of  $h_{m,n}^d$  is ended □

## 6. The case $m = 2^s$ with $s \geq 1$ and $n \geq 2^{s+1}$ even

This section is devoted to the proof of the main result of the paper, which is the following

**Theorem.** *Let  $(M, T)$  be an involution fixing  $K_d P^{2^s} \cup K_d P^n$ , where  $M$  is connected,  $s \geq 1$  and  $n \geq 2^{s+1}$  is even. Then, if  $n > 4$ ,  $(M, T)$  is equivariantly cobordant to  $\Gamma^j(K_d P^{n+2^s+1}, T_{2^s, n})$  for some  $0 \leq j \leq h_{2^s, n}^d$ ; if  $n = 4$  (and thus  $s = 1$ ),  $(M, T)$  is either equivariantly cobordant to  $\Gamma^j(K_d P^7, T_{2, 4})$  for some  $0 \leq j \leq h_{2, 4}^d$ , or equivariantly cobordant to  $\Gamma^{2d}(K_d P^3, T_{0, 2}) \cup (K_d P^5, T_{0, 4})$ .*

This extends the main result of [12], given by the  $s = 1$  and  $d = 1$  case. To prove the result, we need some preliminary lemmas, which are valid in a more general setting; in this way, these lemmas may be useful for studying the general case  $(m, n) = (\text{even}, \text{even})$ . For an arbitrary involution  $(M, T)$  fixing

$K_d P^m \cup K_d P^n$ , with  $M$  connected,  $m$  and  $n$  even and  $m < n$ , we again use the notations of Sections 4 and 5 for the normal bundles and characteristic classes of the projective space bundles over the components  $K_d P^m$  and  $K_d P^n$ ; that is, we have

$$W(RP(\eta)) = (1 + \alpha)^{n+1} \left( \sum_j (1 + c)^{k-dj} \binom{p}{j} \alpha^j \right),$$

$$W(RP(\xi)) = (1 + \beta)^{m+1} \left( \sum_j (1 + e)^{l-dj} \binom{q}{j} \beta^j \right).$$

**Lemma 1.** *Suppose  $0 < m < n$ , where  $m$  and  $n$  are even. Then  $w_d(\xi)$  and  $w_d(\eta)$  are nonzero classes.*

**Proof.** Since  $w_d(\xi) = q\beta$  and  $w_d(\eta) = p\alpha$ , we need to show that  $p$  and  $q$  are odd. One has  $w_d(RP(\eta)) = \binom{k}{d} c^d + (1+p)\alpha$  and  $w_d(RP(\xi)) = \binom{l}{d} e^d + (1+q)\beta$ . Consider the class  $\widehat{w}_d(RP(\ )) = w_d(RP(\ )) + \binom{k}{d} c^d$ . Since  $k+nd = l+md$  and  $m$  and  $n$  are even,  $\binom{k}{d} = \binom{l}{d}$  and thus  $\widehat{w}_d(RP(\eta)) = (1+p)\alpha$ ,  $\widehat{w}_d(RP(\xi)) = (1+q)\beta$ . Now  $m < n$  implies that  $\widehat{w}_d^n(RP(\xi)) = (q+1)\beta^n = 0$ ; thus

$$0 = \widehat{w}_d^n e^{k-1} [RP(\xi)] = \widehat{w}_d^n c^{k-1} [RP(\eta)] = (p+1)\alpha^n c^{k-1} [RP(\eta)] = p+1.$$

Hence  $p$  is odd and, in particular,  $\widehat{w}_d(RP(\eta)) = (p+1)\alpha = 0$ . Therefore

$$q+1 = (q+1)\beta^m e^{l-1} [RP(\xi)] = \widehat{w}_d^m e^{l-1} [RP(\xi)] = \widehat{w}_d^m c^{l-1} [RP(\eta)] = 0,$$

which shows that also  $q$  is odd.  $\square$

As we remarked in Section 2, the crucial point of the classification task is showing that  $p = m+1$  and  $q = n+1$ . The next lemma reduces this task to show only one of the assertions.

**Lemma 2.** *Here we consider  $0 \leq m < n$ .*

a) *If  $p = m+1$ , then  $(M, T)$  is cobordant to  $\Gamma^i(K_d P^{m+n+1}, T_{m,n})$ , with  $0 \leq i \leq h_{m,n}^d$ ;*

b) *If  $p = 1$  and  $k-d \leq h_{0,n}^d$ , then  $0 \leq l-d \leq h_{m,0}^d$  and  $(M, T)$  is cobordant to  $\Gamma^{l-d}(K_d P^{m+1}, T_{m,0}) \cup \Gamma^{k-d}(K_d P^{n+1}, T_{0,n})$ .*

**Proof.** (a) Since  $m+1 \leq n$ ,  $w_{(m+1)d}(\eta) = \alpha^{m+1} \neq 0$ , which means that  $k \geq (m+1)d$ . Thus we can suppose with no loss that  $\eta = (m+1)\lambda_{n,d} \oplus s$ , where  $s = k - (m+1)d$ . Now  $l = k + nd - md = (m+1)d + s + nd - md = (n+1)d + s$  and  $(n+1)d > md$  imply that  $\xi$  can be written as  $\xi = \mu \oplus s$ ,

where  $\mu$  is an  $(n+1)d$ -dimensional bundle over  $K_dP^m$ . By removing sections, if necessary, we conclude that  $\mu \cup (m+1)\lambda_{n,d}$  is the fixed data of an involution, and thus  $(\mu \cup (m+1)\lambda_{n,d}) \cup ((n+1)\lambda_{m,d} \cup (m+1)\lambda_{n,d})$  also is. Then  $(\mu \rightarrow K_dP^m) \cup ((n+1)\lambda_{m,d} \rightarrow K_dP^m)$  is the fixed data of an involution, and from [21] and [5] this fixed data bounds. Then  $\mu$  is cobordant to  $(n+1)\lambda_{m,d}$ , which gives the result.

(b) By Lemma 1) we get  $k \geq d$ , and thus we can suppose with no loss that  $\eta = \lambda_{n,d} \oplus (k-d)$ . Since  $k-d \leq h_{0,n}^d$ , we get that  $((\xi \rightarrow K_dP^m) \cup (\lambda_{n,d} \oplus (k-d) \rightarrow K_dP^n)) \cup ((\lambda_{n,d} \oplus (k-d) \rightarrow K_dP^n) \cup (k+nd \rightarrow \{point\}))$  is the fixed data of an involution, and thus  $(\xi \rightarrow K_dP^m) \cup (k+nd \rightarrow \{point\})$  also is. Using Section 3) we get the result.  $\square$

**Lemma 3.** *Let  $\eta \rightarrow K_dP^n$  be an arbitrary  $k$ -dimensional vector bundle over  $K_dP^n$ , where  $k, n > 0$ . Denote by  $\alpha \in H^d(K_dP^n)$  the generator and set  $W(\lambda_\eta) = 1 + c$ . Then, in  $H^*(RP(\eta))$ , the following relations hold:*

- (a)  $Sq^d(c^{2d}) = 0$ ,  $Sq^d(\alpha^2) = 0$  and  $Sq^d(\alpha c^d) = \alpha^2 c^d + \alpha c^{2d}$ ;
- (b)  $Sq^{2^v d}(\alpha^{2^v} c^d) = \alpha^{2^{v+1}} c^d$ ,  $Sq^{2^v d}(\alpha c^{2^v d}) = \alpha c^{2^{v+1}d}$  and  $Sq^{2^v d}(c^{(2^v+1)d}) = c^{(2^{v+1}+1)d}$ , for each  $v \geq 1$ .

**Proof.** Take  $v \geq 0$  and consider the  $2^v d$ -dimensional bundles  $2^v \lambda_{n,d} \rightarrow K_dP^n$ ,  $2^v d \lambda_\eta \rightarrow RP(\eta)$ . One has  $W(2^v \lambda_{n,d}) = 1 + \alpha^{2^v}$  and  $W(2^v d \lambda_\eta) = 1 + c^{2^v d}$ . Using the Wu formula for these bundles, for each  $0 < i < 2^v d$  we get

$$Sq^i(\alpha^{2^v}) = Sq^i(w_{2^v d}(2^v \lambda_{n,d})) = w_i(2^v \lambda_{n,d}) w_{2^v d}(2^v \lambda_{n,d}) = 0,$$

$$Sq^i(c^{2^v d}) = Sq^i(w_{2^v d}(2^v d \lambda_\eta)) = w_i(2^v d \lambda_\eta) w_{2^v d}(2^v d \lambda_\eta) = 0.$$

In particular,  $Sq^d(\alpha^2) = 0$  and  $Sq^d(c^{2d}) = 0$ . Now, using the Cartan formula, we get

$$Sq^{2^v d}(\alpha^{2^v} c^d) = \sum_{i=0}^{2^v d} Sq^i(\alpha^{2^v}) Sq^{2^v d-i}(c^d) = Sq^0(\alpha^{2^v}) Sq^{2^v d}(c^d) + Sq^{2^v d}(\alpha^{2^v}) Sq^0(c^d),$$

$$Sq^{2^v d}(\alpha c^{2^v d}) = \sum_{i=0}^{2^v d} Sq^i(\alpha) Sq^{2^v d-i}(c^{2^v d}) = Sq^0(\alpha) Sq^{2^v d}(c^{2^v d}) + Sq^{2^v d}(\alpha) Sq^0(c^{2^v d}),$$

$$Sq^{2^v d}(c^{(2^v+1)d}) = Sq^{2^v d}(c^d c^{2^v d}) = \sum_{i=0}^{2^v d} Sq^i(c^d) Sq^{2^v d-i}(c^{2^v d}) =$$

$$= Sq^0(c^d) Sq^{2^v d}(c^{2^v d}) + Sq^{2^v d}(c^d) Sq^0(c^{2^v d}).$$

Setting  $v = 0$  in  $Sq^{2^v d}(\alpha^{2^v} c^d)$  we then get

$$Sq^d(\alpha c^d) = Sq^0(\alpha) Sq^d(c^d) + Sq^d(\alpha) Sq^0(c^d) = \alpha c^{2d} + \alpha^2 c^d,$$

and setting  $v \geq 1$  we get



$$\begin{aligned}
Sq^{2^v d}(\alpha^{2^v} c^d) &= Sq^0(\alpha^{2^v})Sq^{2^v d}(c^d) + Sq^{2^v d}(\alpha^{2^v})Sq^0(c^d) = \alpha^{2^{v+1}} c^d, \\
Sq^{2^v d}(\alpha c^{2^v d}) &= Sq^{2^v d}(c^{2^v d})Sq^0(\alpha) + Sq^0(c^{2^v d})Sq^{2^v d}(\alpha) = \alpha c^{2^{v+1} d}, \\
Sq^{2^v d}(c^{(2^v+1)d}) &= Sq^0(c^d)Sq^{2^v d}(c^{2^v d}) + Sq^{2^v d}(c^d)Sq^0(c^{2^v d}) = c^d c^{2^{v+1} d} = c^{(2^{v+1}+1)d}.
\end{aligned}$$

Hence the lemma is proved.  $\square$

Now we prove the main result. Since  $M$  is connected,  $k > 0$ . Suppose first  $k \leq d$ . Then Lemma 1 says that  $W(\eta) = 1 + \alpha$  and, in particular,  $k = d$ . By Lemma 2-b,  $l - d \leq h_{2^s, 0}^d$  and  $(M, T)$  is cobordant to  $\Gamma^{l-d}(K_d P^{2^s+1}, T_{2^s, 0}) \cup (K_d P^{n+1}, T_{0, n})$ . We have  $l + 2^s d = k + nd$ , and so  $l - d = (n - 2^s)d$ ; also  $n \geq 2^{s+1}$  gives  $n - 2^s \geq 2^s$ . Then, if  $s > 1$ ,  $2^s d \leq h_{2^s, 0}^d = (2^s - 1)d$ , which gives a contradiction. Thus  $s = 1$  and  $l - d = (n - 2)d \leq h_{2, 0}^d = 2d$ , which gives  $n = 4$ . In this way, from now we can assume  $k > d$  and our task is reduced to show that  $(M, T)$  is cobordant to  $\Gamma^j(K_d P^{n+2^s+1}, T_{2^s, n})$  for some  $0 \leq j \leq h_{2^s, n}^d$ . By Lemma 2-a, it suffices to show that  $p = 2^s + 1$ ; to do this, we will analyse the 2-adic expansion of  $p$ , by handling suitable characteristic number equations. Let  $2^u$  be the greatest power of 2 that appears in the 2-adic expansion of  $n$ , that is,  $2^u \leq n < 2^{u+1}$ . We can assume  $p, q < 2^{u+1}$ . One has  $\overline{W}(\eta) = (1 + \alpha)^{p'}$  and  $\overline{W}(\xi) = (1 + \beta)^{q'}$ , where  $p' = 2^{u+1} - p$  and  $q' = 2^{u+1} - q$ . Since  $p$  and  $q$  are odd,  $p'$  and  $q'$  are odd and  $\binom{p}{2^v} + \binom{p'}{2^v} = 1 = \binom{q}{2^v} + \binom{q'}{2^v}$  for every  $1 \leq v \leq u$ .

Rewrite

$$\begin{aligned}
W(RP(\eta)) &= (1 + \alpha)^{n+1} \left( \sum_i (1 + c)^{k-id} \binom{p}{i} \alpha^i \right), \\
W(RP(\xi)) &= (1 + \beta)^{2^s+1} \left( \sum_i (1 + e)^{l-id} \binom{q}{i} \beta^i \right).
\end{aligned}$$

Formally, for each integer  $x$ , consider the class

$$W[x](RP(\quad)) = \frac{W(RP(\quad))}{(1+c)^{k-xd}} = w[x]_1 + w[x]_2 + \dots$$

Then

$$\begin{aligned}
W[x](RP(\eta)) &= (1 + \alpha)^{n+1} \left( \sum_i (1 + c^d)^{x-i} \binom{p}{i} \alpha^i \right) \text{ and} \\
W[x](RP(\xi)) &= (1 + \beta)^{2^s+1} (1 + e^d)^{n-2^s} \left( \sum_i (1 + e^d)^{x-i} \binom{q}{i} \beta^i \right).
\end{aligned}$$

**Fact 1.**  $\binom{p'}{n-2^s} + \binom{p'}{n-2^{s+1}} = 1$ .

**Proof.** One has

$$\begin{aligned}
w[1]_{2d}(RP(\eta)) &= \left[ \binom{n}{2} + \binom{p}{2} + 1 \right] \alpha^2 + \alpha c^d \text{ and} \\
w[1]_{2d}(RP(\xi)) &= \left[ \binom{2^s}{2} + \binom{q}{2} + 1 \right] \beta^2 + \beta e^d + \binom{n-2^s}{2} e^{2d}.
\end{aligned}$$

For each  $j \geq 1$ , consider the polynomial

$$\tilde{w}_{(2^j+1)d} = Sq^{2^{j-1}d} \dots Sq^{2^2d} Sq^d(w[1]_{2d}).$$

By iteratively applying Lemma 3, we get

$$\tilde{w}_{(2^j+1)d}(RP(\eta)) = \alpha^{2^j} c^d + \alpha c^{2^j d} \text{ and } \tilde{w}_{(2^j+1)d}(RP(\xi)) = \beta^{2^j} e^d + \beta e^{2^j d}.$$

For  $j = s$ , we then get

$$\begin{aligned} 1 + \binom{q'}{2^s-1} &= (\beta^{2^s} e^d + \beta e^{2^s d}) e^{k+(n-2^s-1)d-1} [RP(\xi)] \\ &= \tilde{w}_{(2^s+1)d} e^{k+(n-2^s-1)d-1} [RP(\xi)] = \tilde{w}_{(2^s+1)d} c^{k+(n-2^s-1)d-1} [RP(\eta)] \\ &= (\alpha^{2^s} c^d + \alpha c^{2^s d}) c^{k+(n-2^s-1)d-1} [RP(\eta)] = \binom{p'}{n-2^s} + \binom{p'}{n-1}, \end{aligned}$$

and for  $j = s + 1$  we get

$$\begin{aligned} \binom{q'}{2^{s+1}-1} &= (\beta^{2^{s+1}} e^d + \beta e^{2^{s+1}d}) e^{k+(n-2^{s+1}-1)d-1} [RP(\xi)] \\ &= (\alpha^{2^{s+1}} c^d + \alpha c^{2^{s+1}d}) c^{k+(n-2^{s+1}-1)d-1} [RP(\eta)] = \binom{p'}{n-2^{s+1}} + \binom{p'}{n-1}. \end{aligned}$$

We obtain the result by putting together the equations above.  $\square$

**Fact 2.**  $\binom{p'+2^s}{n-2^s} = 1$ .

**Proof.** This follows from Fact 1 and the fact that  $\binom{p'}{n-2^s} + \binom{p'}{n-2^{s+1}} = \binom{p'+2^s}{n-2^s}$ ; to see this, in  $Z_2[X]$  consider the relation  $X^{2^s}(1+X)^{p'} + X^{2^{s+1}}(1+X)^{p'} = X^{2^s}(1+X)^{2^s+p'}$  and see the coefficient of  $X^n$  in each member of this relation.  $\square$

**Fact 3.**  $p \geq 2^s + 1$  and  $k \geq (2^s + 1)d$ .

**Proof.** Note that, if  $i < s$ , then  $2^i$  appears in the 2-adic expansion of  $n - 2^s$  if, and only if, it appears in the 2-adic expansion of  $n - 2^{s+1}$ . Therefore Fact 1 implies that  $\binom{p'}{2^i} = 0$  for some  $s \leq i \leq u$ , and thus  $\binom{p'}{2^i} = 1$ , which shows that  $p \geq 2^s + 1$ . Further, if  $i = s$ , we have  $\binom{p'}{2^{s+1}} = 1$  because  $p$  is odd; thus  $w_{(2^s+1)d}(\eta) = \alpha^{2^s+1} \neq 0$ , which implies that  $k \geq (2^s + 1)d$ . If  $s < i \leq u$ ,  $w_{2^i d}(\eta) = \alpha^{2^i} \neq 0$  and  $k \geq 2^i d > (2^s + 1)d$ .  $\square$

Fact 3 reduces our task to show that  $p \leq 2^s + 1$ . Note that if  $\beta^{p-1}$  is a nonzero class then  $p - 1 \leq 2^s$ , and  $p \leq 2^s + 1$ . Then our strategy will consist in finding a nonzero characteristic number involving  $\beta^{p-1}$ . Let  $2^t$  be the lesser power of 2 that appears in the 2-adic expansion of  $n$ , and set  $f = \text{minimum}\{s, t\}$ . For each  $1 \leq v \leq f$  one has

$$\begin{aligned} w[2^v - 1]_{(2^v+1)d}(RP(\xi)) &= \left[ \binom{q}{2^v} + \binom{2^s}{2^v} \right] \beta^{2^v} e^d + \binom{n-2^s}{2^v} e^{(2^v+1)d} \text{ and} \\ w[2^v - 1]_{(2^v+1)d}(RP(\eta)) &= \left[ \binom{p}{2^v} + \binom{2^t}{2^v} \right] \alpha^{2^v} c^d. \end{aligned}$$

To handle these classes, we need the following

**Fact 4.** For each  $1 \leq v \leq f$  one has that  $\binom{q}{2^v} + \binom{2^s}{2^v} = \binom{p}{2^v} + \binom{2^t}{2^v} = \binom{n-2^s}{2^v}$ .

**Proof.** Take  $1 \leq v \leq f$  and, for each  $j \geq 1$ , consider the polynomial

$$\tilde{w}[2^v - 1]_{(2^{v+j+1})d} = Sq^{2^{v+j-1}d} \dots Sq^{2^v d}(w[2^v - 1]_{(2^v+1)d}).$$

By iteratively applying Lemma 3, we get

$$\begin{aligned} \tilde{w}[2^v - 1]_{(2^{v+j+1})d}(RP(\xi)) &= \left[ \binom{q}{2^v} + \binom{2^s}{2^v} \right] \beta^{2^{v+j}} e^d + \binom{n-2^s}{2^v} e^{(2^{v+j+1})d} \text{ and} \\ \tilde{w}[2^v - 1]_{(2^{v+j+1})d}(RP(\eta)) &= \left[ \binom{p}{2^v} + \binom{2^t}{2^v} \right] \alpha^{2^{v+j}} c^d. \end{aligned}$$

Using the fact that  $\beta^{2^{s+1}} = 0$ , we then get

$$\begin{aligned} (\tilde{w}[2^v - 1]_{(2^{s+1})d} e^{k+(n-2^s-1)d-1} + \tilde{w}[2^v - 1]_{(2^{s+1+1})d} e^{k+(n-2^{s+1}-1)d-1})[RP(\xi)] &= \\ = \left[ \binom{q}{2^v} + \binom{2^s}{2^v} \right] \beta^{2^s} e^{k+(n-2^s)d-1} [RP(\xi)] &= \binom{q}{2^v} + \binom{2^s}{2^v}. \end{aligned}$$

On the other hand, by using Fact, 1 we get

$$\begin{aligned} (\tilde{w}[2^v - 1]_{(2^{s+1})d} e^{k+(n-2^s-1)d-1} + \tilde{w}[2^v - 1]_{(2^{s+1+1})d} e^{k+(n-2^{s+1}-1)d-1})[RP(\eta)] &= \\ = \left[ \binom{p}{2^v} + \binom{2^t}{2^v} \right] \left( \alpha^{2^s} c^{k+(n-2^s)d-1} + \alpha^{2^{s+1}} c^{k+(n-2^{s+1})d-1} \right) [RP(\eta)] &= \\ = \left[ \binom{p}{2^v} + \binom{2^t}{2^v} \right] \left[ \binom{p'}{n-2^s} + \binom{p'}{n-2^{s+1}} \right] &= \binom{p}{2^v} + \binom{2^t}{2^v}. \end{aligned}$$

It follows that  $\binom{q}{2^v} + \binom{2^s}{2^v} = \binom{p}{2^v} + \binom{2^t}{2^v}$ . To complete the argument, first assume by contradiction that  $\binom{n-2^s}{2^v} = 0$  and  $\binom{q}{2^v} + \binom{2^s}{2^v} = \binom{p}{2^v} + \binom{2^t}{2^v} = 1$ . Let  $2^g$  be the lesser power of 2 that appears in the 2-adic expansion of  $n - 2^s$ . Note that  $2^g \geq 2^f \geq 2^v$ . Thus, for  $z = 0$  or  $1$ , we have

$$\begin{aligned} \binom{p'}{n-2^s-z2^g} &= \alpha^{2^s+z2^g} c^{k+(n-2^s-z2^g)d-1} [RP(\eta)] \\ &= \tilde{w}[2^v - 1]_{(2^s+1)d} (\tilde{w}[2^v - 1]_{(2^g+1)d})^z c^{k+(n-2^s-z2^g+z-1)d-1} [RP(\eta)] \\ &= \tilde{w}[2^v - 1]_{(2^s+1)d} (\tilde{w}[2^v - 1]_{(2^g+1)d})^z e^{k+(n-2^s-z2^g+z-1)d-1} [RP(\xi)] \\ &= \beta^{2^s+z2^g} e^{k+(n-2^s-z2^g)d-1} [RP(\xi)] = z + 1 \end{aligned}$$

because  $\beta^{2^s} [K_d P^{2^s}] = 1$  and  $\beta^{2^s+2^g} = 0$ . Thus  $\binom{p'}{n-2^s} = 1$  and  $\binom{p'}{n-2^s-2^g} = 0$ , which is impossible because  $2^g$  appears in the 2-adic expansion of  $n - 2^s$ . Now assume  $\binom{n-2^s}{2^v} = 1$ . We remark that, in this case,  $s \neq t$  and  $v = f$ ; in particular,  $2^f$  appears in the 2-adic expansion of  $n - 2^s$ . By Fact 2,  $\binom{p'+2^s}{n-2^s} = 1$ , and then  $\binom{p'+2^s}{2^f} = 1$ . Because  $2^f \leq 2^s$  we then get

$$1 = \binom{p'}{2^f} + \binom{2^s}{2^f} = \binom{p'}{2^f} + 1 + \binom{2^t}{2^f} = \binom{p}{2^f} + \binom{2^t}{2^f},$$

which gives the result.  $\square$

We return to our task of finding a nonzero characteristic number involving  $\beta^{p-1}$ . First assume  $s = t (= f)$ . We will use the class  $w[1]_{2d}$  used in Fact 1; by Fact 4 with  $v = 1$  we have  $w[1]_{2d}(RP(\xi)) = \beta^2 + \beta e^d$  and  $w[1]_{2d}(RP(\eta)) = \alpha^2 + \alpha c^d$ . We want to calculate characteristic numbers involving  $w[1]_{2d}^{p-1}$ , and this requires that  $2(p-1)d \leq \dim(RP(\eta))$ . The fact that  $\binom{p'+2^s}{n-2^s} = 1$  (Fact 2) gives that  $\binom{p'+2^s}{2^u} = 1$ ; also,  $\binom{p}{2^s} = 1$  implies that  $2^s$  does not appear in the 2-adic expansion of  $p'$ . Thus  $\binom{p'}{2^u} = 1$  and so  $2^u$  does not appear in the 2-adic expansion of  $p$ . Since  $p < 2^{u+1}$ , this gives  $p < 2^u \leq n$ . It follows that  $w_{pd}(\eta) = \alpha^p \neq 0$ , which means that  $k = \dim(\eta) \geq pd$ . In this way,  $2(p-1)d \leq nd + k - 1 = \dim(RP(\eta))$ , as desired. Now,

$$\begin{aligned} \beta^{p-1}(\beta + e^d)^{p-1} e^{nd+k-2(p-1)d-1} [RP(\xi)] &= w[1]_{2d}^{p-1} e^{nd+k-2(p-1)d-1} [RP(\xi)] \\ &= w[1]_{2d}^{p-1} c^{nd+k-2(p-1)d-1} [RP(\eta)] = \binom{p'+p-1}{n-(p-1)} = \binom{2^{u+1}-1}{n-p+1} = 1, \end{aligned}$$

and we conclude in this case that  $p = 2^s + 1$ . Now suppose  $s \neq t$ . In this case, we have  $\binom{n-2^s}{2^f} = 1$  and  $\binom{n-2^s}{2^v} = 0$  for every  $1 \leq v < f$ . Since  $p = 2^{u+1} - p'$ , to show that  $p = 2^s + 1$  it suffices to show that  $p' + 2^s = 2^{u+1} - 1$ , which in turn is equivalent to the fact that  $\binom{p'+2^s}{2^i} = 1$  for each  $1 \leq i \leq u$ . To prove this fact, first consider  $i < f (\leq t)$ . In this case, Fact 4 gives  $\binom{p}{2^i} = \binom{p}{2^i} + \binom{2^t}{2^i} = \binom{n-2^s}{2^i} = 0$  and thus  $\binom{p'}{2^i} = 1$ . Since  $i < f \leq s$ , this gives  $\binom{p'+2^s}{2^i} = 1$ , as desired. Now consider  $f \leq i \leq u$ . We use again the polynomials  $\tilde{w}[2^v - 1]_{(2^{v+j+1})d}$  considered in the proof of Fact 4; for  $v = f$  and taking into account Fact 4 we have

$$\begin{aligned} \tilde{w}[2^f - 1]_{(2^i+1)d}(RP(\eta)) &= \alpha^{2^i} c^d \text{ and} \\ \tilde{w}[2^f - 1]_{(2^i+1)d}(RP(\xi)) &= \beta^{2^i} e^d + e^{(2^i+1)d}. \end{aligned}$$

In particular,

$$\begin{aligned} \tilde{w}[2^f - 1]_{(2^s+1)d}(RP(\eta)) + c^{(2^s+1)d} &= \alpha^{2^s} c^d + c^{(2^s+1)d} = (\alpha + c^d)^{2^s} c^d \text{ and} \\ \tilde{w}[2^f - 1]_{(2^s+1)d}(RP(\xi)) + e^{(2^s+1)d} &= \beta^{2^s} e^d. \end{aligned}$$

By Fact 3,  $k + nd - (2^s + 1)d - 1 \geq 0$ , which makes possible the equation

$$1 = \beta^{2^s} e^d e^{k+nd-(2^s+1)d-1} [RP(\xi)] = (\alpha + c^d)^{2^s} c^d e^{k+nd-(2^s+1)d-1} [RP(\eta)] = \binom{p'+2^s}{n}.$$

Therefore, if  $2^i$  appears in the 2-adic expansion of  $n$ , this gives  $\binom{p'+2^s}{2^i} = 1$  and, in particular,  $\binom{p'+2^s}{2^u} = 1$ . So we can suppose that  $i < u$  and that  $2^i$  does not appear in the 2-adic expansion of  $n$ , which means that  $2^i$  appears in the 2-adic

expansion of  $n - 2^i$ . In this case, Fact 3 gives that  $k + nd - (2^s + 2^i + 2)d - 1 \geq 0$ , which makes possible the equation

$$\begin{aligned} 1 &= \beta^{2^s} e^d (\beta^{2^i} e^d + e^{(2^i+1)d}) e^{k+nd-(2^s+2^i+2)d-1} [RP(\xi)] \\ &= (\alpha + c^d)^{2^s} c^d \alpha^{2^i} c^d e^{k+nd-(2^s+2^i+2)d-1} [RP(\eta)] = \binom{p'+2^s}{n-2^i}. \end{aligned}$$

Therefore  $\binom{p'+2^s}{2^i} = 1$ , which ends the proof.

**7.  $Z_2^k$ -actions fixing  $F = K_d P^{2^s} \cup K_d P^n$  with  $n \geq 2^{s+1}$  even and  $F = \{point\} \cup K_d P^n$ .**

As mentioned in Section 1, the equivariant cobordism classifications of involutions fixing  $F = \{point\} \cup K_d P^n$  (every  $n \geq 1$ ) and  $F = K_d P^{2^s} \cup K_d P^n$  ( $s \geq 1$ ,  $n \geq 2^{s+1}$  even) are automatically extended for  $Z_2^k$ -actions; in general lines, this follows from [14], [15] and [17], where it was shown that, for certain fixed sets  $F$ , the equivariant cobordism classification of  $Z_2^k$ -actions fixing  $F$  is completely determined by the corresponding classification for involutions, which is the  $k = 1$  case. Specifically, for these  $F$ , it was shown that an involution fixing  $F$  gives rise to a special set of  $Z_2^k$ -actions fixing  $F$  which provides the desired classification. This section will be devoted to describe this  $Z_2^k$  classification.

Here,  $Z_2^k$  is the group generated by  $k$  commuting involutions  $T_1, T_2, \dots, T_k$ . Given a closed smooth manifold  $M$  and a smooth action  $\Phi : Z_2^k \times M \rightarrow M$ ,  $\Phi = (T_1, T_2, \dots, T_k)$ , the *fixed data* of  $(M; \Phi)$  is  $\eta = \bigoplus_{\rho} \varepsilon_{\rho} \rightarrow F$ , where  $F = \{x \in M / T_i(x) = x \text{ for all } 1 \leq i \leq k\}$  is the fixed point set of  $\Phi$  and  $\eta = \bigoplus_{\rho} \varepsilon_{\rho}$  is the normal bundle of  $F$  in  $M$ , decomposed into eigenbundles  $\varepsilon_{\rho}$  with  $\rho$  running through the  $2^k - 1$  nontrivial irreducible representations of  $Z_2^k$ . Next we describe the (above mentioned) special set of  $Z_2^k$ -actions fixing  $F$  coming from an involution fixing  $F$ . Let  $(W, T)$  be any involution. For each  $r$  with  $1 \leq r \leq k$  one has a  $Z_2^k$ -action  $\Gamma_r^k(W, T)$ , defined on the cartesian product  $W^{2^{r-1}} = W \times \dots \times W$  ( $2^{r-1}$  factors), and described in the following inductive way: first set  $\Gamma_1^1(W, T) = (W, T)$ . Taking  $k \geq 2$  and supposing by inductive hypothesis one has constructed  $\Gamma_{k-1}^{k-1}(W, T)$ , define  $\Gamma_k^k(W, T) = (W^{2^{k-1}}; T_1, T_2, \dots, T_k)$ , where  $(W^{2^{k-1}}; T_1, T_2, \dots, T_{k-1}) = (W^{2^{k-2}} \times W^{2^{k-2}}; T_1, T_2, \dots, T_{k-1}) = \Gamma_{k-1}^{k-1}(W, T) \times \Gamma_{k-1}^{k-1}(W, T)$  and  $T_k$  acts switching  $W^{2^{k-2}} \times W^{2^{k-2}}$  (here, the product of actions

$(M, \Phi) \times (N, \Psi)$  means the action  $(M \times N, \Phi \times \Psi)$ . This defines  $\Gamma_k^k(W, T)$  for any  $k \geq 1$ . Next, define  $\Gamma_r^k(W, T) = (W^{2^{r-1}}; T_1, T_2, \dots, T_k)$  setting  $(W^{2^{r-1}}; T_1, T_2, \dots, T_r) = \Gamma_r^r(W, T)$  and letting  $T_{r+1}, \dots, T_k$  act trivially. If  $(W, T)$  fixes  $F$  and  $\eta \rightarrow F$  is the normal bundle of  $F$  in  $W$ , then  $\Gamma_r^k(W, T)$  fixes  $F$  and its fixed data consists of  $2^{r-1}$  copies of  $\eta$ ,  $2^{r-1} - 1$  copies of the tangent bundle of  $F$  and  $2^k - 2^r$  copies of the zero-dimensional bundle over  $F$ . In particular,  $\Gamma_r^k(F \times F, twist) = (F^{2^r}; T_1, T_2, \dots, T_k)$ , where  $(T_1, T_2, \dots, T_r)$  is the usual twist  $Z_2^r$ -action on  $F^{2^r}$  which interchanges factors and  $T_{r+1}, \dots, T_k$  act trivially, with the fixed data having in this case  $2^r - 1$  copies of the tangent bundle of  $F$  and  $2^k - 2^r$  zero bundles. In this special case, we allow  $r$  to be zero, setting  $\Gamma_0^k(F \times F, twist) = (F; T_1, T_2, \dots, T_k)$ , where each  $T_i$  is the identity involution.

Now, from a given  $Z_2^k$ -action  $(M; \Phi)$ ,  $\Phi = (T_1, \dots, T_k)$ , we can obtain a collection of new  $Z_2^k$ -actions, described as follows: first, each automorphism  $\sigma : Z_2^k \rightarrow Z_2^k$  yields a new action given by  $(M; \sigma(T_1), \dots, \sigma(T_k))$ ; we denote this action by  $\sigma(M; \Phi)$ . The fixed data of  $\sigma(M; \Phi)$  is obtained from the fixed data of  $(M; \Phi)$  by a permutation of eigenbundles, obviously depending on  $\sigma$ . Next, it was shown in [18] that if  $(M; \Phi)$  has fixed data  $\bigoplus_\rho \varepsilon_\rho \rightarrow F$  and one of the eigenbundles  $\varepsilon_\theta$  is isomorphic to  $\varepsilon'_\theta \oplus 1$ , then there is an action  $(N; \Psi)$  with fixed data  $\bigoplus_\rho \mu_\rho \rightarrow F$ , where  $\mu_\rho = \varepsilon_\rho$  if  $\rho \neq \theta$  and  $\mu_\theta = \varepsilon'_\theta$ . We say in this case that  $(N; \Psi)$  is obtained from  $(M; \Phi)$  *by removing one section*. Thus, the iterative process of removing sections may possibly enlarge the set  $\{\sigma(M; \Phi), \sigma \in Aut(Z_2^k)\}$ . Summarizing, from a given involution  $(W, T)$  fixing  $F$ , we get a set of  $Z_2^k$ -actions fixing  $F$  by applying the operations  $\sigma \Gamma_r^k$  on  $(W, T)$  and next by removing the (possible) sections from the resultant eigenbundles; this is the special set of  $Z_2^k$ -actions considered in [14], [15] and [17].

Now consider fixed sets of the form  $F = \{point\} \cup V^n$ , where  $V^n$  is any connected manifold of dimension  $n$  with  $n > 0$ . In [17] it was shown that, for these fixed sets, the  $Z_2^k$  cobordism classification is determined by the  $Z_2$  classification in the sense discussed above. By combining the results of Section 3, Section 5 and the case  $n = 0$  of Section 4 with the result of [17], one has

the following theorem (in our terminology, we agree that *the set obtained from*  $(M; \Phi)$  *by removing sections* includes  $(M; \Phi)$ ):

**Theorem.** *For  $n \geq 1$ , write  $n = 2^p q$ , where  $p \geq 0$  and  $q \geq 1$  is odd, and suppose  $(M; \Phi)$  a  $Z_2^k$ -action fixing  $F = \{\text{point}\} \cup K_d P^n$ . Then,*

*i) if  $p = 0$ ,  $(M; \Phi)$  is equivariantly cobordant to an action belonging to the set  $\{\sigma \Gamma_r^k(K_d P^{n+1}, T_{0,n}), \sigma \in \text{Aut}(Z_2^k), 1 \leq r \leq k\}$ ;*

*ii) if  $p = 1$ ,  $(M; \Phi)$  is equivariantly cobordant to an action belonging to the set obtained from  $\{\sigma \Gamma_r^k \Gamma^{2d}(K_d P^{n+1}, T_{0,n}), \sigma \in \text{Aut}(Z_2^k), 1 \leq r \leq k\}$  by removing sections; and*

*iii) if  $p > 1$ ,  $(M; \Phi)$  is equivariantly cobordant to an action belonging to the set obtained from  $\{\sigma \Gamma_r^k \Gamma^{(2^p-1)d}(K_d P^{n+1}, T_{0,n}), \sigma \in \text{Aut}(Z_2^k), 1 \leq r \leq k\}$  by removing sections.*

Let  $F^n$  be a connected, smooth and closed  $n$ -dimensional manifold, where  $n > 0$ , satisfying the following property, which we call *property  $\mathcal{H}$*  : if  $N^m$  is any smooth and closed  $m$ -dimensional manifold with  $m > n$  and  $T : N^m \rightarrow N^m$  is a smooth involution whose fixed point set is  $F^n$ , then  $m = 2n$ . From [3], this implies that  $(N^m, T)$  is cobordant to  $(F^n \times F^n, \text{twist})$ , which in particular means that the cobordism classification of involutions whose fixed point set is connected and satisfies property  $\mathcal{H}$  is established. This concept was introduced and studied in Pergher and Oliveira [15], inspired by Conner and Floyd [9; 27.6] (or Conner [10; 29.2]), where it was shown that  $RP^{\text{even}}$  has this property. In [14] and [15] it was shown that for fixed sets of the form  $F = K$  or  $F = K \cup L$ , where  $K$  and  $L$  are manifolds with property  $\mathcal{H}$  and where  $\dim(K) < \dim(L)$ , again the  $Z_2^k$  classification is determined by the  $Z_2$  classification; on the other hand, in [9] (mentioned above) and in [21] it was shown that  $K_d P^n$  has property  $\mathcal{H}$  for  $n$  even and  $d = 1, 2$  and  $4$ . By combining these results with the results of Sections 5 and 6 one has the following

**Theorem.** *Let  $(M; \Phi)$  be a  $Z_2^k$ -action fixing  $K_d P^{2^s} \cup K_d P^n$ , where  $M$  is connected,  $s \geq 1$  and  $n \geq 2^{s+1}$  is even. Then  $(M; \Phi)$  is equivariantly cobordant*

to an action belonging to the set  $A \cup B$ , where the sets  $A$  and  $B$  are described below in terms of  $n$ :

i)  $n - 2^s = 2^p q$ , with  $q$  odd and  $p > 1$ , and where  $n$  is not a power of 2:

$A = \emptyset =$  the empty set;

$B =$  the set obtained from  $\{\sigma \Gamma_r^k \Gamma^{(2^p-1)d}(K_d P^{n+2^s+1}, T_{2^s, n}), \sigma \in \text{Aut}(Z_2^k), 1 \leq r \leq k\}$  by removing sections.

ii)  $n - 2 = 2q$ , where  $n$  is not a power of 2 and  $q$  is odd:

$A = \emptyset$ ;

$B =$  the set obtained from  $\{\sigma \Gamma_r^k \Gamma^{2d}(K_d P^{n+2^s+1}, T_{2^s, n}), \sigma \in \text{Aut}(Z_2^k), 1 \leq r \leq k\}$  by removing sections;

iii)  $n = 2^t$  is a power of 2 with  $t \geq 3$  (and obviously  $t > s$ ):

$A = \{\sigma \Gamma_r^k(K_d P^{2^s} \times K_d P^{2^s}, \text{twist}) \cup \sigma' \Gamma_{r-t+s-1}^k(K_d P^{2^t} \times K_d P^{2^t}, \text{twist}), \sigma, \sigma' \in \text{Aut}(Z_2^k), t - s + 1 \leq r \leq k\}$ ;

$B =$  the set obtained from  $\{\sigma \Gamma_r^k \Gamma^{2d}(K_d P^{2^t+3}, T_{2, 2^t}), \sigma \in \text{Aut}(Z_2^k), 1 \leq r \leq k\}$  by removing sections, if  $s = 1$ , and the set obtained from  $\{\sigma \Gamma_r^k \Gamma^{(2^s-1)d}(K_d P^{2^t+2^s+1}, T_{2^s, 2^t}), \sigma \in \text{Aut}(Z_2^k), 1 \leq r \leq k\}$  by removing sections, if  $s > 1$  (by dimensional reasons, in this case  $A = \emptyset$  if  $t + 1 - s > k$ );

iv)  $n = 4$ : for  $(W^{5d}, T) = \Gamma^{2d}(K_d P^3, T_{0,2}) \cup (K_d P^5, T_{0,4})$ ,

$A = \{\sigma \Gamma_{r+1}^k(K_d P^2 \times K_d P^2, \text{twist}) \cup \sigma' \Gamma_r^k(K_d P^4 \times K_d P^4, \text{twist}), \sigma, \sigma' \in \text{Aut}(Z_2^k), 0 \leq r \leq k - 1\} \cup \{\sigma \Gamma_r^k(W^{5d}, T), \sigma \in \text{Aut}(Z_2^k), 1 \leq r \leq k\}$ ;

$B =$  the set obtained from  $\{\sigma \Gamma_r^k \Gamma^{2d}(K_d P^7, T_{2,4}), \sigma \in \text{Aut}(Z_2^k), 1 \leq r \leq k\}$  by removing sections.

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