

ON THE EXTENSION OF CERTAIN MAPS WITH VALUES IN SPHERES

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ABSTRACT. Let E be an oriented, smooth and closed m -dimensional manifold with $m \geq 2$ and $V \subset E$ an oriented, pathwise connected, smooth and closed $(m - 2)$ -dimensional submanifold which is homologous to zero in E . Consider $S^{n-2} \subset S^n$ the standard inclusion, where S^n is the n -sphere and $n \geq 3$. In this paper we prove the following extension result: if $h : V \rightarrow S^{n-2}$ is a smooth map, then h extends to a smooth map $g : E \rightarrow S^n$ transverse to S^{n-2} and with $g^{-1}(S^{n-2}) = V$. Using this result, we give a new and simpler proof of a theorem of Carlos Biasi related to the *ambiental bordism* question, which consists in studying the possibility of, given a smooth closed n -dimensional manifold E and $V \subset E$ a smooth closed m -dimensional submanifold, to find a compact smooth $m + 1$ -dimensional submanifold $W \subset E$ such that the boundary of W is V .

1. Introduction

The extension problem is concerned with the possibility of, given topological spaces X, Y , a subspace $A \subset X$ and a continuous map $f : A \rightarrow Y$, to find a continuous map $g : X \rightarrow Y$ such that $g|_A = f$. For example, if D^n is the unit n -disk, with boundary $\partial(D^n) = S^{n-1}$ = the unit $(n - 1)$ -sphere, then the identity map $Id : S^{n-1} \rightarrow S^{n-1}$ cannot be extended to a map $g : D^n \rightarrow S^{n-1}$, and this non-extension result has as a consequence the famous Brouwer fixed-point theorem, which asserts that all continuous map $g : D^n \rightarrow D^n$ has a fixed point. In fact, this is a particular case of the stronger non-extension result: let M^n be any n -dimensional, connected and closed manifold and W^{n+1} an $(n + 1)$ -dimensional compact manifold such that its boundary is M^n . Then

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$Id : M^n \rightarrow M^n$ cannot be extended to a map $W^{n+1} \rightarrow M^n$. More generally, the same non-extension result is valid if we replace $Id : M^n \rightarrow M^n$ by a map connecting closed manifolds, $g : M^n \rightarrow V^n$, which induces an isomorphism in homology, $g^* : H_n(M^n) \rightarrow H_n(V^n)$, with any coefficients; for example, if g is a homotopy equivalence. Inspired in this setting, in this paper we prove the following extension result.

Theorem 1. *Let E be an oriented, smooth and closed m -dimensional manifold, with $m \geq 2$, and $V \subset E$ an oriented, pathwise connected, smooth and closed $(m - 2)$ -dimensional submanifold which is homologous (with \mathbb{Z} -coefficients) to zero in E . Consider $S^{n-2} \subset S^n$ the standard inclusion, where $n \geq 3$. Then all smooth map $h : V \rightarrow S^{n-2}$ has a smooth extension $g : E \rightarrow S^n$ transverse to S^{n-2} and with $g^{-1}(S^{n-2}) = V$.*

Theorem 1 gives a method to attack the *ambiental bordism* question, which consists in studying the possibility of, given a smooth closed n -dimensional manifold E and $V \subset E$ a smooth closed m -dimensional submanifold, to find a compact smooth $m + 1$ -dimensional submanifold $W \subset E$ such that the boundary of W is V ; in this case, we say that V *bounds in E* . Concerning this question, if $V = S^m$ and $E = S^{m+2}$ such a W is called a *Seifert surface* for the knot $S^m \rightarrow S^{m+2}$. Hirsch considered a related question in his old paper [5]; specifically, he showed that if V is an m -dimensional connected closed and oriented manifold which bounds, then there exists an embedding from V into R^n which is a boundary in R^n when $n \geq 2m$. In [6], Sato showed that every connected closed and oriented submanifold $V^m \subset S^{m+2}$ bounds in S^{m+2} . In [1], C. Biasi obtained the following result, which in particular gives Sato's result: denote by $i : V \rightarrow E$ the inclusion map and suppose E and V oriented. Suppose that (V, i) bounds as an element in the oriented cobordism group $\Omega_m(E)$. Then V bounds in E in the following cases: i) $n = m + 2$; ii) $m \leq 3$ and $n \geq m + 2$; iii) $m = 4$, $n \geq 6$ and $n \neq 7$ (evidently, $[(V, i)] = 0$ in $\Omega_m(E)$ is always a necessary condition for V to be a boundary in E). Using Theorem 1, we give a new and simpler proof of case i).

2. Proofs of the results

This section is devoted to the proofs of the results stated in Section 1. Homology and cohomology will be understood with Z -coefficients. To simplify notation, if $X \subset Y$ and $\alpha \in H_r(X)$, we use the same notation $\alpha \in H_n(Y)$ to denote the image of α under the homomorphism induced by the inclusion $X \rightarrow Y$. If W is an n -dimensional, oriented and closed manifold, we will denote by $\mu_W \in H_n(W)$ its fundamental homology class.

To prove Theorem 1, denote by $\eta \rightarrow V$ and $\nu \rightarrow S^{n-2}$ the normal bundles of V in E and S^{n-2} in S^n , and by $D(\eta)$, $D(\nu)$, $S(\eta)$ and $S(\nu)$ the associated disk bundles and sphere bundles. Here the symbols D_P , D_L and D_A will be used to denote, respectively, the Poincarè, Lefschetz and Alexander duality isomorphisms, with the convention that the domain of these maps are the cohomology Z -modules. Denote by $D(\eta)_* \subset D(\eta)$ the subset of nonzero vectors, and by $U_\eta \in H^2(D(\eta), D(\eta)_*)$ the Thom class of η . Under excision, U_η can be considered as lying in $H^2(E, E - V)$, and $D_L : H^s(D(\eta), S(\eta)) \cong H^s(D(\eta), D(\eta)_*) \rightarrow H_{m-s}(D(\eta))$ can be seen as an isomorphism $D_L : H^s(E, E - V) \rightarrow H_{m-s}(E)$; in this setting, the inclusion map $i : V \rightarrow E$ can be seen as the zero section. Write $j : E \rightarrow (E, E - V)$ for the inclusion map. If $e \in H^2(V)$ is the Euler class of η , we assert that $j_*i_*(\mu_V) = D_A(e)$ in $H_{m-2}(E, E - V)$. In fact, a basic property of Thom classes (and that is used sometimes as the definition of these classes) is that $U_\eta = D_L^{-1}i_*(\mu_V)$ (see, for example, [3, Chapter 6, Section 11]). Also it is true that the composite homomorphism

$$D_A(ji)^*D_L^{-1} : H_{m-2}(E) \rightarrow H^2(E, E - V) \rightarrow H^2(V) \rightarrow H_{m-2}(E, E - V)$$

coincides with $j_* : H_{m-2}(E) \rightarrow H_{m-2}(E, E - V)$; this follows from the fact that the rules of the duality isomorphisms are essentially the cap product with fundamental homology classes. The Euler class e is given by $e = (ji)^*(U_\eta)$, and thus $D_A(e) = D_A(ji)^*D_L^{-1}i_*(\mu_V) = j_*i_*(\mu_V)$, which shows the assertive. Since from the hypothesis $\mu_V = 0$ in $H_{m-2}(E)$, we get $e = 0$, and we assert that this implies that η is a trivial vector bundle. In fact, it is well known

that 2-dimensional oriented vector bundles over V are in one-to-one correspondence with the set of homotopy classes of maps from V into a classifying space $BSO(2)$, $[V, BSO(2)]$. A model for $BSO(2)$ is the complex projective space $CP^\infty = \lim_n CP^n$ (with the weak topology). CP^∞ is an Eilenberg-MacLane space of type $(Z, 2)$, and so $[V, CP^\infty]$ is in one-to-one correspondence with $H^2(V, Z)$; choosing a generator $\alpha \in H^2(CP^\infty, Z) \cong Z$, this correspondence can be given by $[f] \in [V, CP^\infty] \rightarrow f^*(\alpha) \in H^2(V, Z)$. On the other hand, it is also well known that the Euler class of the oriented 2-dimensional universal vector bundle over CP^∞ (which is the complex canonical line bundle) is either α or $-\alpha$. If $f \in [V, CP^\infty]$ classifies $\eta \rightarrow V$, we then have by naturality of Euler classes that, up to sign, $f^*(\alpha)$ is the Euler class of η . It follows that $f^*(\alpha) = 0$ and thus f is homotopic to a constant map, which gives that η is a trivial bundle (this outline follows from the bundle theory and the material of [3, Chapter 7, Sections 13 and 14]; alternatively, see [7, Part III]).

Since η and ν are trivial bundles, $P := D(\eta)$ and $T := D(\nu)$ are trivial disc (smooth) bundles over V and S^{n-2} , respectively. Moreover, P and T can be considered as tubular neighborhoods of V in E and S^{n-2} in S^n , respectively. Set $M := E - \text{int}(P)$, $A := S(\eta) = \partial(M) = \partial(P)$, $N := S^n - \text{int}(T)$ and $B := S(\nu) = \partial(N) = \partial(T)$. By Proposition 4.3 of [1], there exists a cross section $r : V \rightarrow A$ such that $r_*(\mu_V) = 0$ in $H_{m-2}(M)$. Note that, since S^1 is a Lie group, any (smooth) bundle $X \rightarrow B$ with fiber S^1 has the following property:

for any (smooth) sections $s_1, s_2 : B \rightarrow X$, there exists a (smooth) bundle isomorphism $g : X \rightarrow X$ inducing the identity on B and such that $s_2 = gs_1$.

Consequently, there exists a (smooth) bundle isomorphism $g : A \rightarrow V \times S^1$ such that $g(r(v)) = (v, 1)$ for every $v \in V$, where $1 \in S^1$. Let $G : P \rightarrow V \times D^2$ be a (smooth) bundle isomorphism such that $G|_A = g$ and $G(v) = (v, 0)$ for every $v \in V$, where 0 is the center of D^2 . We identify T with $S^{n-2} \times D^2$ in the standard way; then $B = S^{n-2} \times S^1$ and $N = D^{n-1} \times S^1$, with $\partial(D^{n-1}) = S^{n-2}$. Let $H : V \times D^2 \rightarrow S^{n-2} \times D^2 = T$ be defined by $H(v, w) = (h(v), w)$. Then $HG : P \rightarrow T$ is transverse to S^{n-2} and $(HG)^{-1}(S^{n-2}) = V$. Thus an extension

of $f := (H_{|V \times S^1})g : A \rightarrow B$ to a smooth map $M \rightarrow N$ gives an extension as stated in Theorem 1. To obtain this extension, the first step is finding a continuous extension $M \rightarrow N$.

Since N is an Eilenberg-MacLane space of type $(Z, 1)$, a continuous extension $M \rightarrow N$ of f exists if and only if $\delta(if)^*(\theta) = 0$ in $H^2(M, A)$, where $i : B \rightarrow N$ is the inclusion map, $\delta : H^1(A) \rightarrow H^2(M, A)$ is the coboundary homomorphism and θ is a generator of $H^1(N) \cong Z$ (see [2, Theorem 12, page 428]).

The following diagram

$$\begin{array}{ccc} H^1(A) & \xrightarrow{D_P} & H_{m-2}(A) \\ \delta \downarrow & & \downarrow k_* \\ H^2(M, A) & \xrightarrow{D_L} & H_{m-2}(M), \end{array}$$

where $k : A \rightarrow M$ is the inclusion, is commutative (see [9, page 379]). It follows that $\delta(if)^*(\theta) = 0$ if and only if $k_*(D_P(if)^*(\theta)) = 0$.

Now, we assert that $D_P(if)^*(\theta) = \pm r_*(\mu_V)$. In fact, set $f' = H_{|V \times S^1} : V \times S^1 \rightarrow B$. By Kunneth formula for cohomology, $(if')^*(\theta) = u_1 \times u_2$, where $u_1 \in H^0(V)$ and $u_2 \in H^1(S^1)$ are generators. Moreover, $\mu_{V \times S^1} = \mu_V \times \mu_{S^1}$. By property 21 in [2, page 255], with α a generator of $H_0(S^1)$, we obtain

$$D_P(if')^*(\theta) = (if')^*(\theta) \cap \mu_{V \times S^1} = (u_1 \cap \mu_V) \times (u_2 \cap \mu_{S^1}) = \mu_V \times \alpha.$$

Thus, since $g(r(v)) = (v, 1)$, by Kunneth formula for homology, we obtain that $D_P(if')^*(\theta) = \pm (gr)_*(\mu_V)$. It follows that $D_P(if)^*(\theta) = \pm r_*(\mu_V)$, because $g : A \rightarrow V \times S^1$ is a homeomorphism and $f = f'g$. Since $k_*(r_*(\mu_V)) = 0$, we obtain that $\delta(if)^*(\theta) = 0$, and consequently we get the required continuous extension $M \rightarrow N$ of f . This continuous extension can be slightly modified to give a map $M \rightarrow N$ which is smooth in a collar neighbourhood of A in M . This last map can be approximated, without changing its values in a smaller collar neighbourhood of A in M , by a smooth map $M \rightarrow N$. Together with $HG : P \rightarrow T$, this gives the desired smooth map $E \rightarrow S^n$ (for the approximation theorems for smooth maps used here and in the next corollary, see for example [4] and [8]).

Corollary (C. Biasi, [1]) *Let E be an oriented, smooth and closed m -dimensional manifold with $m \geq 2$, $V \subset E$ an oriented, pathwise connected, smooth and closed $m - 2$ -dimensional submanifold and $i : V \rightarrow E$ the inclusion map. If (V, i) bounds as an element in the oriented cobordism group $\Omega_{m-2}(E)$, then V bounds in E .*

Proof. Consider S^3 as the one-point compactification $\{(x, y, z) \in R^3 / x^2 + y^2 + z^2 < 1\} \cup \{\infty\}$ and $S^1 = \{(x, y, z) \in R^3 / x^2 + y^2 = \frac{1}{4}, z = 0\} \subset S^3$, and take $h : V \rightarrow S^1$ a constant map. Let $j : W^{m-1} \rightarrow E$ be a map that realizes the cobordism of (V, i) in $\Omega_{m-2}(E)$ and set $k : V \rightarrow W^{m-1}$ for the inclusion map. Since $i = jk$ and $\mu_V = 0$ in $H_{m-2}(W^{m-1})$, $\mu_V = 0$ in $H_{m-2}(E)$. Evidently, the inclusion $S^1 \rightarrow S^3$ has the properties of the standard inclusion $S^{n-2} \rightarrow S^n$ used in Theorem 1, hence this theorem applies to $h : V \rightarrow S^1$; as in its proof, denote by P a closed tubular neighbourhood of V in E and by T the closed tubular neighbourhood of S^1 in S^3 given by the product of S^1 and an orthogonal 2-disk of radius $\frac{1}{4}$. In the same way, set $M = E - \text{int}(P)$, $A = \partial(M) = \partial(P)$, $N = S^3 - \text{int}(T)$ and $B = \partial(N) = \partial(T)$. As we have seen, $h : V \rightarrow S^1$ extends to a smooth map $F : E \rightarrow S^3$ transverse to S^1 and with $F^{-1}(S^1) = V$. Consider the Seifert surface $D \subset S^3$ for S^1 , $D = \{(x, y, z) \in R^3 / x^2 + y^2 \leq \frac{1}{4}, z = 0\}$. Because of the construction of F in the proof of Theorem 1, the transversality condition for F and D holds at every point in $F^{-1}(D) \cap P$. Then there exists an ϵ -approximation $F' : E \rightarrow S^3$ for F which is smooth, transverse to D and with $F'|_P = F|_P$. Then $F'|_P^{-1}(S^1) = V$, and for ϵ sufficiently small the points of $E - P$ cannot be mapped by F' into S^1 . By the Thom transversality theorem one then has that $F'^{-1}(D) = W$ is an $(m - 1)$ -dimensional submanifold of E whose boundary is $F'^{-1}(S^1) = V$, and the proof is ended. \square

REFERENCES

- [1] C. Biasi, *On ambient bordism*, Pacific Journal of Math. 163, N. 1, (1994), 73-80.
- [2] E. H. Spanier, *Algebraic Topology*, McGraw-Hill Book Company, New York (1966).
- [3] G. E. Bredon, *Topology and Geometry*, GTM 139, Springer-Verlag, New York (1993).
- [4] J. W. Milnor, *Topology from the differentiable viewpoint*, Charlottesville, The University Press of Virginia (1965).
- [5] M. Hirsch, *On embedding of bounding manifolds in euclidean space*, Ann. of Math. 74, N. 2, (1961), 494-497.

- [6] N. Sato, *Cobordism of semiboundary links* , Topology Appl. 18, (1984), 225-234.
- [7] N. E. Steenrod, *The topology of fibre bundles* , Princeton Univ. Press, Princeton, New Jersey, (1951).
- [8] R. Thom, *Quelques propriétés globales de variétés différentiable* , Comment. Math. Helv. 28, (1954), 17-86.
- [9] W. S. Massey, *A basic course in Algebraic Topology*, Springer-Verlag, New York (1991).

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