

Boundary triples, Sobolev traces and self-adjoint extensions in multiply connected domains

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September 17, 2008

Abstract

A simple variation of the concept of boundary triples for hermitian operators is proposed. Then it is used to find self-adjoint extensions of some operators; in particular, with the concept of Sobolev traces, for Schrödinger operators in the plane with a hole (and bounded smooth boundary); the strategy may be adapted and generalized to a large class of situations.

1 Introduction

A simple variation of the rather recent approach to self-adjoint extensions via boundary triples (see [3, 6, 12] for descriptions of the usual procedure) is presented. The method is first exemplified with simple models, and then applied to find self-adjoint extensions of a quantum particle hamiltonian in a multiply connected domain through Sobolev traces. The examples are chosen in order to reduce technicalities so that the method becomes as clear as possible. Surely, generalizations and more general applications can be handled in a similar way. Currently it is been used to find self-adjoint extensions of the two-dimensional Aharonov-Bohm Schrödinger operator with nonzero radius solenoid [5].

In Section 2 details of the proposed variation of the usual boundary forms and boundary triples is discussed in the abstract setting of hermitian operators in a Hilbert space, including how they can be used to characterize the self-adjoint extensions; the proof presented here is simpler when compared with the usual approach [3, 6]. In Section 3 the method is exemplified by recovering the self-adjoint extensions of simple known operators. In Section 4 such boundary triples and Sobolev traces are employed to find self-adjoint extensions of the laplacian plus a simple potential (i.e., a Schrödinger operator) in the plane with a circular hole, a particular case of multiply connected domain with smooth boundary; as mentioned above, this example can be

easily generalized to similar situations with bounded and smooth boundaries. To author's the best knowledge this is the simplest characterization of those self-adjoint extensions.

Some notation. Every hermitian operator T is supposed to have domain $\text{dom } T$ dense in the complex Hilbert space \mathcal{H} (which inner product $\langle \cdot, \cdot \rangle$ is linear in the second variable), T^* will denote its adjoint, $\overline{T} = T^{**}$ its closure and $\text{K}_{\pm}(T) = \text{N}(T^* \pm iI)$, $n_{\pm}(T) = \dim \text{K}_{\pm}(T)$, the deficiency spaces and indices of T , respectively. I denotes the identity operator and/or matrix. Given a subset $A \subset B$, then $A \sqsubseteq B$ will indicate that A is dense in B (in the underlying topology). $\mathcal{H}^p(\Lambda)$, $p \geq 1$, indicates the standard Sobolev space of order p of functions in $L^2(\Lambda)$. As usual, "iff" will abbreviate "if and only if."

2 Boundary forms and triples

If T, S are hermitian operators in \mathcal{H} and $T \subset S$ (i.e., S is an extension of T), one has $T \subset S \subset S^* \subset T^*$, and so any hermitian extension of T is a hermitian restriction of T^* . In general, the larger the domain of a hermitian operator the smaller the domain of its adjoint. The choice of the domain of S has to be properly adjusted in order to get a self-adjoint extension of T ; further, a self-adjoint operator is maximal, in the sense that it has no proper hermitian extensions. Recall an important and well-known result of von Neumann used ahead: with respect to the graph inner product of T^* one has

$$\text{dom } T^* = \text{dom } \overline{T} \oplus_{T^*} \text{K}_+(T) \oplus_{T^*} \text{K}_-(T).$$

Thus, if $\zeta \in \text{dom } T^*$, then there are $\eta \in \text{dom } \overline{T}$ and $\eta_{\pm} \in \text{K}_{\pm}(T)$ so that $\zeta = \eta + \eta_+ + \eta_-$, and $T^*\zeta = \overline{T}\eta - i\eta_+ + i\eta_-$.

2.1 Boundary forms

This subsection presents some preparatory results about boundary forms necessary to discuss boundary triples; the discussion follows [12].

Definition 1. Let T be a hermitian operator. The *boundary form* of T is the sesquilinear map $\Gamma = \Gamma_{T^*} : \text{dom } T^* \times \text{dom } T^* \rightarrow \mathbb{C}$ given by

$$\Gamma(\xi, \eta) := \langle T^*\xi, \eta \rangle - \langle \xi, T^*\eta \rangle, \quad \xi, \eta \in \text{dom } T^*.$$

In case T^* is known, Γ can be used to find the closure of T , that is, \overline{T} . Since $\overline{T} = T^{**} \subset T^*$, by the definition of the adjoint operator T^{**} one has $\xi \in \text{dom } \overline{T}$ iff there is $\eta \in \mathcal{H}$ with

$$\langle \xi, T^*\zeta \rangle = \langle \eta, \zeta \rangle, \quad \forall \zeta \in \text{dom } T^*,$$

and $\eta = \overline{T}\xi$. Since $\overline{T} \subset T^*$ one has $\eta = T^*\xi$ and so the above relation is equivalent to

$$0 = \Gamma(\xi, \zeta) = \langle T^*\xi, \zeta \rangle - \langle \xi, T^*\zeta \rangle, \quad \forall \zeta \in \text{dom } T^*,$$

which is a (anti)linear equation for $\xi \in \text{dom } \overline{T}$.

Since $\Gamma(\xi, \eta) = 0, \forall \xi, \eta \in \text{dom } T^*$, iff T^* is self-adjoint, that is, iff T is essentially self-adjoint, then Γ quantifies the “lack of self-adjointness” of T^* .

Proposition 1. *If T is hermitian then*

$$\text{dom } \overline{T} = \{\xi \in \text{dom } T^* : \Gamma(\xi, \eta_{\pm}) = 0, \forall \eta_{\pm} \in \text{K}_{\pm}(T)\}.$$

Proof. If $\zeta \in \text{dom } T^*$, then by von Neumann result $\zeta = \eta + \eta_+ + \eta_-$, with $\eta \in \text{dom } \overline{T}$, and $\eta_{\pm} \in \text{K}_{\pm}(T)$. Since $\Gamma(\xi, \eta) = 0$ for all $\xi \in \text{dom } T^*, \eta \in \text{dom } \overline{T}$, it follows that $\xi \in \text{dom } \overline{T}$ iff for all $\zeta \in \text{dom } T^*$

$$0 = \Gamma(\xi, \zeta) = \Gamma(\xi, \eta + \eta_+ + \eta_-) = \Gamma(\xi, \eta_+ + \eta_-).$$

The result follows. \square

Let $\zeta^1 = \eta^1 + \eta_+^1 + \eta_-^1$ and $\zeta^2 = \eta^2 + \eta_+^2 + \eta_-^2$, with $\eta^1, \eta^2 \in \text{dom } \overline{T}$, $\eta_+^1, \eta_+^2 \in \text{K}_+(T)$, $\eta_-^1, \eta_-^2 \in \text{K}_-(T)$, be general elements of $\text{dom } T^*$; since $T^*\eta_{\pm} = \mp i\eta_{\pm}$, it follows that

$$\Gamma(\zeta^1, \zeta^2) = \Gamma(\eta_+^1 + \eta_-^1, \eta_+^2 + \eta_-^2) = 2i (\langle \eta_+^1, \eta_+^2 \rangle - \langle \eta_-^1, \eta_-^2 \rangle).$$

It is then clear that the nonvanishing of Γ is related to the deficiency subspaces. Boundary forms can be used to find self-adjoint extensions of T by noting that such extensions are restrictions of T^* on suitable domains \mathcal{D} so that $\Gamma(\xi, \eta) = 0, \forall \xi, \eta \in \mathcal{D}$ (Lemma 1). Recall another von Neumann result: each self-adjoint extension of T is related to a unitary operator $\mathcal{U} : \text{K}_-(T) \rightarrow \text{K}_+(T)$ onto $\text{K}_+(T)$ (thus $n_-(T) = n_+(T)$); denote by $T_{\mathcal{U}}$ the corresponding self-adjoint extension, whose domain is $\text{dom } T_{\mathcal{U}} = \{\eta = \zeta + \eta_- - \mathcal{U}\eta_- : \zeta \in \text{dom } \overline{T}, \eta_- \in \text{K}_-(T)\}$. Then, explicitly one has

Lemma 1. *The boundary form Γ_{T^*} restricted to $\text{dom } T_{\mathcal{U}}$ vanishes identically.*

Proof. For any two elements $\eta = \zeta_1 + \eta_- - \mathcal{U}\eta_-$ and $\xi = \zeta_2 + \xi_- - \mathcal{U}\xi_-$ in $\text{dom } T_{\mathcal{U}}$ ($\zeta_1, \zeta_2 \in \text{dom } \overline{T}$) one has

$$\Gamma(\xi, \eta) = 2i (\langle \mathcal{U}\xi_-, \mathcal{U}\eta_- \rangle - \langle \xi_-, \eta_- \rangle) = 0,$$

since \mathcal{U} is unitary. \square

Proposition 2. *Assume that T has self-adjoint extensions. Then each self-adjoint extension of T is of the form*

$$\text{dom } T_{\mathcal{U}} = \{\xi \in \text{dom } T^* : \Gamma(\xi, \eta_- - \mathcal{U}\eta_-) = 0, \forall \eta_- \in \mathbf{K}_-(T)\},$$

$$T_{\mathcal{U}}\xi = T^*\xi, \xi \in \text{dom } T_{\mathcal{U}}.$$

Proof. If $T_{\mathcal{U}}$ is a self-adjoint extension of T , then $\text{dom } T_{\mathcal{U}} = \{\eta = \zeta + \eta_- - \mathcal{U}\eta_- : \zeta \in \text{dom } \bar{T}, \eta_- \in \mathbf{K}_-(T)\}$; since Γ restricted to $\text{dom } T_{\mathcal{U}}$ vanishes, by Proposition 1 one has, for $\xi \in \text{dom } T_{\mathcal{U}}$,

$$0 = \Gamma(\xi, \zeta + \eta_- - \mathcal{U}\eta_-) = \Gamma(\xi, \eta_- - \mathcal{U}\eta_-), \quad \forall \eta_- \in \mathbf{K}_-.$$

Hence, $\text{dom } T_{\mathcal{U}} \subset A := \{\xi \in \text{dom } T^* : \Gamma(\xi, \eta_- - \mathcal{U}\eta_-) = 0, \forall \eta_- \in \mathbf{K}_-(T)\}$.

Now, given \mathcal{U} , consider the linear equation for $\zeta + \xi_- + \xi_+ = \xi \in \text{dom } T^*$ (of course $\xi_{\pm} \in \mathbf{K}_{\pm}(T)$)

$$0 = \Gamma(\xi, \eta_- - \mathcal{U}\eta_-), \quad \forall \eta_- \in \mathbf{K}_-(T).$$

By Lemma 1, any $\xi \in \text{dom } T_{\mathcal{U}}$ is a solution of this equation. Let $\xi \in \text{dom } T^*$ be a solution and write

$$\xi = \zeta + \xi_- - \mathcal{U}\xi_- + \xi_+ + \mathcal{U}\xi_-;$$

thus

$$\begin{aligned} 0 &= \Gamma(\xi, \eta_- - \mathcal{U}\eta_-) = \Gamma(\xi_- - \mathcal{U}\xi_- + \xi_+ + \mathcal{U}\xi_-, \eta_- - \mathcal{U}\eta_-) \\ &= 2i (\langle (\xi_+ + \mathcal{U}\xi_-) - \mathcal{U}\xi_-, -\mathcal{U}\eta_- \rangle - \langle \xi_-, \eta_- \rangle) \\ &= 2i (\langle \xi_+ + \mathcal{U}\xi_-, -\mathcal{U}\eta_- \rangle + \langle \mathcal{U}\xi_-, \mathcal{U}\eta_- \rangle - \langle \xi_-, \eta_- \rangle) \\ &= 2i \langle \xi_+ + \mathcal{U}\xi_-, -\mathcal{U}\eta_- \rangle, \quad \forall \eta_- \in \mathbf{K}_-(T). \end{aligned}$$

Since $\text{rng } \mathcal{U} = \mathbf{K}_+$, it follows that $\xi_+ + \mathcal{U}\xi_- = 0$, or $\xi_+ = -\mathcal{U}\xi_-$; thus $\xi = \zeta + \xi_- - \mathcal{U}\xi_- \in \text{dom } T_{\mathcal{U}}$ so that $A \subset \text{dom } T_{\mathcal{U}}$. Therefore $\text{dom } T_{\mathcal{U}} = A$, and the proposition is proved. \square

Remark 1. *Note that the specification of the self-adjoint extensions $T_{\mathcal{U}}$ in Proposition 2 does not require the explicit knowledge of \bar{T} .*

2.2 Boundary triples

A boundary triple is an abstraction of the notion of boundary values in function spaces; this idea comes back to Calkin in 1939 [4] and Vishik in 1952 [11]. The following proposed definition is a variation of the usual one [6, 3] which, in some sense, can be considered natural.

Definition 2. Let T be a hermitian operator with $n_-(T) = n_+(T)$. A *modified boundary triple* $(\mathbf{h}, \rho_1, \rho_2)$ for T is composed of a Hilbert space \mathbf{h} and two linear maps $\rho_1, \rho_2 : \text{dom } T^* \rightarrow \mathbf{h}$ with dense ranges and so that

$$a\Gamma_{T^*}(\xi, \eta) = \langle \rho_1(\xi), \rho_1(\eta) \rangle - \langle \rho_2(\xi), \rho_2(\eta) \rangle, \quad \forall \xi, \eta \in \text{dom } T^*,$$

for some constant $0 \neq a \in \mathbb{C}$. Note that $\langle \cdot, \cdot \rangle$ is also denoting the inner product in \mathbf{h} .

In general, given a hermitian operator T with equal deficiency indices, different boundary triples (usually, the adjective “modified” will be kept understood) can be associated with it. Since for $\zeta^1, \zeta^2 \in \text{dom } T^*$ (by using the above notation)

$$\Gamma(\zeta^1, \zeta^2) = 2i (\langle \eta_+^1, \eta_+^2 \rangle - \langle \eta_-^1, \eta_-^2 \rangle),$$

only the deficiency subspaces effectively appear in the boundary form, and so one may take either $\mathbf{h} = \mathbf{K}_-(T)$ or $\mathbf{h} = \mathbf{K}_+(T)$ (with ρ properly chosen); in this case, say $\mathbf{h} = \mathbf{K}_-(T)$, by von Neumann theory it is known that self-adjoint extensions are in one-to-one relation with unitary operators $\mathcal{U} : \mathbf{K}_-(T) \rightarrow \mathbf{K}_+(T)$. However, it is convenient to allow a general \mathbf{h} with $\dim \mathbf{h} = n_+(T)$ (recall that two Hilbert spaces are unitarily equivalent iff they have the same dimension), and Theorem 1 will adapt von Neumann theory to this situation.

Again, self-adjoint extensions of T are restrictions of T^* on suitable domains \mathcal{D} so that $\Gamma(\xi, \eta) = 0, \forall \xi, \eta \in \mathcal{D}$, and given a boundary triple for T , such \mathcal{D} are related to isometric maps $\hat{\mathcal{U}} : \mathbf{h} \rightarrow \mathbf{h}$ (which can be taken to be onto; extend it by continuity, if necessary) so that $\hat{\mathcal{U}}\rho_1(\xi) = \rho_2(\xi)$ and

$$\langle \rho_1(\xi), \rho_1(\eta) \rangle = \langle \rho_2(\xi), \rho_2(\eta) \rangle = \langle \hat{\mathcal{U}}\rho_1(\xi), \hat{\mathcal{U}}\rho_1(\eta) \rangle, \quad \forall \xi, \eta \in \mathcal{D}.$$

Next the linearity of $\hat{\mathcal{U}}$ will be established.

Lemma 2. *Each isometry $\hat{\mathcal{U}}$ above is a linear and unitary map.*

Proof. Note that $\text{rng } \hat{\mathcal{U}} = \mathbf{h}$ and it will suffice to show that this operator is invertible and linear. To simplify the notation, ρ_1 and ρ_2 will not appear in what follows.

If $\hat{\mathcal{U}}(\xi) = \hat{\mathcal{U}}(\eta)$, then

$$\begin{aligned} 0 &= \langle \hat{\mathcal{U}}(\xi) - \hat{\mathcal{U}}(\eta), \hat{\mathcal{U}}(\xi) - \hat{\mathcal{U}}(\eta) \rangle \\ &= \langle \hat{\mathcal{U}}(\xi), \hat{\mathcal{U}}(\xi) \rangle - \langle \hat{\mathcal{U}}(\xi), \hat{\mathcal{U}}(\eta) \rangle - \langle \hat{\mathcal{U}}(\eta), \hat{\mathcal{U}}(\xi) \rangle + \langle \hat{\mathcal{U}}(\eta), \hat{\mathcal{U}}(\eta) \rangle \\ &= \langle \xi, \xi \rangle - \langle \xi, \eta \rangle - \langle \eta, \xi \rangle + \langle \eta, \eta \rangle = \|\xi - \eta\|^2; \end{aligned}$$

therefore $\xi = \eta$ and so $\hat{\mathcal{U}}$ is injective and $\hat{\mathcal{U}}^{-1} : \mathbf{h} \rightarrow \mathbf{h}$ exists.

If $\hat{\mathcal{U}}^{-1}(\xi_1) = \xi$ and $\hat{\mathcal{U}}^{-1}(\eta_1) = \eta$, since by hypothesis $\langle \hat{\mathcal{U}}(\xi), \hat{\mathcal{U}}(\eta) \rangle = \langle \xi, \eta \rangle$, $\forall \xi, \eta$, then $\langle \xi_1, \eta_1 \rangle = \langle \hat{\mathcal{U}}^{-1}(\xi_1), \hat{\mathcal{U}}^{-1}(\eta_1) \rangle$; since $\hat{\mathcal{U}}$ is bijective such relation holds for every vector in the space. In this relation, if $\xi_1 = \hat{\mathcal{U}}(\xi_2)$, then $\langle \hat{\mathcal{U}}(\xi_2), \eta_1 \rangle = \langle \xi_2, \hat{\mathcal{U}}^{-1}(\eta_1) \rangle$, again for all vectors of \mathbf{h} .

Now, for all $\eta, \xi, \zeta \in \mathbf{h}$ and $a, b \in \mathbb{C}$, one has

$$\begin{aligned} \langle \hat{\mathcal{U}}(a\xi + b\eta), \zeta \rangle &= \langle a\xi + b\eta, \hat{\mathcal{U}}^{-1}(\zeta) \rangle \\ &= a \langle \xi, \hat{\mathcal{U}}^{-1}(\zeta) \rangle + b \langle \eta, \hat{\mathcal{U}}^{-1}(\zeta) \rangle \\ &= a \langle \hat{\mathcal{U}}(\xi), \zeta \rangle + b \langle \hat{\mathcal{U}}(\eta), \zeta \rangle = \langle a\hat{\mathcal{U}}(\xi) + b\hat{\mathcal{U}}(\eta), \zeta \rangle, \end{aligned}$$

showing that $\hat{\mathcal{U}}(a\xi + b\eta) = a\hat{\mathcal{U}}(\xi) + b\hat{\mathcal{U}}(\eta)$, that is, $\hat{\mathcal{U}}$ is linear. \square

Theorem 1. *Let T be a hermitian operator with equal deficiency indices. If $(\mathbf{h}, \rho_1, \rho_2)$ is a (modified) boundary triple for T , then the self-adjoint extensions $T_{\hat{\mathcal{U}}}$ of T are precisely*

$$\text{dom } T_{\hat{\mathcal{U}}} = \{\xi \in \text{dom } T^* : \rho_2(\xi) = \hat{\mathcal{U}}\rho_1(\xi)\}, \quad T_{\hat{\mathcal{U}}}\xi = T^*\xi,$$

for each unitary map $\hat{\mathcal{U}} : \mathbf{h} \rightarrow \mathbf{h}$.

Proof. A necessary condition for the restriction of T^* to a domain \mathcal{D} be self-adjoint is that the corresponding boundary form vanishes identically on \mathcal{D} . Given the boundary triple, taking into account Lemma 2 and the discussion that precedes it, Lemma 1 and Proposition 2, such \mathcal{D} 's are necessarily obtained through unitary maps $\hat{\mathcal{U}} : \mathbf{h} \rightarrow \mathbf{h}$ and it is enough to check that actually each $T_{\hat{\mathcal{U}}}$ is self-adjoint.

Clearly $T_{\hat{\mathcal{U}}}$ is a hermitian extension of T . If $\eta \in \text{dom } T_{\hat{\mathcal{U}}}^*$ one has

$$\langle T_{\hat{\mathcal{U}}}^*\eta, \xi \rangle = \langle \eta, T_{\hat{\mathcal{U}}}\xi \rangle = \langle \eta, T_{\hat{\mathcal{U}}}^*\xi \rangle, \quad \forall \xi \in \text{dom } T_{\hat{\mathcal{U}}}.$$

Then,

$$\begin{aligned}
0 &= \Gamma_{T_{\hat{\mathcal{U}}}}(\eta, \xi) = \langle T_{\hat{\mathcal{U}}}^* \eta, \xi \rangle - \langle \eta, T_{\hat{\mathcal{U}}}^* \xi \rangle \\
&= \langle \rho_1(\eta), \rho_1(\xi) \rangle - \langle \rho_2(\eta), \rho_2(\xi) \rangle \\
&= \langle \rho_1(\eta), \rho_1(\xi) \rangle - \langle \rho_2(\eta), \hat{\mathcal{U}} \rho_1(\xi) \rangle \\
&= \langle \rho_1(\eta), \rho_1(\xi) \rangle - \langle \hat{\mathcal{U}}^* \rho_2(\eta), \rho_1(\xi) \rangle \\
&= \langle \rho_1(\eta) - \hat{\mathcal{U}}^* \rho_2(\eta), \rho_1(\xi) \rangle, \quad \forall \xi \in \text{dom } T_{\hat{\mathcal{U}}}.
\end{aligned}$$

Since ρ_1 has dense range in \mathbf{h} , it follows that $\rho_1(\eta) - \hat{\mathcal{U}}^* \rho_2(\eta) = 0$, that is, $\rho_2(\eta) = \hat{\mathcal{U}} \rho_1(\eta)$ and $\eta \in \text{dom } T_{\hat{\mathcal{U}}}$. Therefore, $T_{\hat{\mathcal{U}}}$ is self-adjoint. \square

Often a boundary triple for differential operators gives self-adjoint extensions in terms of boundary conditions, and different choices of the triple correspond to different parametrizations of such extensions. In applications sometimes it is convenient to distinguish the spaces $\rho_1(\mathbf{h})$ from $\rho_2(\mathbf{h})$ by different symbols.

3 Simple examples

In this section boundary forms will be used to get explicitly self-adjoint extensions of unidimensional Schrödinger operators. The main goal is to illustrate the method, but such examples will also indicate the way of dealing with more involved situations—see Section 4.

Example 1. [Free particle in a half-line] The initial energy operator is $H\psi = -\psi''$, $\text{dom } H = C_0^\infty(0, \infty) \subseteq L^2(0, \infty)$, which has $n_- = n_+ = 1$. Also H^* has the same action as H but with $\text{dom } H^* = \mathcal{H}^2[0, \infty)$ and, for $\psi, \varphi \in \text{dom } H^*$, on integrating by parts, the boundary form is

$$\Gamma(\psi, \varphi) = \overline{\psi'(0)}\varphi(0) - \overline{\psi(0)}\varphi'(0),$$

since the elements of $\text{dom } H^*$ vanish at infinity. Now define the vector spaces $X := \{\Psi = \psi(0) - i\psi'(0) : \psi \in \text{dom } H^*\}$ and the map $Y = \rho(X) := \{\rho(\Psi) = \psi(0) + i\psi'(0) : \Psi \in X\}$, and observe that

$$\langle \Psi, \Phi \rangle - \langle \rho(\Psi), \rho(\Phi) \rangle = 2i\Gamma(\psi, \varphi)$$

(of course $\Phi = \varphi(0) - i\varphi'(0)$), so that a boundary triple was found (think of $X = \rho_1(\text{dom } H^*)$ and $Y = \rho_2(\text{dom } H^*)$, with $\rho_2 = \rho$ above.)

Now, according to Theorem 1, a domain \mathcal{D} so that the restriction $H^*|_{\mathcal{D}}$ is self-adjoint is characterized by unitary maps between X and Y . Since X and Y are unidimensional, such unitary maps are multiplication by $e^{i\theta}$ for fixed $0 \leq \theta < 2\pi$. Therefore, the domain of self-adjoint extensions of H are so that $\Psi = e^{i\theta} \rho(\Psi)$ for all $\Psi \in X$. Writing out such relation

$$\psi(0) - i\psi'(0) = e^{i\theta} (\psi(0) + i\psi'(0)),$$

and so $(1 - e^{i\theta})\psi(0) = i(1 + e^{i\theta})\psi'(0)$; if $\theta \neq 0$ one has

$$\psi(0) = \lambda\psi'(0), \quad \lambda = i\frac{(1 + e^{i\theta})}{(1 - e^{i\theta})} \in \mathbb{R}.$$

Therefore the self-adjoint extensions H_λ of H are characterized by the following boundary conditions

$$\text{dom } H_\lambda = \{\psi \in \mathcal{H}^2[0, \infty) : \psi(0) = \lambda\psi'(0)\}, \quad H_\lambda\psi = -\psi'',$$

for each $\lambda \in \mathbb{R} \cup \{\infty\}$. The value $\lambda = \infty$ is for including $\theta = 0$, which corresponds to Neumann boundary condition $\psi'(0) = 0$. Dirichlet boundary condition occurs for $\lambda = 0$. Thus, this well-known example has been recovered via such boundary triple.

Example 2. [Free particle on an interval] The initial energy operator is $H\psi = -\psi''$, $\text{dom } H = C_0^\infty(0, 1)$; one has $n_- = n_+ = 2$. Also H^* has the same action as H but with $\text{dom } H^* = \mathcal{H}^2[0, 1]$ and the boundary form is, for $\psi, \varphi \in \text{dom } H^*$, again upon integrating by parts,

$$\Gamma(\psi, \varphi) = \overline{\psi(1)}\varphi'(1) - \overline{\psi'(1)}\varphi(1) - \overline{\psi(0)}\varphi'(0) + \overline{\psi'(0)}\varphi(0).$$

Based on Example 1, define the two-dimensional vector spaces of elements

$$\Psi = \begin{pmatrix} \psi'(0) - i\psi(0) \\ \psi'(1) + i\psi(1) \end{pmatrix}, \quad \rho(\Psi) = \begin{pmatrix} \psi'(0) + i\psi(0) \\ \psi'(1) - i\psi(1) \end{pmatrix},$$

for $\psi \in \text{dom } H^*$. A direct evaluation of inner products leads to

$$\langle \Psi, \Phi \rangle - \langle \rho(\Psi), \rho(\Phi) \rangle = -2i\Gamma(\psi, \varphi),$$

and a boundary triple was found.

By Theorem 1, a domain \mathcal{D} so that $H^*|_{\mathcal{D}}$ is self-adjoint is characterized by a unitary 2×2 matrix \hat{U} so that $\Psi = \hat{U}\rho(\Psi)$ for all Ψ . Writing out such relation one obtains the boundary conditions

$$(I - \hat{U}) \begin{pmatrix} \psi'(0) \\ \psi'(1) \end{pmatrix} = -i(I + \hat{U}) \begin{pmatrix} -\psi(0) \\ \psi(1) \end{pmatrix}$$

and the domain of the corresponding self-adjoint extension $H_{\hat{U}}$ of H is composed of the elements $\psi \in \mathcal{H}^2[0, 1]$ so that the above boundary conditions are satisfied; also $H_{\hat{U}}\psi = -\psi''$.

In case $(I + \hat{U})$ is invertible (similarly if $(I - \hat{U})$ is invertible) one can write the above boundary conditions as

$$A \begin{pmatrix} \psi'(0) \\ \psi'(1) \end{pmatrix} = \begin{pmatrix} -\psi(0) \\ \psi(1) \end{pmatrix},$$

where $A = i \left(I + \hat{U} \right)^{-1} \left(I - \hat{U} \right)$ is a self-adjoint 2×2 matrix. By allowing some entries of A taking the value ∞ it is possible to recover some cases $\left(I + \hat{U} \right)$ is not invertible; nevertheless, it is not always a simple task to recover all such cases, so that the boundary conditions in terms of \hat{U} seems preferable.

The choices for the matrix \hat{U}

$$\text{a) } I, \quad \text{b) } -I, \quad \text{c) } \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \text{d) } \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix},$$

impose, respectively, the boundary conditions: a) $\psi(0) = 0 = \psi(1)$ (Dirichlet); b) $\psi'(0) = 0 = \psi'(1)$ (Neumann); c) $\psi(0) = \psi(1)$ and $\psi'(0) = \psi'(1)$ (periodic); d) $\psi(0) = -\psi(1)$ and $\psi'(0) = -\psi'(1)$ (antiperiodic).

The end point a is *regular* for the differential operator

$$H = -\frac{d^2}{dx^2} + V(x), \quad \text{dom } H = C_0^\infty(a, b) \subseteq L^2(a, b),$$

with $-\infty < a < b < +\infty$ if for some $c \in (a, b)$ (and so for all such c) one has $\int_a^c |V(x)| dx := \lim_{d \rightarrow a^+} \int_d^c |V(x)| dx < \infty$; b is regular for H if $\int_c^b |V(x)| dx := \lim_{d \rightarrow b^-} \int_c^d |V(x)| dx < \infty$. If an end point is not regular it is called *singular*.

From the theory of differential equations [10] it is known that the space of solutions of the K_\mp -equation

$$H^* \psi = -\psi'' + V\psi = \pm i\psi, \quad \psi \in \text{dom } H^*,$$

is two-dimensional and if a is a regular point then any solution ψ has finite limits $\psi(a) := \psi(a^+) = \lim_{x \rightarrow a^+} \psi(x)$ and $\psi'(a) := \psi'(a^+) = \lim_{x \rightarrow a^+} \psi'(x)$; if a is singular then such limits can be divergent. Ahead, $W_x[\psi, \varphi] = \overline{\psi(x)}\varphi'(x) - \overline{\psi'(x)}\varphi(x)$ is the wronskian of ψ, φ at $x \in (a, b)$, and $W_a[\psi, \varphi] := \lim_{x \rightarrow a^+} W_x[\psi, \varphi]$, $W_b[\psi, \varphi] := \lim_{x \rightarrow b^-} W_x[\psi, \varphi]$.

Example 3. If the potential V is such that both end points $0, 1$ are regular, then the deficiency indices of $H\psi = -\psi'' + V\psi$, $\text{dom } H = C_0^\infty(0, 1)$, are equal to 2, and for any $\psi \in \text{dom } H^*$ the boundary values $\psi(0), \psi(1), \psi'(0), \psi'(1)$ are well defined. Thus, its self-adjoint extensions can be characterized in the same way as in Example 2, through the same boundary conditions. Particular cases are

$$V(x) = a \ln x, \quad V(x) = a/x^\alpha, \quad \alpha < 1, \quad a \in \mathbb{R}.$$

For singular endpoints the limit values of ψ, ψ' could not exist, so that the strategy presented in the above examples is not guaranteed to work. However, in some cases it is possible to properly adapt that strategy in order to get self-adjoint extensions. This will be illustrated in the next example.

Example 4. The self-adjoint extensions of $\text{dom } H = C_0^\infty(0, 1) \subseteq L^2(0, 1)$,

$$(H\psi)(x) = -\psi''(x) - \frac{1}{4x^2}\psi(x), \quad \psi \in \text{dom } H,$$

will be found. If Ω is an open subset of \mathbb{R} , $\text{AC}(\Omega)$ indicates the set of absolutely continuous functions in every bounded and closed subinterval of Ω .

If $\psi \in \text{dom } H^* = \{\psi \in L^2(0, 1) : \psi, \psi' \in \text{AC}(0, 1), (-\psi'' - \psi/(4x^2)) \in L^2(0, 1)\}$ one has

$$H^*\psi = -\psi'' - \frac{1}{4x^2}\psi := u \in L^2(0, 1),$$

which is a nonhomogeneous second order linear differential equation for ψ ; the general solution of the corresponding homogeneous equation $H^*\psi = 0$ is $b_1\psi_1(x) + b_2\psi_2(x)$, $b_1, b_2 \in \mathbb{C}$, with $\psi_1(x) = \sqrt{x}$ and $\psi_2(x) = \sqrt{x} \ln x$, whose wronskian is $W_x[\psi_1, \psi_2] = 1$, $\forall x \in [0, 1]$. Introduce $\varphi = \psi/\sqrt{x}$ so that

$$(x\varphi')' = x\varphi'' + \varphi' = -\sqrt{x}u,$$

and since $\sqrt{x}u \in L^1[0, 1]$, on integrating one gets

$$\varphi'(x) = \frac{b_2}{x} - \frac{1}{x} \int_0^x \sqrt{s}u(s) ds.$$

By Cauchy-Schwarz, the function $x \mapsto \frac{1}{x} \int_0^x \sqrt{s}u(s) ds$ is also integrable in $[0, 1]$, so that

$$\varphi(x) = b_1 + b_2 \log x - \int_0^x \frac{ds}{s} \int_0^s \sqrt{t}u(t) dt$$

and finally $\psi(x) = b_1\sqrt{x} + b_2\sqrt{x} \log x + v_\psi(x)$, (note that $b_j = b_j(\psi)$, $j = 1, 2$) with

$$v_\psi(x) = -\sqrt{x} \int_0^x \frac{ds}{s} \int_0^s \sqrt{t}u(t) dt.$$

By Cauchy-Schwarz again,

$$|v_\psi(x)| \leq \sqrt{x} \int_0^x \frac{ds}{s} \left| \int_0^s \sqrt{t}u(t) \right| dt$$

$$\leq \sqrt{x} \int_0^x \frac{ds}{s} \frac{s}{\sqrt{2}} \|u\|_2 = \frac{x^{3/2}}{\sqrt{2}} \|u\|_2,$$

so that v_ψ is differentiable, $v_\psi(x) \sim x^{3/2}$, $v'_\psi(x) \sim x^{1/2}$ for $x \rightarrow 0$. The above procedure is an alternative to the use of the variation of parameters formula.

The boundary form is, for $\psi, \varphi \in \text{dom } H^*$, $\psi(x) = b_1(\psi)\sqrt{x} + b_2(\psi)\sqrt{x} \log x + v_\psi(x)$ and $\varphi(x) = b_1(\varphi)\sqrt{x} + b_2(\varphi)\sqrt{x} \log x + v_\varphi(x)$,

$$\begin{aligned} \Gamma(\psi, \varphi) &= W_1[\psi, \varphi] - W_0[\psi, \varphi] \\ &= \overline{\psi(1)}\varphi'(1) - \overline{\psi'(1)}\varphi(1) + \lim_{x \rightarrow 0^+} \left(-\overline{\psi(x)}\varphi'(x) + \overline{\psi'(x)}\varphi(x) \right) \\ &= \overline{\psi(1)}\varphi'(1) - \overline{\psi'(1)}\varphi(1) - \overline{b_1(\psi)}b_2(\varphi) + b_1(\varphi)\overline{b_2(\psi)}. \end{aligned}$$

Based on Example 1, define the two-dimensional vector spaces of elements

$$\Psi = \begin{pmatrix} b_2(\psi) - ib_1(\psi) \\ \psi'(1) + i\psi(1) \end{pmatrix}, \quad \rho(\Psi) = \begin{pmatrix} b_2(\psi) + ib_1(\psi) \\ \psi'(1) - i\psi(1) \end{pmatrix},$$

for $\psi \in \text{dom } H^*$. A direct evaluation of inner products leads to

$$\langle \Psi, \Phi \rangle - \langle \rho(\Psi), \rho(\Phi) \rangle = -2i\Gamma(\psi, \varphi),$$

and a boundary triple was found. The self-adjoint extensions $H_{\hat{U}}$ of H are associated with 2×2 unitary matrices \hat{U} that entail the following boundary conditions

$$\left(I - \hat{U} \right) \begin{pmatrix} b_2(\psi) \\ \psi'(1) \end{pmatrix} = -i \left(I + \hat{U} \right) \begin{pmatrix} -b_1(\psi) \\ \psi(1) \end{pmatrix},$$

that is, the domain of the self-adjoint extension $H_{\hat{U}}$ of H is composed of the elements $\psi \in \text{dom } H^*$ so that the above boundary conditions are satisfied; also $H_{\hat{U}}\psi = H^*\psi$, $\psi \in \text{dom } H_{\hat{U}}$. The reader can play with different choices of \hat{U} in order to get explicit self-adjoint extensions.

Example 5. Let $T = -id/dx$ with

$$\text{dom } T = C_0^\infty(\mathbb{R} \setminus \{0\}) = C_0^\infty(-\infty, 0) \oplus C_0^\infty(0, \infty).$$

One point was removed and the self-adjoint extensions are obtained from $\text{dom } T^*$ through suitable matching conditions at the origin (recall that in case the domain is $C_0^\infty(\mathbb{R})$ the operator T is essentially self-adjoint). Set $T_1 = T|_{C_0^\infty(-\infty, 0)}$ and $T_2 = T|_{C_0^\infty(0, \infty)}$, so that $T = T_1 \oplus T_2$. One has $\text{dom } T^* = \{\psi \in \text{AC}(\mathbb{R} \setminus \{0\}) : \psi' \in L^2(\mathbb{R})\}$, $T^*\psi = -i\psi'$. Further,

$$\text{dom } T_1^* = \{\psi \in \text{AC}(-\infty, 0) : \psi' \in L^2(-\infty, 0]\},$$

$$\text{dom } T_2^* = \{\psi \in \text{AC}(0, \infty) : \psi' \in L^2[0, \infty)\},$$

and T_1^*, T_2^* with the same action as T^* .

In order to find the deficiency indices consider the K_{\pm} -equations

$$(T_2^* \pm iI)\psi_{\pm} = 0,$$

whose solutions are proportional to $\psi_{\pm}(x) = e^{\pm x}$. Similarly for T_1 . Hence $n_-(T_1) = 0 = n_+(T_2)$, $n_-(T_2) = 1 = n_+(T_1)$, and combining these values one obtains $n_-(T) = 1 = n_+(T)$. Now, for $\psi, \varphi \in \text{dom } T^*$ the following boundary form is found (on integrating by parts):

$$\begin{aligned} \Gamma(\psi, \varphi) &= \langle T^* \psi, \varphi \rangle - \langle \psi, T^* \varphi \rangle \\ &= \left(\int_{-\infty}^{0^-} + \int_{0^+}^{\infty} \right) dx \left(\overline{(-i\psi'(x))} \varphi(x) - \overline{\psi(x)} (-i\varphi'(x)) \right) \\ &= i \left(\overline{\psi(0^+)} \varphi(0^+) - \overline{\psi(0^-)} \varphi(0^-) \right). \end{aligned}$$

Introduce the one-dimensional vector spaces $X = \{\psi(0^+) : \psi \in \text{dom } T^*\}$ and $Y = \{\psi(0^-) = \rho(\psi(0^+)) : \psi \in \text{dom } T^*\}$ and note that $\Gamma(\psi, \varphi) = 0$ is equivalent to the equality of inner products

$$\langle \psi(0^+), \varphi(0^+) \rangle = \langle \rho(\psi(0^+)), \rho(\varphi(0^+)) \rangle.$$

Self-adjoint extensions are got on domains $\mathcal{D} \subset \text{dom } T^*$ so that $\Gamma(\psi, \varphi) = 0$, $\forall \psi, \varphi \in \mathcal{D}$, that is, X and Y are related by unitary maps $e^{i\theta}$, $0 \leq \theta < 2\pi$; explicitly $\psi(0^+) = e^{i\theta} \psi(0^-)$.

Therefore, the family of operators

$$\text{dom } T_{\theta} = \{\psi \in \text{AC}(\mathbb{R} \setminus \{0\}) : \psi' \in L^2(\mathbb{R}), \psi(0^+) = e^{i\theta} \psi(0^-)\},$$

$$T_{\theta} \psi = -i\psi',$$

constitutes the self-adjoint extensions of T . The case $\theta = 0$ agrees with the momentum operator P with initial domain $C_0^{\infty}(\mathbb{R})$.

4 A multiply connected domain

Some self-adjoint extensions of a Schrödinger operator with infinite deficiency index will be found. It will combine the spherical symmetry with the topological property of multiply connectedness. Some specific results on Sobolev traces will be recalled in a suitable way. For simplicity, the underlying space will be the plane with a circular hole, but it is clear that this can be adapted to spaces with boundaries so that the trace construction

applies (e.g., smooth and compact boundaries); in fact, it is far from having exhausted all the possibilities this method can be applied.

Let $B(0; a)$ be the open ball centered at the origin and radius $a > 0$ in \mathbb{R}^2 , $\Lambda = \mathbb{R}^2 \setminus \overline{B(0; a)}$, and its closure $\overline{\Lambda} = \mathbb{R}^2 \setminus B(0; a)$; thus Λ is the real plane with a circular hole and its boundary $\partial\Lambda$ is the circumference $S = \{(x_1, x_2) \in \mathbb{R}^2 : r = (x_1^2 + x_2^2)^{\frac{1}{2}} = a\}$.

For simplicity the potential will be a bounded continuous function $V : \overline{\Lambda} \rightarrow \mathbb{R}$, with $V(x) = V(r)$ (i.e., V is spherically symmetric), and the initial hamiltonian is the hermitian operator

$$H = -\Delta + V, \quad \text{dom } H = C_0^\infty(\Lambda).$$

Polar coordinates $x_1 = r \cos \varphi$, $x_2 = r \sin \varphi$ are introduced so that $L^2(\overline{\Lambda})$ is unitarily equivalent to $L_{rdr}^2([a, \infty)) \otimes L_{d\varphi}^2(S)$, and consider the functions $e_m(\varphi) = e^{im\varphi}/\sqrt{2\pi}$, $0 \leq \varphi \leq 2\pi$, $m \in \mathbb{Z}$, so that for $\psi(r, \varphi) = R(r)e_m(\varphi)$,

$$H(Re_m) = \left(-\frac{d^2}{dr^2} - \frac{1}{r} \frac{d}{dr} + \frac{m^2}{r^2} + V(r) \right) Re_m,$$

and after the unitary transformation $\tau : L_{rdr}^2([a, \infty)) \rightarrow L_{dr}^2([a, \infty))$, $(\tau R)(r) = \sqrt{r}R(r)$, one obtains for $\tau H \tau^{-1}$ restricted to the subspace spanned by e_m

$$\hat{H}_m = \left(-\frac{d^2}{dr^2} + \frac{m^2 - 1/4}{r^2} + V(r) \right), \quad \text{dom } \hat{H}_m = C_0^\infty(a, \infty).$$

The original problem is thus reduced to the study of infinitely many Schrödinger operators on $[a, \infty)$ with potentials

$$\hat{V}(r) = (m^2 - 1/4)/r^2 + V(r), \quad m \in \mathbb{Z}.$$

One then rather easily checks that, for all m , the deficiency indices of H_m are equal to 1 (the point here is that $a > 0$), so that $n_+(H) = \infty = n_-(H)$.

Although a $\psi(r, \varphi) \in \mathcal{H}^1(\Lambda)$ is not necessarily continuous, it is possible to give a meaning to the restriction $\psi(a, \varphi) = \psi|_{\partial\Lambda}(\varphi) \in L^2(S)$ via the so-called *trace* (more properly, it should be called *Sobolev trace*) of ψ ; see ahead. It turns out that there is a continuous linear map $\gamma : C_0^1(\mathbb{R}^2) \subset \mathcal{H}^1(\Lambda) \rightarrow L^2(S)$, $\gamma(\phi(r, \varphi)) = \phi(a, \varphi)$, that is, there is $C > 0$ so that

$$\|\gamma\phi\|_{L^2(S)} = \|\phi(a, \varphi)\|_{L^2(S)} \leq C \|\phi\|_{\mathcal{H}^1(\Lambda)}, \quad \phi \in C_0^1(\mathbb{R}^2).$$

Note that for $\phi \in C_0^1(\mathbb{R}^2)$ the boundary values $\phi(a, \varphi)$ are well defined for any angle φ . By density, this map has a unique continuous extension (keeping the same notation) $\gamma : \mathcal{H}^1(\Lambda) \rightarrow L^2(S)$, called the *trace map* (see chapters 1 and 2 of [9] and also [1, 2]), and one defines $\psi(a, \varphi) := \gamma(\psi)$ for all $\psi \in \mathcal{H}^1(\Lambda)$. Some important properties of the trace $\gamma(\psi)$ are as follows (since its smooth boundary $\partial\Lambda$ is compact).

- (i) For $\psi \in \mathcal{H}^1(\Lambda)$ the trace is not defined in a pointwise manner, only as a function in $L^2(S)$. General elements of $L^2(\Lambda)$ do not have a trace defined.
- (ii) the range $\text{rng } \gamma \subseteq L^2(S)$ and the following Green formula holds

$$\int_{\Lambda} \frac{\partial \psi(x)}{\partial x_j} \phi(x) dx + \int_{\Lambda} \psi(x) \frac{\partial \phi(x)}{\partial x_j} dx = a \int_0^{2\pi} \psi(a, \varphi) \phi(a, \varphi) d\varphi,$$

for all $\psi, \phi \in \mathcal{H}^1(\Lambda)$, $j = 1, 2$.

- (iii) The kernel of the trace operator is the Hilbert space

$$\mathcal{H}_0^1(\Lambda) := \{\psi \in \mathcal{H}^1(\Lambda) : \gamma(\psi) = \psi(a, \varphi) = 0\},$$

which can also be defined as the closure of $C_0^\infty(\Lambda)$ in $\mathcal{H}^1(\Lambda)$.

- (iv) In a similar way, if $\psi \in \mathcal{H}^2(\Lambda)$ one has a well-defined trace $\gamma(\partial\psi/\partial r)$, which will be denoted by $\partial\psi/\partial r(a, \varphi)$. Note that $\partial\psi/\partial r$ stands for the normal (with respect to $\partial\Lambda$) derivative, and this derivative belongs to $\mathcal{H}^1(\Lambda)$.
- (v) The ranges of both trace maps $\mathcal{H}^2(\Lambda) \ni \psi \mapsto \psi(a, \varphi)$ and $\mathcal{H}^2(\Lambda) \ni \psi \mapsto \partial\psi/\partial r(a, \varphi)$ are dense in $L^2(S)$, and the following Green formula holds

$$\int_{\Lambda} \Delta\psi(x) \phi(x) dx + \int_{\Lambda} \nabla\psi(x) \nabla\phi(x) dx = a \int_0^{2\pi} \frac{\partial\psi}{\partial r}(a, \varphi) \phi(a, \varphi) d\varphi,$$

for all $\psi, \phi \in \mathcal{H}^2(\Lambda)$.

Now a subtle point must be mentioned. At first sight one could (wrongly) guess that the domain of the adjoint H^* is $\mathcal{H}^2(\Lambda)$. However, in \mathbb{R}^n , $n \geq 2$, for open sets $\Omega \subset \mathbb{R}^n$, $\Omega \neq \mathbb{R}^n$, there are functions $\psi \in L^2(\Omega)$ with distributional laplacian $\Delta\psi \in L^2(\Omega)$ that do not belong to $\mathcal{H}^2(\Omega)$; the point is that other derivatives, as first derivatives, of ψ need not exist as functions! It turns out that

$$\text{dom } H^* = \{\psi \in L^2(\Lambda) : (-\Delta\psi + V\psi) \in L^2(\Lambda)\}$$

and $H^*\psi = -\Delta\psi + V\psi$, $\psi \in \text{dom } H^*$, and this domain is strictly larger than $\mathcal{H}^2(\Lambda)$. Further, $\text{dom } H^*$ is not contained in any $\mathcal{H}^s(\Lambda)$, $s > 0$! See [7], [8] and references therein.

By using the above characterization of H^* , some self-adjoint extensions of H will be found via suitable restrictions of H^* . The boundary form of H , for $\psi, \phi \in \text{dom } H^*$, is

$$\Gamma(\psi, \phi) := \langle (-\Delta + V)\psi, \phi \rangle - \langle \psi, (-\Delta + V)\phi \rangle.$$

By restricting to those self-adjoint extensions whose domains are contained in $\mathcal{H}^2(\Lambda)$, Sobolev traces can be invoked, the continuity of the potential guarantees that $V|_{\partial\Lambda} = V(a)$ is well posed and the above Green formula can be used to compute, for $\psi, \phi \in \mathcal{H}^2(\Lambda)$,

$$\Gamma(\psi, \phi) = a \int_0^{2\pi} \left(\overline{\psi(a, \varphi)} \frac{\partial \phi}{\partial r}(a, \varphi) - \overline{\frac{\partial \psi}{\partial r}(a, \varphi)} \phi(a, \varphi) \right) d\varphi.$$

Introduce $\rho_j : \mathcal{H}^2(\Lambda) \rightarrow L^2(S)$, $j = 1, 2$, by

$$\rho_1(\psi) = \psi(a, \varphi) + i \frac{\partial \psi}{\partial r}(a, \varphi),$$

$$\rho_2(\psi) = \psi(a, \varphi) - i \frac{\partial \psi}{\partial r}(a, \varphi),$$

and so

$$(2i/a) \Gamma(\psi, \phi) = \langle \rho_1(\psi), \rho_1(\phi) \rangle_{L^2(S)} - \langle \rho_2(\psi), \rho_2(\phi) \rangle_{L^2(S)}.$$

A boundary triple for H in the Sobolev space $\mathcal{H}^2(\Lambda)$ has been found with $\mathbf{h} = L^2(S)$. As before (i.e., by Theorem 1), from this boundary triple the self-adjoint extensions H_U of H are characterized by unitary operators $U : L^2(S) \leftrightarrow L^2(S)$ so that $\rho_1(\psi) = U \rho_2(\psi)$, $\forall \psi \in \text{dom } H_U$, and $H_U \psi = H^* \psi$. After writing out this relation one finds

$$(\mathbf{I}_d - U) \psi(a, \varphi) = -i(\mathbf{I}_d + U) \frac{\partial \psi}{\partial r}(a, \varphi).$$

Therefore, all self-adjoint extensions of H with domain in $\mathcal{H}^2(\Lambda)$ were found and they are realized through suitable boundary conditions on $\partial\Lambda$; such boundary conditions are in terms of traces of elements of $\mathcal{H}^2(\Lambda)$. Below some explicit self-adjoint extensions are described.

1. $U = -\mathbf{I}_d$.

In this case

$$\text{dom } H_U = \{\psi \in \mathcal{H}^2(\Lambda) : \psi(a, \varphi) = 0\} = \mathcal{H}^2(\Lambda) \cap \mathcal{H}_0^1(\Lambda),$$

$H_U \psi = (-\Delta + V)\psi$, $\psi \in \text{dom } H_U$. This is the so-called Dirichlet realization (of the laplacian if $V = 0$) in Λ .

2. $U = \mathbf{I}_d$.

In this case $\text{dom } H_U = \{\psi \in \mathcal{H}^2(\Lambda) : \partial\psi/\partial r(a, \varphi) = 0\}$, $H_U \psi = (-\Delta + V)\psi$. This is the so-called Neumann realization.

3. $(I_d + U)$ is invertible.

In this case one gets that for each self-adjoint operator $A : \text{dom } A \subseteq L^2(S) \rightarrow L^2(S)$ corresponds a self-adjoint extension H^A . In fact, first pick a unitary operator U_A so that $A = -i(I_d - U_A)(I_d + U_A)^{-1}$, $\text{dom } A = \text{rng } (I_d - U_A)$ and $\text{rng } A = (I_d - U_A)$; remind of Cayley transform. Now, $\text{dom } H^A$ is the set of $\psi \in \mathcal{H}^2(\Lambda)$ with “ $\partial\psi/\partial r(a, \cdot) = A\psi(a, \cdot)$,” prudently understood in the sense that

$$(I_d - U_A) \psi(a, \varphi) = -i(I_d + U_A) \frac{\partial\psi}{\partial r}(a, \varphi),$$

in order to avoid domain questions. Of course the quotation marks can be removed in case the operator A is bounded.

Similarly, for each self-adjoint B acting in $L^2(S)$ there corresponds a unitary U_B , and if $(I_d - U_B)$ is invertible, then it corresponds the self-adjoint extension H^B of H with $\text{dom } H^B$ the set of $\psi \in \mathcal{H}^2(\Lambda)$ so that “ $\psi(a, \cdot) = B \frac{\partial\psi}{\partial r}(a, \cdot)$,” in the sense that

$$(I_d - U_B) \psi(a, \varphi) = -i(I_d + U_B) \frac{\partial\psi}{\partial r}(a, \varphi).$$

Again the quotation marks can be removed in case the operator B is bounded.

Note that 4. ahead are, in fact, particular cases of 3. in which $A = \mathcal{M}_f$ and $B = \mathcal{M}_g$.

4. U is a multiplication operator.

Given a real-valued (measurable) function $u(\varphi)$ put $U = \mathcal{M}_{e^{iu(\varphi)}}$. If $\{\varphi : \exp(iu(\varphi)) = -1\}$ has measure zero, then

$$f(\varphi) = -i \frac{1 - e^{iu(\varphi)}}{1 + e^{iu(\varphi)}}$$

is (measurable) well defined and real valued. The domain of the corresponding self-adjoint extension is

$$\text{dom } H_U = \{\psi \in \mathcal{H}^2(\Lambda) : \partial\psi/\partial r(a, \varphi) = f(\varphi)\psi(a, \varphi)\}.$$

Similarly, if $\{\varphi : \exp(iu(\varphi)) = 1\}$ has measure zero,

$$g(\varphi) = i \frac{1 + e^{iu(\varphi)}}{1 - e^{iu(\varphi)}}$$

is real valued and the domain of the subsequent self-adjoint extension is

$$\text{dom } H_U = \{\psi \in \mathcal{H}^2(\Lambda) : \psi(a, \varphi) = g(\varphi)\partial\psi/\partial r(a, \varphi)\}.$$

Special cases are given by constant functions f, g .

5. $A = -id/d\varphi$

with domain $\mathcal{H}^1(S) = \{u \in \mathcal{H}^1(0, 2\pi) : u(0) = u(2\pi)\}$. The corresponding self-adjoint extension has domain

$$\left\{ \psi \in \mathcal{H}^2(\Lambda) : \psi(a, \varphi) \in \mathcal{H}^1(S), \left. \frac{\partial \psi}{\partial r}(a, \varphi) = -i \frac{d\psi}{d\varphi}(a, \varphi) \right\}.$$

Since the deficiency indices of H are infinite, there is a plethora of self-adjoint extensions of the laplacian in the multiply connected domain $\Lambda = \mathbb{R}^2 \setminus \overline{B}(0; a)$. Some of them can be quite unusual and hard to understand from the physical and mathematical points of view.

Remark 2. *The choice of $\Lambda = \mathbb{R}^2 \setminus \overline{B}(0; a)$ was for notational convenience. In a similar way one finds expressions for the boundary form of $H = -\Delta + V$ with domain $C_0^\infty(\mathbb{R}^2 \setminus \overline{\Omega})$, with $\Omega \subset \mathbb{R}^2$ an open set with compact boundary $\partial\Omega$ of class C^1 ; when restricted to domains in $\mathcal{H}^2(\mathbb{R}^2 \setminus \Omega)$, Sobolev traces (as well as similar properties as above) are properly defined in this setting, and one can also consider \mathbb{R}^n , $n \geq 2$. For such more general multiply connected regions, one must consider the normal derivative $\partial\psi/\partial\mathbf{n}$ at the boundary $\partial\Omega$, instead of $\partial\psi/\partial r$, and also the corresponding modifications in the expressions of Green formulae [2], [9].*

Remark 3. *By using a continuous extension of the trace maps to the dual Sobolev spaces $\mathcal{H}^{-1/2}(\partial\Lambda)$ and $\mathcal{H}^{-3/2}(\partial\Lambda)$, in [7] one finds references and comments to her previous works on all self-adjoint extensions of the laplacian in terms of self-adjoint operators from closed subspaces of $\mathcal{H}^{-1/2}(\partial\Lambda)$.*

Example 6. Let $0 < a < b < \infty$ and

$$\Omega_{ab} = \{(x_1, x_2) \in \mathbb{R}^2 : a \leq (x_1^2 + x_2^2)^{\frac{1}{2}} \leq b\}$$

be a closed annulus in \mathbb{R}^2 ; set $\Lambda_{ab} = \mathbb{R}^2 \setminus \Omega$ and consider the laplacian $H = -\Delta$, $\text{dom } H = C_0^\infty(\Lambda_{ab})$. This operator is hermitian and its adjoint is

$$\text{dom } H^* = \{\psi \in L^2(\Lambda_{ab}) : \Delta\psi \in L^2(\Lambda_{ab})\},$$

$H^*\psi = -\Delta\psi$; this domain is strictly larger than $\mathcal{H}^2(\Lambda_{ab})$. Under the restriction of $\psi, \phi \in \mathcal{H}^2(\Lambda_{ab})$, the boundary form $\Gamma(\psi, \phi)$, in terms of Sobolev traces, equals

$$\begin{aligned} & b \int_0^{2\pi} \left(\overline{\psi(b, \varphi)} \frac{\partial \phi}{\partial r}(b, \varphi) - \overline{\frac{\partial \psi}{\partial r}(b, \varphi)} \phi(b, \varphi) \right) d\varphi \\ & - a \int_0^{2\pi} \left(\overline{\psi(a, \varphi)} \frac{\partial \phi}{\partial r}(a, \varphi) - \overline{\frac{\partial \psi}{\partial r}(a, \varphi)} \phi(a, \varphi) \right) d\varphi. \end{aligned}$$

With elements of $L^2(S_a) \oplus L^2(S_b)$, where S_a and S_b are the circumferences in \mathbb{R}^2 , centered at the origin and radii a and b , respectively (so $\partial\Lambda_{ab} = S_a \cup S_b$), introduce the vector spaces of functions

$$\Psi = \begin{pmatrix} \sqrt{a} \frac{\partial \psi}{\partial r}(a, \varphi) - i\sqrt{a}\psi(a, \varphi) \\ \sqrt{b} \frac{\partial \psi}{\partial r}(b, \varphi) + i\sqrt{b}\psi(b, \varphi) \end{pmatrix}, \quad \rho(\Psi) = \begin{pmatrix} \sqrt{a} \frac{\partial \psi}{\partial r}(a, \varphi) + i\sqrt{a}\psi(a, \varphi) \\ \sqrt{b} \frac{\partial \psi}{\partial r}(b, \varphi) - i\sqrt{b}\psi(b, \varphi) \end{pmatrix},$$

and analogous for ϕ , denoted by Φ and $\rho(\Phi)$. By using the auxiliary space $L^2(S_1)$, evaluation the inner products gives

$$\langle \Psi, \Phi \rangle_{L^2(S_1)} - \langle \rho(\Psi), \rho(\Phi) \rangle_{L^2(S_1)} = -2i\Gamma(\psi, \varphi),$$

and a (modified) boundary triple is found.

The self-adjoint extensions H_U of H , with domains in $\mathcal{H}^2(\Lambda_{ab})$, are characterized by unitary operators $U : L^2(S_a) \oplus L^2(S_b) \leftrightarrow$ so that $\Psi = U\rho(\Psi)$, $\forall \psi \in \text{dom } H_U$, and $H_U\psi = H^*\psi$. Writing out this relation one finds the boundary conditions

$$(I - U) \begin{pmatrix} \sqrt{a} \frac{\partial \psi}{\partial r}(a, \varphi) \\ \sqrt{b} \frac{\partial \psi}{\partial r}(b, \varphi) \end{pmatrix} = -i(I + U) \begin{pmatrix} -\sqrt{a}\psi(a, \varphi) \\ \sqrt{b}\psi(b, \varphi) \end{pmatrix}.$$

As before, the choice of the identity operator $U = I$ gives the Dirichlet realization of the laplacian in this domain. Other choices of U being similar to previous discussion. Such simple characterization of these self-adjoint extensions also appears to have been reported here for the first time.

Acknowledgments

The author acknowledges correspondences with Profs. T. Arbogast and H. Brezis about Sobolev traces, and with Prof. G. Grubb about Sobolev space details. This work was partly supported by CNPq (Brazil).

References

- [1] R. A. Adams and J. J. F. Fournier, *Sobolev spaces*. Second edition. PAM **140** (Elsevier/Academic Press, Amsterdam, 2003).
- [2] H. Brezis, *Analyse Fonctionnelle: Théorie et Applications* (Dunod, Paris, 1999).
- [3] J. Brüning, V. Geyler, and K. Pankrashkin, Spectra of self-adjoint extensions and applications to solvable Schrödinger operators, Preprint (arXiv:math-ph/0611088) (2006).

- [4] J. W. Calkin, *Trans. Amer. Math. Soc.* **45**, 369 (1939).
- [5] C. R. de Oliveira and M. Pereira (in preparation).
- [6] V. I. Gorbachuk and M. L. Gorbachuk, *Boundary Value Problems for Operator Differential Equations*. Mathematics and its Applications (Soviet Series) **48** (Kluwer, Dordrecht, 1991).
- [7] G. Grubb, *Bull. Amer. Math. Soc. (N.S.)* **43**, 227–230 (2006).
- [8] G. Grubb, *Distributions and Operators*. GTM **252** (Springer-Verlag, Berlin, 2009).
- [9] J.-L. Lions and E. Magenes, *Non-Homogeneous Boundary Value Problems and Applications* (3 vol.) (Springer-Verlag, New York, 1973).
- [10] M. A. Naimark, *Theory of Linear Differential Operators* (Nauka, Moscow, 1969).
- [11] M. I. Vishik, *Amer. Math. Soc. Transl., II Ser.*, **24**, 107 (1963) (Original in Russian (1952)).
- [12] B. L. Voronov, D. M. Gitman and I. V. Tyutin, Self-adjoint differential operators associated with self-adjoint differential expressions. Preprint (arXiv:quant-ph/0603187v2) (2006).