

Well-posed p -laplacian problems with large diffusion

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Abstract

In this work we study the asymptotic behavior of p -laplacian parabolic problems of the form $u_t - D\Delta_p u + |u|^{p-2}u = B(u)$ in a bounded smooth domain in \mathbb{R}^n and Neumann boundary conditions when the diffusion coefficient D becomes large. We prove, under suitable assumptions, that the family of attractors behaves lower and upper semicontinuously as the diffusion increases to infinity.

Keywords: p -laplacian ; large diffusion; attractors

1 Introduction

Let Ω be a bounded smooth domain and consider the following reaction-diffusion problem

$$\begin{aligned} \frac{\partial u}{\partial t} &= D\Delta u + f(u) \\ \frac{\partial u}{\partial n} &= 0 \quad \text{in } \partial\Omega \end{aligned} \tag{1.1}$$

where $u = (u_1, \dots, u_m)$, $D = \text{diag}(d_1, \dots, d_m)$ and n is the outward normal vector. It is a well known fact that if the $\inf_{1 \leq i \leq m} d_i$ goes to infinity then the solutions of (1.1) quickly approach solutions of

$$\dot{u} = f(u) \tag{1.2}$$

(see [3], [5], [7], [10], [11] and [12]). The physical reason for this lays in the fact that if the diffusion is very large the concentrations quickly homogenize and the term $D\Delta u$ becomes a small perturbation. Mathematically, this is justified from the fact that the subspace of constant functions is an invariant manifold for (1.1) where it reduces to (1.2) and, as D becomes large there is a gap between the zero eigenvalue of the Neumann Laplacian and its first positive eigenvalue, which ensures that this invariant manifold is exponentially attracting.

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In this work we prove that the same homogenization process occurs in similar p -laplacian problems if $p > 2$. Let us consider the problem

$$(P_D) \begin{cases} \frac{\partial u^D}{\partial t}(t) - D\Delta_p u^D(t) + |u^D(t)|^{p-2}u^D(t) = B(u^D(t)), & t > 0 \\ u^D(0) = u_0^D, \end{cases}$$

under homogeneous Neumann boundary conditions, where $p > 2$, $1 \leq D \in \mathbb{R}$, $u_0 \in H = L^2(\Omega)$, $B : L^2(\Omega) \rightarrow L^2(\Omega)$ is a globally Lipschitz map with Lipschitz constant $L \geq 0$, and Ω is a bounded smooth domain in \mathbb{R}^n . Using the definition and the properties of the operator

$$A^D u \doteq -\operatorname{div}(D|\nabla u|^{p-2}\nabla u) + |u|^{p-2}u = -D\Delta_p u + |u|^{p-2}u,$$

with Neumann boundary conditions we have that A^D is a maximal monotone operator in H [14], and by Proposition 1 in [4] the problem (P_D) determines a continuous semigroup of nonlinear operators $S^D(t) : H \rightarrow H$, where $S^D(t)u_0^D$ is the global weak solution of (P_D) . By Lemmas 3, 4, 5 and Theorem 1 in [4], the semigroup associate with the problem (P_D) has a maximal compact invariant global attractor \mathcal{A}_D in H . We prove that the family of attractors $\{\mathcal{A}_D\}$ is upper and lower semicontinuous at infinity, that means,

$$\max \left\{ \sup_{a^D \in \mathcal{A}_D} \operatorname{dist}_H(a^D, \mathcal{A}^\infty), \sup_{a \in \mathcal{A}^\infty} \operatorname{dist}_H(a, \mathcal{A}_D) \right\} \rightarrow 0 \text{ as } D \rightarrow +\infty,$$

where \mathcal{A}^∞ is the attractor of the limit problem (P_L)

$$(P_L) \begin{cases} \dot{u}(t) + |u(t)|^{p-2}u(t) = \tilde{B}(u(t)), & t > 0 \\ u(0) = u_0 \in \mathbb{R}, \end{cases}$$

with $\tilde{B} \doteq B|_{\mathbb{R}}$.

We use the next section to obtain the necessary uniform estimates for solutions of (P_D) . In Section 3 we discuss all relevant questions about the limit problem (P_L) and in the last section we prove the continuity of the attractors. The lower semicontinuity in this case is a trivial fact, once the attractor \mathcal{A}^∞ is a subset of \mathcal{A}_D for each $D \geq 1$.

2 Uniform Estimates

In this section we obtain $L^2(\Omega)$ and $W^{1,p}(\Omega)$ estimates for the solutions u^D 's of the problem (P_D) , uniformly on $D \geq 1$. As a consequence we conclude that $\overline{\bigcup_{D \geq 1} \mathcal{A}_D}$ is a compact subset of H .

Lemma 2.1 *If u^D is a solution of (P_D) in $(0, \infty)$, then there are positive constants r_0, t_0 such that $\|u^D(t)\|_H \leq r_0$, for each $t \geq t_0$ and $D \geq 1$.*

Proof: It comes from the equation in (P_D) that

$$\begin{aligned}
& \frac{1}{2} \frac{d}{dt} \|u^D(\tau)\|_H^2 + D \|\nabla u^D(\tau)\|_p^p + \|u^D(\tau)\|_p^p = \langle B(u^D(\tau)), u^D(\tau) \rangle = \\
& = \langle B(u^D(\tau)) - B(0), u^D(\tau) \rangle_H + \langle B(0), u^D(\tau) \rangle_H \\
& \leq \|B(u^D(\tau)) - B(0)\|_H \|u^D(\tau)\|_H + \|B(0)\|_H \|u^D(\tau)\|_H \\
& \leq L \|u^D(\tau)\|_H^2 + c \|u^D(\tau)\|_H,
\end{aligned}$$

τ -a.e.in $(0, \infty)$, where $c = \|B(0)\|_H$ is a constant. Then

$$\frac{1}{2} \frac{d}{dt} \|u^D(\tau)\|_H^2 \leq -C_p \|u^D(\tau)\|_H^p + L \|u^D(\tau)\|_H^2 + c \|u^D(\tau)\|_H,$$

where $C_p = C(p, \Omega) > 0$. From the Young's inequality we obtain that

$$\frac{1}{2} \frac{d}{dt} \|u^D(\tau)\|_H^2 \leq -\frac{C_p}{2} \|u^D(\tau)\|_H^p + c_1,$$

where $c_1 = c_1(p) > 0$ is a real number. Then by Lemma 5.1 in [15],

$$\|u^D(t)\|_H^2 \leq \left(\frac{2c_1}{C_p}\right)^{2/p} + \left(\frac{1}{(C_p)}\left(\frac{p}{2} - 1\right)t\right)^{-\frac{2}{p-2}}, \quad \forall t > 0.$$

Fix $t_0 > 0$ (for example $t_0 \doteq 1$) and choose

$$r_0 \doteq \left[\left(\frac{2c_1}{C_p}\right)^{2/p} + \left(\frac{1}{(C_p)}\left(\frac{p}{2} - 1\right)t_0\right)^{-\frac{2}{p-2}} \right]^{1/2}.$$

Then, $\|u^D(t)\|_H \leq r_0$, whenever $t \geq t_0$. ■

Remark 2.1 *Observe that the constants r_0, t_0 in Lemma 2.1 depend neither on the initial data nor on D .*

Remark 2.2 *For each fixed $D \geq 1$, as an easy consequence of Gronwall-Bellman inequality, there exists a positive constant $\tilde{r}_0(u_0^D, t_0)$ such that $\|u^D(t)\|_H < \tilde{r}_0(u_0^D, t_0), \forall t \in [0, t_0]$ and, for initial conditions in bounded subsets of H , we have that $\|u^D(t)\|_H < \tilde{r}_0, \forall D \geq 1$ and $\forall t \in [0, t_0]$.*

Corollary 2.1 *There is a bounded set B_0 in H such that $\mathcal{A}_D \subset B_0, \forall D \geq 1$.*

Proof: It is an immediate consequence of the above lemma and the invariance of \mathcal{A}_D . ■

Now repeating the same arguments used in the proof of the Lemma 2.2 in [9] we obtain the $W^{1,p}(\Omega)$ uniform estimates:

Lemma 2.2 *If u^D is a solution of (P_D) in $(0, \infty)$, then there exist positive constants $r_1 > 0$ and $t_1 > t_0$ such that $\|u^D(t)\|_{W^{1,p}(\Omega)} \leq r_1$, for each $t \geq t_1$ and $D \geq 1$, with t_0 as in the Lemma 2.1.*

Proof: We consider

$$\varphi^D(v) \doteq \begin{cases} \frac{1}{p} \left[D \int_{\Omega} |\nabla v|^p dx + \int_{\Omega} |v|^p dx \right], & v \in W^{1,p}(\Omega) \\ +\infty, & \text{otherwise.} \end{cases}$$

We have that φ^D is a convex, proper and lower semicontinuous map, and A^D is the subdifferential $\partial\varphi^D$ of φ^D . Let be u^D a solution of (P_D) . We have that

$$\begin{aligned} \frac{d}{d\tau} \varphi^D(u^D(\tau)) &= \langle \partial\varphi^D(u^D(\tau)), u^D(\tau) \rangle = \langle B(u^D(\tau)) - u^D(\tau), u^D(\tau) \rangle \\ &= \langle B(u^D(\tau)) - u^D(\tau), u^D(\tau) - B(u^D(\tau)) + B(u^D(\tau)) \rangle \\ &= -\|B(u^D(\tau)) - u^D(\tau)\|_H^2 + \langle B(u^D(\tau)) - u^D(\tau), B(u^D(\tau)) \rangle, \end{aligned}$$

τ -a.e.in $(0, \infty)$. Therefore

$$\frac{1}{2} \|B(u^D(\tau)) - u^D(\tau)\|_H^2 + \frac{d}{d\tau} \varphi^D(u^D(\tau)) \leq \frac{1}{2} \|B(u^D(\tau))\|_H^2$$

In particular

$$\frac{d}{d\tau} \varphi^D(u^D(\tau)) \leq \frac{1}{2} \|B(u^D(\tau))\|_H^2 < \frac{1}{2} k_1^2, \quad \forall D \geq 1 \text{ and } \tau \geq t_0 \text{ a.e.}, \quad (2.3)$$

where k_1 depends on L (the Lipschitz constant of B) and, according Remark 2.1, k_1 can be uniformly chosen on D .

By definition of subdifferential we have the following inequality

$$\varphi^D(u^D(\tau)) \leq \langle \partial\varphi^D(u^D(\tau)), u^D(\tau) \rangle \quad (2.4)$$

Thus,

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|u^D(\tau)\|_H^2 + \varphi^D(u^D(\tau)) &= \langle u_t^D(\tau), u^D(\tau) \rangle + \varphi^D(u^D(\tau)) \\ &\leq \langle u_t^D(\tau), u^D(\tau) \rangle + \langle \partial\varphi^D(u^D(\tau)), u^D(\tau) \rangle \\ &= \langle B(u^D(\tau)), u^D(\tau) \rangle \\ &\leq \|B(u^D(\tau))\|_H \|u^D(\tau)\|_H \leq k_1 r_0, \end{aligned} \quad (2.5)$$

$\forall D \geq 1$ and $\tau \geq t_0$ a.e..

Let $t \geq t_0$ and $r > 0$. Integrating (2.5) from t to $t + r$ we obtain that

$$\int_t^{t+r} \varphi^D(u^D(\tau)) d\tau \leq \frac{1}{2} \|u^D(t)\|_H^2 + k_1 r_0 r \leq \frac{1}{2} r_0^2 + k_1 r_0 r \doteq a_3, \quad \forall D \geq 1. \quad (2.6)$$

From 2.3, 2.6 and the Uniform Gronwall Lemma ([15]), we obtain

$$\varphi^D(u^D(t+r)) \leq \frac{a_3}{r} + \frac{1}{2} k_1^2 r \doteq \tilde{r}_1, \quad \forall t \geq t_0 \text{ and } \forall D \geq 1. \quad (2.7)$$

Therefore

$$\frac{1}{p} \|u^D(\ell)\|_{W^{1,p}(\Omega)}^p \leq \varphi^D(u^D(\ell)) \leq \tilde{r}_1, \quad \forall \ell \geq t_0 + r \text{ and } \forall D \geq 1. \quad (2.8)$$

Considering $r_1 \doteq (p\tilde{r}_1)^{1/p}$ and $t_1 \doteq t_0 + r$, we conclude that $\|u^D(\ell)\|_{W^{1,p}(\Omega)} \leq r_1$, for each $\ell \geq t_1$ and $D \geq 1$. \blacksquare

As an important consequence of Lemma 2.2 it follows that $\bigcup_{D \geq 1} \mathcal{A}_D$ is a bounded subset of $W^{1,p}(\Omega)$ and once $W^{1,p}(\Omega) \subset\subset L^2(\Omega)$, we can conclude:

Corollary 2.2 $\mathcal{A} \doteq \overline{\bigcup_{D \geq 1} \mathcal{A}_D}$ is a compact subset of H .

3 The Limit Problem and Convergence Properties

Our objective in this section is to prove that the asymptotic dynamic of the problem (P_D) is described by an ordinary differential equation as D increases to infinity. Firstly we observe that the gradients of the solutions u^D converge in norm to zero as $D \rightarrow \infty$, which allows us to guess the limit problem (P_L) . Then we observe that (P_L) is well posed and possesses a global compact attractor.

Lemma 3.1 *If for each $D \geq 1$, u^D is the solution of (P_D) in $(0, \infty)$, then for each $t > t_1$, the sequence of real numbers $\{\|\nabla u^D(t)\|_H\}_{D \geq 1}$ has a subsequence $\{\|\nabla u^{D_\ell}(t)\|_H\}$ which converges to zero as $\ell \rightarrow +\infty$, where t_1 is the positive constant of Lemma 2.2.*

Proof: Let $T > t_1$ and $t \in (t_1, T)$, where t_1 is as in Lemma 2.2. From the equation in (P_D) it comes that

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \|u^D(\tau)\|_H^2 + D \|\nabla u^D(\tau)\|_p^p + \|u^D(\tau)\|_p^p = \langle B(u^D(\tau)), u^D(\tau) \rangle = \\ & = \langle B(u^D(\tau)) - B(0), u^D(\tau) \rangle_H + \langle B(0), u^D(\tau) \rangle_H \\ & \leq \|B(u^D(\tau)) - B(0)\|_H \|u^D(\tau)\|_H + \|B(0)\|_H \|u^D(\tau)\|_H \\ & \leq L \|u^D(\tau)\|_H^2 + c \|u^D(\tau)\|_H \leq k_2, \quad \tau - a.e. \text{ in } (t_1, T) \end{aligned} \quad (3.9)$$

where $k_2 > 0$ is a constant which is independent of D . Integrating (3.9) from t_1 to T , we obtain

$$\begin{aligned} & \frac{1}{2} \|u^D(T)\|_H^2 + D \int_{t_1}^T \|\nabla u^D(\tau)\|_p^p d\tau + \int_{t_1}^T \|u^D(\tau)\|_p^p d\tau \leq \\ & \leq \int_{t_1}^T k_2 d\tau + \frac{1}{2} \|u^D(t_1)\|_H^2 \leq k_2 T + \frac{1}{2} r_0^2 \doteq k_3(T). \end{aligned}$$

In particular

$$D \int_{t_1}^T \|\nabla u^D(\tau)\|_p^p d\tau \leq k_3(T),$$

that implies

$$\int_{t_1}^T \|\nabla u^D(\tau)\|_p^p d\tau \leq \frac{1}{D} k_3(T) \rightarrow 0 \text{ as } D \rightarrow +\infty,$$

that means,

$$\| \|\nabla u^D(\cdot)\|_p^p - 0 \|_{L^1(t_1, T; \mathbb{R})} = \int_{t_1}^T | \|\nabla u^D(\tau)\|_p^p - 0 | d\tau \rightarrow 0 \text{ as } D \rightarrow +\infty.$$

Therefore there is a subsequence $\{ \|\nabla u^{D_\ell}(\tau)\|_p^p \}$ such that

$$\|\nabla u^{D_\ell}(\tau)\|_p^p \rightarrow 0 \text{ as } \ell \rightarrow +\infty, \tau - a.e. \text{ in } (t_1, T),$$

and so there exists a subset $J \subset (t_1, T)$ with Lebesgue measure $m((t_1, T)/J) = 0$ such that

$$\|\nabla u^{D_\ell}(\tau)\|_p^p \rightarrow 0 \text{ as } \ell \rightarrow +\infty, \forall \tau \in J.$$

Given $t \in (t_1, T)$ we claim that there is at least one $s \in J$ with $s < t$, on the contrary we would have $(t_1, t) \cap J = \emptyset$, so $m((t_1, T)/J) > 0$ which is a contradiction. Now pick one $s \in J$ with $t_1 < s < t$ and let $h = t - s$. Let $\varepsilon > 0$ and $\ell_0 = \ell(\varepsilon)$ be such that if $\ell > \ell_0$ then

$$\|\nabla u^{D_\ell}(s)\|_p^p < \frac{\varepsilon}{2}.$$

We have from (P_D)

$$\frac{d}{d\tau} \varphi^{D_\ell}(u^{D_\ell}(s + \tau)) = \langle \partial \varphi^{D_\ell}(u^{D_\ell}(s + \tau)), u_\tau^{D_\ell}(s + \tau) \rangle, \tau - a.e. \text{ in } (0, T).$$

where

$$\varphi^{D_\ell}(v) \doteq \begin{cases} \frac{1}{p} \left[D_\ell \int_\Omega |\nabla v|^p dx + \int_\Omega |v|^p dx \right], & v \in W^{1,p}(\Omega) \\ +\infty, & \text{otherwise.} \end{cases}$$

Therefore

$$\begin{aligned}
& \frac{1}{p}[D_\ell\|\nabla u^{D_\ell}(s+h)\|_p^p + \|u^{D_\ell}(s+h)\|_p^p] - \frac{1}{p}[D_\ell\|\nabla u^{D_\ell}(s)\|_p^p + \|u^{D_\ell}(s)\|_p^p] \\
&= \varphi^{D_\ell}(u^{D_\ell}(s+h)) - \varphi^{D_\ell}(u^{D_\ell}(s)) \\
&= \int_0^h \frac{d}{d\tau} \varphi^{D_\ell}(u^{D_\ell}(s+\tau)) d\tau = \int_0^h \langle \partial \varphi^{D_\ell}(u^{D_\ell}(s+\tau)), u_\tau^{D_\ell}(s+\tau) \rangle d\tau \\
&= \int_0^h \langle Bu^{D_\ell}(s+\tau) - u_\tau^{D_\ell}(s+\tau), u_\tau^{D_\ell}(s+\tau) \rangle d\tau \\
&= \int_0^h \langle Bu^{D_\ell}(s+\tau), u_\tau^{D_\ell}(s+\tau) \rangle d\tau - \int_0^h \langle u_\tau^{D_\ell}(s+\tau), u_\tau^{D_\ell}(s+\tau) \rangle d\tau \\
&\leq \int_0^h \|Bu^{D_\ell}(s+\tau)\|_H \|u_\tau^{D_\ell}(s+\tau)\|_H d\tau - \int_0^h \|u_\tau^{D_\ell}(s+\tau)\|_H^2 d\tau \\
&\leq \frac{1}{2} \int_0^h \|Bu^{D_\ell}(s+\tau)\|_H^2 d\tau + \frac{1}{2} \int_0^h \|u_\tau^{D_\ell}(s+\tau)\|_H^2 d\tau - \int_0^h \|u_\tau^{D_\ell}(s+\tau)\|_H^2 d\tau \\
&= \frac{1}{2} \int_0^h \|Bu^{D_\ell}(s+\tau)\|_H^2 d\tau - \frac{1}{2} \int_0^h \|u_\tau^{D_\ell}(s+\tau)\|_H^2 d\tau \\
&\leq \frac{1}{2} \int_0^h \|Bu^{D_\ell}(s+\tau)\|_H^2 d\tau \\
&\leq \frac{1}{2} \int_0^h k_1 d\tau = \frac{1}{2} k_1 h.
\end{aligned}$$

where k_1 depends on L (the Lipschitz constant of B) and, according Remark 2.1, k_1 can be uniformly chosen on D_ℓ . Thus,

$$\|\nabla u^{D_\ell}(s+h)\|_p^p \leq \|\nabla u^{D_\ell}(s)\|_p^p + \frac{1}{D_\ell} \|u^{D_\ell}(s)\|_p^p + \frac{p}{2D_\ell} k_1 h \quad (3.10)$$

and so,

$$\|\nabla u^{D_\ell}(s+h)\|_p^p \leq \|\nabla u^{D_\ell}(s)\|_p^p + \frac{1}{D_\ell} r_1^p + \frac{p}{2D_\ell} k_1 |T - t_1|, \quad (3.11)$$

where r_1 is the positive constant which appears in Lemma 2.2.

Now we choose $\ell_1 = \ell_1(\varepsilon)$ sufficiently large such that

$$\frac{1}{D_\ell} r_1^p + \frac{p}{2D_\ell} k_1 |T - t_1| < \varepsilon/2,$$

whenever $\ell > \ell_1$ and we consider $\ell_2 = \ell_2(\varepsilon) = \max\{\ell_0, \ell_1\}$. For $\ell > \ell_2$ we have

$$\begin{aligned} \|\nabla u^{D_\ell}(t)\|_p^p &= \|\nabla u^{D_\ell}(s+t-s)\|_p^p \\ &\leq \|\nabla u^{D_\ell}(s)\|_p^p + \frac{1}{D_\ell} r_1^p + \frac{p}{2D_\ell} k_1 |T - t_1| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

As $p > 2$,

$$\|\nabla u^{D_\ell}(t)\|_H^p \leq [m(\Omega)^{1/2-1/p}]^p \|\nabla u^{D_\ell}(t)\|_p^p \leq [m(\Omega)^{1/2-1/p}]^p \varepsilon.$$

So,

$$\|\nabla u^{D_\ell}(t)\|_H \rightarrow 0 \text{ as } \ell \rightarrow +\infty.$$

■

As we can see the above Lemma is telling us that the equation

$$(P_L) \begin{cases} \dot{u}(t) + |u(t)|^{p-2}u(t) = \tilde{B}(u(t)), & t > 0 \\ u(0) = u_0 \in \mathbb{R}, \end{cases}$$

with $\tilde{B} \doteq B|_{\mathbb{R}}$ is a good candidate for the limit problem. For this reason we give below two results where we guarantee that (P_L) is a well posed problem which possesses a global attractor.

Lemma 3.2 *The problem (P_L) has an unique global solution.*

Proof: We consider $\phi_p : \mathbb{R} \rightarrow \mathbb{R}$ and $\psi : \mathbb{R} \rightarrow \mathbb{R}$ given by $\phi_p(v) \doteq |v|^{p-2}v$ and $\psi(v) \doteq \int_0^v |s|^{p-2}s \, ds$, respectively.

We have that ψ is a convex, proper and continuous map, so $\partial\psi : \mathbb{R} \rightarrow \mathbb{R}$ is a maximal monotone operator in \mathbb{R} (see Example 2.3.4. in [1]). From Fundamental Theorem of Calculus we have $\psi'(v) = |v|^{p-2}v = \phi_p(v)$. Using the definition of subdifferential and lateral derivatives we obtain $\partial\psi(v) = \psi'(v) = \phi_p(v)$. Therefore $\phi_p : \mathbb{R} \rightarrow \mathbb{R}$ is a maximal monotone operator with $\mathcal{D}(\phi_p) = \mathbb{R}$.

Since $\tilde{B} : \mathbb{R} \rightarrow \mathbb{R}$ is globally Lipschitz map, from Theorem 3.17 and Remark 3.14 in [1] we can conclude that the problem (P_L) has an unique global solution. ■

Theorem 3.1 *The problem (P_L) defines a semigroup of class \mathcal{K} which is B -dissipative, and so there exists a global B -attractor \mathcal{A}^∞ associated with it. Moreover, the attractor \mathcal{A}^∞ is equal to the union of all the bounded complete trajectories in \mathbb{R} .*

Proof: We define $S(t) : \mathbb{R} \rightarrow \mathbb{R}$ by $S(t)v_0 = v(t)$, with v being the unique global solution of the problem (P_L) and $v(0) = v_0$. It is easy to see that $S(t)$ verifies the semigroup properties.

We will show that $S(t)$ is of class \mathcal{K} and also it is B-dissipative. In fact, multiplying the equation $v_t + \phi_p(v) = \tilde{B}(v)$ by v and using the Young's Inequality we obtain

$$\frac{1}{2} \frac{d}{dt} |v(t)|^2 \leq -\frac{1}{2} |v(t)|^p + c,$$

where $c > 0$ is a constant. Therefore, the map $y(t) \doteq |v(t)|^2$ satisfies the inequality

$$\frac{d}{dt} y(t) \leq -(y(t))^{p/2} + 2c.$$

So, by Lemma 5.1 in [15],

$$|v(t)|^2 \leq \left(\frac{2c}{1}\right)^{2/p} + \left(1\left(\frac{p}{2} - 1\right)t\right)^{-\frac{2}{p-2}}, \quad \forall t > 0.$$

Fix $\tau_0 \doteq 1$ and choose $s_0 \doteq \left[\left(\frac{2c}{1}\right)^{2/p} + \left(1\left(\frac{p}{2} - 1\right)\tau_0\right)^{-\frac{2}{p-2}}\right]^{1/2}$. Then,

$$|v(t)| \leq s_0, \quad \text{whenever } t \geq \tau_0 = 1. \quad (3.12)$$

The inequality (3.12) guarantees that the semigroup is B-dissipative.

Now, once we have that

$$\frac{1}{2} \frac{d}{dt} |v(t)|^2 + \phi_p(v(t))v(t) = \tilde{B}(v(t))v(t)$$

and since $\phi_p(v(t))v(t) = |v(t)|^p \geq 0$, then

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} |v(t)|^2 &\leq \left| \tilde{B}(v(t)) - \tilde{B}(0) \right| |v(t) - 0| + |\tilde{B}(0)| |v(t)| \\ &\leq L |v(t)|^2 + \frac{1}{2} |\tilde{B}(0)|^2 + \frac{1}{2} |v(t)|^2, \quad \forall t > 0, \end{aligned}$$

where L is the Lipschitz's constant of \tilde{B} . Integrating the above inequality from 0 to τ , $\tau \in [0, \tau_0]$, we obtain

$$|v(\tau)|^2 \leq |v_0|^2 + |\tilde{B}(0)|^2 + \int_0^\tau (2L + 1) |v(t)|^2 dt, \quad \forall \tau \in [0, \tau_0].$$

So, by Gronwall-Bellman's Lemma it follows that

$$|v(\tau)|^2 \leq (|v_0|^2 + |\tilde{B}(0)|^2) e^{(2L+1)\tau}, \quad \forall \tau \in [0, \tau_0]. \quad (3.13)$$

Using (3.12) and (3.13) we conclude that for each $t > 0$, $S(t)$ maps bounded sets into bounded sets. As a result we conclude that for each $t > 0$ the operator $S(t) : \mathbb{R} \rightarrow \mathbb{R}$ is compact.

Thus, the Theorem 2.2 and the Proposition 2.2 in [13], guarantee that the semigroup $S(t)$ has a maximal compact invariant global B-attractor \mathcal{A}^∞ , given as union of all bounded complete trajectories in \mathbb{R} . \blacksquare

The next result guarantees that (P_L) is in fact the limit problem for (P_D) , as $D \rightarrow +\infty$.

Theorem 3.2 *For each $D \geq 1$, let u^D be the solution of (P_D) with $u^D(0) = u_0^D$ and let u be the solution of (P_L) with $u(0) = u_0$. If $u_0^D \rightarrow u_0$ in H as $D \rightarrow +\infty$, then for each $T > 0$, $u^D \rightarrow u$ in $C([0, T]; H)$ as $D \rightarrow +\infty$.*

Proof: Let $T > 0$ fixed and suppose that $u_0^D \rightarrow u_0$ in H as $D \rightarrow +\infty$.

Subtracting the two equations in (P_D) and (P_L) and making the inner product with $u^D - u$ we obtain

$$\langle u_t^D - u_t, u^D - u \rangle_H + \langle A^D u^D - \phi_p(u), u^D - u \rangle_H = \langle B(u^D) - B(u), u^D - u \rangle_H.$$

Using the Tartar's Inequality [16] we obtain that there are constants $C_1, C_2 \geq 0$, both independent of D , such that

$$\langle A^D u^D - \phi_p(u), u^D - u \rangle \geq DC_1 \|\nabla u^D - \nabla u\|_p^p + C_2 \|u^D - u\|_p^p \geq 0.$$

Thus,

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|u^D - u\|_H^2 &= \langle u_t^D - u_t, u^D - u \rangle \\ &\leq \langle B(u^D) - B(u), u^D - u \rangle \\ &\leq \|B(u^D) - B(u)\|_H \|u^D - u\|_H \\ &\leq L \|u^D - u\|_H^2, \text{ a.e. in } (0, T). \end{aligned}$$

Integrating from 0 to t , $t \leq T$, we obtain

$$\|u^D(t) - u(t)\|_H^2 \leq \|u_0^D - u_0\|_H^2 + \int_0^t 2L \|u^D(s) - u(s)\|_H^2 ds$$

So, by Gronwall-Bellman's Lemma we obtain

$$\|u^D(t) - u(t)\|_H^2 \leq \|u_0^D - u_0\|_H^2 e^{2LT}, \quad \forall t \in [0, T].$$

Therefore $u^D \rightarrow u$ in $C([0, T]; H)$ as $D \rightarrow +\infty$, whenever $u_0^D \rightarrow u_0$ in H as $D \rightarrow +\infty$. \blacksquare

4 Continuity of Attractors

In this section we prove that not only $u^D \rightarrow u$ as $D \rightarrow \infty$ but we have also that the family of attractors $\{\mathcal{A}_D\}$ behaves continuously as the diffusion parameter increases to infinity. We start by introducing the following lemma which guarantees that the relevant elements to describe the asymptotic behaviour of these problems are around its own spatial average if D is large enough.

Lemma 4.1 *If for each $D \geq 1$ $u_0^D \in \mathcal{A}_D$ and $u_0 \doteq \lim_{D \rightarrow +\infty} u_0^D$ in H , then u_0 is a constant function.*

Proof: Let $\{u_0^D\}$ be a sequence with $u_0^D \in \mathcal{A}_D$, for each $D \geq 1$, and let $u_0 \doteq \lim_{D \rightarrow +\infty} u_0^D$ in H . Since each $\mathcal{A}_D \subset W^{1,p}(\Omega) \subset W^{1,2}(\Omega)$, from Poincaré-Wirtinger's Inequality ([2]) we have

$$\|u_0^D - \overline{u_0^D}\|_H \leq C \|\nabla u_0^D\|_H, \quad (4.14)$$

where $C > 0$ is a constant and $\overline{u_0^D} \doteq \frac{1}{|\Omega|} \int_{\Omega} u_0^D(x) dx$.

Consider $t_1 > 0$ as in Lemma 2.2 and let $\bar{t} > t_1$. Therefore, once attractors are invariant sets under the flow, for each $D \geq 1$,

$$u_0^D = u^D(\bar{t}) \doteq S^D(\bar{t})(v_0), \quad \text{for some } v_0 \in \mathcal{A}_D.$$

From Lemma 3.1 we have that the sequence of real numbers $\{\|\nabla u^D(\bar{t})\|_H\}_{D \geq 1}$ has a subsequence $\{\|\nabla u^{D_\ell}(\bar{t})\|_H\}$ which converges to zero as $\ell \rightarrow +\infty$. Therefore

$$\|\nabla u_0^{D_\ell}\|_H = \|\nabla u^{D_\ell}(\bar{t})\|_H \rightarrow 0 \quad \text{as } \ell \rightarrow +\infty.$$

From (4.14)

$$\|u_0^{D_\ell} - \overline{u_0^{D_\ell}}\|_H \rightarrow 0 \quad \text{as } \ell \rightarrow +\infty.$$

Then, using the Triangular Inequality, we conclude that

$$\|u_0 - \overline{u_0^{D_\ell}}\|_H \rightarrow 0 \quad \text{as } \ell \rightarrow +\infty.$$

Since $u_0^{D_\ell} \rightarrow u_0$ in H implies $\overline{u_0^{D_\ell}} \rightarrow u_0$ in H , we consider the characteristic function $\chi_\Omega \in L^2(\Omega)$ to obtain that $\overline{u_0^{D_\ell}} \rightarrow \overline{u_0}$ (convergence of real numbers) and this implies that $\overline{u_0^{D_\ell}} \rightarrow \overline{u_0}$ in $H = L^2(\Omega)$. So we conclude that $u_0 = \overline{u_0}$ and then u_0 is a constant function. \blacksquare

Now we are ready to prove the next:

Theorem 4.1 *The family of global B-attractors $\{\mathcal{A}_D; D \geq 1\}$ of the problem (P_D) is upper semi-continuous at infinity.*

Proof: Let $\{u_0^D\}$ be an arbitrary sequence with $u_0^D \in \mathcal{A}_D$, for each $D \geq 1$, and $D \rightarrow +\infty$. Using the Lemma 1.1 in [6], to get upper semicontinuity of the family of global B-attractors it is sufficient to guarantee that $\{u_0^D\}$ has a convergent subsequence whose limit belongs to the B-attractor \mathcal{A}^∞ of the limit problem. By Corollary 2.2, $\mathcal{A} \doteq \overline{\bigcup_{D \geq 1} \mathcal{A}_D}$ is a compact subset in H . Therefore there

exists a subsequence $\{u_0^{D_j}\}$ of $\{u_0^D\}$ such that $u_0^{D_j} \rightarrow u_0$ in H as $j \rightarrow +\infty$. By Lemma 4.1, $u_0 \in \mathbb{R}$.

Our objective is to show that $u_0 \in \mathcal{A}^\infty$. Using the Proposition 2.2 in [13], it is sufficient to construct a bounded complete trajectory $\varphi : \mathbb{R} \rightarrow \mathbb{R}$ through u_0 .

Let $\{S^D(\tau)\}$ and $\{S(\tau)\}$ the semigroups associated with (P_D) and (P_L) respectively. For each $j \in \mathbb{N}$, consider $t_j > j$, $t_1 < t_2 < \dots < t_j < \dots$ and $x_j \in \mathcal{A}_{D_j}$ such that $S^{D_j}(t_j)(x_j) = u_0^{D_j}$.

From Theorem 3.2 we obtain

$$S^{D_j}(t_j + t)(x_j) \rightarrow S(t)u_0 \text{ in } H \text{ as } j \rightarrow +\infty, \forall t \geq 0.$$

Now consider the sequence $\{S^{D_j}(t_j - 1)(x_j)\} \subset \bigcup_{D \geq 1} \mathcal{A}_D$. We know from Corollary 2.2 that, passing to a subsequence if necessary,

$$S^{D_j}(t_j - 1)(x_j) \rightarrow z_1 \text{ in } H \text{ as } j \rightarrow +\infty.$$

From Lemma 4.1 we have that $z_1 \in \mathbb{R}$. Appealing once more to Theorem 3.2, we obtain that

$$S^{D_j}(t_j - 1 + t)(x_j) \rightarrow S(t)z_1 \text{ in } H \text{ as } j \rightarrow +\infty, \forall t \geq 0.$$

Notice that $S(1)z_1 = u_0$.

Proceeding inductively, we find for each $r = 0, 1, 2, \dots$, a real number z_r such that $S(1)z_{r+1} = z_r$. Given $t \in \mathbb{R}$, we define $\varphi(t)$ to be the common value of $S(t+r)z_r$ for $r > -t$. Then we have that φ is a bounded complete trajectory through u_0 . ■

Remark 4.1 Notice that each bounded complete trajectory of the limit problem (P_L) is in particular a bounded complete trajectory of the problem (P_D) . Therefore $\mathcal{A}^\infty \subset \mathcal{A}_D$, $\forall D \geq 1$. Consequently we obtain that the family of attractors $\{\mathcal{A}_D; D \geq 1\}$ of the problem (P_D) is lower semicontinuous at infinity i.e.,

$$\sup_{x \in \mathcal{A}^\infty} \text{dist}_H(x, \mathcal{A}_D) \rightarrow 0 \text{ as } D \rightarrow +\infty.$$

Thus, using the Theorem 4.1 and the Remark 4.1, we obtain that the family of attractors $\{\mathcal{A}_D; D \geq 1\}$ of the problem (P_D) is continuous at infinity.

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